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# From the Prékopa-Leindler inequality to modified logarithmic Sobolev inequality

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## Abstract

We develop in this paper an improvement of the method given by S. Bobkov and M. Ledoux in [BL00]. Using the Prékopa-Leindler inequality, we prove a modified logarithmic Sobolev inequality adapted for all measures on  $\mathbb{R}^n$ , with a strictly convex and super-linear potential. This inequality implies modified logarithmic Sobolev inequality, developed in [GGM05, GGM06], for all uniform strictly convex potential as well as the Euclidean logarithmic Sobolev inequality.

## Résumé

Nous montrons dans cet article une amélioration de la méthode donnée par S. Bobkov et M. Ledoux dans [BL00]. En utilisant l'inégalité de Prékopa-Leindler, nous prouvons une inégalité de Sobolev logarithmique modifiée adaptée à toutes les mesures sur  $\mathbb{R}^n$  possédant un potentiel strictement convexe et super-linéaire. Cette inégalité implique en particulier une inégalité de Sobolev logarithmique modifiée, développée dans [GGM05, GGM06], pour les mesures ayant un potentiel uniformément strictement convexe aussi bien que l'inégalité de Sobolev logarithmique de type euclidien.

## 1 Introduction

The Prékopa-Leindler inequality is the functional form of Brunn-Minkowski inequality. Let  $a, b$  be some positive reals such that  $a + b = 1$ , and  $u, v, w$  be some non negative measurable functions on  $\mathbb{R}^n$ . Assume that, for any  $x, y \in \mathbb{R}^n$ , we have

$$u(x)^a v(y)^b \leq w(ax + by),$$

then

$$\left( \int u(x) dx \right)^a \left( \int v(x) dx \right)^b \leq \int w(x) dx. \quad (1)$$

If we apply inequality (1) to characteristic functions of bounded measurable sets  $A$  and  $B$  in  $\mathbb{R}^n$ , we get the multiplicative form of the Brunn-Minkowski inequality

$$\text{vol}(A)^a \text{vol}(B)^b \leq \text{vol}(aA + bB),$$

where  $aA + bB = \{ax_A + bx_B, x_A \in A, x_B \in B\}$  and  $\text{vol}(A)$  is the Lebesgue measure of the set  $A$ . One can see for example two interesting reviews on this topic [Gup80, Mau04].

Bobkov and Ledoux in [BL00] use the Prékopa-Leindler inequality to prove some functional inequalities like Brascamp-Lieb, Logarithmic Sobolev and Transportation inequalities.

More precisely, let  $\varphi$  be a  $C^1$  strictly convex function on  $\mathbb{R}^n$  and let

$$d\mu_\varphi(x) = e^{-\varphi(x)} dx \quad (2)$$

be a probability measure on  $\mathbb{R}^n$  ( $\int e^{-\varphi(x)} dx = 1$ ). The function  $\varphi$  is called *the potential* of the measure  $\mu_\varphi$ . Bobkov and Ledoux obtained in particular the following two results:

- (Proposition 2.1 of [BL00]) Brascamp-Lieb inequality: assume that  $\varphi$  is a  $C^2$  function on  $\mathbb{R}^n$ , then for all smooth enough functions  $g$ ,

$$\mathbf{Var}_{\mu_\varphi}(g) := \int \left( g - \int g d\mu_\varphi \right)^2 d\mu_\varphi \leq \int \nabla g \cdot \text{Hess}(\varphi)^{-1} \nabla g d\mu_\varphi, \quad (3)$$

where  $\text{Hess}(\varphi)^{-1}$  is the inverse of the Hessian of  $\varphi$ .

- (Proposition 3.1 of [BL00]) Assume that for some  $c > 0$  and  $p \geq 2$ , for all  $t, s > 0$  with  $t + s = 1$ , and for all  $x, y \in \mathbb{R}^n$ ,  $\varphi$  satisfies, as  $s$  goes to 0,

$$t\varphi(x) + s\varphi(y) - \varphi(tx + sy) \geq \frac{c}{p}(s + o(s))\|x - y\|^p, \quad (4)$$

where  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^n$ . Then for all smooth enough functions  $g$ ,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) := \int e^g \log \frac{e^g}{\int e^g d\mu_\varphi} d\mu_\varphi \leq c \int \|\nabla g\|^q e^g d\mu_\varphi, \quad (5)$$

where  $1/p + 1/q = 1$ . They give as an example the function  $\varphi(x) = \|x\|^p + Z_\varphi$  (where  $Z_\varphi$  is a normalization constant) which satisfies inequality (4) for some constant  $c > 0$ .

The main result of this paper is to prove a modified logarithmic Sobolev inequality for the measure  $\mu_\varphi$  with a potential strictly convex and super-linear. We obtain, for all smooth enough functions  $g$  on  $\mathbb{R}^n$ ,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int \{x \cdot \nabla g(x) - \varphi^*(\nabla \varphi(x)) + \varphi^*(\nabla \varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\varphi(x), \quad (6)$$

where  $\varphi^*$  is the Fenchel-Legendre transform of  $\varphi$ ,  $\varphi^*(x) := \sup_{z \in \mathbb{R}^n} \{x \cdot z - \varphi(z)\}$ .

