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To cite this version:

Kossivi Adjamagbo, Jean-Yves Charbonnel, Arno van den Essen. On ring homomorphisms of Azumaya algebras. 2005. hal-00008556

HAL Id: hal-00008556

https://hal.science/hal-00008556v1

Preprint submitted on 8 Sep 2005

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ccsd-00008556, version 1 - 8 Sep 2005

On ring homomorphisms of Azumaya algebras

Kossivi Adjamagbo, Jean-Yves Charbonnel and Arno van den Essen

September 8, 2005

Abstract: *The main theorem (Theorem 4.1) of this paper claims that any ring morphism from an Azumaya algebra of constant rank over a commutative ring to another one of the same constant rank and over a reduced commutative ring induces a ring morphism between the centers of these algebras. The second main theorem (Theorem 5.3) implies (through its Corollary 5.4) that a ring morphism between two Azumaya algebras of the same constant rank is an isomorphism if and only if it induces an isomorphism between their centers. As preliminary to these theorems we also prove a theorem (Theorem 2.8) describing the Brauer group of a commutative Artin ring and give an explicit proof of the converse of the Artin-Procesi theorem on Azumaya algebras (Theorem 2.6).*

1 Introduction

According to A. Grothendieck's point of view on algebraic geometry, as explained by J. Dieudonné in [DIE], the essence of algebraic geometry is the study of scheme morphisms and not of schemes themselves. From this point of view, which attaches more importance to the structure's morphisms than to the structures themselves, the most distinguished property of Azumaya algebras is that any algebra endomorphism of an Azumaya algebra is an automorphism. This is a well-known fact since 1960 thanks to the historical paper [AG] of M. Auslander and O. Goldman, which has clearly been the source

of inspiration of the famous Dixmier conjecture formulated in 1968 [DIX] for the first Weyl algebra over a field of characteristic zero. This conjecture in its present general formulation claims that any algebra endomorphism of a Weyl algebra of index n over such a field is an automorphism.

This historical link between Azumaya algebras and Weyl algebras has been confirmed since 1973 by a remarkable theorem of P. Revoy in [R] asserting that any Weyl algebra of index n over a field of positive characteristic is a free Azumaya algebra over its center and that the latter is isomorphic as algebra over this field to the algebra of polynomials in $2n$ indeterminates over this field. This link is not formal but substantial and fecund, for it suggests naturally a strategy and many tactics to investigate the Dixmier Conjecture with the help of tools from the theory of Azumaya algebras which have already been proved and which are to be proved.

Given an endomorphism f of a Weyl algebra $A_n(k)$ over a field k of characteristic zero, this strategy consists in reducing the problem first to an endomorphism f_R of a Weyl algebra $A_n(R)$ over a suitable finitely generated subring R of k, then to an endomorphism f_p of a Weyl algebra $A_n(R_p)$ over a suitable reduction R_p modulo a suitable prime integer p of the ring R in such a way that R_p is a field of characteristic p. From then, tools of the theory of Azumaya algebras applied to the Azumaya algebra $A_n(R_p)$ suggest many tactics to prove that f_p is an automorphism. It remains only to find a good tactic to conclude that f is an automorphism.

This strategy has been partially applied since 2002 by Y. Tsuchimoto in [T], but without the clarity and the power of the tools of Azumaya algebra theory.

The aim of the present paper is to prove some new tools of Azumaya algebra theory needed for the investigation of the Dixmier Conjecture by the evocated strategy, and more generally for the deepening, the completeness and the beauty of this theory which has already benefited from the contributions of an impressive army of researchers among which A. Grothendieck, M. Artin and O. Gabber, as explicited in [V]. As indicated in the title of the present paper, these new tools concern ring morphisms of Azumaya algebras, unlike the cited fundamental theorem on algebra endomorphisms of these algebras.

The main theorem (Theorem 4.1) of this paper claims that any ring morphism from an Azumaya algebra of constant rank over a commutative ring to another one of the same constant rank and over a reduced commutative ring induces a ring morphism between the centers of these algebras. The second

main theorem (Theorem 5.3) implies (through its Corollary 5.4) that a ring morphism between two Azumaya algebras of the same constant rank is an isomorphism if and only if it induces an isomorphism between their centers. As preliminary to these theorems we also prove a theorem (Theorem 2.8) describing the Brauer group of a commutative Artin ring which implies the fact, well-known to the specialist, that the Brauer group of a finite ring is trivial, hence that an Azumaya algebra over such a ring is isomorphic to an algebra of matrices over this ring (Corollary 2.9). As another preliminary to these theorems, we also give an explicit proof of the converse of the Artin-Procesi theorem on Azumaya algebras (Theorem 2.6). Using this result we sketch an alternative proof of the main theorem.

