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Abstract

We develop in this paper an amelioration of the method given by S. Bobkov and M. Ledoux in [BL00]. We prove by Prékopa-Leindler Theorem an optimal modified logarithmic Sobolev inequality adapted for all log-concave measure on $\mathbb{R}^n$. This inequality implies results proved by Bobkov and Ledoux, the Euclidean Logarithmic Sobolev inequality generalized in the last years and it also implies some convex logarithmic Sobolev inequalities for large entropy.

Résumé

Dans cet article nous proposons une amélioration de la méthode développée par S. Bobkov et M. Ledoux dans [BL00]. Nous prouvons par le théorème de Prékopa-Leindler une inégalité de Sobolev logarihmique, optimale et adaptée à toutes les mesures log-concaves sur $\mathbb{R}^n$. Cette inégalité implique les résultats de Bobkov et Ledoux, les inégalités de Sobolev logarithlmique de type Euclidien généralisées ces dernières années et enfin cetaines inégalités de Sobolev logarithmiques de type convexe pour les grandes entropies.

1 Introduction

Prékopa-Leindler is the functional form of Brunn-Minkowski inequality. Let $a, b > 0$, $a+b = 0$, and u, v, w three non negative measurable functions on $\mathbb{R}^n$. Assume that, for any $x, y \in \mathbb{R}^n$, we have

$$u(x)^a v(y)^b \leq w(ax + by),$$

then

$$\left(\int u(x)dx \right)^a \left(\int v(x)dx \right)^b \leq \int w(x)dx. \tag{1}$$

If you applied inequality (1) to characteristic functions of bounded measurable sets A and B in $\mathbb{R}^n$, it yields the multiplicative form of the Brunn-Minkowski inequality

$$vol(A)^a vol(B)^b \leq vol(aA + bB),$$

where $aA + bB = \{ax_A + bx_B,\ x_A \in A, x_B \in B\}$. One can see for example two interesting reviews on this topic [Gup80, Mau04].
Bobkov and Ledoux in [BL00] use Prékopa-Leindler Theorem to prove some functional inequalities like Brascamp-Lieb, Logarithmic Sobolev and Transportation inequalities.

More precisely, let φ be a $\mathcal{C}^1$ strictly convex function on $\mathbb{R}^n$ and let

$$d\mu_\varphi(x) = e^{-\varphi(x)}dx,$$

the probability measure on $\mathbb{R}^n$ (assume that $\int e^{-\varphi(x)}dx = 1$). Bobkov-Ledoux prove in particular the following two results:

- (Proposition 2.1) Brascamp-Lieb inequality: assume that φ is a $\mathcal{C}^2$ function then for all smooth enough g,

$$\mathbf{Var}_{\mu_\varphi}(g) := \int \left(g - \int g d\mu_\varphi\right)^2 d\mu_\varphi \leq \int \nabla g \cdot \mathrm{Hess}(\varphi)^{-1} \nabla g d\mu_\varphi, \tag{2}$$

  where $\mathrm{Hess}(\varphi)^{-1}$ is the inverse of the Hessian of φ.

- (Proposition 3.1) Assume that for some $c > 0$ and $p \geqslant 2$, all $t, s > 0$ with $t + s = 1$, and for all $x, y \in \mathbb{R}^n$, φ satisfies

$$t\varphi(x) + s\varphi(y) - \varphi(tx + sy) \geqslant \frac{c}{p}(s + o(s))\|x - y\|^p, \tag{3}$$

  where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. Then for all smooth enough function g,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) := \int e^g \log \frac{e^g}{\int e^g d\mu_\varphi} d\mu_\varphi \leq c \int \|\nabla g\|^q e^g d\mu_\varphi, \tag{4}$$

  where $1/p + 1/q = 1$. They give the example of the function $\varphi(x) = \|x\|^p + Z_\varphi$ (Z_φ is a normalization constant) which satisfies inequality (3) for some constant c.

In this article, we prove also with Prékopa-Leindler Theorem, some optimal logarithmic Sobolev inequality for log-concave measure without conditions like inequality (3). We obtain, for all smooth enough function g on $\mathbb{R}^n$,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\varphi(x), \tag{5}$$

where φ^* is the Frenchel-Legendre transform of φ, $\varphi^*(x) := \sup_{z\in\mathbb{R}^n}\{x \cdot z - \varphi(z)\}$.