The main application of this result is to extend the modified logarithmic Sobolev inequalities presented in [GGM05, GGM06] for probability measures on  $\mathbb{R}$  satisfying a uniform strictly convexity condition. It is well known that if the potential  $\varphi$  is  $C^2$  on  $\mathbb{R}$  such that for all  $x \in \mathbb{R}$ ,  $\varphi''(x) \geq \lambda > 0$ , then the measure  $\mu_\varphi$  defined on (2) verifies the logarithmic Sobolev inequality introduced by Gross in [Gro75], for all smooth enough functions  $g$ , namely

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \frac{1}{2\lambda} \int g'^2 e^g d\mu_\varphi.$$

This result comes from the  $\mathbf{E}_2$ -criterion of D. Bakry and M. Émery, see [BÉ85] or [ABC<sup>+</sup>00] for a review. We then improve the classical logarithmic Sobolev inequality of Gross, in the situation where if the potential is even with  $\varphi(0) = 0$  and satisfies

$$\forall x \in \mathbb{R}, \quad \varphi''(x) \geq \lambda \text{ and } \lim_{|x| \rightarrow \infty} \varphi''(x) = \infty.$$

Adding a technical hypothesis (see Section 3.1), we show that for all smooth functions  $g$ ,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int H_\varphi(g') e^g d\mu_\varphi,$$

where

$$H_\varphi(x) = \begin{cases} C' \varphi^*(x) & \text{if } |x| > C \\ \frac{1}{2\lambda} x^2 & \text{if } |x| \leq C, \end{cases}$$

for some constants  $C, C' > 0$  depending on  $\varphi$ . Remark that we always have

$$\forall x \in \mathbb{R}, \quad H_\varphi(x) \leq \frac{1}{2\lambda}x^2.$$

This inequality implies concentration inequalities which are more adapted to the measure studied, as we will see in Section 3.1.

The next section is divided into two subsections. In the first one we state the main theorem of this article, inequality (6). In the second subsection, we explain how this result improves results of [BL00]. In particular, one gets modified logarithmic inequality (5) or Brascamp-Lieb inequality (3). Section 3 deals with some applications. The first one is an improvement of the  $\mathbf{I}_2$ -criterion of Bakry-Émery for measures on  $\mathbb{R}$ . We obtain then a global view of modified logarithmic Sobolev inequality for log-concave measures as introduced in a joint work with A. Guillin and L. Miclo in [GGM05, GGM06]. Finally, we explain how the main theorem is equivalent to the Euclidean logarithmic Sobolev inequality. As a consequence, a short proof of the generalization given in [DPD03, Gen03, AGK04] is obtained.

## 2 Modified logarithmic Sobolev inequality for log-concave measures

### 2.1 The main theorem

**Theorem 2.1** *Let  $\varphi$  be a  $C^1$  strictly convex function on  $\mathbb{R}^n$ , such that*

$$\lim_{\|x\| \rightarrow \infty} \frac{\varphi(x)}{\|x\|} = \infty. \quad (7)$$

*Denote by  $\mu_\varphi(dx) = e^{-\varphi(x)}dx$  a probability measure on  $\mathbb{R}^n$ , where  $dx$  is the Lebesgue measure on  $\mathbb{R}^n$ , ( $\int e^{-\varphi(x)}dx = 1$ ).*

*If  $\varphi^*$  is the Fenchel-Legendre transform of  $\varphi$ ,  $\varphi^*(x) := \sup_{z \in \mathbb{R}^n} \{x \cdot z - \varphi(z)\}$ , then for all smooth enough functions  $g$  on  $\mathbb{R}^n$ , one gets,*

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int \{x \cdot \nabla g(x) - \varphi^*(\nabla \varphi(x)) + \varphi^*(\nabla \varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\varphi(x). \quad (8)$$

The proof of this result is based on the following lemma:

**Lemma 2.2** *Let  $g$  be a  $C^\infty$  function with a compact support on  $\mathbb{R}^n$ . Let  $s, t \geq 0$  with  $t + s = 1$  and denote,*

$$\forall z \in \mathbb{R}^n, \quad g_s(z) = \sup_{z=tx+sy} (g(x) - (t\varphi(x) + s\varphi(y) - \varphi(tx + sy))).$$

*Then we get, when  $s$  goes to 0,*

$$g_s(z) = g(z) + s\{z \cdot \nabla g(z) - \varphi^*(\nabla \varphi(z)) + \varphi^*(\nabla \varphi(x) - \nabla g(x))\} + \left( ((z - y_0) \cdot \nabla(g + \varphi)(z) + \|z - y_0\|^2) \right) O(s^2),$$

*uniformly on  $z \in \mathbb{R}^n$ , where  $y_0 \in \mathbb{R}^n$  (and depends on  $z$ ).*

#### **Proof**

◁ Let  $s \in ]0, 1/2[$  and note  $x = z/t - (s/t)y$ , hence

$$g_s(z) = \varphi(z) + \sup_{y \in \mathbb{R}^n} \left( g\left(\frac{z}{t} - \frac{s}{t}y\right) - t\varphi\left(\frac{z}{t} - \frac{s}{t}y\right) - s\varphi(y) \right).$$