Before getting to the heart of the matter, we would like to express our deep gratitude to M. Kontsevich for a fruitful and hearty discussion which led us to guess and prove the main theorem below, inspired by his conviction of the truth of this statement in the case of matrix rings over fields. We also insist to thank our colleague Alberto Arabia from the University Paris 7 for a crucial idea in the proof of Lemma 3.2 below.

2 Preliminaries on Azumaya algebras.

First we recall some well-known facts about Azumaya algebras. Then we add some new ones. For details we refer to [AG], [DI] and [KO].

Definition 2.1 Let R be a commutative ring and A an R-algebra given by a ring homomorphism $\phi : R \to A$ (A need not be commutative). Then A is called a central R-algebra if A is a faithful R-module i.e. $Ann_R(A) = 0$ and the center of A, denoted $Z(A)$, equals $R.1_A = \phi(R)$, where 1_A denotes the unit element of A. Finally, A is called an Azumaya algebra over R if A is a finitely generated faithful R-module such that for each $\mathfrak{m} \in Max(R)$ $A/\mathfrak{m}A$ is a central simple $R/\mathfrak{m}$-algebra.

Example 2.2 (1) Since the ring of $n \times n$ matrices over a field is central simple, it follows easily that all matrix rings $M_n(R)$ are Azumaya algebras over R.

(2) Other examples of Azumaya algebras are given by the Weyl algebras $A_n(k)$ over a field k of characteristic $p > 0$. Indeed, it is proved in [R] that $A_n(k) = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, with the relations $[y_i, x_j] = \delta_{ij}$, is an

Azumaya algebra and a free module over its center $k[x_1{}^p, \ldots, x_n{}^p, y_1{}^p, \ldots, y_n{}^p]$ which is isomorphic to the k-algebra of polynomials in $2n$ indeterminates.

Remark 2.3 The main well-known properties of Azumaya algebras we need are the following (notations as above) :

(1) A is a central R-algebra and a finitely generated projective R-module which contains R as a direct summand.

(2) There is a 1-1 correspondence between the ideals I of R and the two-sided ideals J of A given by $I \to IA$ and $J \to \phi^{-1}(J)$.

(3) Consequently, if $(\mathfrak{a}_i)_{i\in I}$ is a family of ideals in R, then $\cap_i(\mathfrak{a}_i A) = (\cap \mathfrak{a}_i)A$, since $\phi^{-1}(\cap(\mathfrak{a}_i A)) = \cap\phi^{-1}(\mathfrak{a}_i A) = \cap\mathfrak{a}_i$.

(4) If S is a commutative R-algebra, then $A\otimes_R S$ is an Azumaya algebra over S.

(5) Hence, for each $\mathfrak{p} \in Spec(R)$, $A\otimes_R R_{\mathfrak{p}}$ is a finite free $R_{\mathfrak{p}}$-module of rank $r_{\mathfrak{p}}(A)$ depending only on the connected component of $\mathfrak{p}$ in $Spec(R)$ and called the rank of A at $\mathfrak{p}$.

(6) If A_1 and A_2 are Azumaya algebras over R, then so is $A_1\otimes_R A_2$.

(7) If A is an Azumaya algebra over R, then so is A^0 and $A\otimes_R A^0 \simeq End_R(A)$ (here A^0 denotes the opposite algebra of A i.e. $A^0 := A$ as a set, it has the same addition as A and its multiplication is given by $a \star b = b.a$).

(8) If $A_2 \subset A_1$ are Azumaya algebras over R, then the commutant of A_2 in A_1 i.e.

$$A_1^{A_2} := \{a_1 \in A_1 \,|\, a_1 a_2 = a_2 a_1 \, for\, all\, a_2 \in A_2\}$$

is an Azumaya algebra over R. Furthermore, the canonical map

$$A_2 \otimes_R A_1^{A_2} \to A_1,\, a_2 \otimes a_1 \to a_1 a_1$$

is an isomorphism.

