



**HAL**  
open science

# On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows

Claude Viterbo

► **To cite this version:**

Claude Viterbo. On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows. 2005. hal-00008538v2

**HAL Id: hal-00008538**

**<https://hal.science/hal-00008538v2>**

Preprint submitted on 29 Sep 2005 (v2), last revised 22 Nov 2006 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows

CLAUDE VITERBO

Centre de Mathématiques Laurent Schwartz

UMR 7640 du CNRS

École Polytechnique - 91128 Palaiseau, France

viterbo@math.polytechnique.fr

29th September 2005

## Abstract

We show that if a sequence of Hamiltonian flows has a  $C^0$  limit, and if the generating Hamiltonians of the sequence have a limit, this limit is uniquely determined by the limiting  $C^0$  flow. This answers a question by Y.G. Oh in [Oh04].

## 1 Introduction

Let  $H_k$  be a sequence of Hamiltonians with flow  $\varphi_k^t$  on a symplectic manifold  $(M, \omega)$ , such that  $\lim_k H_k = H$  and  $\lim_k \varphi_k^t = \varphi^t$  where limits are intended as  $C^0$  limits. Can we say that  $\varphi$  is the flow of  $H$ ? In the case where  $H$  has a flow (e.g.  $H$  is  $C^{1,1}$ ) this has been proved in [Vit92], and one could alternatively use the methods of [Hof90]. However if  $H$  is only  $C^0$ , it is not easy to make sense of this question, since the flow of a  $C^0$  hamiltonian is not defined.

This question is not as artificial as the reader may think, and has apparently been asked by Y.G. Oh in the framework of  $C^0$ -Hamiltonians ([Oh04], this seems to be related to Question 3.11 or 3.20). It is sufficient to solve this question for the case where  $H$  is continuous and  $\varphi = Id$ . Do we necessarily have  $H = 0$  in this case?

Of course we could have  $H(t, x) = h(t)$  and we exclude this case by normalizing the Hamiltonian, either by

$$\int_M H(t, x) \omega^n = 0$$

for  $M$  compact, or by assuming it has compact support in the non-compact case.

Our aim in this short note is to give a positive answer to Oh's question.

**Theorem 1.1.** *Let  $H_n(t, z)$  be a sequence of  $C^{1,1}$  Hamiltonians on  $(M, \omega)$ , such that  $H_n$  converges in the  $C^0$  topology to some continuous function  $H(t, z)$ . Let  $\varphi_n^t$  be the flow of  $H_n$ . Then if  $\varphi_n^t$  converges to  $\text{Id}$  in the  $C^0$  topology, we have  $H = 0$ .*

*Remark 1.2.* a) Throughout the paper, by  $C^0$  convergence of  $\varphi_n^t$  to  $\varphi^t$ , we always mean  $C^0$  convergence uniform in  $t$ , for  $t$  in a compact interval. In other words,

$$\forall \varepsilon > 0 \exists N_0 \forall t \in I \forall N > N_0, \|\varphi_n^t - \varphi^t\|_{C^0} \leq \varepsilon$$

b) Note that if we do not assume that  $H_n$  converges, the theorem does not hold. Indeed, consider a non-zero Hamiltonian  $H_0$  supported in the unit ball. Then the sequence  $nH_0(nz)$  does not converge, but the time one flow  $C^0$ , does converge to the identity.

c) According to Y.G. Oh, one can adapt the proof of the theorem to the case where convergence is in Hofer norm, i.e. the norm given by

$$\|H\| = \int_0^1 \left[ \max_{x \in M} H(t, x) - \min_{x \in M} H(t, x) \right] dt$$

As a Corollary we get

**Corollary 1.3.** *Let  $\varphi_n^t, \psi_n^t$  be sequences of Hamiltonian flows associated to  $H_n, K_n$ . Assume*

$$\lim_n H_n = H, \lim_n K_n = K$$

and

$$\lim_n \varphi_n^t = \lim_n \psi_n^t = \rho^t$$

where all limits are intended as  $C^0$  limits. Then

$$H = K$$

*Proof of Corollary, assuming the theorem.* Indeed,  $(\varphi_n^t)^{-1} \circ (\psi_n^t)$   $C^0$  converges to the identity, and is generated by  $(K_n - H_n)(t, \varphi_n^t(z))$ , hence  $C^0$  converges to  $(H - K)(t, \rho^t(z))$ . Thus, according to the theorem we have  $(H - K)(t, \rho^t(z)) = 0$  hence  $H = K$ .