Due to the fact that  $g$  has a compact support and by property (7) there exists  $y_s \in \mathbb{R}^n$  such that

$$\sup_{y \in \mathbb{R}^n} \left( g\left(\frac{z}{t} - \frac{s}{t}y\right) - t\varphi\left(\frac{z}{t} - \frac{s}{t}y\right) - s\varphi(y) \right) = g\left(\frac{z}{t} - \frac{s}{t}y_s\right) - t\varphi\left(\frac{z}{t} - \frac{s}{t}y_s\right) - s\varphi(y_s).$$

Moreover,  $y_s$  satisfies

$$\nabla g\left(\frac{z}{t} - \frac{s}{t}y_s\right) - t\nabla\varphi\left(\frac{z}{t} - \frac{s}{t}y_s\right) + t\nabla\varphi(y_s) = 0. \quad (9)$$

The function  $\varphi$  is a strictly convex function, so that there is a unique solution  $y_0$  of the equation

$$\nabla\varphi(y_0) = \nabla\varphi(z) - \nabla g(z), \quad y_0 = (\nabla\varphi)^{-1}(\nabla\varphi(z) - \nabla g(z)). \quad (10)$$

We prove now that  $\lim_{s \rightarrow 0} y_s = y_0$ .

First we show that there exists  $R \geq 0$  such that  $\forall s \in [0, 1/2]$ ,  $\|y_s\| \leq R$ . Indeed, if the function  $y_s$  is not bounded one can find  $(s_k)_{k \in \mathbb{N}}$  such that  $s_k \rightarrow 0$  and  $\|y_{s_k}\| \rightarrow \infty$ . By property (7)  $\lim_{\|x\| \rightarrow \infty} \varphi(x) = \infty$ , and since  $g$  is bounded we obtain  $s_k y_{s_k} = O(1)$ . Due to the strict convexity of  $\varphi$ , the last assertion is in contradiction with equation (9).

Let  $\hat{y}$  an accumulation point of the function  $y_s$ , when  $s$  tends to 0. Then  $\hat{y}$  satisfies equation (10). By unicity of the solution of (10) we get  $\hat{y} = y_0$ . Therefore we have proved that  $\lim_{s \rightarrow 0} y_s = y_0$ .

Using Taylor's formula and the continuity of  $y_s$  at  $s = 0$  we get

$$\varphi\left(\frac{z}{t} - \frac{s}{t}y_s\right) = \varphi(z) + s(z - y_0) \cdot \nabla\varphi(z) + \left( ((z - y_0) \cdot \nabla\varphi(z) + \|z - y_0\|^2) \right) O(s^2),$$

and

$$g\left(\frac{z}{t} - \frac{s}{t}y_s\right) = g(z) + s(z - y_0) \cdot \nabla g(z) + \left( ((z - y_0) \cdot \nabla g(z) + \|z - y_0\|^2) \right) O(s^2).$$

As a consequence,

$$g_s(z) = g(z) + s\{\varphi(z) - \varphi(y_0) + (z - y_0) \cdot (\nabla g(z) - \nabla\varphi(z))\} + \left( ((z - y_0) \cdot \nabla(g + \varphi)(z) + \|z - y_0\|^2) \right) O(s^2).$$

Using equation (10) and the expression of the Fenchel-Legendre transformation for a strictly convex function

$$\forall x \in \mathbb{R}^n, \quad \varphi^*(x) = x \cdot (\nabla\varphi)^{-1}(x) - \varphi\left((\nabla\varphi)^{-1}(x)\right),$$

and

$$\forall x \in \mathbb{R}^n, \quad \varphi^*(\nabla\varphi(z)) = \nabla\varphi(z) \cdot z - \varphi(z),$$

we get the result.  $\triangleright$

We are now to deduce our main result: **Proof of Theorem 2.1**

$\triangleleft$  The proof is based on the proof of Theorem 3.2 of [BL00]. First we prove inequality (8) for all functions  $g$ ,  $C^\infty$  with compact support on  $\mathbb{R}^n$ .

Let  $t, s \geq 0$  with  $t + s = 1$  and denote for  $z \in \mathbb{R}^n$ ,

$$g_t(z) = \sup_{z=tx+sy} (g(x) - (t\varphi(x) + s\varphi(y) - \varphi(tx + sy))).$$

We apply Prékopa-Leindler inequality to the functions

$$u(x) = \exp\left(\frac{g(x)}{t} - \varphi(x)\right), \quad v(y) = \exp(-\varphi(y)), \quad w(z) = \exp(g_s(z) - \varphi(z)),$$

and get

$$\left( \int \exp(g/t) d\mu_\varphi \right)^t \leq \int \exp(g_s) d\mu_\varphi.$$

The differentiation of the  $L^p$  norm gives the entropy, and thanks to a Taylor's formula we get

$$\left( \int \exp(g/t) d\mu_\varphi \right)^t = \int e^g d\mu_\varphi + s \mathbf{Ent}_{\mu_\varphi}(e^g) + O(s^2).$$

Then applying Lemma 2.2 yields

$$\begin{aligned} \int \exp(g_s) d\mu_\varphi = \\ \int e^g d\mu_\varphi + s \int \{z \cdot \nabla g(z) - \varphi^*(\nabla \varphi(z)) + \varphi^*(\nabla \varphi(z) - \nabla g(z))\} e^{g(z)} d\mu_\varphi(z) + O(s^2). \end{aligned}$$

When  $s$  goes to 0, inequality (8) arises and can be extended extend it for all smooth enough functions  $g$ .  $\triangleright$

If  $\varphi(x) = \|x\|^2/2 + (n/2) \log(2\pi)$  we obtain the classical logarithmic Sobolev of Gross for the canonical Gaussian measure on  $\mathbb{R}^n$ , with the optimal constant.

