Counting integer points by Teichmüller discs in $\Omega\mathcal{M}(2)$

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COUNTING INTEGER POINTS BY TEICHMÜLLER DISCS IN $\Omega\mathcal{M}(2)$.

SAMUEL LELIÈVRE AND EMMANUEL ROYER

A . We prove formulae for the countings by orbit of square-tiled surfaces of genus two with one singularity. These formulae were conjectured by Hubert & Lelièvre. We show that these countings admit quasimodular forms as generating functions.

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1. I

The main result of this paper is the proof of a conjecture of Hubert & Lelièvre.

Theorem 1. *For odd n, the countings by orbit of primitive square-tiled surfaces of the stratum $\Omega\mathcal{M}(2)$ tiled with n squares are the following. Orbit A contains*

$$a_n^{\mathrm{p}} = \frac{3}{16}(n-1)n^2 \prod_{p|n}(1-\frac{1}{p^2})$$

primitive surfaces with n squares and orbit B contains

$$b_n^{\mathrm{p}} = \frac{3}{16}(n-3)n^2 \prod_{p|n}(1-\frac{1}{p^2})$$

primitive surfaces with n squares.

A square-tiled (translation) surface is a translation surface that can be tiled by unit squares. It is therefore a collection of unit squares with pairwise identifications of parallel sides [Mas05]. Because of the identifications, one can associate to each vertex an angle which is a multiple of 2π. If there is only one vertex with angle equal to 6π, all the other vertices having angle 2π, then the surface is said to belong to the stratum $\Omega\mathcal{M}(2)$. Figure 1 is an example of square-tiled surface, with 30 unit squares, belonging to the stratum $\Omega\mathcal{M}(2)$. The only one vertex with angle different from 2π is the dotted one. A square-tiled surface is said to be primitive if it cannot be tiled with a pattern made of several unit squares, whose induced identifications

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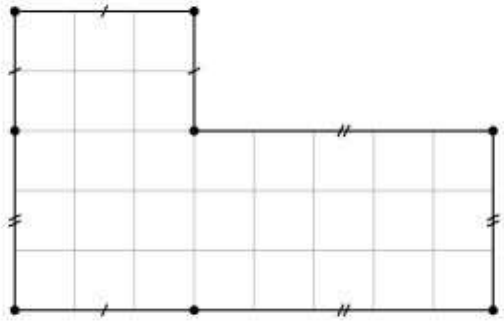
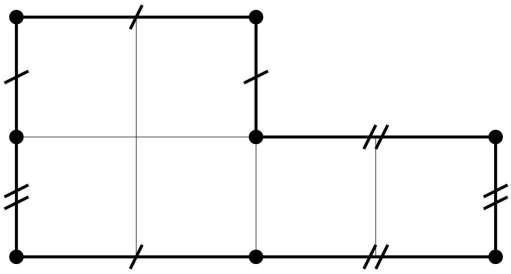
would produce a torus (*i.e.* a parallelogram with parallel sides indentified). Figure 1 exhibits a primitive surface while figure 2 exhibits a non primitive one. In this last example, a pattern responsible for the non primitivity is a rectangle made of two unit squares.



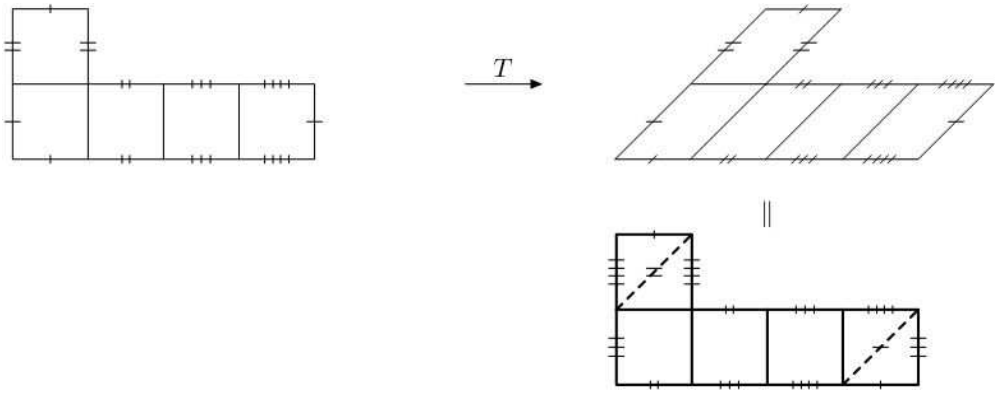
F 1. A primitive square-tiled surface in $\Omega\mathcal{M}(2)$



F 2. A non primitive square-tiled surface in $\Omega\mathcal{M}(2)$

It is immediate that a square-tiled surface of genus at least 2 (in particular, a square-tiled surface in $\Omega\mathcal{M}(2)$) with a prime number of unit squares is primitive. By cutting and gluing, using identifications, a square-tiled surface may be viewed as made of squares and triangles (see figure 3). This is useful when considering the action of $\mathrm{SL}(2,\mathbb{Z})$ we describe now. For any integer n, the linear action of $\mathrm{SL}(2,\mathbb{Z})$ induces an action of primitive square-tiled surfaces with n squares. For an example of transform by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, see figure 3.

An equivalent definition is that a square-tiled surface is a pair (X, ω) where X is a Riemann surface which is a ramified cover of the standard torus $\mathbf{T}^2 = \mathbb{C}/\mathbb{Z}^2$, ramified only over the origin, and ω is the pull-back by the covering map of the



F 3. Action of T

1-form dz on $\mathbf{T}^2$ induced by the 1-form dz on $\mathbb{C}$. They are in the stratum $\Omega\mathcal{M}(2)$ when X of genus 2 and ω has a double zero.

Square-tiled surfaces are the integer points of moduli spaces of translation surfaces. On these moduli spaces, there is a natural action of $\mathrm{SL}(2,\mathbb{R})$ whose orbits are called Teichmüller discs. The $\mathrm{SL}(2,\mathbb{R})$-action induces the $\mathrm{SL}(2,\mathbb{Z})$-action on square-tiled surfaces mentioned above.

