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On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows

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Abstract

We show that if a sequence of Hamiltonian flows has a C^0 limit, and if the generating Hamiltonians of the sequence have a limit, this limit is uniquely determined by the limiting C^0 flow. This answers a question by Y.G. Oh in [Oh04].

1 Introduction

Let H_k be a sequence of Hamiltonians with flow φ_k^t on a symplectic manifold (M, ω) , such that $\lim_k H_k = H$ and $\lim_k \varphi_k^t = \varphi^t$ where limits are intended as C^0 limits. Can we say that φ is the flow of H ? In the case where H has a flow (e.g. H is $C^{1,1}$) this has been proved in [Vit92], and one could alternatively use the methods of [Hof90]. However if H is only C^0 , it is not easy to make sense of this question, since the flow of a C^0 hamiltonian is not defined.

This question is not as artificial as the reader may think, and has apparently been asked by Y.G. Oh in the framework of C^0 -Hamiltonians ([Oh04], this seems to be related to Question 3.11 or 3.20). It is sufficient to solve this question for the case where H is continuous and $\varphi = Id$. Do we necessarily have $H = 0$ in this case?

Of course we could have $H(t, x) = h(t)$ and we exclude this case by normalizing the Hamiltonian, either by

$$\int_M H(t, x) \omega^n = 0$$

for M compact, or by assuming it has compact support in the non-compact case.

Our aim in this short note is to give a positive answer to Oh's question.

Theorem 1.1. *Let $H_n(t, z)$ be a sequence of $C^{1,1}$ Hamiltonians on (M, ω) , such that H_n converges in the C^0 topology to some continuous function $H(t, z)$. Let φ_n^t be the flow of H_n . Then if φ_n^t converges to Id in the C^0 topology, we have $H = 0$.*

Remark 1.2. a) Throughout the paper, by C^0 convergence of φ_n^t to φ^t , we always mean C^0 convergence uniform in t , for t in a compact interval. In other words,

$$\forall \varepsilon > 0 \exists N_0 \forall t \in I \forall N > N_0, \|\varphi_n^t - \varphi^t\|_{C^0} \leq \varepsilon$$

b) Note that if we do not assume that H_n converges, the theorem does not hold. Indeed, consider a non-zero Hamiltonian H_0 supported in the unit ball. Then the sequence $nH_0(nz)$ does not converge, but the time one flow C^0 , does converge to the identity.

c) As Y.G. Oh pointed out, the proof of the theorem has a straightforward adaptation to the convergence in Hofer norm, i.e. the norm given by

$$\|H\| = \int_0^1 \left[\max_{x \in M} H(t, x) - \min_{x \in M} H(t, x) \right] dt$$

As a Corollary we get

Corollary 1.3. *Let φ_n^t, ψ_n^t be sequences of Hamiltonian flows associated to H_n, K_n . Assume*

$$\lim_n H_n = H, \lim_n K_n = K$$

and

$$\lim_n \varphi_n^t = \lim_n \psi_n^t = \rho^t$$

where all limits are intended as C^0 limits. Then

$$H = K$$

Proof of Corollary, assuming the theorem. Indeed, $(\varphi_n^t)^{-1} \circ (\psi_n^t)$ C^0 converges to the identity, and is generated by $(K_n - H_n)(t, \varphi_n^t(z))$, hence C^0 converges to $(H - K)(t, \rho^t(z))$. Thus, according to the theorem we have $(H - K)(t, \rho^t(z)) = 0$ hence $H = K$.

□

We thank Albert Fathi for drawing our attention to this problem, and Y.G. oh for raising the question and for some useful comments.

2 Proof of the theorem

If $H(t, x)$ is a Hamiltonian, we set

$$\Psi^s : (t, \tau, z) \longrightarrow (t, \tau - sH(st, \varphi^{st}(z)), \varphi^{st}(z))$$

This is the flow of $K(s; t, \tau, z) = sH(st, z)$ on $T^*\mathbb{R} \times M$, with s the new time variable.