The $\mathbf{\Gamma_2}$-criterion of Bakry-Emery implies that if $\mathrm{Hess}(\varphi) \geqslant \lambda Id$ in the sense of symmetric matrix with $\lambda > 0$, then the probability measure μ_φ satisfies classical logarithmic Sobolev inequality, for all smooth function g,

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \frac{1}{2\lambda} \int \|\nabla g\|^2 e^g d\mu_\varphi. \tag{6}$$

This inequality is proved by Gross in [Gro75], one can see also [ABC+00] for a review about this inequality and the related fields. Inequality (5) is then a generalization of the classical logarithmic Sobolev inequality of Gross, adapted for all log-concave measure on $\mathbb{R}^n$ which does'nt satisfies $\mathbf{\Gamma_2}$-criterion. We get an optimal modified logarithmic Sobolev inequality for log-concave measures.

The next section is divided into two subsections. In the first one we give the main theorem of this paper: inequality (5). In the second subsection we explain how the theorem implies results of [BL00]. In particular one find again Brascamp-Lieb inequality (2) or modified logarithmic Sobolev inequality for some function φ, inequality (4). In section 3 we prove that inequality (5) is equivalent to the Euclidean logarithmic Sobolev inequality. In particular it gives a short proof of the generalization given in [DPD03, Gen03, AGK04]. In section 4 we give a convex inequality for large entropy. In particular we obtain a n-dimensional version for large entropy of inequalities prove in [GGM05b, GGM05a].

2 Logarithmic Sobolev inequality

2.1 The main theorem

Theorem 2.1 *Let φ be a $\mathcal{C}^1$ strictly convex function on $\mathbb{R}^n$, such that*

$$\lim_{|x|\to\infty} \frac{\varphi(x)}{\|x\|} = \infty. \tag{7}$$

We note the probability measure

$$\mu_\varphi(dx) = e^{-\varphi(x)}dx,$$

where dx is the Lebesgue measure on $\mathbb{R}^n$, assume that $\int e^{-\varphi(x)}dx = 1$.

Then for all function g on $\mathbb{R}^n$, smooth enough such that integrals used exits we have

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int \{x\cdot\nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\}e^{g(x)}d\mu_\varphi(x). \tag{8}$$

Lemma 2.2 *Let g be a $\mathcal{C}^\infty$ function with a compact support on $\mathbb{R}^n$. Let $s,t \geqslant 0$ with $t+s=1$ and we note for $z\in\mathbb{R}^n$,*

$$g_s(z) = \sup_{z=tx+sy} (g(x) - (t\varphi(x) + s\varphi(y) - \varphi(tx+sy))).$$

Then we get

$$g_s(z) = g(z) + s\{z\cdot\nabla g(z) - \varphi^*(\nabla\varphi(z)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} + O\Big(((z-y_0)\cdot\nabla(g+\varphi)(z) + \|z-y_0\|^2)s^2\Big),$$

where $y_0\in\mathbb{R}^n$.

Proof

$\lhd$ Let $s\in]0,1/2[$ and note $x = z/t - (s/t)y$, hence

$$g_s(z) = \varphi(z) + \sup_{y\in\mathbb{R}^n}\Big(g\Big(\frac{z}{t} - \frac{s}{t}y\Big) - t\varphi\Big(\frac{z}{t} - \frac{s}{t}y\Big) - s\varphi(y)\Big).$$

Due to the fact that g has a compact support and by the property (7) there exists $y_s\in\mathbb{R}^n$ such that

$$\sup_{y\in\mathbb{R}^n}\Big(g\Big(\frac{z}{t} - \frac{s}{t}y\Big) - t\varphi\Big(\frac{z}{t} - \frac{s}{t}y\Big) - s\varphi(y)\Big) = g\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) - t\varphi\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) - s\varphi(y_s).$$

Moreover y_s satisfies

$$\nabla g\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) - t\nabla\varphi\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) + t\nabla\varphi(y_s) = 0. \tag{9}$$

The function φ is a strictly convex function then there is a unique solution y_0 of the equation

$$\nabla\varphi(y_0) = \nabla\varphi(z) - \nabla g(z), \quad y_0 = (\nabla\varphi)^{-1}(\nabla\varphi(z) - \nabla g(z)). \tag{10}$$

We prove now that $\lim_{s\to 0} y_s = y_0$.