In this paper, we consider this action on surfaces in the stratum $\Omega\mathcal{M}(2)$. Hubert & Lelièvre [HL05b] and McMullen [McM05] have shown that, for any odd $n \geq 5$, there are two types of surfaces, called A and B. We briefly explain (see [HL05b] for the details) how to determine the type of the surfaces. The matrix $-I$ induces an application of any surface $(X,\omega) \in \Omega\mathcal{M}(2)$ to itself which admits 6 fix points, called Weierstrass points. These points have half-integer coordinates in the squares. If the number of squares is odd, then the number of Weierstrass points with integer coordinates is either 1 (and (X,ω) is then said to be of type A) or 3 (and (X,ω) is then said to be of type B). If n is odd ≥ 5, the set of primitive square-tiled surfaces with n squares splits into 2 orbits under $\mathrm{SL}(2,\mathbb{Z})$: one of the orbits consists in the surfaces of type A, the other one consists in the surfaces of type B. Our aim is to compute the respective cardinalities a_n^{p} and b_n^{p} of these orbits. We shall also count the total number of square-tiled surfaces of each type (primitive or not), a_n and b_n.

Eskin, Masur and Schmoll [EMS03] give two formulae for the number of primitive square-tiled surfaces tiled by a given number of squares in $\Omega\mathcal{M}(2)$, one under the form of sums over several parameters and the other one in closed form. Hubert and Lelièvre [HL05b] suggest as a conjecture a closed formula for the countings by orbit when there are two orbits (called orbit A and orbit B). Our theorem 1 asserts the validity of this conjecture.

We concentrate on a_n^{p} since b_n^{p} can be deduced by the formula (consequence of the one of Eskin, Masur & Schmoll):

$$a_n^{\mathrm{p}} + b_n^{\mathrm{p}} = \frac{3}{8} n^2 (n-2) \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

The numbers a_n and a_n^{p} are related by the formula

$$a_n = \sum_{d|n} \sigma_1\left(\frac{n}{d}\right) a_d^{\mathrm{p}}$$

where $\sigma_1(k)$ is the sum of divisors of k.

Remark. One has

$$\prod_{p|n}\left(1-\frac{1}{p^2}\right)=\sum_{d|n}\frac{\mu(d)}{d^2}$$

with μ the Möbius function. Here and in the following, a product over p means a product over prime numbers p and a sum over divisors means a sum over positive divisors.

Theorem 1 can also be expressed in terms of quasimodularity of the generating functions of the countings. More precisely:

Corollary 2. *Assume $n \geq 5$ is an odd integer. The number of non necessarily primitive square-tiled surfaces in $\Omega\mathcal{M}(2)$ of type A tiled with n unit squares is the nth coefficient of the quasimodular form*

$$\sum_{n=1}^{+\infty} a_n \exp(2i\pi nz)=\frac{1}{1280}\left[E_4(z)+10\frac{\mathrm{d}}{2i\pi\,\mathrm{d}z}E_2(z)\right]$$

of weight 4, *depth* 2 *on* $\mathrm{SL}(2,\mathbb{Z})$.

Remark. Functions E_2 and E_4 are the usual Eisenstein series of weight 2 and 4 respectively. They are precisely defined in equations (7) and (11).

Since the coefficients a_n have no geometric meaning for even n, it makes sense to consider only the odd part of the Fourier series. Considering the odd part is the same as considering the Fourier series twisted by a Dirichlet character of modulus 2 (see section 3.4). It is then natural to expect that, similarly to the case of modular forms (see [Iwa97, Theorem 7.4]), the odd part of the Fourier series is a quasi-modular form on the congruence subgroup $\Gamma_0(4)$. Actually, we will prove this is the case. Let Φ_2 and Φ_4 be the two modular forms of respective weights 2 and 4, defined on $\Gamma_0(4)$ as in (12) and (13).

Theorem 3. *The Fourier series*

$$\sum_{n\in 2\mathbb{Z}_{\geq 0}+1} a_n \exp(2i\pi nz)$$

is the quasimodular form of weight 4 *and depth* 1 *on* $\Gamma_0(4)$ *defined by*

$$\frac{1}{1280}\left[E_4(z)-9E_4(2z)+8E_4(4z)-15\frac{\mathrm{d}}{2i\pi\,\mathrm{d}z}\Phi_2(z)+15\frac{\mathrm{d}}{2i\pi\,\mathrm{d}z}\Phi_4(z)\right].$$

Remark. This theorem will be proved in section 5. It is interesting to note that forgetting the artificial terms of even order leads to a decreasing of the depth, that is to a simplification of the modular situation (since a modular form is a quasimodular form of depth 0, the depth may be seen as a measure of complexity).

In particular, one obtains

n	5	7	9	11	13	15	17	19	21	23	25	27
a_n^{p}	18	54	108	225	378	504	864	1215	1440	2178	2700	3159
a_n	18	54	120	225	378	594	864	1215	1680	2178	2808	3630

The main lines of the proof of theorem 1 are the following. In section 2, we evaluate the number a_n^{p} in terms of sums over sets defined by complicated arithmetic conditions. In section 4, we relate these coefficients a_n^{p} to sums of sums of divisors of the form

$$\sum_{\substack{(a,b)\in\mathbb{Z}^2_{>0}\\ ka+b=n}} \sigma_1(a)\sigma_1(b).$$

For the computation of these sums, we use, in section 3, the notion of quasimodular forms on congruence subgroups (introduced by Kaneko & Zagier in [KZ95]) and we take advantage of the fact that the spaces of quasimodular forms have finite dimension to linearize the above sums. Here, linearizing means expressing them as linear combinations of sums of powers of divisors. Having obtained a series whose odd coefficients are the numbers a_n, we introduce the notion of twist of a quasimodular form by a Dirichlet character, to construct a new quasimodular form generating series without artificial Fourier coefficients.

Remark. We learned that Matt Bainbridge was also able to prove the conjecture, using different methods.

2. S -

2.1. **Primitive square-tiled surfaces.** The lattice of periods $\mathrm{Per}(\omega)$ of a square-tiled surface (X,ω) in $\Omega\mathcal{M}(2)$ is the module over $\mathbb{Z}$ generated by the geodesic segments (for the flat metric) on S with both endpoints at the 6π point.

The condition for a square-tiled surface (X,ω) to be primitive is the assumption that its lattice of periods $\mathrm{Per}(\omega)$ is $\mathbb{Z}^2$ (the surface is tiled by a fundamental parallelogram of $\mathrm{Per}(\omega)$). In other words, the surface is primitive if the covering map $(X,\omega)\longrightarrow(\mathbf{T}^2,\mathrm{d}z)$ does not factor through a square-tiled torus bigger that $\mathbf{T}^2$.