Now if $H_n \rightarrow H$ and $\varphi_n^t \rightarrow \text{Id}$, we have $\Psi_n^s \rightarrow \Psi^s$, where Ψ^s is given by $\Psi^s(t, \tau, z) = (t, \tau - sH(st, z), z)$, all limits being in the C^0 topology. Let us show that Ψ^s cannot be the C^0 limit of a sequence of Hamiltonian maps, unless $H = 0$.

More generally, consider the problem of a topological submanifold (i.e. C^0) L in T^*N , that would be a C^0 limit of C^1 Lagrangian submanifolds. According to [LS94], if L is C^1 , it is necessarily Lagrangian. When L is C^0 , what ‘‘Lagrangian’’ means is unclear. However when L is a graph in the cotangent bundle, $L = \{(x, p(x)) \mid x \in N\}$, requiring that $p(x)dx$ is closed makes sense even if $p \in C^0$. Indeed we may interpretate this as meaning that $p(x)dx$ is closed in the distribution sense, as suggested by Michael Herman ([Her89] definition 8.13 page 60). In our case we wish to prove that if L is a non Lagrangian C^0 graph, we may not approximate it by Lagrangian submanifolds. Our crucial assertion is

Proposition 2.1. *Let N be a closed manifold that is the total space of an S^1 fibration. Let p be a continuous section of T^*N which, considered as a one-form, is not closed in the sense of distributions. Then, there exists $f \in C^\infty$ such that $p(x) - df(x)$ does not vanish on N .*

Corollary 2.2. *Let L_n be a sequence of exact Lagrangians submanifolds in T^*N . If L is the C^0 graph of a one form p , and L_n converges C^0 to L , that is for any neighbourhood U of L , and n large enough, L_n is in U , then p is closed in the sense of distributions.*

Remark 2.3. (a). Note that once we have a C^1 solution, f , for any C^1 close function g we will have that $p(x) - dg(x)$ does not vanish. Thus, we can replace f by a smooth approximation. In the sequel we shall thus not bother about the smoothness of the solution.

(b). Note that a smooth fibration with fiber S^1 , that is a principal bundle with group $Diff(S^1)$ is equivalent to a principal bundle with fiber the group S^1 , since the inclusion of S^1 into $Diff(S^1)$ induces an isomorphism of homotopy groups.

Under the above assumptions, let V be the base of the circle fibration, and $y : N \rightarrow V$ be the projection. For y in a domain of trivialization of the fibration, we consider coordinates (θ, y) where $\theta \in S^1$. Note that we may always assume to be given an invariant measure (by the circle action) on N . We also set $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$ where $\theta \in S^1, y \in V, \pi(\theta, y) \in \mathbb{R}$ and r is a section of T^*V parametrized by $\theta \in S^1$.

Lemma 2.4. *Assume for some $y \in V$ we have*

$$P(y) = \int_{S^1} \pi(\theta, y) d\theta \neq 0$$

Then there exists $f \in C^1(S^1 \times V, \mathbb{R})$ such that $p(\theta, y) - df(\theta, y)$ does not vanish. The same holds for N the total space of a circle fibration..

Proof. Let $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$ be a one form on $S^1 \times V$ that is a section of $T^*(S^1 \times V)$. We look for a function $f(\theta, y)$ such that $p(\theta, y) - df(\theta, y)$ never vanishes. Let us first try to solve

$$\pi(\theta, y) - \frac{\partial f}{\partial \theta}(\theta, y) = \varepsilon(\theta, y)$$

Then this is solvable if and only if

$$\int_{S^1} (\pi(\theta, y) - \varepsilon(\theta, y)) d\theta = 0$$

and thus, denoting by $P(y) = \int_{S^1} \pi(\theta, y) d\theta$, we can choose ε non vanishing outside a neighbourhood of the set $Z = \{(\theta, y) \mid P(y) = 0\}$ (e.g. take $\varepsilon(\theta, y) = P(y)$). In general we cannot choose ε non-zero in such a neighbourhood.