Definition 2.4 (1) With the notations above, A is said to have constant rank r if $r = r_{\mathfrak{p}}(A)$ for each $\mathfrak{p} \in Spec(R)$. It is for example the case if the reduced ring of R is integral, in particular if R is integral.

(2) On the set of Azumaya algebras one defines an equivalence relation by saying that two Azumaya algebras A_1 and A_2 are equivalent if $A_1\otimes_R A_2 \simeq$

$End_R(P)$ for some faithful finitely generated projective R-module P. It then follows from the properties above that the operation $\otimes_R$ defines a group structure on the set of equivalence classes of Azumaya algebras: the class of R is the neutral element and the class of A^0 is the inverse of the class of A. This abelian group is called the Brauer group of R and is denoted by $Br(R)$.

Remark 2.5 (1) Since obviously A is equivalent to R iff $A \simeq A \otimes_R R \simeq End_R(P)$ for some faithful finitely generated projective R-module P, it follows that $Br(R)$ is trivial iff each Azumaya algebra A over R is isomorphic to $End_R(P)$ for some faithful finitely generated projective R-module P.

(2) It follows that if R is a local ring, then $Br(R)$ is trivial iff for each Azumaya algebra A over R there exists an $n \in \mathbb{N}$ such that $A \simeq M_n(R)$.

(3) It also follows that if R is a semi-local ring with a trivial Brauer group, then for each Azumaya algebra A of constant rank over R there exists $n \in \mathbb{N}$ such that $A \simeq M_n(R)$, according to the liberty of projective modules of constant rank over semi-local rings (see for instance [BO1], Ch.II,§5, no. 3, Proposition. 5).

(4) It is proved in [AG], th. 6.5 that the Brauer group of any complete local commutative ring is isomorphic to the one of its residue field.

(5) It is easy to check that the Brauer group of the direct sum of a finite number of commutative rings is the direct sum of their Brauer groups (see for instance [O],p. 58, exercise(i))

(6) One of the deepest results on Azumaya algebras, the Artin-Procesi theorem, in its version explicitely proved by the authors and their followers, claims that if a ring satisfies all the $\mathbb{Z}$-identities of the ring of n by n matrices over $\mathbb{Z}$ and if no nonzero homomorphic image of this ring satisfies all the $\mathbb{Z}$-identities of the ring of $n-1$ by $n-1$ matrices over $\mathbb{Z}$, then this ring is an Azumaya algebra of constant rank n^2 over its center (see [AR] and [P]). A recent and short proof can be found in [DR], Theorem 9.3.

(7) The following theorem is the converse of this formulation of the Artin-Procesi theorem (see Theorem 9.5 of [DR]). It has been claimed without proof in [AR] and [P] and many followers. It seems that no explicit proof has been published. Also the proof given in [DR] is based on unproven remarks concerning matrix rings over commutative rings.

Theorem 2.6 *Any Azumaya algebra of constant rank n^2 satisfies all the $\mathbb{Z}$-identities of the ring $M_n(\mathbb{Z})$ and no nonzero homomorphic image of this ring satisfies all the $\mathbb{Z}$-identities of the ring $M_{n-1}(\mathbb{Z})$.*

Proof. Let A be such an algebra. According to the Amitsur-Levitzki theorem ([RO], Theorem 1.4.5 and Corollary 1.8.43), no nonzero homomorphic image of A satisfies the standard identity of degree $2n-2$. Hence, according to the Amitsur-Levitzki theorem ([RO], Theorem 1.4.1), no nonzero homomorphic image of A satisfies all the $\mathbb{Z}$-identities of the ring $M_{n-1}(\mathbb{Z})$. Finally, according to Rowen's version of the Artin-Procesi theorem ([RO], Theorem 1.4.1) and to the MacConnell-Robson version of the Artin-Procesi theorem ([M], Theorem 7.14 and Corollary 6.7), A satisfies all the $\mathbb{Z}$-identities of the ring $M_n(R)$, where R is the ring of polynomials with coefficients in $\mathbb{Z}$ generated by a family of indeterminates indexed by $\{1, ..., n\}^2 \times \mathbb{N}$. As R is a domain, it follows from [M], Theorem 7.14, Corollary 6.7 and 2.2(1) that $M_n(R)$ satisfies all the $\mathbb{Z}$-identities of the ring $M_n(\mathbb{Z})$. So we can conclude that A satisfies all the $\mathbb{Z}$-identities of the ring $M_n(\mathbb{Z})$, as desired.