Consider a square-tiled surface (X,ω), then its lattice of periods $\mathrm{Per}(\omega)$ is a sublattice of $\mathbb{Z}^2$ and the covering map $\pi\colon (X,\omega)\longrightarrow(\mathbf{T}^2,\mathrm{d}z)$, whose degree n is the number of square tiles of S, can be factored as $(X,\omega)\longrightarrow(\mathbb{R}^2/\mathrm{Per}(\omega)\,,\mathrm{d}z)\longrightarrow(\mathbf{T}^2,\mathrm{d}z)$, where π_2 is the isogeny whose degree is the index of $\mathrm{Per}(\omega)$ in $\mathbb{Z}^2$.

Now, calling P a fundamental domain for the lattice $\mathrm{Per}(\omega)$, the surface (X,ω) is a "primitive P-tiled surface". If $\mathrm{Per}(\omega)$ has index d in $\mathbb{Z}^2$, then d is a divisor of n, P has area d, and S is tiled by n/d copies of P.

The number of d-square-tiled tori is the number of index d sublattices of $\mathbb{Z}^2$. Such a lattice can always be given a basis $((r,0),(t,h))$ where r, h, t are integers satisfying $rh=d$ and $0\leqslant t<r$. Such a basis is of course unique. Each choice of r (a divisor of d) determines h and leaves r choices for t, whence the number of d-square-tiled tori is $\sigma_1(d)=\sum_{r|d} r$. Note that if d is odd, then so are r and h.

This leads to the following relation between countings h_n of square-tiled surfaces in $\Omega\mathcal{M}(2)$ and countings h_n^{p} of primitive surfaces in $\Omega\mathcal{M}(2)$:

$$h_n=\sum_{d|n}\sigma_1(d)h^{\mathrm{p}}_{n/d}. \tag{1}$$

We find it convenient to introduce an intermediate class of square-tiled surfaces, which we call "primitive in height". Square-tiled surfaces that are primitive in height are those for which the basis $((r,0),(t,h))$ of $\mathrm{Per}(\omega)$ has $h=1$. (Primitive surfaces are those for which $r=h=1$.)

The relation between general countings and height-primitive countings is now:

$$h_n = \sum_{d|n} h^{\mathrm{ph}}_{n/d} \tag{2}$$

(given a height-primitive surface with n/d squares, associate to it the surface built by replacing each square by a rectangle of dimensions $1 \times d$).

Equation (2) can be inverted as $h^{\mathrm{ph}}_n = \sum_{d|n} \mu(d) h_{n/d}$.

From (1), (2) and the relation $\mu * \sigma_1 = \mathrm{id}$, we have the following relation between height-primitive and primitive countings:

$$h^{\mathrm{ph}}_n = \sum_{d|n} d h^{\mathrm{p}}_{n/d}$$

which can be inverted as $h^{\mathrm{p}}_n = \sum_{d|n} d\mu(d) h^{\mathrm{ph}}_{n/d}$.

2.2. **Orbitwise countings of square-tiled surfaces in $\Omega\mathcal{M}(2)$.** In [HL05a], for odd n, A_n and B_n were used to denote the $\mathrm{SL}(2,\mathbb{Z})$-orbits of primitive n-square-tiled surfaces in $\Omega\mathcal{M}(2)$, and a_n and b_n were used to denote their cardinalities.[1]

We change the notation a little bit with respect to the notation of [HL05a]. Here, for odd n, we denote by a_n (resp. b_n) the total number of n-square-tiled surfaces in $\Omega\mathcal{M}(2)$ (primitive or not) with 1 (resp. 3) integer Weierstrass points, and by a^{p}_n and b^{p}_n the number of primitive such surfaces. So the a_n and b_n of [HL05a] are now a^{p}_n and b^{p}_n. We also use a^{ph}_n and b^{ph}_n for the corresponding countings of height-primitive surfaces.

Assume n is odd. The invariant "number of integer Weierstrass points" is well-behaved. Write $\mathrm{Per}(\omega) = a\mathbb{Z} \oplus w\mathbb{Z}$ where $a = (r, 0)$ and $w = (t, h)$. Then a Weierstrass point modulo $\mathrm{Per}(\omega)$ is one of the four following points:

$$0, \qquad \frac{1}{2}a + w, \qquad a + \frac{1}{2}w, \qquad \frac{1}{2}a + \frac{1}{2}w.$$

and either one or three of the Weierstrass points are 0. Since r and h are odd, only 0 is 0 modulo $\mathbb{Z}^2$.

Therefore the formulae relating h_n, h^{ph}_n and h^{p}_n also hold when we replace h by either a or b.

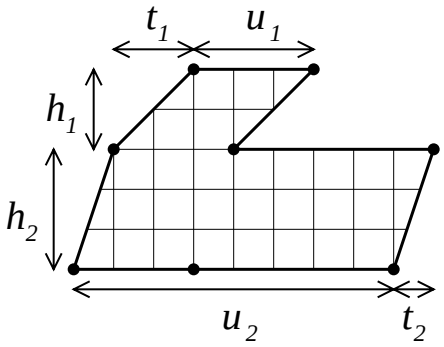
Lemma 4. *For any odd $n \geq 5$,*

$$a_n = \sum_{d|n} \sigma_1(d) a^{\mathrm{p}}_{n/d}, \qquad a_n = \sum_{d|n} a^{\mathrm{ph}}_{n/d}, \qquad a^{\mathrm{ph}}_n = \sum_{d|n} d a^{\mathrm{p}}_{n/d}.$$

Square-tiled surfaces in $\Omega\mathcal{M}(2)$ can have either one or two horizontal cylinders, and there is a corresponding parametrization by height, width and twist parameters of the cylinders (see [Zor02] and figure 4).

As analyzed in [HL05b, § 5.1], for odd $n \geq 5$, the parities of the parameters determine whether a given surface is of type A or B, as follows. (By the remark about parallelograms of odd area, this remains valid not only for primitive surfaces, but also for all surfaces, or for height-primitive surfaces.) One-cylinder surfaces with horizontal sides of lengths a, b, c are of type A if a, b, c are all odd, of type B otherwise; for each (a, b, c) up to cyclic permutation, there are n possible

[1] In terms of the notations in [McM05], A_n is the set of square-tiled surfaces in $\Omega W^0_{n^2}$ and B_n the set of those in $\Omega W^1_{n^2}$.