## 2.2 Remarks and examples

In the next corollary we recall a classical result of perturbation. If  $\Phi$  is a function on  $\mathbb{R}^n$  such that  $\int e^{-\Phi} dx < \infty$  we note the probability measure  $\mu_\Phi$  by

$$d\mu_\Phi(x) = \frac{e^{-\Phi(x)}}{Z_\Phi} dx, \quad (11)$$

where  $Z_\Phi = \int e^{-\Phi(x)} dx$ .

**Corollary 2.3** *Assume that  $\varphi$  is a  $C^1$ , strictly convex function on  $\mathbb{R}^n$  such that*

$$\lim_{\|x\| \rightarrow \infty} \frac{\varphi(x)}{\|x\|} = \infty.$$

*Let  $\Phi = \varphi + U$ , where  $U$  is a bounded function on  $\mathbb{R}^n$  and denote by  $\mu_\Phi$  the measure defined by (11). Then for all smooth enough functions  $g$  on  $\mathbb{R}^n$ , one has*

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq e^{2\text{osc}(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla \varphi(x)) + \varphi^*(\nabla \varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\Phi(x), \quad (12)$$

where  $\text{osc}(U) = \sup(U) - \inf(U)$ .

### **Proof**

$\triangleleft$  First we observe that

$$e^{-\text{osc}(U)} \leq \frac{d\mu_\Phi}{d\mu_\varphi} \leq e^{\text{osc}(U)}. \quad (13)$$

Moreover we have for all probability measures  $\nu$  on  $\mathbb{R}^n$ ,

$$\mathbf{Ent}_\nu(e^g) = \inf_{a \geq 0} \left\{ \int \left( e^g \log \frac{e^g}{a} - e^g + a \right) d\nu \right\}.$$

Using the fact that for all  $x, a > 0$ ,  $x \log \frac{x}{a} - x + a \geq 0$ , we get

$$e^{-\text{osc}(U)} \mathbf{Ent}_{\mu_\Phi}(e^g) \leq \mathbf{Ent}_{\mu_\varphi}(e^g) \leq e^{\text{osc}(U)} \mathbf{Ent}_{\mu_\Phi}(e^g).$$

Then if  $g$  a smooth enough function on  $\mathbb{R}^n$  we have

$$\begin{aligned} \mathbf{Ent}_{\mu_\Phi}(e^g) &\leq e^{\text{osc}(U)} \mathbf{Ent}_{\mu_\varphi}(e^g) \\ &\leq e^{\text{osc}(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\varphi(x). \end{aligned}$$

The convexity of  $\varphi^*$  on  $\mathbb{R}^n$  and the relation  $\nabla\varphi^*(\nabla\varphi(x)) = x$  lead to

$$\forall x \in \mathbb{R}^n, \quad x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x)) \geq 0.$$

Finally by (13) we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq e^{2\text{osc}(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^g d\mu_\Phi.$$

▷

**Remark 2.4** *It is not necessary to state a tensorization result, as we may obtain exactly the same expression when computing directly with a product measure.*

Using Theorem 2.1 we find also the examples given in [BL00] and [BZ05].

**Corollary 2.5** *Let  $p \geq 2$  and let  $\Phi(x) = \|x\|^p/p$  where  $\|\cdot\|$  is Euclidean norm in  $\mathbb{R}^n$ . Then there exists  $c > 0$ , we get for all smooth enough functions  $g$ , one has*

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq c \int \|\nabla g\|^q e^g d\mu_\Phi, \quad (14)$$

where  $1/p + 1/q = 1$  and  $\mu_\Phi$  is defined on (11).

**Proof**

◁ Using Theorem 2.1, we just have to prove that there exists  $c > 0$  such that,

$$\forall x, y \in \mathbb{R}^n, \quad x \cdot y - \Phi^*(\nabla\Phi(x)) + \Phi^*(\nabla\Phi(x) - y) \leq c\|y\|^q.$$

Assume that  $y \neq 0$  and define the function  $\psi$  by,

$$\psi(x, y) = \frac{x \cdot y - \Phi^*(\nabla\Phi(x)) + \Phi^*(\nabla\Phi(x) - y)}{\|y\|^q}.$$

Then  $\psi$  is a bounded function. We know that  $\Phi^*(x) = \|x\|^q/q$ . Choosing  $z = x\|x\|^{p-2}/\|y\|$  and denoting  $e = y/\|y\|$ , we obtain

$$\psi(x, y) = \bar{\psi}(z, e) = z \cdot e \|z\|^{q-2} - \frac{1}{q} \|z\|^q + \frac{1}{q} \|z - e\|^q.$$

Taylor's formula then yields  $\bar{\psi}(z, e) = O(\|z\|^{q-2})$ . But  $p \geq 2$  implies that  $q \leq 2$ , so that  $\bar{\psi}$  is a bounded function. We then get the result with  $c = \sup \bar{\psi} = \sup \psi$ . ▷

In Proposition 2.1 of [BL00], Bobkov and Ledoux prove that the Prékopa-Leindler inequality implies Brascamp-Lieb inequality. In our case, Theorem 2.1 also implies Brascamp-Lieb inequality, as we can see in the next corollary.

