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# On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows

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## Abstract

We show that if a sequence of Hamiltonian flows has a  $C^0$  limit, and if the generating Hamiltonians of the sequence have a limit, this limit is uniquely determined by the limiting  $C^0$  flow. This answers a question by Y.G. Oh in [Oh04].

## 1 Introduction

Let  $H_k$  be a sequence of Hamiltonians with flow  $\varphi_k^t$  on a symplectic manifold  $(M, \omega)$ , such that  $\lim_k H_k = H$  and  $\lim_k \varphi_k^t = \varphi^t$  where limits are intended as  $C^0$  limits. Can we say that  $\varphi$  is the flow of  $H$ ? In the case where  $H$  has a flow (e.g.  $H$  is  $C^{1,1}$ ) this has been proved in [Vit92], and one could alternatively use the methods of [Hof90]. However if  $H$  is only  $C^0$ , it is not easy to make sense of this question, since the flow of a  $C^0$  hamiltonian is not defined.

This question is not as artificial as the reader may think, and has apparently been asked by Y.G. Oh in the framework of  $C^0$ -Hamiltonians ([Oh04], this seems to be related to Question 3.11 or 3.20). It is sufficient to solve this question for the case where  $H$  is continuous and  $\varphi = Id$ . Do we necessarily have  $H = 0$  in this case?

Of course we could have  $H(t, x) = h(t)$  and we exclude this case by normalizing the Hamiltonian, either by

$$\int_M H(t, x) \omega^n = 0$$

for  $M$  compact, or by assuming it has compact support in the non-compact case.

Our aim in this short note is to give a positive answer to Oh's question.

**Theorem 1.1.** *Let  $H_n(t, z)$  be a sequence of  $C^{1,1}$  Hamiltonians on  $(M, \omega)$ , such that  $H_n$  converges in the  $C^0$  topology to some continuous function  $H(t, z)$ . Let  $\varphi_n^t$  be the flow of  $H_n$ . Then if  $\varphi_n^t$  converges to  $\text{Id}$  in the  $C^0$  topology, we have  $H = 0$ .*

*Remark 1.2.* a) Throughout the paper, by  $C^0$  convergence of  $\varphi_n^t$  to  $\varphi^t$ , we always mean  $C^0$  convergence uniform in  $t$ , for  $t$  in a compact interval. In other words,

$$\forall \varepsilon > 0 \exists N_0 \forall t \in I \forall N > N_0, \|\varphi_n^t - \varphi^t\|_{C^0} \leq \varepsilon$$

b) Note that if we do not assume that  $H_n$  converges, the theorem does not hold. Indeed, consider a non-zero Hamiltonian  $H_0$  supported in the unit ball. Then the sequence  $nH_0(nz)$  does not converge, but the time one flow  $C^0$ , does converge to the identity.

c) As Y.G. Oh pointed out, the proof of the theorem has a straightforward adaptation to the convergence in Hofer norm, i.e. the norm given by

$$\|H\| = \int_0^1 \left[ \max_{x \in M} H(t, x) - \min_{x \in M} H(t, x) \right] dt$$

As a Corollary we get

**Corollary 1.3.** *Let  $\varphi_n^t, \psi_n^t$  be sequences of Hamiltonian flows associated to  $H_n, K_n$ . Assume*

$$\lim_n H_n = H, \lim_n K_n = K$$

and

$$\lim_n \varphi_n^t = \lim_n \psi_n^t = \rho^t$$

where all limits are intended as  $C^0$  limits. Then

$$H = K$$

*Proof of Corollary, assuming the theorem.* Indeed,  $(\varphi_n^t)^{-1} \circ (\psi_n^t)$   $C^0$  converges to the identity, and is generated by  $(K_n - H_n)(t, \varphi_n^t(z))$ , hence  $C^0$  converges to  $(H - K)(t, \rho^t(z))$ . Thus, according to the theorem we have  $(H - K)(t, \rho^t(z)) = 0$  hence  $H = K$ .

□

We thank Albert Fathi for drawing our attention to this problem, and Y.G. oh for raising the question and for some useful comments.

## 2 Proof of the theorem

If  $H(t, x)$  is a Hamiltonian, we set

$$\Psi^s : (t, \tau, z) \longrightarrow (t, \tau - sH(st, \varphi^{st}(z)), \varphi^{st}(z))$$

This is the flow of  $K(s; t, \tau, z) = sH(st, z)$  on  $T^*\mathbb{R} \times M$ , with  $s$  the new time variable.