F 4. Parametrization of a surface

twists. For two-cylinder surfaces with parameters h_1, h_2, u_1, u_2, t_1, t_2, such that $h_1u_1 + h_2u_2 = n$ (odd):

- all such surfaces with h_1 and h_2 odd are of type A;
- all such surfaces with u_1 and u_2 odd are of type B;
- exactly half of the remaining surfaces (with different parity for u_1 and u_2 and for h_1 and h_2) are of type A, and half are of type B;

for each (h_1, h_2, u_1, u_2), there are u_1u_2 possible twists (if we had required primitivity instead of height-primitivity, some of them would not be admissible). In the case of different parities for h_1 and h_2 and for u_1 and u_2, the product u_1u_2 is even and exactly half of the possible twists corresponds to each type. The height-primitivity condition is just that the heights of the cylinders have greatest common divisor equal to one, so we obtain the following sum-type formulae for a_n^{ph}:

$$a_n^{\mathrm{ph}} = \frac{1}{3} \sum_{\substack{(a,b,c)\in(2\mathbb{Z}_{\geq 0}+1)^3 \\ a+b+c=n}} n + \sum_{\substack{h_1,h_2,u_1,u_2\in\mathbb{Z}_{\geq 0} \\ h_1u_1+h_2u_2=n \\ (h_1,h_2)=1 \\ h_1,h_2 \text{ odd} \\ u_1<u_2}} u_1u_2 + \frac{1}{2} \sum_{\substack{h_1,h_2,u_1,u_2\in\mathbb{Z}_{\geq 0} \\ h_1u_1+h_2u_2=n \\ (h_1,h_2)=1 \\ h_1\not\equiv h_2 \pmod 2 \\ u_1<u_2 \\ u_1u_2 \text{ even}}} u_1u_2. \tag{3}$$

From there we obtain $a_n = \sum_{d|n} d a_{n/d}^{\mathrm{ph}}$ and

$$a_n^{\mathrm{p}} = \sum_{d|n} d\mu(d) a_{n/d}^{\mathrm{ph}}. \tag{4}$$

3. Q

3.1. **Motivation.** Let ℓ be a positive integer. The function σ_ℓ is defined by

$$\sigma_\ell(n) = \begin{cases} \sum_{d|n} d^\ell & \text{if } n \in \mathbb{Z}_{>0} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The aim of this part is the computation of sums of type

$$S_k(n) = \sum_{\substack{(a,b)\in\mathbb{Z}_{>0}^2 \\ ka+b=n}} \sigma_1(a)\sigma_1(b) \tag{6}$$

with $k \in \mathbb{Z}_{>0}$. We shall study only the cases $k \in \{1, 2, 4\}$ but the method should apply to every other value of k.

Useful to the study of these sums is the weight 2 Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) e(nz) \tag{7}$$

where

$$e(\tau) = \exp(2i\pi\tau) \qquad (\Im m\, \tau > 0).$$

Defining

$$H_k(z) = E_2(z) E_2(kz),$$

one gets

$$H_k(z) = 1 - 24 \sum_{n=1}^{+\infty} \left[\sigma_1(n) + \sigma_1\left(\frac{n}{k}\right) \right] e(nz) + 576 \sum_{n=1}^{+\infty} S_k(n) e(nz). \tag{8}$$

We shall achieve the linearization of H_k using the theory of quasimodular forms, developed by Kaneko & Zagier. The computation of $S_k(n)$ will be deduced for each n.

3.2. **Definition.** Let us therefore begin by surveying our needs on quasimodular forms, referring to [MR05, §17] for the details. Define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4,\ ad - bc = 1,\ N \mid c \right\}$$

for all integers $N \geq 1$. In particular, $\Gamma_0(1)$ is $\mathrm{SL}(2,\mathbb{Z})$. Denote by $\mathcal{H}$ the Poincaré upper half plane:

$$\mathcal{H} = \{z \in \mathbb{C} :\ \Im m\, z > 0\}.$$

Definition 5. *Let $N \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$. A holomorphic function*

$$f : \mathcal{H} \to \mathbb{C}$$

is a quasimodular form of weight k, depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz+d}\right)^i \tag{9}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

Here is what is meant by the requirement for f_s to be holomorphic at the cusps. One can show [MR05, Lemme 119] that if f satisfies the quasimodularity condition (9), then f_s satisfies the modularity condition

$$(cz+d)^{-(k-2s)} f_s\left(\frac{az+b}{cz+d}\right) = f_s(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Asking f_s to be holomorphic at the cusps is asking that, for all $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(1)$, the function

$$z \mapsto (\gamma z + \delta)^{-(k-2s)} f_s\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

has a Fourier expansion of the form

$$\sum_{n=0}^{+\infty} \widehat{f}_{s,M}(n) e\left(\frac{nz}{u_M}\right)$$

where

$$u_M = \inf\{u \in \mathbb{Z}_{>0} \colon T^u \in M^{-1}\Gamma_0(N)M\}.$$

In other words, f_s is automatically a modular function and is required to be more than that, a modular form of weight $k - 2s$ on $\Gamma_0(N)$. It follows that if f is a quasimodular form of weight k and depth s, non identically vanishing, then k is even and $s \leq k/2$.

Remark. Let χ be a Dirichlet character (see § 3.4). If f satisfies all of what is needed to be a quasimodular form except (9) being replaced by

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \chi(d) \sum_{i=0}^{n} f_i(z) \left(\frac{c}{cz + d}\right)^i,$$

then, one says that f is a quasimodular form of weight k, depth s and character χ on $\Gamma_0(N)$.

The Eisenstein series E_2 transforms as

$$(cz + d)^{-2} E_2\left(\frac{az + b}{cz + d}\right) = E_2(z) + \frac{6}{i\pi} \frac{c}{cz + d}$$

under the action of any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(1)$. Hence, E_2 is a quasimodular form of weight 2 and depth 1 on $\Gamma_0(1)$. Defining

$$E_{N,2}(z) = E_2(Nz),$$

one has

$$(cz + d)^{-2} E_{N,2}\left(\frac{az + b}{cz + d}\right) = E_{N,2}(z) + \frac{6}{i\pi N} \frac{c}{cz + d}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Hence, $E_{N,2}$ is a quasimodular form of weight 2 and depth 1 on $\Gamma_0(N)$. One denotes by $\widetilde{\mathrm{M}}_k[\Gamma_0(N)]^{\leq s}$ the space of quasimodular forms of weight k and depth less than or equal to s on $\Gamma_0(N)$. The space $\widetilde{\mathrm{M}}_k[\Gamma_0(N)]^{\leq 0}$ is the space $\mathrm{M}_k[\Gamma_0(N)]$ of modular forms of weight k on $\Gamma_0(N)$. A recurrence on the depth implies [MR05, Corollaire 121] the equality

$$\widetilde{\mathrm{M}}_k[\Gamma_0(N)]^{\leq s} = \bigoplus_{i=0}^{s} \mathrm{M}_{k-2i}[\Gamma_0(N)] E_2^i. \tag{10}$$