**Corollary 2.6** *Let  $\varphi$  satisfy the conditions of Theorem 2.1 and assume that  $\varphi$  is  $\mathcal{C}^2$  on  $\mathbb{R}^n$ . Then for all smooth enough functions  $g$  we get,*

$$\mathbf{Var}_{\mu_\varphi}(g) \leq \int \nabla g \cdot \text{Hess}(\varphi)^{-1} \nabla g d\mu_\varphi,$$

where  $\text{Hess}(\varphi)^{-1}$  denote the inverse of the Hessian of  $\varphi$ .

**Proof**

◁ Assume that  $g$  is a  $\mathcal{C}^\infty$  function with a compact support and apply inequality (8) with the function  $\epsilon g$  where  $\epsilon > 0$ . Using Taylor's formula we get

$$\mathbf{Ent}_{\mu_\varphi}(\exp \epsilon g) = \frac{\epsilon^2}{2} \mathbf{Var}_{\mu_\varphi}(g) + o(\epsilon^2),$$

and

$$\int \{x \cdot \nabla g(x) - \varphi^*(\nabla \varphi(x)) + \varphi^*(\nabla \varphi(x) - \nabla g(x))\} e^g(x) d\mu_\varphi(x) = \int \frac{\epsilon^2}{2} \nabla g \cdot \text{Hess}(\varphi^*)(\nabla \varphi) \nabla g d\mu_\varphi + o(\epsilon^2).$$

Because of  $\nabla \varphi^*(\nabla \varphi(x)) = x$ , one has that  $\text{Hess}(\varphi^*)(\nabla \varphi) = \text{Hess}(\varphi)^{-1}$  and the corollary is proved.

▷

**Remark 2.7** Let  $\varphi$  satisfy the conditions of Theorem 2.1, and  $L$  be defined by

$$\forall x, y \in \mathbb{R}^n, \quad L(x, y) = \varphi(y) - \varphi(x) + (y - x) \cdot \nabla \varphi(x).$$

The convexity of  $\varphi$  implies that  $L(x, y) \geq 0$  for all  $x, y \in \mathbb{R}^n$ . Let  $F$  be a density of probability with respect to the measure  $\mu_\varphi$ , we defined the following Wasserstein distance with the cost function  $L$  by

$$W_L(F d\mu_\varphi, d\mu_\varphi) = \inf \left\{ \int L(x, y) d\pi(x, y) \right\},$$

where the infimum is taken over all probability measures  $\pi$  on  $\mathbb{R}^n \times \mathbb{R}^n$  with marginal distributions  $F d\mu_\varphi$  and  $d\mu_\varphi$ . Bobkov and Ledoux proved in [BL00] the following transportation inequality

$$W_L(F d\mu_\varphi, d\mu_\varphi) \leq \mathbf{Ent}_{\mu_\varphi}(F). \quad (15)$$

The main theorem of Otto and Villani in [OV00] is the following: Classical logarithmic Sobolev inequality of Gross (when  $\varphi(x) = \|x\|^2/2 + (n/2) \log(2\pi)$ ) implies the transportation inequality (15) for all functions  $F$ , density of probability with respect to  $\mu_\varphi$  (see also [BGL01] for another proof). The method developed in [BGL01], enables to extend the property for  $\varphi(x) = \|x\|^p + Z_\varphi$  ( $p \geq 2$ ). In the general case exposed here, we do not know if the modified logarithmic Sobolev inequality (8) implies transportation inequality (15).

## 3 Applications

### 3.1 Application to modified logarithmic Sobolev inequalities

In [GGM05, GGM06], is given a modified logarithmic Sobolev inequality for measure  $\mu_\varphi$  on  $\mathbb{R}$  is given with a potential between  $|x|$  and  $x^2$ . More precisely let  $\Phi$  be a function on the real line and assume that  $\Phi$  is even and satisfies the following property: there exists  $M \geq 0$  and  $0 < \epsilon \leq 1/2$  such that,

$$\forall x \geq M, \quad (1 + \epsilon)\Phi(x) \leq x\Phi'(x) \leq (2 - \epsilon)\Phi(x). \quad (\text{H})$$

Then there exist  $A, B, D > 0$  such that for all smooth functions  $g$  we have

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq A \int H_\Phi(g') e^g d\mu_\Phi, \quad (16)$$

where

$$H_\Phi(x) = \begin{cases} \Phi^*(Bx) & \text{if } |x| \geq D, \\ x^2 & \text{if } |x| \leq D, \end{cases}$$



and  $\mu_\Phi$  is defined on (11).

The proof of inequality (16) is rather technical and is divided in two parts: the large and the small entropy. Using Theorem 2.1 one obtains two results in this direction. In the next theorem, we extend (16) in the case where the potential is “bigger” than  $x^2$ .