Now if  $H_n \rightarrow H$  and  $\varphi_n^t \rightarrow \text{Id}$ , we have  $\Psi_n^s \rightarrow \Psi^s$ , where  $\Psi^s$  is given by  $\Psi^s(t, \tau, z) = (t, \tau - sH(st, z), z)$ , all limits being in the  $C^0$  topology. Let us show that  $\Psi^s$  cannot be the  $C^0$  limit of a sequence of Hamiltonian maps, unless  $H = 0$ .

More generally, consider the problem of a topological submanifold (i.e.  $C^0$ )  $L$  in  $T^*N$ , that would be a  $C^0$  limit of  $C^1$  Lagrangian submanifolds. According to [LS94], if  $L$  is  $C^1$ , it is necessarily Lagrangian. When  $L$  is  $C^0$ , what ‘‘Lagrangian’’ means is unclear. However when  $L$  is a graph in the cotangent bundle,  $L = \{(x, p(x)) \mid x \in N\}$ , requiring that  $p(x)dx$  is closed makes sense even if  $p \in C^0$ . Indeed we may interpretate this as meaning that  $p(x)dx$  is closed in the distribution sense, as suggested by Michael Herman ([Her89] definition 8.13 page 60). In our case we wish to prove that if  $L$  is a non Lagrangian  $C^0$  graph, we may not approximate it by Lagrangian submanifolds. Our crucial assertion is

**Proposition 2.1.** *Let  $N$  be a closed manifold that is the total space of an  $S^1$  fibration. Let  $p$  be a continuous section of  $T^*N$  which, considered as a one-form, is not closed in the sense of distributions. Then, there exists  $f \in C^\infty$  such that  $p(x) - df(x)$  does not vanish on  $N$ .*

**Corollary 2.2.** *Let  $L_n$  be a sequence of exact Lagrangians submanifolds in  $T^*N$ . If  $L$  is the  $C^0$  graph of a one form  $p$ , and  $L_n$  converges  $C^0$  to  $L$ , that is for any neighbourhood  $U$  of  $L$ , and  $n$  large enough,  $L_n$  is in  $U$ , then  $p$  is closed in the sense of distributions.*

*Remark 2.3.* (a). Note that once we have a  $C^1$  solution,  $f$ , for any  $C^1$  close function  $g$  we will have that  $p(x) - dg(x)$  does not vanish. Thus, we can replace  $f$  by a smooth approximation. In the sequel we shall thus not bother about the smoothness of the solution.

(b). Note that a smooth fibration with fiber  $S^1$ , that is a principal bundle with group  $Diff(S^1)$  is equivalent to a principal bundle with fiber the group  $S^1$ , since the inclusion of  $S^1$  into  $Diff(S^1)$  induces an isomorphism of homotopy groups.

Under the above assumptions, let  $V$  be the base of the circle fibration, and  $y : N \rightarrow V$  be the projection. For  $y$  in a domain of trivialization of the fibration, we consider coordinates  $(\theta, y)$  where  $\theta \in S^1$ . Note that we may always assume to be given an invariant measure (by the circle action) on  $N$ . We also set  $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$  where  $\theta \in S^1, y \in V, \pi(\theta, y) \in \mathbb{R}$  and  $r$  is a section of  $T^*V$  parametrized by  $\theta \in S^1$ .

**Lemma 2.4.** *Assume for some  $y \in V$  we have*

$$P(y) = \int_{S^1} \pi(\theta, y) d\theta \neq 0$$

*Then there exists  $f \in C^1(S^1 \times V, \mathbb{R})$  such that  $p(\theta, y) - df(\theta, y)$  does not vanish. The same holds for  $N$  the total space of a circle fibration..*

*Proof.* Let  $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$  be a one form on  $S^1 \times V$  that is a section of  $T^*(S^1 \times V)$ . We look for a function  $f(\theta, y)$  such that  $p(\theta, y) - df(\theta, y)$  never vanishes. Let us first try to solve

$$\pi(\theta, y) - \frac{\partial f}{\partial \theta}(\theta, y) = \varepsilon(\theta, y)$$

Then this is solvable if and only if

$$\int_{S^1} (\pi(\theta, y) - \varepsilon(\theta, y)) d\theta = 0$$

and thus, denoting by  $P(y) = \int_{S^1} \pi(\theta, y) d\theta$ , we can choose  $\varepsilon$  non vanishing outside a neighbourhood of the set  $Z = \{(\theta, y) \mid P(y) = 0\}$  (e.g. take  $\varepsilon(\theta, y) = P(y)$ ). In general we cannot choose  $\varepsilon$  non-zero in such a neighbourhood.

Note that  $f$  is well-defined up to a function of  $y$ .