It is known that $\mathrm{M}_2[\Gamma_0(1)] = \{0\}$. However, if $N > 1$, one deduces from

$$\mathbb{C}E_2 \oplus \mathbb{C}E_{N,2} \subset \widetilde{\mathrm{M}}_2[\Gamma_0(N)]^{\leq 1} = \mathrm{M}_2[\Gamma_0(N)] \oplus \mathbb{C}E_2$$

that $\dim \mathrm{M}_2[\Gamma_0(N)] \geq 1$. By the way, for every family $(c_d)_{d|N}$ such that

$$\sum_{d|N} \frac{c_d}{d} = 0$$

one has

$$\left[z \mapsto \sum_{d|N} c_d E_2(dz)\right] \in \mathrm{M}_2[\Gamma_0(N)].$$

Denote by D the differential operator

$$D = \frac{1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}z}.$$

It defines a linear application from $\widetilde{\mathrm{M}}_k[\Gamma_0(N)]^{\leq s}$ to $\widetilde{\mathrm{M}}_{k+2}[\Gamma_0(N)]^{\leq s+1}$. This application is injective and strictly increases the depth if $k > 0$. This property allows to linearize the basis given in (10).

Lemma 6. *Let $k \geq 2$ even. Then,*

$$\widetilde{\mathrm{M}}_k[\Gamma_0(N)]^{\leq k/2} = \bigoplus_{i=0}^{k/2-1} D^i \mathrm{M}_{k-2i}[\Gamma_0(N)] \oplus \mathbb{C}D^{k/2-1}E_2.$$

3.3. **Sums of sums of divisors.** Lemma 6 allows to reach our goal by expressing the sums S_1, S_2 and S_4 introduced in (6) as follows.

Proposition 7. *Let $n \geq 1$. Then,*

$$S_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n),$$

$$S_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right)$$

and

$$S_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right).$$

Proof. We detail the proof for the expression of S_4. The function H_4, introduced in (8), is a quasimodular form of weight 4 and depth 2 on $\Gamma_0(4)$. Lemma 6 gives

$$\widetilde{\mathrm{M}}_4[\Gamma_0(4)]^{\leq 2} = \mathrm{M}_4[\Gamma_0(4)] \oplus D\mathrm{M}_2[\Gamma_0(4)] \oplus \mathbb{C}DE_2.$$

The space $\mathrm{M}_4[\Gamma_0(4)]$ has dimension 3 and contains the linearly independent functions

$$E_4(z) = 1 + 240\sum_{n=1}^{+\infty}\sigma_3(n)e(nz) \tag{11}$$

$$E_{2,4}(z) = E_4(2z)$$

$$E_{4,4}(z) = E_4(4z).$$

The space $\mathrm{M}_2[\Gamma_0(4)]$ has dimension 2 and is generated by

$$\Phi_2(z) = 2E_2(2z) - E_2(z) \tag{12}$$

$$\Phi_4(z) = \frac{4}{3}E_2(4z) - \frac{1}{3}E_2(z). \tag{13}$$

Hence, by the computations of the first seven Fourier coefficients, one gets

$$H_4 = \frac{1}{20}E_4 + \frac{3}{20}E_{2,4} + \frac{4}{5}E_{4,4} + \frac{9}{2}D\Phi_4 + 3DE_2.$$

The computation of S_4 is then obtained by comparison of the Fourier coefficients of this equality. The computation of S_2 is obtained along the same lines via the equality

$$H_2 = \frac{1}{5}E_4 + \frac{4}{5}E_{2,4} + 3D\Phi_2 + 6DE_2$$

between forms of $\widetilde{\mathrm{M}}_4[\Gamma_0(2)]^{\leq 2}$. At last, the expression of S_1 is deduced from the equality

$$E_2^2 = E_4 + 12DE_2$$

between forms of $\widetilde{\mathrm{M}}_4[\Gamma_0(1)]^{\leq 2}$. □

Remark. The computation of H_4, which lies in the dimension 6 vector space with basis $\{E_4, E_{2,4}, E_{4,4}, D\Phi 2, D\Phi_4, DE_2\}$, required working on *seven* consecutive Fourier coefficients. We briefly explain why, mentioning that any sequence of 6 consecutive coefficients is not sufficient. For any function

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n)e(nz)$$

and any integer $i \geq 0$, define

$$c(f,i) = \left(\widehat{f}(i), \widehat{f}(i+1), \widehat{f}(i+2), \widehat{f}(i+3), \widehat{f}(i+4), \widehat{f}(i+5)\right),$$

and let

$$\begin{aligned} v_1(i) &= c(E_4, i) & v_2(i) &= c(E_{2,4}, i) & v_3(i) &= c(E_{4,4}, i) \\ v_4(i) &= c(D\Phi_2, i) & v_5(i) &= c(D\Phi_4, i) & v_6(i) &= c(DE_2, i). \end{aligned}$$

Then, for each i, there exists an explicitly computable linear relation between $v_2(i)$, $v_3(i)$, $v_4(i)$, $v_5(i)$ and $v_6(i)$. One could think of using a basis of $\widetilde{\mathrm{M}}_4[\Gamma_0(\Gamma_0(4))]^{\leq 2}$ echelonized by increasing powers of $e(z)$. However the same phenomenon would appear when changing to such a basis and expressing the new basis elements in terms of the original basis.

3.4. **Twist by a Dirichlet character.** Recall that a Dirichlet character χ is a character of a multiplicative group $\left(\mathbb{Z}/q\mathbb{Z}\right)^\times$ extended to a function on $\mathbb{Z}$ by defining

$$\chi(n) = \begin{cases} \chi\,(n \pmod q) & \text{if } (n,q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(see *e.g.* [IK04, Chapter 3]).

A quasimodular form admits a Fourier expansion

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n)e(nz). \tag{14}$$

Since we shall need to compute the odd part of a quasimodular form, we introduces the notion of twist of a quasimodular form by a Dirichlet character.

Definition 8. *Let χ be a Dirichlet character. Let f be a function having Fourier expansion of the form* (14)*. The twist of f by χ is the function $f \otimes \chi$ defined by the Fourier expansion*

$$f \otimes \chi(z) = \sum_{n=0}^{+\infty} \chi(n)\widehat{f}(n)e(nz).$$

The interest of this definition is that it allows to build new quasimodular forms, as stated in the next proposition.