□

We thank Albert Fathi for drawing our attention to this problem, and Y.G. oh for raising the question and for some useful comments.

## 2 Proof of the theorem

If  $H(t, x)$  is a Hamiltonian, we set

$$\Psi^s : (t, \tau, z) \longrightarrow (t, \tau - sH(st, \varphi^{st}(z)), \varphi^{st}(z))$$

This is the flow of  $K(s; t, \tau, z) = sH(st, z)$  on  $T^*\mathbb{R} \times M$ , with  $s$  the new time variable.

Now if  $H_n \rightarrow H$  and  $\varphi_n^t \rightarrow \text{Id}$ , we have  $\Psi_n^s \rightarrow \Psi^s$ , where  $\Psi^s$  is given by  $\Psi^s(t, \tau, z) = (t, \tau - sH(st, z), z)$ , all limits being in the  $C^0$  topology. Let us show that  $\Psi^s$  cannot be the  $C^0$  limit of a sequence of Hamiltonian maps, unless  $H = 0$ .

More generally, consider the problem of a topological submanifold (i.e.  $C^0$ )  $L$  in  $T^*N$ , that would be a  $C^0$  limit of  $C^1$  Lagrangian submanifolds. According to [LS94], if  $L$  is  $C^1$ , it is necessarily Lagrangian. When  $L$  is  $C^0$ , what ‘‘Lagrangian’’ means is unclear. However when  $L$  is a graph in the cotangent bundle,  $L = \{(x, p(x)) \mid x \in N\}$ , requiring that  $p(x)dx$  is closed makes sense even if  $p \in C^0$ . Indeed we may interpretate this as meaning that  $p(x)dx$  is closed in the distribution sense, as suggested by Michael Herman ([Her89] definition 8.13 page 60). In our case we wish to prove that if  $L$  is a non Lagrangian  $C^0$  graph, we may not approximate it by Lagrangian submanifolds. Our crucial assertion is

**Proposition 2.1.** *Let  $N$  be a closed manifold that is the total space of an  $S^1$  fibration. Let  $p$  be a continuous section of  $T^*N$  which, considered as a one-form, is not closed in the sense of distributions. Then, there exists  $f \in C^\infty$  such that  $p(x) - df(x)$  does not vanish on  $N$ .*

**Corollary 2.2.** *Let  $L_n$  be a sequence of exact Lagrangians submanifolds in  $T^*N$ . If  $L$  is the  $C^0$  graph of a one form  $p$ , and  $L_n$  converges  $C^0$  to  $L$ , that is for any neighbourhood  $U$  of  $L$ , and  $n$  large enough,  $L_n$  is in  $U$ , then  $p$  is closed in the sense of distributions.*

*Remark 2.3.* (a). Note that once we have a  $C^1$  solution,  $f$ , for any  $C^1$  close function  $g$  we will have that  $p(x) - dg(x)$  does not vanish. Thus, we can replace  $f$  by a smooth approximation. In the sequel we shall thus not bother about the smoothness of the solution.

(b). Note that a smooth fibration with fiber  $S^1$ , that is a principal bundle with group  $Diff(S^1)$  is equivalent to a principal bundle with fiber the group  $S^1$ , since the inclusion of  $S^1$  into  $Diff(S^1)$  induces an isomorphism of homotopy groups.