Now we need to find  $h$  such that  $p(\theta, y) - df(\theta, y) - dh(y)$  does not vanish on  $Z$ . But if the projection of  $Z$  on  $V, U$ , is not all of  $V$ , we can find a function  $h$  on  $V$  with no critical point in  $U$ . Multiplying  $h$  by a large constant, we

may assume  $dh$  to be arbitrarily large. Then  $p(\theta, y) - df(\theta, y) - dh(y)$  will not vanish for all  $(\theta, y)$  with  $y \in U$ , and thus for  $(\theta, y) \in Z$ . □

**Lemma 2.5.** *Assume  $p$  is  $C^0$  on  $N$ , and consider a smooth circle fibration of  $N$ . Assume for any curve  $\gamma$ ,  $C^\infty$  close to a fiber of the fibration, we have*

$$\int_{S^1} p(\gamma(t)) \dot{\gamma}(t) dt = 0$$

*Then  $p$  is closed in the sense of distributions.*

*Proof.* We shall take local coordinates  $(\theta, y)$  in the neighbourhood of a fiber. Now let  $\eta(\theta, y)$  be a smooth vector field on  $N$ .

Let  $\alpha$  be a one-form. We wish to compute the integral of  $\alpha$  over the curve  $t \rightarrow (t, y + \varepsilon\eta(t, y))$ . This will be

$$\int_{S^1} \alpha(\theta, y + \varepsilon\eta(\theta, y)) (1, \varepsilon \frac{\partial}{\partial \theta} \eta(\theta, y)) d\theta$$

Writing  $\alpha = \alpha_\theta d\theta + \alpha_y dy$  we can rewrite the above as

$$\int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta$$

Now averaging this over  $y \in V$ , and differentiating with respect to  $\varepsilon$  at  $\varepsilon = 0$ , we get

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon} \int_V \int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta dy = \\ & - \int_V \int_{S^1} \left[ \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) \nabla_y \cdot \eta(\theta, y) + \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) + \right. \\ & \quad \left. \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \wedge d_y \eta(\theta, y) \right] d\theta dy \end{aligned}$$

This is obtained by applying the change of variable  $(\theta', y') = (\theta, y + \varepsilon\eta(\theta, y))$  to the

$$\int_{V \times S^1} f(\theta, y + \varepsilon\eta(\theta, y)) d\theta dy$$

to get

$$\int_{V \times S^1} f(\theta, y') dy$$

where

$$dy' = dy + \varepsilon\eta(\theta, y') + o(\varepsilon)$$

so that

$$dy = dy' - \varepsilon\eta(\theta, y') + o(\varepsilon)$$

and

$$\det(dy') = \det(dy) - \varepsilon\nabla_y \cdot \eta(\theta, y)$$

remembering that  $\text{trace } d\eta = \nabla \cdot \eta$

$$\int_{V \times S^1} f(\theta, y') dy = \int_{V \times S^1} f(\theta, y') dy' - \varepsilon \int_{V \times S^1} f(\theta, y') \frac{\partial}{\partial y'} \eta(\theta, y') dy'$$

and denoting by  $\nabla_y$  the nabla operator with respect to the  $y$  variables. where all derivatives should be understood in the distributional sense. Now computing the above for  $\varepsilon = 0$ , we get

$$\int_V \int_{S^1} \left[ (\alpha_\theta \nabla_y \cdot \eta)(\theta, y) + \alpha_y(\theta, y) \frac{\partial}{\partial \theta} \eta(\theta, y) \right] d\theta dy$$

Integrating by parts we get

$$\begin{aligned} \int_V \int_{S^1} \left[ (\nabla_y \alpha_\theta)(\theta, y) \eta(\theta, y) - \left( \frac{\partial}{\partial \theta} \alpha_y \right)(\theta, y) \eta(\theta, y) \right] d\theta dy = \\ \int_V \int_{S^1} \left[ (\nabla_y \alpha_\theta)(\theta, y) - \frac{\partial}{\partial \theta} \alpha_y(\theta, y) \right] \eta(\theta, y) d\theta dy \end{aligned}$$

The above line is exactly the integration of  $d\alpha$  against the bivector  $\frac{\partial}{\partial \theta} \wedge (0, \eta)$ . As this vanishes for all  $\eta$ , means that  $\iota_{\frac{\partial}{\partial \theta}} d\alpha$  vanishes as a distribution (or current).