Proposition 9. *Let χ be a primitive Dirichlet character of conductor m. Let f be a quasimodular form of weight k and depth s on $\Gamma_0(N)$. Then $f \otimes \chi$ is a quasimodular form of weight k, depth less than or equal s and character χ^2 on $\Gamma_0\left(\mathrm{lcm}(N, m^2)\right)$.*

Proof. The proof is an adaptation of the corresponding result for modular forms (see *e.g.* [Iwa97, Theorem 7.4]). We consider for each k the following action of $\mathrm{SL}_2(\mathbb{R})$ on holomorphic functions on $\mathcal{H}$:

$$(f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Let $\tau(\chi)$ be the Gauss sum

$$\tau(\chi) = \sum_{u \pmod m} \chi(u) e\left(\frac{u}{m}\right).$$

Since χ is primitive, this sum is not zero. One has

$$\tau(\overline{\chi}) g \otimes \chi = \sum_{v \pmod m} \overline{\chi}(v) (g|_k \left(\begin{smallmatrix} 1 & v/m \\ 0 & 1 \end{smallmatrix}\right)) \tag{15}$$

as soon as g has a Fourier expansion of the form (14). Define $M = \mathrm{lcm}(N, m^2)$. Let $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(M)$. The matrix

$$\begin{pmatrix} 1 & v/m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & v\delta^2/m \\ 0 & 1 \end{pmatrix}^{-1}$$

being in $\Gamma_0(N)$, one deduces from the level N quasimodularity of f and (15) that

$$\tau(\overline{\chi})(f \otimes \chi|_k \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right))(z) = \sum_{i=0}^{s} \sum_{v \pmod m} \overline{\chi}(v) (f_i |_{k-2i} \left(\begin{smallmatrix} 1 & \delta^2 v/m \\ 0 & 1 \end{smallmatrix}\right))(z) \left[\frac{\gamma}{\gamma\left(z + \frac{\delta^2 v}{m}\right) + \delta - \frac{\gamma\delta^2 v}{m}}\right]^i.$$

Since the functions f_i are themselves quasimodular forms (see [MR05, Lemme 119]), they admit a Fourier expansion. Hence, from (15),

$$\tau(\overline{\chi})(f \otimes \chi|_k \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right))(z) = \tau(\overline{\chi}) \chi(\delta)^2 \sum_{i=0}^{s} f_i \otimes \chi(z) \left(\frac{\gamma}{\gamma z + \delta}\right)^i.$$

It follows that $f \otimes \chi$ satisfies the quasimodularity condition. There remains to prove the holomorphy at the cusps, which is quite delicate since $f_s \otimes \chi$ may be 0 even though f_s is not. Actually, lemma 6 and the fact that the twist of a modular form on $\Gamma_0(N)$ by a primitive Dirichlet character of conductor m is a modular form on $\Gamma_0(M)$ show that the proposition is proved as soon as it is proved for $f = D^{k/2-1} E_2$. In that case, $s = k/2$ and $f_s \otimes \chi$ is not 0 (see [MR05, Lemma 118], hence f_s being a modular form implies that $f_s \otimes \chi$ is also one.

□

4. P H & L `

The aim of this part is the proof of theorem 1. In all this part, n is assumed to be odd. Define

$$\alpha_1(n,r) = \sum_{(h_1,u_1,h_2,u_2) \in \mathcal{A}_1(n,r)} u_1 u_2$$

$$\alpha_2(n,r) = \frac{1}{2} \sum_{(h_1,u_1,h_2,u_2) \in \mathcal{A}_2(n,r)} u_1 u_2$$

$$\alpha_3(n,r) = \frac{n}{3} \sum_{(u_1,u_2,u_3) \in \mathcal{A}_3(n,r)} 1$$

with

$$\mathcal{A}_1(n,r) = \left\{(h_1,u_1,h_2,u_2)\in\mathbb{Z}^4_{>0}\colon \left|\begin{array}{c}(h_1,h_2)=1,\\ h_1 \text{ and } h_2 \text{ odd,}\end{array}\quad u_1<u_2,\ h_1u_1+h_2u_2=\frac{n}{r}\right.\right\}$$

$$\mathcal{A}_2(n,r) = \left\{(h_1,u_1,h_2,u_2)\in\mathbb{Z}^4_{>0}\colon \left|\begin{array}{c}(h_1,h_2)=1,\\ h_1 \text{ or } h_2 \text{ even,}\end{array}\right.\left|\begin{array}{c}u_1<u_2,\\ u_1 \text{ or } u_2 \text{ even,}\end{array}\quad h_1u_1+h_2u_2=\frac{n}{r}\right.\right\}$$

and

$$\mathcal{A}_3(n,r) = \left\{(u_1,u_2,u_3)\in(2\mathbb{Z}_{\geq 0}+1)^3\colon u_1+u_2+u_3=\frac{n}{r}\right\}.$$

By equations (4) and (3), our goal is the computation of

$$a_n^{\mathrm{p}} = \sum_{r|n}\mu(r)\left[r\alpha_1(n,r)+r\alpha_2(n,r)+\alpha_3(n,r)\right]. \tag{16}$$

4.1. **Two cylinders and odd heights.** Here, we compute the sum

$$\sum_{r|n} r\mu(r)\alpha_1(n,r).$$

Write

$$\alpha_1(n,r) = \gamma_1(n,r) - \widetilde{\alpha}_1(n,r) \tag{17}$$

with

$$\gamma_1(n,r) = \sum_{(h_1,u_1,h_2,u_2)\in C(n,r)} u_1u_2$$

and

$$\widetilde{\alpha}_1(n,r) = \sum_{(h_1,u_1,h_2,u_2)\in\widetilde{\mathcal{A}}_1(n,r)} u_1u_2$$

where

$$C(n,r) = \left\{(h_1,u_1,h_2,u_2)\in\mathbb{Z}^4_{>0}\colon (h_1,h_2)=1,\ u_1<u_2,\ h_1u_1+h_2u_2=\frac{n}{r}\right\}$$

and (recalling that n is odd)

$$\widetilde{\mathcal{A}}_1(n,r) = \left\{(h_1,u_1,h_2,u_2)\in\mathbb{Z}^4_{>0}\colon \left|\begin{array}{c}(h_1,h_2)=1,\\ h_1 \text{ or } h_2 \text{ even,}\end{array}\quad u_1<u_2,\ h_1u_1+h_2u_2=\frac{n}{r}\right.\right\}.$$

Note that the sum

$$\sum_{r|n} r\mu(r)\gamma_1(n,r)$$

is the total number of primitive surfaces with two cylinders.