Example 2.7 (See for instance [BO2], Ch.VIII,§10, n. 4 or [O], Th. 6.36)
(1) It follows from two theorems of Wedderburn that the Brauer group of any finite field is trivial.

(2) It follows from a theorem of Wedderburn that the Brauer group of any algebraically closed field is trivial.

Theorem 2.8 *The Brauer group of a commutative Artin ring is isomorphic to the direct sum of the residue fields of its maximal ideals.*

Proof. Let R be such a ring. According to the structure theorem for commutative Artin rings (see for instance [AT], Ch.8, th. 8.7), R is the direct sum of complete local rings R_i with the same residue fields k_i as of the maximal ideals of R. So according to 2.5(5) the Brauer group of R is the direct sum of the ones of the R_i's. But according to 2.5(4) the Brauer group of R_i is isomorphic to the one of k_i. The conclusion follows.

Corollary 2.9 *The Brauer group of a finite commutative ring or more generally of a commutative Artin ring which is such that each of its residue fields is either finite or algebraically closed,is trivial. Hence, any Azumaya algebra of constant rank over such a ring is isomorphic to an algebra of matrices over this ring.*

Proof. Follows from 2.8, 2.5(3) and the examples 2.7(1),(2).

Remark 2.10 (1) Considering an algebraic closure of the residue field of a prime ideal of a commutative ring, it follows from 2.7(2) that the rank of any Azumaya algebra at a prime ideal of its center is the square of an integer.

(2) To conclude this section let us recall for the purpose of the proof of the main theorem below, that a commutative ring is called a Jacobson ring if each prime ideal is the intersection of a family of maximal ideals.

(3) A typical example of such a ring is any finitely generated $\mathbb{Z}$-algebra. In this case the residue field of any of its maximal ideals is finite (see for instance [BO1], V,§3, no. 4, Th.3).

3 Ring homomorphisms between matrix rings

In order to prepare for the proof of the main theorem we first prove the following special case. Recall that a ring is reduced if it has no non-zero nilpotent elements.

Theorem 3.1 *Let R and $R^{'}$ be commutative rings and assume that $R^{'}$ is reduced. If $\phi : M_n(R) \to M_n(R^{'})$ is a ring homomorphism, then $\phi(Z(M_n(R))) \subset Z(M_n(R^{'}))$.*

Lemma 3.2 *Let $\phi : M_n(R) \to M_{n'}(R^{'})$ be a ring homomorphism. Then $n \leq n^{'}$.*

Proof i) Let $\mathfrak{p}$ be a prime ideal in R' and denote by $\pi : M_{n'}(R') \to M_{n'}(R'/\mathfrak{p})$ the canonical map sending each matrix (a_{ij}) to $(\overline{a_{ij}})$. Replacing ϕ by $\pi \circ \phi$ we may assume that R' is a domain.
ii) We may also assume that ϕ is a injective: namely let $J := ker\phi$. Then J is a two-sided ideal in the Azumaya algebra $A := M_n(R)$, whence by 2.3(2), $J = IA$ for some ideal I in R. So we get an induced injective ring homomorphism $M_n(R)/IM_n(R) \to M_{n'}(R^{'})$. Since $M_n(R)/IM_n(R) \simeq M_n(R/I)$, we get an injective ring homomorphism from $M_n(R/I) \to M_{n'}(R^{'})$.
ii) So let ϕ be injective and let $a \in M_n(R)$ be the standard Jordan cell of maximal rank i.e. the first column of a is zero and for each $i \geq 2$ the i-th column of a is equal to the $i-1$-th standard basis vector e_{i-1}. So $a^n = 0$ but $a^{n-1} \neq 0$. Consequently $\phi(a)^n = 0$ and $\phi(a)^{n-1} \neq 0$ since ϕ is injective.

If now $n' < n$ it follows from the fact that R' is a domain that $\phi(a)^{n-1} = 0$, contradiction.