Note that f is well-defined up to a function of y .

Now we need to find h such that $p(\theta, y) - df(\theta, y) - dh(y)$ does not vanish on Z . But if the projection of Z on V, U , is not all of V , we can find a function h on V with no critical point in U . Multiplying h by a large constant, we

may assume dh to be arbitrarily large. Then $p(\theta, y) - df(\theta, y) - dh(y)$ will not vanish for all (θ, y) with $y \in U$, and thus for $(\theta, y) \in Z$. □

Lemma 2.5. *Assume p is C^0 on N , and consider a smooth circle fibration of N . Assume for any curve γ , C^∞ close to a fiber of the fibration, we have*

$$\int_{S^1} p(\gamma(t)) \dot{\gamma}(t) dt = 0$$

Then p is closed in the sense of distributions.

Proof. We shall take local coordinates (θ, y) in the neighbourhood of a fiber. Now let $\eta(\theta, y)$ be a smooth vector field on N .

Let α be a one-form. We wish to compute the integral of α over the curve $t \rightarrow (t, y + \varepsilon\eta(t, y))$. This will be

$$\int_{S^1} \alpha(\theta, y + \varepsilon\eta(\theta, y)) (1, \varepsilon \frac{\partial}{\partial \theta} \eta(\theta, y)) d\theta$$

Writing $\alpha = \alpha_\theta d\theta + \alpha_y dy$ we can rewrite the above as

$$\int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta$$

Now averaging this over $y \in V$, and differentiating with respect to ε at $\varepsilon = 0$, we get

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon} \int_V \int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) d\theta dy = \\ & - \int_V \int_{S^1} \left[\alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) \nabla_y \cdot \eta(\theta, y) + \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \frac{\partial}{\partial \theta} \eta(\theta, y) + \right. \\ & \quad \left. \varepsilon \alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \wedge d_y \eta(\theta, y) \right] d\theta dy \end{aligned}$$

This is obtained by applying the change of variable $(\theta', y') = (\theta, y + \varepsilon\eta(\theta, y))$ to the

$$\int_{V \times S^1} f(\theta, y + \varepsilon\eta(\theta, y)) d\theta dy$$

to get

$$\int_{V \times S^1} f(\theta, y') dy$$

where

$$dy' = dy + \varepsilon\eta(\theta, y') + o(\varepsilon)$$

so that

$$dy = dy' - \varepsilon\eta(\theta, y') + o(\varepsilon)$$

and

$$\det(dy') = \det(dy) - \varepsilon\nabla_y \cdot \eta(\theta, y)$$

remembering that $\text{trace } d\eta = \nabla \cdot \eta$

$$\int_{V \times S^1} f(\theta, y') dy = \int_{V \times S^1} f(\theta, y') dy' - \varepsilon \int_{V \times S^1} f(\theta, y') \frac{\partial}{\partial y'} \eta(\theta, y') dy'$$

and denoting by ∇_y the nabla operator with respect to the y variables. where all derivatives should be understood in the distributional sense. Now computing the above for $\varepsilon = 0$, we get

$$\int_V \int_{S^1} \left[(\alpha_\theta \nabla_y \cdot \eta)(\theta, y) + \alpha_y(\theta, y) \frac{\partial}{\partial \theta} \eta(\theta, y) \right] d\theta dy$$

Integrating by parts we get

$$\begin{aligned} \int_V \int_{S^1} \left[(\nabla_y \alpha_\theta)(\theta, y) \eta(\theta, y) - \left(\frac{\partial}{\partial \theta} \alpha_y \right)(\theta, y) \eta(\theta, y) \right] d\theta dy = \\ \int_V \int_{S^1} \left[(\nabla_y \alpha_\theta)(\theta, y) - \frac{\partial}{\partial \theta} \alpha_y(\theta, y) \right] \eta(\theta, y) d\theta dy \end{aligned}$$

The above line is exactly the integration of $d\alpha$ against the bivector $\frac{\partial}{\partial \theta} \wedge (0, \eta)$. As this vanishes for all η , means that $\iota_{\frac{\partial}{\partial \theta}} d\alpha$ vanishes as a distribution (or current).