4.1.1. *Surfaces with two cylinders.* Using Möbius inversion formula, one obtains

$$\gamma_1(n,r) = \sum_{d|n/r} \mu(d) \sum_{\substack{(i_1,u_1,i_2,u_2)\in\mathbb{Z}_{>0}^4 \\ u_1<u_2 \\ i_1u_1+i_2u_2=n/(rd)}} u_1u_2$$

$$= \gamma_{1,1}(n,r) - \gamma_{1,2}(n,r) \tag{18}$$

with

$$\gamma_{1,1}(n,r) = \frac{1}{2}\sum_{d|n/r} \mu(d) \sum_{\substack{(i_1,u_2,i_2,u_2)\in\mathbb{Z}_{>0}^4 \\ i_1u_1+i_2u_2=n/(rd)}} u_1u_2$$

and

$$\gamma_{1,2}(n,r) = \frac{1}{2}\sum_{d|n/r} \mu(d) \sum_{\substack{(i_1,i_2,u)\in\mathbb{Z}_{>0}^3 \\ (i_1+i_2)u=n/(rd)}} u^2.$$

One has

$$\gamma_{1,1}(n,r) = \frac{1}{2}\sum_{d|n/r} \mu(d) \sum_{\substack{(v_1,v_2)\in\mathbb{Z}_{>0}^2 \\ v_1+v_2=n/(rd)}} \sum_{w_1|v_1} w_1 \sum_{w_2|v_2} w_2$$

$$= \frac{1}{2}\sum_{d|n/r} \mu(d) \sum_{v_1=1}^{n/(rd)-1} \sigma_1(v_1)\sigma_1\left(\frac{n}{rd}-v_1\right).$$

By proposition 7, this can be linearized to

$$\gamma_{1,1}(n,r) = \frac{5}{24}\sum_{d|n/r}\mu(d)\sigma_3\left(\frac{n}{rd}\right) - \frac{n}{4r}\sum_{d|n/r}\frac{\mu(d)}{d}\sigma_1\left(\frac{n}{rd}\right) + \frac{1}{24}\sum_{d|n/r}\mu(d)\sigma_1\left(\frac{n}{rd}\right)$$

so as to obtain

$$\sum_{r|n} r\mu(r)\gamma_{1,1}(n,r) = \left(\frac{5}{24}n^3 - \frac{1}{4}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2} \tag{19}$$

thanks to the following lemmas 10 and 11.

Lemma 10. *Let $n \geq 1$.*

$$\sum_{r|n} r\mu(r)\sum_{d|n/r}\mu(d)\sigma_k\left(\frac{n}{rd}\right) = n^k\sum_{r|n}\frac{\mu(r)}{r^{k-1}}.$$

Proof. The left-hand side is

$$\sum_{der|n} r\mu(r)\mu(d)e^k = \sum_{er|n} r\mu(r)e^k \sum_{d|n/(er)}\mu(d) = \sum_{r|n} r\mu(r)\left(\frac{n}{r}\right)^k = n^k\sum_{r|n}\frac{\mu(r)}{r^{k-1}}.$$

□

Lemma 11.

$$\sum_{r|n}\mu(r)\sum_{d|n/r}\frac{\mu(d)}{d}\sigma_1\left(\frac{n}{rd}\right) = n\sum_{d|n}\frac{\mu(d)}{d^2}.$$

Proof. The left-hand side is

$$\sum_{red|n}\mu(r)\frac{\mu(d)}{d}e=\sum_{de|n}\mu(d)\frac{e}{d}\sum_{r|n/(de)}\mu(r)=\sum_{d|n}\frac{\mu(d)}{d}\frac{n}{d}=n\sum_{d|n}\frac{\mu(d)}{d^2}.$$

□

Next, one has

$$\gamma_{1,2}(n,r)=\frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{v|n/(rd)}v^2\left(\frac{n}{rdv}-1\right)$$

so that

$$\begin{aligned}\sum_{r|n}r\mu(r)\gamma_{1,2}(n,r)&=\frac{n}{2}\sum_{r|n}\mu(r)\sum_{d|n/r}\frac{\mu(d)}{d}\sigma_1\left(\frac{n}{rd}\right)\\&\qquad-\frac{1}{2}\sum_{r|n}r\mu(r)\sum_{d|n/r}\mu(d)\sigma_2\left(\frac{n}{rd}\right)\\&=\frac{n^2}{2}\sum_{r|n}\frac{\mu(r)}{r^2}-\frac{n}{2}\varphi(n)\end{aligned}\tag{20}$$

by lemmas 10 and 11.

Finally, reporting (20) and (19) in (18) leads to

$$\sum_{r|n}r\mu(r)\gamma_1(n,r)=\left(\frac{5}{24}n^3-\frac{3}{4}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2}+\frac{n}{2}\varphi(n).\tag{21}$$

4.1.2. *Even heights.* Let us now compute the contribution of $\widetilde{\alpha}_1(n,r)$. Write

$$\widetilde{\alpha}_1(n,r)=\widetilde{\alpha}_{1,1}(n,r)-\widetilde{\alpha}_{1,2}(n,r)\tag{22}$$

with, recalling again that n is odd,

$$\widetilde{\alpha}_{1,1}(n,r)=\frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{(i_1,u_1,i_2,u_2)\in\widetilde{\mathcal{A}}_{1,1}(n,r)}u_1u_2$$

and

$$\widetilde{\alpha}_{1,2}(n,r)=\frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{(i_1,i_2,u)\in\widetilde{\mathcal{A}}_{1,2}(n,r)}u^2$$

where

$$\widetilde{\mathcal{A}}_{1,1}=\left\{(i_1,u_1,i_2,u_2)\in\mathbb{Z}^4_{>0}\colon i_1\text{ or }i_2\text{ even},\ i_1u_1+i_2u_2=\frac{n}{dr}\right\}$$

and

$$\widetilde{\mathcal{A}}_{1,2}=\left\{(i_1,i_2,u)\in\mathbb{Z}^3_{>0}\colon i_1\text{ or }i_2\text{ even},\ (i_1+i_2)u=\frac{n}{dr}\right\}.$$