Remark 3.3 The main interest of the above proof of 3.3 is that it is elementary. A shorter but less elementary proof consists in applying the converse of the Artin-Procesi theorem (2.6), using 2.2(1).

Proof of theorem 3.1
i) First assume that $R^{'}$ is a domain. Then $R^{'} \subset Q(R^{'}) \subset k$, where k is an algebraic closure of the quotient field $Q(R^{'})$ of $R^{'}$. Observe that $M_n(R^{'}) \subset M_n(k)$ and that $M_n(R^{'}) \cap Z(M_n(k)) \subset Z(M_n(R^{'}))$. So we may assume that $R^{'} = k$. Now let $c \in Z(M_n(R))$ and let λ be an eigenvalue of $\phi(c)$. Put

$$V_\lambda := \{v \in k^n | \phi(c)(v) = \lambda v\}.$$

Since $c \in Z(M_n(R))$ we get that $\phi(c)\phi(x) = \phi(x)\phi(c)$ for all $x \in M_n(R)$, which implies that $\phi(x)V_\lambda \subset V_\lambda$ for all $x \in M_n(R)$. So we get a ring homomorphism

$$M_n(R) \ni x \to \phi(x)_{|V_\lambda} \in End(V_\lambda) \simeq M_{n'}(k)$$

where $n^{'}$:=dim$V_\lambda \leq n$. Since by (3.2) $n \leq n^{'}$ we get $n = n^{'}$. So dim$V_\lambda =$ dim k^n i.e. $V_\lambda = k^n$, whence $\phi(c) = \lambda I_n \in Z(M_n(k))$.
ii) Now let $R^{'}$ be a reduced ring, $c \in Z(M_n(R))$ and $a \in M_n(R^{'})$. We need to show that $[\phi(c), a] = 0$. Therefore it suffices to show that for each $\mathfrak{p} \in Spec(R^{'})$ all entries of the matrix $[\phi(c), a]$ belong to $\mathfrak{p}$ (for then these entries belong to $\cap\mathfrak{p} = r(0) = (0)$ since $R^{'}$ is reduced, so $[\phi(c), a] = 0$). So let $\mathfrak{p} \in Spec(R^{'})$ and denote by $\pi : R^{'} \to R^{'}/\mathfrak{p}$ the canonical map. Then extending π to $M_n(R^{'})$ in the obvious way we get

$$\pi([\phi(c), a]) = [(\pi \circ \phi)(c), \pi(a)] = 0$$

by i), since $\pi \circ \phi : M_n(R) \to M_n(R^{'}/\mathfrak{p})$ is a ring homomorphism, $c \in Z(M_n(R))$ and $R^{'}/\mathfrak{p}$ is a domain. So all entries of $[\phi(c), a]$ belong to $ker\pi = \mathfrak{p}$, as desired.

4 The main theorem

The main result of this paper is the following theorem.

Theorem 4.1 *Let $\phi : A \to A^{'}$ be a ring homomorphism between Azumaya algebras of constant ranks over R respectively $R^{'}$.*
(1) Then the rank of A is lower or equal than the rank of $A^{'}$.
(2) If furthermore they are equal and $R^{'}$ is reduced, then ϕ sends the center of A to the center of $A^{'}$.

Proof i) Let us first assume in addition that R and $R^{'}$ are finitely generated $\mathbb{Z}$-algebras, and let $\mathfrak{m} \in Max(R^{'})$. According to 2.3(2), there exists an ideal I of R such that $\phi^{-1}(\mathfrak{m}A^{'}) = IA$. So we get an injective map $\psi : A/IA \to A^{'}/\mathfrak{m}A^{'}$ induced by ϕ . Now observe that $A^{'}/\mathfrak{m}A^{'} = A^{'} \otimes_{R^{'}} R^{'}/\mathfrak{m}$ is finite, since $R^{'}/\mathfrak{m}$ is finite according to 2.10(2) and $A^{'} \otimes_{R^{'}} R^{'}/\mathfrak{m}R^{'}$ is finitely generated over $R^{'}/\mathfrak{m}$ according to 2.3(1). Since ψ is injective it follows that A/IA is finite. By 2.3(4) both A/IA and $A^{'}/\mathfrak{m}A^{'}$ are Azumaya algebras over R/I respectively $R^{'}/\mathfrak{m}R^{'}$. So by 2.9 and the hypothesis on the rank of the Azumaya algebras A and $A^{'}$ it follows that $A/IA \simeq M_n(R/I)$ and $A^{'}/\mathfrak{m}A^{'} \simeq M_{n^{'}}(R^{'}/\mathfrak{m}R^{'})$ for some integers $n \geq 1$ and $n^{'} \geq 1$. So statement (1) follows from 3.3.
ii) Furthermore by 3.1 we deduce that $\psi(Z(A/IA)) \subset Z(A^{'}/\mathfrak{m}A^{'})$. Now let $c \in Z(A)$ and $a^{'} \in A^{'}$. It follows that $[\phi(c), a^{'}] \in \mathfrak{m}A^{'}$ for all $\mathfrak{m} \in Max(R^{'})$. Hence by 2.3(3)