Under the above assumptions, let  $V$  be the base of the circle fibration, and  $y : N \rightarrow V$  be the projection. For  $y$  in a domain of trivialization of the fibration, we consider coordinates  $(\theta, y)$  where  $\theta \in S^1$ . Note that we may always assume to be given an invariant measure (by the circle action) on  $N$ . We also set  $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$  where  $\theta \in S^1, y \in V, \pi(\theta, y) \in \mathbb{R}$  and  $r$  is a section of  $T^*V$  parametrized by  $\theta \in S^1$ .

**Lemma 2.4.** *Assume for some  $y \in V$  we have*

$$P(y) = \int_{S^1} \pi(\theta, y) d\theta \neq 0$$

*Then there exists  $f \in C^1(S^1 \times V, \mathbb{R})$  such that  $p(\theta, y) - df(\theta, y)$  does not vanish. The same holds for  $N$  the total space of a circle fibration..*

*Proof.* Let  $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$  be a one form on  $S^1 \times V$  that is a section of  $T^*(S^1 \times V)$ . We look for a function  $f(\theta, y)$  such that  $p(\theta, y) - df(\theta, y)$  never vanishes. Let us first try to solve

$$\pi(\theta, y) - \frac{\partial f}{\partial \theta}(\theta, y) = \varepsilon(\theta, y)$$

Then this is solvable if and only if

$$\int_{S^1} (\pi(\theta, y) - \varepsilon(\theta, y)) d\theta = 0$$

and thus, denoting by  $P(y) = \int_{S^1} \pi(\theta, y) d\theta$ , we can choose  $\varepsilon$  non vanishing outside a neighbourhood of the set  $Z = \{(\theta, y) \mid P(y) = 0\}$  (e.g. take  $\varepsilon(\theta, y) = P(y)$ ). In general we cannot choose  $\varepsilon$  non-zero in such a neighbourhood.

Note that  $f$  is well-defined up to a function of  $y$ . Also, we can assume  $f$  to be smooth, provides we took care to choose  $\varepsilon(\theta, y) - \pi(\theta, y)$  smooth.

Now we need to find  $h$  such that  $p(\theta, y) - df(\theta, y) - dh(y)$  does not vanish on  $Z$ . But if the projection of  $Z$  on  $V, U$ , is not all of  $V$ , so we can find

a function  $h$  on  $V$  with no critical point in  $U'$ . Multiplying  $h$  by a large constant, we may assume  $dh$  to be arbitrarily large. Then  $p(\theta, y) - df(\theta, y) - dh(y)$  will not vanish for all  $(\theta, y)$  with  $y \in U$ , and thus for  $(\theta, y) \in Z$ .  $\square$

**Lemma 2.5.** *Assume  $p$  is  $C^0$  on  $N$ , and consider a smooth circle fibration of  $N$ . Assume for any curve  $\gamma$ ,  $C^\infty$  close to a fiber of the fibration, we have*

$$\int_{S^1} p(\gamma(t))\dot{\gamma}(t)dt = 0$$

*Then  $p$  is closed in the sense of distributions.*

*Proof.* We shall take local coordinates  $(\theta, y)$  in the neighbourhood of a fiber. Now let  $\eta(\theta, y)$  be a smooth vector field on  $N$ .

Let  $\alpha$  be a one-form. We wish to compute the integral of  $\alpha$  over the curve  $t \rightarrow (t, y + \varepsilon\eta(t, y))$ . This will be

$$\int_{S^1} \alpha(\theta, y + \varepsilon\eta(\theta, y))(1, \varepsilon \frac{\partial}{\partial \theta} \eta(\theta, y)) d\theta$$

Writing  $\alpha = \alpha_\theta d\theta + \alpha_y dy$  we can rewrite the above as

$$\int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta$$

Now averaging this over  $y \in V$ , and differentiating with respect to  $\varepsilon$  at  $\varepsilon = 0$ , we get