Since i_1 and i_2 are not simultaneously even, one has

$$\begin{aligned}\widetilde{\alpha}_{1,1}(n,r)&=\sum_{d|n/r}\mu(d)\sum_{\substack{(v_1,v_2)\in\mathbb{Z}^2_{>0}\\ v_1+v_2=n/(dr)}}\sum_{\substack{i_1|v_1\\ i_1\text{ even}}}\sum_{i_2|v_2}i_2\\&=\sum_{d|n/r}\mu(d)S_2\left(\frac{n}{dr}\right)\end{aligned}$$

the sum S_2 having been defined in (6). Using proposition 7 and lemmas 10 and 11, one obtains

$$\sum_{r|n} r\mu(r)\widetilde{\alpha}_{1,1}(n,r) = \left(\frac{1}{12}n^3 - \frac{1}{8}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2}. \tag{23}$$

Next,

$$\widetilde{\alpha}_{1,2}(n,r) = \frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{u|n/(dr)} u^2\left(\frac{n}{rdu} - 1\right).$$

Lemmas 10 and 11 give

$$\sum_{r|n} r\mu(r)\widetilde{\alpha}_{1,2}(n,r) = \frac{n^2}{2}\sum_{r|n}\frac{\mu(r)}{r^2} - \frac{n}{2}\varphi(n). \tag{24}$$

Reporting (24) and (23) in (22) leads to

$$\sum_{r|n} r\mu(r)\widetilde{\alpha}_{1}(n,r) = \left(\frac{1}{12}n^3 - \frac{5}{8}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2} + \frac{n}{2}\varphi(n). \tag{25}$$

Finally (25), (21) and (17) give

$$\sum_{r|n} r\mu(r)\alpha_{1}(n,r) = \left(\frac{1}{8}n^3 - \frac{1}{8}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2}. \tag{26}$$

4.2. **Two cylinders with even product of heights.** Compute at last the sum

$$\sum_{r|n} r\mu(r)\alpha_2(n,r).$$

Since n is odd, one has

$$\alpha_2(n,r) = \frac{1}{4}\sum_{d|n/r}\mu(d)\sum_{(i_1,u_1,i_2,u_2)\in\widehat{\mathcal{A}}_{1,2}(n,r,d)} u_1u_2$$

with

$$\widehat{\mathcal{A}}_{1,2}(n,r,d) = \left\{(i_1,u_1,i_2,u_2)\in\mathbb{Z}_{>0}^4 : \left|\begin{array}{c} i_1 \text{ and } u_1 \text{ even,} \\ \text{or} \\ i_2 \text{ and } u_2 \text{ even,}\end{array}\right. \quad i_1u_1 + i_2u_2 = \frac{n}{rd}\right\}.$$

Hence,

$$\begin{aligned}\alpha_2(n,r) &= \frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{\substack{(i_1,u_1,i_2,u_2)\in\mathbb{Z}_{>0}^4 \\ i_1 \text{ and } u_1 \text{ even} \\ i_1u_1+i_2u_2=n/(rd)}} u_1u_2 \\ &= \sum_{d|n/r}\mu(d)S_4\left(\frac{n}{rd}\right)\end{aligned}$$

the sum S_4 having been defined in (6). Finally, proposition 7 and lemmas 10 and 11 give

$$\sum_{r|n} r\mu(r)\alpha_{2}(n,r) = \left(\frac{1}{48}n^2 - \frac{1}{16}n^2\right)\sum_{r|n}\frac{\mu(r)}{r^2}. \tag{27}$$

4.3. **One cylinder.** The counting in that case is more direct. One actually has

$$\begin{aligned}\sum_{r|n}\mu(r)\alpha_3(n,r) &= \frac{n}{3}\sum_{r|n}\mu\left(\frac{n}{r}\right)\sum_{\substack{(v_1,v_2,v_3)\in\mathbb{Z}^3_{\geq 0}\\ v_1+v_2+v_3=(r-3)/2}} 1\\ &= \frac{n}{3}\sum_{r|n}\mu\left(\frac{n}{r}\right)\frac{r^2-1}{8}\\ &= \frac{1}{24}n^3\sum_{r|n}\frac{\mu(r)}{r^2}. \qquad (28)\end{aligned}$$

4.4. **Conclusion and computation of a generating series.** Reporting (28), (27) and (26) in (16), one gets the number of primitive surfaces of type A:

$$a_n^{\mathrm{p}} = \frac{3}{16}n^2(n-1)\sum_{r|n}\frac{\mu(r)}{r^2}.$$

This ends the proof of theorem 1.

The number of non necessarily primitive surfaces with an odd number n of squares of type A is given by

$$a_n = \sum_{d|n}\sigma_1\left(\frac{n}{d}\right)a_d^{\mathrm{p}}.$$

Even though this does not have any geometric sense, one can define numbers a_n^{p} and a_n by these formulae for even $n \geq 2$. We will compute the Fourier series attached to the resulting sequence $(a_n)_{n\in\mathbb{Z}_{>0}}$. Corollary 2 follows directly from the following proposition.

Proposition 12. *Let $n \geq 1$. Then*

$$a_n = \frac{3}{16}[\sigma_3(n) - n\sigma_1(n)].$$

Proof. One has

$$a_n = \frac{3}{16}(\alpha_n - \beta_n)$$

where

$$\alpha_n = \sum_{d|n}\sigma_1\left(\frac{n}{d}\right)d^3\sum_{r|d}\frac{\mu(r)}{r^2}$$

and

$$\beta_n = \sum_{d|n}\sigma_1\left(\frac{n}{d}\right)d^2\sum_{r|d}\frac{\mu(r)}{r^2}.$$

Opening σ_1 gives

$$\alpha_n = \sum_{\ell rq|n}\ell r q^3\mu(r) = \sum_{s|n}s\sum_{q|s}q^2\sum_{r|s/q}\mu(r) = \sigma_3(n)$$

and

$$\beta_n = \sum_{\ell rq|n}\ell q^2\mu(r) = n\sigma_1(n).$$

□

5. T F

Recall that the two weight 2 modular forms Φ_2 and Φ_4 on $\Gamma_0(2)$ and $\Gamma_0(4)$ respectively have been defined in (12) and (13). In this section, we prove theorem 3. Since we want to eliminate the coefficients of even order, it is natural to consider the Fourier series obtained by twisting all coefficients by a modulus 2 character. By proposition 9, one obtains a quasimodular form of weight 4, depth less than or equal to 2 on $\Gamma_0(4)$, hence a linear combination of E_4, $E_{2,4}$, $E_{4,4}$, $D\Phi_2$, $D\Phi_4$ and DE_2 (see the proof of proposition 7). The coefficients of this combination are found by computation of the first seven Fourier coefficients.

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