$$[\phi(c), a^{'}] \in \cap_{\mathfrak{m} \in Max(R^{'})} \mathfrak{m}A^{'} = (\cap \mathfrak{m}_{\mathfrak{m} \in Max(R^{'})})A^{'}.$$

Since $R^{'}$ is Jacobson by 2.10(2), and reduced by hypothesis we have that

$$\cap_{\mathfrak{m} \in Max(R^{'})} \mathfrak{m} = \cap_{\mathfrak{p} \in Spec(R^{'})} \mathfrak{p} = (0)$$

It follows that $[\phi(c), a^{'}] = 0$, as desired.
iii) In the general case of R and $R^{'}$, let us consider $c \in Z(A)$. According to Proposition 5.7, p.97 in [KO] there exist subrings R_0, $R_0^{'}$ of R respectively $R^{'}$ which are finitely generated $\mathbb{Z}$-algebras and Azumaya algebras A_0 over R_0 and $A_0^{'}$ over $R_0^{'}$ such that $A \simeq A_0 \otimes_{R_0} R$ and $A^{'} \simeq A_0^{'} \otimes_{R_0^{'}} R^{'}$. So there is a subring R_1 of R which contains R_0 and is finitely generated over $\mathbb{Z}$ and there is a subring $R_1^{'}$ of $R^{'}$ which contains $R_0^{'}$ and is finitely generated over $\mathbb{Z}$ such that c belongs to the image of the canonical embedding of $A_0 \otimes_{R_0} R_1$ into A and ϕ maps this image to the image of the canonical embedding of $A_0^{'} \otimes_{R_0^{'}} R_1^{'}$ into $A^{'}$. Since $A \simeq (A_0 \otimes_{R_0} R_1) \otimes_{R_1} R$ and $A^{'} \simeq (A_0^{'} \otimes_{R_0^{'}} R_1^{'}) \otimes_{R_1^{'}} R'$ statement (1) of 4.1 follows from i).
Finally it follows from ii) that $\phi(c)$ belongs to the canonical image in $A^{'}$ of

the center of $A_0^{'} \otimes_{R_0^{'}} R_1^{'}$. According to 2.3(4), this means that $\phi(c)$ belongs to the center of $A^{'}$, as desired.

Remark 4.2 The main interest of the proof above is that it uses only tools of the classical theory of Azumaya algebras as exposed in [AG], [DI] and [KO]. The following proof is more direct and short, but less elementary. It does not need a reduction to finitely generated $\mathbb{Z}$-algebras and to finite rings. It consists in combining the converse of the Artin-Procesi theorem (2.6) with the eigenvalue argument in the proof of 3.1.