We thus proved that if for all  $\eta$  the integration of  $\alpha$  over the loop  $t \rightarrow (y + \varepsilon\eta(t, y), t)$  has vanishing derivative, we must have that  $\iota_{\frac{\partial}{\partial \theta}} d\alpha$  is identically zero. Now if we slightly modify our fibration, and apply the same argument, we get that  $\iota_Z d\alpha = 0$  for any vector field  $Z$  tangent to the fiber of a circle fibration of  $N$ , close to the given one. The next lemma allows us to conclude the proof.

**Lemma 2.6.** *Assume  $\alpha$  is a continuous form such that for all  $Z$  vector field tangent to a fibration close to  $Z_0$ , we have  $i_Z d\alpha = 0$ . Then  $d\alpha = 0$  as a distribution.*

*Proof.* Indeed, it is enough to show that our assumption implies that  $i_Z d\alpha$  vanishes for all vector fields  $Z$ .

First of all, the problem is local: using a partition of unity, it is enough to show that  $i_Z d\alpha = 0$  holds for any  $Z$  supported in a small set, tangent to a fibration close to  $Z_0$ .

Now since  $Z_0$  does not vanish, any vector field  $C^1$  close to  $Z_0$  has a flow box near  $z_0$ , hence a small diffeomorphism makes it tangent to  $Z_0$ . Thus locally, the set of  $Z$  such that  $i_Z d\alpha = 0$  is open in the  $C^\infty$  topology, and thus, by considering  $i_{Z-z_0}$ , any small vector field supported in the neighbourhood of  $z_0$  satisfies  $i_Z d\alpha = 0$ .  $\square$

$\square$

*Proof of the proposition.* According to the second lemma, if  $p$  is not closed, using a vector field, we may smoothly perturb the fibration,  $\pi$  so that one of the fibers satisfies  $\int_{\pi^{-1}(y)} p \neq 0$ . Then, using this new fibration and the first lemma, we see that there is a function  $f$  such that  $p(x) - df(x)$  does not vanish.  $\square$

$\square$

*Proof of Corollary, following [LS94].* First of all if  $L_n$  converges to  $L$ , then  $L_n \times 0_{S^1} \subset T^*(N \times S^1)$  converges to  $L \times 0_{S^1}$ , and this will be the graph of  $p$ , considered as a one-form on  $N \times S^1$ . Now if  $p$  is closed on  $N$ , its extension to  $N \times S^1$  is also closed, since

$$\int_{N \times S^1} \left[ \frac{\partial}{\partial x_i} p_j(x) - \frac{\partial}{\partial x_j} p_i(x) \right] \varphi(x, \theta) dx d\theta$$

defined as

$$- \int_{N \times S^1} \left[ p_j(x) \frac{\partial}{\partial x_i} \varphi(x, \theta) - p_i(x) \frac{\partial}{\partial x_j} \varphi(x, \theta) \right] dx d\theta$$

is equal to

$$\int_{N \times S^1} \left[ p_j(x) \frac{\partial}{\partial x_i} \bar{\varphi}(x) - p_i(x) \frac{\partial}{\partial x_j} \bar{\varphi}(x) \right] dx d\theta$$

where we set  $\bar{\varphi}(x) = \int_{S^1} \varphi(x, \theta) d\theta$ , so that  $p$  is closed (in the sense of distributions) as a one form on  $N$  if and only if it is closed (in the sense of distributions) as a one form on  $N \times S^1$ .

According to the above lemma, we see that we may, using a Hamiltonian symplectomorphism, send  $L$  away from the zero section (by  $(x, p) \rightarrow (x, p - df(x))$ ) and thus any Lagrangian submanifold  $L_n$  in a neighbourhood of  $L$  will also be sent to  $T^*N \setminus 0_N$  and thus may be disjointed from itself by a small



Hamiltonian isotopy, since  $(x, p) \rightarrow (x, \lambda p)$  is conformal, and thus induces a Hamiltonian isotopy on exact Lagrangians. But this is impossible according to Gromov's theorem ([Gro85] p. 330).  $\square$

Note in particular that if  $L_n = \varphi_n(V)$  for some Lagrangian submanifold  $V$  such that  $H_1(V) \rightarrow H_1(N)$  is injective the assumptions of the corollary are satisfied. Indeed, in this case, let  $\alpha$  be the class of the Liouville form on  $V$ , and assume  $\alpha = \pi^*(\beta)$  where  $\beta \in H^1(N)$ , then applying the map  $(x, p) \rightarrow (x, p - \beta(x))$  which is symplectic, we send  $L$  to  $L'$  which is Lagrangian if and only if  $L$  is, and the image  $V'$  of  $V$  is now exact. Then the  $\varphi_n(V')$  now converge to  $L'$ , and if  $L'$  is not closed, we may send  $L'$  away from the zero section by a symplectic map, and thus, the same holds for  $V'$ . But  $V'$  is exact, a contradiction.