**Theorem 3.1** *Let  $\varphi$  be a  $C^2$  function on  $\mathbb{R}$ . Assume that  $\varphi$  is even,  $\varphi(0) = 0$  and satisfies, for some  $\lambda > 0$ ,*

$$\forall x \in \mathbb{R}, \quad \varphi''(x) \geq \lambda \text{ and } \lim_{|x| \rightarrow \infty} \varphi''(x) = \infty. \quad (17)$$

*Assume also that there exists  $A > 1$  such that for  $|x|$  large enough,*

$$A\varphi(x) \leq x\varphi'(x). \quad (18)$$

*Then there exists  $C > 0$  such that for all smooth enough functions  $g$ ,*

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int H_\varphi(g') e^g d\mu_\varphi, \quad (19)$$

where

$$H_\varphi(x) = \begin{cases} \frac{2A}{A-1}\varphi^*(x) & \text{if } |x| > C \\ \frac{1}{2\lambda}x^2 & \text{if } |x| \leq C. \end{cases} \quad (20)$$

The proof of this theorem is a straightforward application of the following lemma:

**Lemma 3.2** *Assume that  $\varphi$  satisfies conditions of Theorem 3.1, then we get*

$$\forall x, y \in \mathbb{R}, \quad xy - \varphi^*(\varphi'(x)) + \varphi^*(\varphi'(x) - y) \leq H_\varphi(y). \quad (21)$$

**Proof**

◁ We know that  $x = \varphi^{*'}(\varphi'(x))$ , and the convexity of  $\varphi^*$  yields

$$xy - \varphi^*(\varphi'(x)) + \varphi^*(\varphi'(x) - y) \leq y \left( \varphi^{*'}(\varphi'(x)) - \varphi^{*'}(\varphi'(x) - y) \right). \quad (22)$$

Let fixed  $y \in \mathbb{R}$  and note  $\psi_y(x) = \varphi^{*'}(x+y) - \varphi^{*'}(x)$ . The function  $\varphi$  is convex, so one gets for all  $x \in \mathbb{R}$ ,  $\varphi^{*''}(\varphi'(x))\varphi''(x) = 1$ , and the maximum of  $\psi_y(x)$  is reached on  $x_0 \in \mathbb{R}$  which satisfies the condition  $\varphi^{*''}(x_0) = \varphi^{*''}(x_0 + y)$ .

Conditions on  $\varphi$  easily imply that there exists  $C > 0$  such that

$$\forall y \geq C, \quad \sup_{t \geq x+y} \varphi^{*''}(t) < \inf_{t \leq x} \varphi^{*''}(t),$$

so that if  $y \geq C$ , then  $x_0 \in [-y, 0]$ . One obtains that

$$\forall y \geq C, \forall x \in \mathbb{R}, \quad \varphi^{*'}(x+y) - \varphi^{*'}(x) \leq 2\varphi^{*'}(y),$$

and hence

$$\forall |y| \geq C, \forall x \in \mathbb{R}, \quad xy - \varphi^*(\varphi'(x)) + \varphi^*(\varphi'(x) - y) \leq y2\varphi^{*'}(y).$$

By (18) one gets

$$\forall |y| \geq C, \forall x \in \mathbb{R}, \quad xy - \varphi^*(\varphi'(x)) + \varphi^*(\varphi'(x) - y) \leq \frac{2A}{A-1}\varphi^*(y). \quad (23)$$

A Taylor's formula then leads to

$$\forall x, y \in \mathbb{R}, \quad xy - \varphi^*(\varphi'(x)) + \varphi^*(\varphi'(x) - y) \leq \frac{y^2}{2}\varphi^{*''}(\varphi'(x) - \theta y) \leq \frac{y^2}{2\lambda},$$

for some  $\theta \in (0, 1)$ , and (21) follows (23). ▷

**Remark 3.3** • *The last theorem improved the Bakry-Émery criterion. In fact when a probability measure is more log-concave than the Gaussian measure, we obtain a modified logarithmic Sobolev inequality sharper than the classical inequality of Gross. Using a such inequality then one obtains concentration inequality which is more adapted to the probability measure studied.*

- *Theorem 3.1 is more precise than Corollary 2.5 proved by Bobkov, Ledoux and Zegarlinski in [BL00, BZ05]. The particularity of the function  $H_\varphi$  defined on (20) is its behaviour around the origin. If a probability measure satisfies inequality (19) then it satisfies a Poincaré inequality with constant  $1/\lambda$ .*
- *Note also than this method can not be applied for measures with a concentration between  $e^{-|x|}$  and  $e^{-x^2}$  described in [GGM05, GGM06]. In particular Lemma 3.2.*
- *Note finally that the condition (18) is simply a technical condition, which is satisfied for almost functions.*

A natural application of Theorem 3.1 is a concentration inequality in the spirit of Talagrand, see [Tal95].