Sketch of an alternative proof of theorem 4.1

i) Statement (1) of the main theorem follows directly from the converse of the Artin-Procesi theorem since for any pair of positive integers m and n such that $m \leq n$, any element of $M_m(\mathbb{Z})$ satisfies all $\mathbb{Z}$-identities of $M_n(\mathbb{Z})$ (since $M_m(\mathbb{Z})$ embeds canonically into $M_n(\mathbb{Z})$).
ii) Let $\phi : A \to A^{'}$ be a ring morphism between Azumaya algebras of the same constant rank n^2 with center R respectively $R^{'}$ such that is $R^{'}$ reduced. First assume that $R^{'}$ is a domain. Then $R^{'} \subset Q(R^{'}) \subset k$, where k is an algebraic closure of the quotient field $Q(R^{'})$ of $R^{'}$. Since the canonical map from R' to k is injective, it follows from the flatness of A' over R', (2.3(1)), that the canonical map f from $A' = A^{'} \otimes_{R'} R^{'}$ to $A^{'} \otimes_{R'} k$ is injective. On the other hand by 2.5(1) and 2.7(2), there exists an isomorphism g of k-algebras from $A^{'} \otimes_{R'} k$ to $M_n(k)$. So if we put $\phi' = g \circ f \circ \phi$ we ghet a ring homomorphism $\phi' : A \to M_n(k)$. Furthermore, according to the injectivity of $g \circ f$ we get that $A^{'} \cap Z(M_n(k)) \subset Z(A')$. So we may assume that $A^{'} = M_n(k)$ and $R^{'} = k$. Now let $c \in R$ and λ be an eigenvalue of $\phi'(c)$. Following part i) of the proof of 3.1 (replacing everywhere $M_n(R)$ by A and ϕ by $\phi^{'}$) we get a ring homomorphism

$$A \ni x \to \phi'(x)_{|V_\lambda} \in End(V_\lambda) \simeq M_{n'}(k)$$

where $n^{'} := \dim V_\lambda \leq n$. According to statement (1) of 4.1 and 2.2(1), we have $n \leq n'$. So $n = n^{'}$ and as in the proof of 3.1 we deduce that $\phi'(c) = \lambda I_n \in Z(M_n(k))$.
iii) Finally, let R' be reduced, $c \in Z(A)$ and $a' \in A'$. We need to show that $[\phi(c), a'] = 0$. So let $\mathfrak{p} \in Spec(R^{'})$ and $\pi : A^{'} \to A^{'}/\mathfrak{p}A^{'}$ be the canonical map. Since by 2.3(4) $A^{'}/\mathfrak{p}A^{'}$ is an Azumaya algebra over $R^{'}/\mathfrak{p}R^{'}$, which is a domain, it follows from ii) that $\pi \circ \phi : A \to A^{'}/\mathfrak{p}A^{'}$ sends the center of A to

the center of $A^{'}/\mathfrak{p}A^{'}$. So $[\pi \circ \phi(c), \overline{(a)}^{'}] = 0$, for all $\mathfrak{p} \in Spec(R')$, whence by 2.3(3)

$$[\phi(c), a^{'}] \in \cap(\mathfrak{p}A^{'}) = (\cap\mathfrak{p})A^{'} = (0)$$

since $R^{'}$ is reduced. So $[\phi(c), a'] = 0$ i.e. $\phi(c) \in Z(A^{'})$, as desired.

Corollary 4.3 *There is no algebra morphism from a Weyl algebra over a field k to another Weyl algebra of strictly lower dimension over k.*

Proof i) If k has positive characteristic it follows from 4.1 and 2.2(2).
ii) So assume that k has characteristic zero and let $\phi : A_n(k) \to A_{n'}(k)$ be a morphism of k-algebras. Then there exists a finitely generated $\mathbb{Z}$-algebra R contained in k such that ϕ induces a morphism of R-algebra $\phi_R : A_n(R) \to A_{n'}(R)$. Since k has characteristic zero there exists a prime number p whose canonical image in R belongs to a maximal ideal $\mathfrak{m}'$ of R (see for instance [ESS]. Prop.4.1.5). Let k' be the residue field of $\mathfrak{m}'$ and $\phi_{k'} : A_n(k') \to A_{n'}(k')$ the morphism of k'-algebras induced by ϕ_R. Since the characteristic of k' is p, it follows from i) that $n \leq n'$ as desired.

5 Isomorphisms of Azumaya algebras

In this section we consider the following question: when is a ring homomorphism ϕ between Azumaya algebras an isomorphism ?
Apart from a technical assumption, the answer given below states that this is the case if and only if ϕ induces an isomorphism between the centers of the Azumaya algebras. The proof is based on the following easy generalization of a result given in [AG], whose proof we follow closely.

Proposition 5.1 *Let A and $A^{'}$ be Azumaya algebras over R respectively $R^{'}$ and $\phi : A \to A^{'}$ a ring homomorphism such that the restriction of ϕ to R induces an isomorphism between R and $R^{'}$. If*

(*) $rk_{R_{\mathfrak{m}}} A_{\mathfrak{m}} = rk_{R^{'}_{\phi(\mathfrak{m})}} A^{'}_{\phi(\mathfrak{m})}$ for all $\mathfrak{m} \in Max(R)$

then ϕ is an isomorphism.