We thus proved that if for all η the integration of α over the loop $t \rightarrow (y + \varepsilon\eta(t, y), t)$ has vanishing derivative, we must have that $\iota_{\frac{\partial}{\partial \theta}} d\alpha$ is identically zero. Now if we slightly modify our fibration, and apply the same argument, we get that $\iota_Z d\alpha = 0$ for any vector field Z tangent to the fiber of a circle fibration of N , close to the given one. The next lemma allows us to conclude the proof.

Lemma 2.6. *Assume α is a continuous form such that for all Z vector field tangent to a fibration close to Z_0 , we have $i_Z d\alpha = 0$. Then $d\alpha = 0$ as a distribution.*

Proof. Indeed, it is enough to show that our assumption implies that $i_Z d\alpha$ vanishes for all vector fields Z .

First of all, the problem is local: using a partition of unity, it is enough to show that $i_Z d\alpha = 0$ holds for any Z supported in a small set, tangent to a fibration close to Z_0 .

Now since Z_0 does not vanish, any vector field C^1 close to Z_0 has a flow box near z_0 , hence a small diffeomorphism makes it tangent to Z_0 . Thus locally, the set of Z such that $i_Z d\alpha = 0$ is open in the C^∞ topology, and thus, by considering i_{Z-z_0} , any small vector field supported in the neighbourhood of z_0 satisfies $i_Z d\alpha = 0$. \square

\square

Proof of the proposition. According to the second lemma, if p is not closed, using a vector field, we may smoothly perturb the fibration, π so that one of the fibers satisfies $\int_{\pi^{-1}(y)} p \neq 0$. Then, using this new fibration and the first lemma, we see that there is a function f such that $p(x) - df(x)$ does not vanish. \square

\square

Proof of Corollary, following [LS94]. First of all if L_n converges to L , then $L_n \times 0_{S^1} \subset T^*(N \times S^1)$ converges to $L \times 0_{S^1}$, and this will be the graph of p , considered as a one-form on $N \times S^1$. Now if p is closed on N , its extension to $N \times S^1$ is also closed, since

$$\int_{N \times S^1} \left[\frac{\partial}{\partial x_i} p_j(x) - \frac{\partial}{\partial x_j} p_i(x) \right] \varphi(x, \theta) dx d\theta$$

defined as

$$- \int_{N \times S^1} \left[p_j(x) \frac{\partial}{\partial x_i} \varphi(x, \theta) - p_i(x) \frac{\partial}{\partial x_j} \varphi(x, \theta) \right] dx d\theta$$

is equal to

$$\int_{N \times S^1} \left[p_j(x) \frac{\partial}{\partial x_i} \bar{\varphi}(x) - p_i(x) \frac{\partial}{\partial x_j} \bar{\varphi}(x) \right] dx d\theta$$

where we set $\bar{\varphi}(x) = \int_{S^1} \varphi(x, \theta) d\theta$, so that p is closed (in the sense of distributions) as a one form on N if and only if it is closed (in the sense of distributions) as a one form on $N \times S^1$.

According to the above lemma, we see that we may, using a Hamiltonian symplectomorphism, send L away from the zero section (by $(x, p) \rightarrow (x, p - df(x))$) and thus any Lagrangian submanifold L_n in a neighbourhood of L will also be sent to $T^*N \setminus 0_N$ and thus may be disjointed from itself by a small

Hamiltonian isotopy, since $(x, p) \rightarrow (x, \lambda p)$ is conformal, and thus induces a Hamiltonian isotopy on exact Lagrangians. But this is impossible according to Gromov's theorem ([Gro85] p. 330). \square