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon} \int_V \int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta dy = \\ & - \int_V \int_{S^1} \left[ \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) \nabla_y \cdot \eta(\theta, y) + \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) + \right. \\ & \quad \left. \varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \wedge d_y \eta(\theta, y) \right] d\theta dy \end{aligned}$$

This is obtained by applying the change of variable  $(\theta', y') = (\theta, y + \varepsilon\eta(\theta, y))$  to the

$$\int_{V \times S^1} f(\theta, y + \varepsilon\eta(\theta, y)) d\theta dy$$

to get

$$\int_{V \times S^1} f(\theta, y') dy$$

where

$$dy' = dy + \varepsilon\eta(\theta, y') + o(\varepsilon)$$

so that

$$dy = dy' - \varepsilon\eta(\theta, y') + o(\varepsilon)$$

and

$$\det(dy') = \det(dy) - \varepsilon\nabla_y \cdot \eta(\theta, y)$$

remembering that  $\text{trace } d\eta = \nabla \cdot \eta$

$$\int_{V \times S^1} f(\theta, y') dy = \int_{V \times S^1} f(\theta, y') dy' - \varepsilon \int_{V \times S^1} f(\theta, y') \frac{\partial}{\partial y'} \eta(\theta, y') dy'$$

and denoting by  $\nabla_y$  the nabla operator with respect to the  $y$  variables. where all derivatives should be understood in the distributional sense. Now computing the above for  $\varepsilon = 0$ , we get

$$\int_V \int_{S^1} \left[ (\alpha_\theta \nabla_y \cdot \eta)(\theta, y) + \alpha_y(\theta, y) \frac{\partial}{\partial \theta} \eta(\theta, y) \right] d\theta dy$$

Integrating by parts we get

$$\begin{aligned} \int_V \int_{S^1} \left[ (\nabla_y \alpha_\theta)(\theta, y) \eta(\theta, y) - \left( \frac{\partial}{\partial \theta} \alpha_y \right)(\theta, y) \eta(\theta, y) \right] d\theta dy = \\ \int_V \int_{S^1} \left[ (\nabla_y \alpha_\theta)(\theta, y) - \frac{\partial}{\partial \theta} \alpha_y(\theta, y) \right] \eta(\theta, y) d\theta dy \end{aligned}$$

The above line is exactly the integration of  $d\alpha$  against the bivector  $\frac{\partial}{\partial \theta} \wedge (0, \eta)$ . As this vanishes for all  $\eta$ , means that  $\iota_{\frac{\partial}{\partial \theta}} d\alpha$  vanishes as a distribution (or current).

We thus proved that if for all  $\eta$  the integration of  $\alpha$  over the loop  $t \rightarrow (y + \varepsilon\eta(t, y), t)$  has vanishing derivative, we must have that  $\iota_{\frac{\partial}{\partial \theta}} d\alpha$  is identically zero. Now if we slightly modify our fibration, and apply the same argument, we get that  $\iota_Z d\alpha = 0$  for any vector field  $Z$  tangent to the fiber of a circle fibration of  $N$ , close to the given one. The next lemma allows us to conclude the proof.

**Lemma 2.6.** *Assume  $\alpha$  is a continuous form such that for all  $Z$  vector field tangent to a fibration close to  $Z_0$ , we have  $i_Z d\alpha = 0$ . Then  $d\alpha = 0$  as a distribution.*

*Proof.* Indeed, it is enough to show that our assumption implies that  $i_Z d\alpha$  vanishes for all vector fields  $Z$ .

First of all, the problem is local: using a partition of unity, it is enough to show that  $i_Z d\alpha = 0$  holds for any  $Z$  supported in a small set, tangent to a fibration close to  $Z_0$ .