Proof. i) Observe that $\ker\phi$ is a two-sided ideal in A. So by 2.3(2) it is of the form IA for some ideal I in R. Let $i \in I$. Then $\phi(i.1) = 0$, so $i.1 = 0$ since $\phi : R \to R^{'}$ is injective. So $IA = 0$ i.e. ϕ is injective.
Consequently $\phi : A \to A_2 := \phi(A)$ is an isomorphism and since $R' = \phi(R)$,

A_2 is an Azumaya algebra over R' contained in $A_1 := A'$ which is also an Azumaya algebra over R'. So by 2.2(8) $A_1^{A_2}$ is an Azumaya algebra over R' and $\tau : A_2 \otimes_{R'} A_1^{A_2} \to A_1$ is an isomorphism and by 2.2(1) $A_1^{A_2} = R' \oplus L$ for some R'-submodule of L. If we can show that $L = 0$ it follows that $A_1^{A_2} = R'$. Since $R' = \phi(R)$ and τ is surjective this implies that $\phi(A) = A'$.
To see that $L = 0$ let $\mathfrak{m}' \in Max(R')$. Observe that by 2.3(1) A', $\phi(A)$ and $A_1^{A_2}$ are finitely generated projective R'-modules, and hence so is L. Consequently the localizations of these modules with respect to $\mathfrak{m}'$ are all free $R'_{\mathfrak{m}'}$-modules of finite rank, since $R'_{\mathfrak{m}'}$ is local. Now let $\mathfrak{m} = \phi^{-1}(\mathfrak{m}')$, so $\mathfrak{m} \in Max(R)$. Since $\phi : R \to R'$ is an isomorphism and $A \simeq \phi(A)$, because ϕ is injective, we get that $\mathrm{rk}_{R_\mathfrak{m}} A_\mathfrak{m} = \mathrm{rk}_{R'_{\mathfrak{m}'}} \phi(A)_{\mathfrak{m}'}$. Since by hypothesis $\mathrm{rk}_{R_\mathfrak{m}} A_\mathfrak{m} = \mathrm{rk}_{R'_{\mathfrak{m}'}} A'_{\mathfrak{m}'}$ we get that $\mathrm{rk}_{R'_{\mathfrak{m}'}} \phi(A)_{\mathfrak{m}'} = \mathrm{rk}_{R'_{\mathfrak{m}'}} A'_{\mathfrak{m}'}$. Then localizing the isomorphism τ with respect to $\mathfrak{m}'$ and comparing the ranks of the $R'_{\mathfrak{m}'}$-modules, we get that $\mathrm{rk}_{R'_{\mathfrak{m}'}} (A_1^{A_2})_{\mathfrak{m}'} = 1$. Localizing the equation $A_1^{A_2} = R' \oplus L$ with respect to $\mathfrak{m}'$ it follows that $L_{\mathfrak{m}'} = 0$ for all $\mathfrak{m}' \in Max(R')$. So $L = 0$ as desired.

Now we are able to give the main results of this section

Corollary 5.2 *Any endomorphism of Azumaya algebras is an automorphism.*

Theorem 5.3 *A ring morphism ϕ of an Azumaya algebra A over R to another Azumaya algebra A' over R' is an isomorphism if and only if it induces an isomorphism between R and R' and satisfies the condition (*) above.*

Proof. If $\phi : A \to A'$ is a ring isomorphism then ϕ is surjective. Hence it sends the center of A to the center of A'. So ϕ induces a morphism from R to R'. Similarly ϕ^{-1} induces an morphism from R' to R. Consequently ϕ induces an isomorphism from R to R' and satisfies (*). Conversely, if ϕ induces an isomorphism from R to R' and satisfies (*) then it follows from 5.1 that $\phi : A \to A'$ is an isomorphism.

Corollary 5.4 *Let $\phi : A \to A'$ be a ring morphism between Azumaya algebras over R respectively R' of the same constant rank. Then ϕ is an isomorphism if and only if ϕ induces an isomorphism between R and R'.*

Proof. One easily verifies that our hypothesis on the ranks implies condition (*). So we can apply 5.3.

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