*Remark 2.7.* However any submanifold can be approximated in the Hausdorff topology by a (non-exact) Lagrangian one. Indeed, given  $V$ , we may approximate it by a union of small Lagrangian tori, each being contained in a Darboux chart near  $V$ . On the union of such tori, we may perform a Polterovich surgery, in order to obtain a connected Lagrangian submanifold. Quite obviously the map  $H_1(V) \rightarrow H_1(N)$  is not injective in this case: each torus produces a lot of 1-cycles of  $V$ , which go to zero in  $N$ .

*End of proof of the theorem.* Let us consider the graph  $L_n$  of  $\Psi_n^1$  in  $(T^*\mathbb{R} \times M) \times (T^*\mathbb{R} \times M)$ . Then  $L_n$  is a sequence of exact Lagrangian submanifold, converging to  $V$ , the graph of the map  $\Psi^1$ , which is given by

$$\{(t, \tau, z, t, \tau - H(t, z), z) \mid (t, \tau, z) \in T^*(\mathbb{R}) \times M\}$$

Now the identification of  $T^*\mathbb{R} \times T^*\mathbb{R}$  with the cotangent of the diagonal,  $T^*\Delta_{\mathbb{R}^2}$  given by the map

$$(t_1, \tau_1, t_2, \tau_2) \longrightarrow \left( \frac{t_1 + t_2}{2}, \frac{\tau_1 + \tau_2}{2}, \tau_1 - \tau_2, t_1 - t_2 \right)$$

extends to a symplectic map from  $(T^*\mathbb{R} \times M) \times \overline{(T^*\mathbb{R} \times M)}$  to  $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$

The graph  $\Gamma$  of  $\Psi^1$  when regarded over  $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$  is given by

$$(t, \tau, z) \rightarrow \left( t, H(t, z), \tau - \frac{1}{2}H(t, z), H(t, z), z, z \right)$$

Note that  $\Gamma$  is in a neighbourhood of  $T^*(\Delta_{\mathbb{R}^2}) \times \Delta_M$ , hence according to Weinstein's tubular neighbourhood's theorem, it is a submanifold of  $T^*(\Delta_{\mathbb{R}^2}) \times T^*\Delta_M = T^*(\Delta_{\mathbb{R}^2 \times M})$ .

A change of variables in the base, given by  $(t, \tau, z) \rightarrow (t, \tau - \frac{1}{2}H(t, z), z)$  induces on the cotangent bundle a symplectic map such that  $\Gamma$  is now the image of the map

$$(t, \sigma, z) \rightarrow (t, -H(t, z), \sigma, 0, z, z)$$

This is a graph over the diagonal denoted by  $V$ . Since  $\Psi^1$  is not symplectic, in the sense that the Liouville form  $(\tau - H(t, x, p))dt + pdx - \tau dt - pdx = -H(t, x, p)dt$  is not closed in the sense of distributions, unless  $H(t, x, p) = h(t)$  which we excluded by the normalization assumption. Thus  $\Gamma$  is not Lagrangian, we get according to Corollary 2.2 that  $L_n$  cannot converge to  $V$ . This concludes our proof.  $\square$

*Remark 2.8.* One would like to know whether proposition 2.1 still holds for  $N$  a general compact manifold. This does not seem to follow literally from [LS94], even though their method may be useful.

*Remark 2.9.* We could have also used the ideas from [Sik91] for most of our proof. We think however that proposition 2.1 is of independent interest.

## References

- [Gro85] Misha Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.
- [Her89] M. R. Herman. Inégalités “à priori” pour des tores lagrangiens invariants. *Publ. Math. I.H.E.S.*, 70:47–101, 1989.
- [Hof90] Helmut Hofer. On the topological properties of symplectic maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):25–38, 1990.
- [LS94] François Laudenbach and Jean-Claude Sikorav. Hamiltonian disjunction and limits of lagrangian submanifolds. *Intern. Math. Res. Notes*, 4:161–168, 1994.
- [Oh04] Y. G. Oh. The group of hamiltonian homeomorphisms and  $c^0$  symplectic topology. ArXiv:math.SG/0402210, v2, August 2005.
- [Sik91] Jean-Claude Sikorav. Quelques propriétés des plongements lagrangiens. *Mém. S.M.F. n° 46, suppl. au Bull; S.M.F. 119 (1991).*, pages 151–167, 1991.
- [Vit92] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Annalen*, 292:685–710, 1992.