Note in particular that if $L_n = \varphi_n(V)$ for some Lagrangian submanifold V such that $H_1(V) \rightarrow H_1(N)$ is injective the assumptions of the corollary are satisfied. Indeed, in this case, let α be the class of the Liouville form on V , and assume $\alpha = \pi^*(\beta)$ where $\beta \in H^1(N)$, then applying the map $(x, p) \rightarrow (x, p - \beta(x))$ which is symplectic, we send L to L' which is Lagrangian if and only if L is, and the image V' of V is now exact. Then the $\varphi_n(V')$ now converge to L' , and if L' is not closed, we may send L' away from the zero section by a symplectic map, and thus, the same holds for V' . But V' is exact, a contradiction.

Remark 2.7. However any submanifold can be approximated in the Hausdorff topology by a (non-exact) Lagrangian one. Indeed, given V , we may approximate it by a union of small Lagrangian tori, each being contained in a Darboux chart near V . On the union of such tori, we may perform a Polterovich surgery, in order to obtain a connected Lagrangian submanifold. Quite obviously the map $H_1(V) \rightarrow H_1(N)$ is not injective in this case: each torus produces a lot of 1-cycles of V , which go to zero in N .

End of proof of the theorem. Let us consider the graph L_n of Ψ_n^1 in $(T^*\mathbb{R} \times M) \times (T^*\mathbb{R} \times M)$. Then L_n is a sequence of exact Lagrangian submanifold, converging to V , the graph of the map Ψ^1 , which is given by

$$\{(t, \tau, z, t, \tau - H(t, z), z) \mid (t, \tau, z) \in T^*(\mathbb{R}) \times M\}$$

Now the identification of $T^*\mathbb{R} \times T^*\mathbb{R}$ with the cotangent of the diagonal, $T^*\Delta_{\mathbb{R}^2}$ given by the map

$$(t_1, \tau_1, t_2, \tau_2) \longrightarrow \left(\frac{t_1 + t_2}{2}, \frac{\tau_1 + \tau_2}{2}, \tau_1 - \tau_2, t_1 - t_2 \right)$$

extends to a symplectic map from $(T^*\mathbb{R} \times M) \times \overline{(T^*\mathbb{R} \times M)}$ to $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$

The graph Γ of Ψ^1 when regarded over $T^*\Delta_{\mathbb{R}^2} \times (M \times \overline{M})$ is given by

$$(t, \tau, z) \rightarrow \left(t, H(t, z), \tau - \frac{1}{2}H(t, z), H(t, z), z, z \right)$$

Note that Γ is in a neighbourhood of $T^*(\Delta_{\mathbb{R}^2}) \times \Delta_M$, hence according to Weinstein's tubular neighbourhood's theorem, it is a submanifold of $T^*(\Delta_{\mathbb{R}^2}) \times T^*\Delta_M = T^*(\Delta_{\mathbb{R}^2 \times M})$.

A change of variables in the base, given by $(t, \tau, z) \rightarrow (t, \tau - \frac{1}{2}H(t, z), z)$ induces on the cotangent bundle a symplectic map such that Γ is now the image of the map

$$(t, \sigma, z) \rightarrow (t, -H(t, z), \sigma, 0, z, z)$$

This is a graph over the diagonal denoted by V . Since Ψ^1 is not symplectic, in the sense that the Liouville form $(\tau - H(t, x, p))dt + pdx - \tau dt - pdx = -H(t, x, p)dt$ is not closed in the sense of distributions, unless $H(t, x, p) = h(t)$ which we excluded by the normalization assumption. Thus Γ is not Lagrangian, we get according to Corollary 2.2 that L_n cannot converge to V . This concludes our proof. \square

Remark 2.8. One would like to know whether proposition 2.1 still holds for N a general compact manifold. This does not seem to follow literally from [LS94], even though their method may be useful.

Remark 2.9. We could have also used the ideas from [Sik91] for most of our proof. We think however that proposition 2.1 is of independent interest.

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