Now since  $Z_0$  does not vanish, any vector field  $C^1$  close to  $Z_0$  has a flow box near  $z_0$ , hence a small diffeomorphism makes it tangent to  $Z_0$ . Thus locally, the set of  $Z$  such that  $i_Z d\alpha = 0$  is open in the  $C^\infty$  topology, and thus, by considering  $i_{Z-z_0}$ , any small vector field supported in the neighbourhood of  $z_0$  satisfies  $i_Z d\alpha = 0$ .  $\square$

$\square$

*Proof of the proposition.* According to the second lemma, if  $p$  is not closed, using a vector field, we may smoothly perturb the fibration,  $\pi$  so that one of the fibers satisfies  $\int_{\pi^{-1}(y)} p \neq 0$ . Then, using this new fibration and the first lemma, we see that there is a function  $f$  such that  $p(x) - df(x)$  does not vanish.  $\square$

$\square$

*Proof of Corollary, following [LS94].* First of all if  $L_n$  converges to  $L$ , then  $L_n \times 0_{S^1} \subset T^*(N \times S^1)$  converges to  $L \times 0_{S^1}$ , and this will be the graph of  $p$ , considered as a one-form on  $N \times S^1$ . Now if  $p$  is closed on  $N$ , its extension to  $N \times S^1$  is also closed, since

$$\int_{N \times S^1} \left[ \frac{\partial}{\partial x_i} p_j(x) - \frac{\partial}{\partial x_j} p_i(x) \right] \varphi(x, \theta) dx d\theta$$

defined as

$$- \int_{N \times S^1} \left[ p_j(x) \frac{\partial}{\partial x_i} \varphi(x, \theta) - p_i(x) \frac{\partial}{\partial x_j} \varphi(x, \theta) \right] dx d\theta$$

is equal to

$$\int_{N \times S^1} \left[ p_j(x) \frac{\partial}{\partial x_i} \bar{\varphi}(x) - p_i(x) \frac{\partial}{\partial x_j} \bar{\varphi}(x) \right] dx d\theta$$

where we set  $\bar{\varphi}(x) = \int_{S^1} \varphi(x, \theta) d\theta$ , so that  $p$  is closed (in the sense of distributions) as a one form on  $N$  if and only if it is closed (in the sense of distributions) as a one form on  $N \times S^1$ .

According to the above lemma, we see that we may, using a Hamiltonian symplectomorphism, send  $L$  away from the zero section (by  $(x, p) \rightarrow (x, p - df(x))$ ) and thus any Lagrangian submanifold  $L_n$  in a neighbourhood of  $L$  will also be sent to  $T^*N \setminus 0_N$  and thus may be disjointed from itself by a small



Hamiltonian isotopy, since  $(x, p) \rightarrow (x, \lambda p)$  is conformal, and thus induces a Hamiltonian isotopy on exact Lagrangians. But this is impossible according to Gromov's theorem ([Gro85] p. 330).  $\square$

Note in particular that if  $L_n = \varphi_n(V)$  for some Lagrangian submanifold  $V$  such that  $H_1(V) \rightarrow H_1(N)$  is injective the assumptions of the corollary are satisfied. Indeed, in this case, let  $\alpha$  be the class of the Liouville form on  $V$ , and assume  $\alpha = \pi^*(\beta)$  where  $\beta \in H^1(N)$ , then applying the map  $(x, p) \rightarrow (x, p - \beta(x))$  which is symplectic, we send  $L$  to  $L'$  which is Lagrangian if and only if  $L$  is, and the image  $V'$  of  $V$  is now exact. Then the  $\varphi_n(V')$  now converge to  $L'$ , and if  $L'$  is not closed, we may send  $L'$  away from the zero section by a symplectic map, and thus, the same holds for  $V'$ . But  $V'$  is exact, a contradiction.

*Remark 2.7.* However any submanifold can be approximated in the Hausdorff topology by a (non-exact) Lagrangian one. Indeed, given  $V$ , we may approximate it by a union of small Lagrangian tori, each being contained in a Darboux chart near  $V$ . On the union of such tori, we may perform a Polterovich surgery, in order to obtain a connected Lagrangian submanifold. Quite obviously the map  $H_1(V) \rightarrow H_1(N)$  is not injective in this case: each torus produces a lot of 1-cycles of  $V$ , which go to zero in  $N$ .