**Corollary 3.4** *Assume that  $\varphi$  satisfies the conditions of Theorem 3.1 and there exists  $B > 1$  such that for  $|x|$  large enough,*

$$x\varphi'(x) \leq B\varphi(x). \quad (24)$$

*Then there exists constants  $C_1, C_2, C_3 \geq 0$ , independent of  $n$  such that: if  $F$  is a function on  $\mathbb{R}^n$  such that  $\forall i, \|\partial_i F\|_\infty \leq \zeta$ , then we get for  $\lambda \geq 0$ ,*

$$\mu^{\otimes n}(|F - \mu^{\otimes n}(F)| \geq \lambda) \leq \begin{cases} 2 \exp\left(-nC_1\Phi\left(C_2\frac{\lambda}{n\zeta}\right)\right) & \text{if } \lambda > nC_3\zeta, \\ 2 \exp\left(-C_1\frac{\lambda^2}{n\zeta^2}\right) & \text{if } 0 \leq \lambda \leq nC_3\zeta. \end{cases} \quad (25)$$

**Proof**

◁ Using the additional hypothesis (24), the proof of (25) is the same as for Proposition 3.2 of [GGM06]. ▷

When the function  $\varphi$  is strictly convex but does not satisfies (17) then a  $n$ -dimensional version of (19) is available but only for large entropy.

**Theorem 3.5** *Let  $\Phi$  be a  $C^1$ , strictly convex and even function on  $\mathbb{R}^n$ , such that*

$$\lim_{\|x\| \rightarrow \infty} \frac{\Phi(x)}{\|x\|} = \infty.$$

*Assume that  $\Phi \geq 0$  and  $\Phi(0) = 0$  (it implies that 0 is the unique minimum of  $\Phi$ ).*

*Assume that*

$$\lim_{\alpha \rightarrow 0, \alpha \in [0,1]} \sup_{x \in \mathbb{R}^n} \left\{ (1 - \alpha) \frac{\Phi^*\left(\frac{x}{1-\alpha}\right)}{\Phi^*(x)} \right\} = 1, \quad (26)$$

*and also that there exists  $A > 0$  such that*

$$\forall x \in \mathbb{R}^n, \quad x \cdot \nabla \Phi(x) \leq (A + 1)\Phi(x). \quad (27)$$

*Then there exist  $C_1, C_2 \geq 0$  such that for all smooth enough function  $g$  such that  $\int e^g d\mu_\Phi = 1$  and  $\mathbf{Ent}_{\mu_\Phi}(e^g) \geq 1$  we get*

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq C_1 \int \Phi^*(C_2 \nabla g) e^g d\mu_\Phi. \quad (28)$$

**Proof**

◁ Let apply Theorem 2.1 with  $\varphi = \Phi + \log Z_\Phi$ , one has

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq \int \{x \cdot \nabla g(x) - \Phi^*(\nabla\Phi(x)) + \Phi^*(\nabla\Phi(x) - \nabla g(x))\} e^g d\mu_\Phi.$$

Let  $\alpha \in [0, 1]$ ,  $\Phi^*$  is convex so that

$$\forall x \in \mathbb{R}^n, \quad \Phi^*(\nabla\Phi(x) - \nabla g(x)) \leq (1 - \alpha)\Phi^*\left(\frac{\nabla\Phi(x)}{1 - \alpha}\right) + \alpha\Phi^*\left(\frac{-\nabla g(x)}{\alpha}\right). \quad (29)$$

Recall that  $\Phi^*$  is also an even function. Young's inequality implies that

$$\forall x \in \mathbb{R}^n, \quad x \cdot \frac{\nabla g(x)}{\alpha} \leq \Phi(x) + \Phi^*\left(\frac{\nabla g(x)}{\alpha}\right). \quad (30)$$

Using (29) and (30) we get

$$\begin{aligned} \mathbf{Ent}_{\mu_\Phi}(e^g) &\leq 2\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \alpha \int \Phi(x) e^g d\mu_\Phi + \\ &\quad \int \left( (1 - \alpha)\Phi^*\left(\frac{\nabla\Phi(x)}{1 - \alpha}\right) - \Phi^*(\nabla\Phi(x)) \right) e^g d\mu_\Phi. \end{aligned}$$

We have  $\Phi^*(\nabla\Phi(x)) = x \cdot \nabla\Phi(x) - \Phi(x)$ , then inequality (27) implies that  $\Phi^*(\nabla\Phi(x)) \leq A\Phi(x)$ . Because of  $\Phi(0) = 0$  one has  $\Phi^* \geq 0$ , so that

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq \alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + (\alpha + A|\psi(\alpha) - 1|) \int \Phi e^g d\mu_\Phi,$$

where

$$\psi(\alpha) = \sup_{x \in \mathbb{R}^n} \left\{ (1 - \alpha) \frac{\Phi^*\left(\frac{x}{1 - \alpha}\right)}{\Phi^*(x)} \right\}. \quad (31)$$

Let  $\lambda > 0$ , making use of  $\int e^g d\mu_\Phi = 1$  then yields

$$\int \Phi e^g d\mu_\Phi \leq \lambda \left( \mathbf{Ent}_{\mu_\Phi}(e^g) + \log \int e^{\Phi/\lambda} d\mu_\Phi \right).$$