End of proof of the theorem. Let us consider the graph  $L_n$  of  $\Psi_n^1$  in  $(T^*\mathbb{R} \times M) \times (T^*\mathbb{R} \times M)$ . Then  $L_n$  is a sequence of exact Lagrangian submanifold, converging to  $V$ , the graph of the map  $\Psi^1$ , which is given by

$$\{(t, \tau, z, t, \tau - H(t, z), z) \mid (t, \tau, z) \in T^*(\mathbb{R}) \times M\}$$

Now the identification of  $T^*\mathbb{R} \times T^*\mathbb{R}$  with the cotangent of the diagonal,  $T^*\Delta_{\mathbb{R}^2}$  given by the map

$$(t_1, \tau_1, t_2, \tau_2) \longrightarrow \left( \frac{t_1 + t_2}{2}, \frac{\tau_1 + \tau_2}{2}, \tau_1 - \tau_2, t_1 - t_2 \right)$$

extends to a symplectic map from  $(T^*\mathbb{R} \times M) \times \overline{(T^*\mathbb{R} \times M)}$  to  $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$

The graph  $\Gamma$  of  $\Psi^1$  when regarded over  $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$  is given by

$$(t, \tau, z) \rightarrow \left( t, H(t, z), \tau - \frac{1}{2}H(t, z), H(t, z), z, z \right)$$

Note that  $\Gamma$  is in a neighbourhood of  $T^*(\Delta_{\mathbb{R}^2}) \times \Delta_M$ , hence according to Weinstein's tubular neighbourhood's theorem, it is a submanifold of  $T^*(\Delta_{\mathbb{R}^2}) \times T^*\Delta_M = T^*(\Delta_{\mathbb{R}^2 \times M})$ .

A change of variables in the base, given by  $(t, \tau, z) \rightarrow (t, \tau - \frac{1}{2}H(t, z), z)$  induces on the cotangent bundle a symplectic map such that  $\Gamma$  is now the image of the map

$$(t, \sigma, z) \rightarrow (t, -H(t, z), \sigma, 0, z, z)$$

This is a graph over the diagonal denoted by  $V$ . Since  $\Psi^1$  is not symplectic, in the sense that the Liouville form  $(\tau - H(t, x, p))dt + p dx - \tau dt - p dx = -H(t, x, p)dt$  is not closed in the sense of distributions, unless  $H(t, x, p) = h(t)$  which we excluded by the normalization assumption. Thus  $\Gamma$  is not Lagrangian, we get according to Corollary 2.2 that  $L_n$  cannot converge to  $V$ . This concludes our proof.  $\square$

*Remark 2.8.* One would like to know whether proposition 2.1 still holds for  $N$  a general compact manifold. This does not seem to follow literally from [LS94], even though their method may be useful.

*Remark 2.9.* We could have also used the ideas from [Sik91] for most of our proof. We think however that proposition 2.1 is of independent interest.

## References

- [Gro85] Misha Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.
- [Her89] M. R. Herman. Inégalités “à priori” pour des tores lagrangiens invariants. *Publ. Math. I.H.E.S.*, 70:47–101, 1989.
- [Hof90] Helmut Hofer. On the topological properties of symplectic maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):25–38, 1990.
- [LS94] François Laudenbach and Jean-Claude Sikorav. Hamiltonian disjunction and limits of lagrangian submanifolds. *Intern. Math. Res. Notes*, 4:161–168, 1994.
- [Oh04] Y. G. Oh. The group of hamiltonian homeomorphisms and  $c^0$  symplectic topology. ArXiv:math.SG/0402210, v2, August 2005.
- [Sik91] Jean-Claude Sikorav. Quelques propriétés des plongements lagrangiens. *Mém. S.M.F. n° 46, suppl. au Bull; S.M.F. 119 (1991).*, pages 151–167, 1991.
- [Vit92] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Annalen*, 292:685–710, 1992.