One has  $\lim_{\lambda \rightarrow \infty} \log \int e^{\Phi/\lambda} d\mu_\Phi = 0$ . Let then let now choose  $\lambda$  large enough so that  $\log \int e^{\Phi/\lambda} d\mu_\Phi \leq 1$ . Using the property (26), taking  $\alpha$  such that  $(\alpha + A|\psi(\alpha) - 1|)\lambda \leq 1/4$  implies

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq 2\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \frac{1}{4}(\mathbf{Ent}_{\mu_\Phi}(e^g) + 1).$$

Then a consequence of  $\mathbf{Ent}_{\mu_\Phi}(e^g) \geq 1$  is that

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq 4\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi.$$

▷

### 3.2 Application to Euclidean logarithmic Sobolev inequality

**Theorem 3.6** *Assume that the function  $\varphi$  satisfies conditions of Theorem 2.1. Then for all  $\lambda > 0$  and for all smooth enough functions  $g$  on  $\mathbb{R}^n$ ,*

$$\mathbf{Ent}_{dx}(e^g) \leq -n \log(\lambda e) \int e^g dx + \int \varphi^*(-\lambda \nabla g) e^g dx. \quad (32)$$

*This inequality is optimal in the sense that if  $g = -C(x - \bar{x})$  with  $\bar{x} \in \mathbb{R}^n$  and  $\lambda = 1$  we get an equality.*

**Proof**

◁ Integrating by parts in the second term of (8) yields for all  $g$  smooth enough

$$\int x \cdot \nabla g(x) e^{g(x)} d\mu_\varphi(x) = \int (-n + x \cdot \nabla \varphi(x)) e^{g(x)} d\mu_\varphi(x).$$

Then using the equality  $\varphi^*(\nabla \varphi) = x \cdot \nabla \varphi(x) - \varphi(x)$  we get for all smooth enough functions  $g$ ,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int (-n + \varphi + \varphi^*(\nabla \varphi - \nabla g)) e^g d\mu_\varphi,$$

Let now take  $g = f + \varphi$  to obtain

$$\mathbf{Ent}_{dx}(e^f) \leq \int (-n + \varphi^*(-\nabla g)) e^g dx.$$

Finally, let  $\lambda > 0$  and take  $f(x) = g(\lambda x)$ , we get then

$$\mathbf{Ent}_{dx}(e^g) \leq -n \log(\lambda e) \int e^g dx + \int \varphi^*(-\lambda \nabla g) e^g dx,$$

which proves (32).

If now  $g = -C(x - \bar{x})$  with  $\bar{x} \in \mathbb{R}^n$  an easy computation proves that if  $\lambda = 1$  the equality holds. ▷

In the inequality (32), there exists an optimal  $\lambda_0 > 0$  and when  $C$  is homogeneous, we can improve the last result. We find an inequality called *Euclidean logarithmic Sobolev inequality* which is explained on the next corollary.

**Corollary 3.7** *Let  $C$  be a strictly convex function on  $\mathbb{R}^n$  and assume that  $C$  is  $q$ -homogeneous,*

$$\forall \lambda \geq 0 \quad \text{and} \quad \forall x \in \mathbb{R}^n, \quad C(\lambda x) = \lambda^q C(x).$$

*Then for all smooth enough function  $g$  in  $\mathbb{R}^n$  we get*

$$\mathbf{Ent}_{dx}(e^g) \leq \frac{n}{p} \int e^g dx \log \left( \frac{p}{n e^{p-1} \mathcal{L}^{p/n}} \frac{\int C^*(-\nabla g) e^g dx}{\int e^g dx} \right), \quad (33)$$

where  $\mathcal{L} = \int e^{-C} dx$  and  $1/p + 1/q = 1$ .

**Proof**

◁ Let apply Theorem 3.6 with  $\varphi = C + \log \mathcal{L}$ . Then  $\varphi$  satisfies conditions of Theorem 3.6 and we get then

$$\mathbf{Ent}_{dx}(e^g) \leq -n \log \left( \lambda e \mathcal{L}^{1/n} \right) \int e^g dx + \int C^*(-\lambda \nabla g) e^g dx.$$

Due to the fact that  $C$  is  $q$ -homogeneous an easy computation proves that  $C^*$  is  $p$ -homogeneous where  $1/p + 1/q = 1$ . An optimization over  $\lambda > 0$  gives inequality (33). ▷

Inequality (33) is called Euclidean logarithmic Sobolev inequality and computations of this section is the generalization of the work of Carlen in [Car91]. This inequality with  $p = 2$ , appears in the work of Weissler in [Wei78]. It was discussed and extended to this last version in many articles see [Bec99, DPD03, Gen03, AGK04].

**Remark 3.8** As explained in the introduction, computation used in Corollary 3.7 clearly proves that inequality (33) is equivalent to inequality (32). Agueh, Ghoussoub and Kang, in [AGK04], used Monge-Kantorovich theory for mass transport to prove inequalities (32) and (33). Their approach gives another way to establish Theorem 2.1.

Optimal transportation is also used by Cordero-Erausquin, Gangbo and Houdré in [CEGH04] to prove the particular case of the inequality (14).

Note finally that inequality (33) is optimal, extremal functions are given by  $g(x) = -bC(x - \bar{x})$ , with  $\bar{x} \in \mathbb{R}^n$  and  $b > 0$ . If they are only ones is still an open question.

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