First we prove that there exists $R\geqslant 0$ such that $\forall s\in[0,1/2]$, $\|y_s\|\leq R$. Indeed, if the function y_s is not bounded one can found $(s_k)_{k\in\mathbb{N}}$ such that $s_k\to 0$ and $\|y_{s_k}\|\to\infty$. By property (7) $\lim_{\|x\|\to\infty}\varphi(x) = \infty$ then since g is bounded we obtain $s_k y_{s_k} = O(1)$. Due to to the strictly convexity of φ, the last assertion is in contradiction with equation (9).
Let $\hat{y}$ a value of adherence at $s=0$ of the function y_s then $\hat{y}$ satisfies equation (10). By unicity of the solution of (10) we get $\hat{y} = y_0$. Then we have proved that $\lim_{s\to 0} y_s = y_0$.

By Taylor formula and the continuity of y_s at $s = 0$ we get

$$\varphi\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) = \varphi(z) + s(z - y_0)\cdot\nabla\varphi(z) + O\Big(((z - y_0)\cdot\nabla\varphi(z) + \|z - y_0\|^2)s^2\Big),$$

and

$$g\Big(\frac{z}{t} - \frac{s}{t}y_s\Big) = g(z) + s(z - y_0)\cdot\nabla g(z) + O\Big(((z - y_0)\cdot\nabla g(z) + \|z - y_0\|^2)s^2\Big).$$

Then

$$g_s(z) = g(z) + s\{\varphi(z) - \varphi(y_0) + (z - y_0)\cdot(\nabla g(z) - \nabla\varphi(z))\} + O\Big(((z - y_0)\cdot\nabla(g + \varphi)(z) + \|z - y_0\|^2)s^2\Big).$$

Using equation (10) and the expression of the Frenchel-Legendre transformation for a strictly convex function

$$\varphi^*(x) = x\cdot(\nabla\varphi)^{-1}(x) - \varphi\Big((\nabla\varphi)^{-1}(x)\Big),$$

and

$$\varphi^*(\nabla\varphi(z)) = \nabla\varphi(z)\cdot z - \varphi(z),$$

we get the result. $\triangleright$

Proof of Theorem 2.1

$\triangleleft$ The proof is based on the proof of Theorem 3.2 of [BL00]. First we prove inequality (8) for all function g, $\mathcal{C}^\infty$ with a compact support on $\mathbb{R}^n$.
Let $t, s \geqslant 0$ with $t + s = 1$ and we note for $z \in \mathbb{R}^n$,

$$g_t(z) = \sup_{z=tx+sy}(g(x) - (t\varphi(x) + s\varphi(y) - \varphi(tx + sy))).$$

We apply Prékopa-Leindler theorem to the functions

$$u(x) = \exp\left(\frac{g(x)}{t} - \varphi(x)\right), \quad v(y) = \exp(-\varphi(y)), \quad w(z) = \exp(g_s(z) - \varphi(z)),$$

to get

$$\left(\int \exp(g/t)d\mu_\varphi\right)^t \leq \int \exp(g_s)d\mu_\varphi.$$

The derivation of the L^p norm gives the entropy, then using Taylor formula we get

$$\left(\int \exp(g/t)d\mu_\varphi\right)^t = \int e^g\mu_\varphi + s\mathbf{Ent}_{\mu_\varphi}(e^g) + O(s^2).$$

Then apply Lemma 2.2 to get

$$\int \exp(g_s)d\mu_\varphi = \int e^g\mu_\varphi + s\int\{z\cdot\nabla g(z) - \varphi^*(\nabla\varphi(z)) + \varphi^*(\nabla\varphi(z) - \nabla g(z))\}e^{g(z)}d\mu_\varphi(z) + O(s^2).$$

Then when s goes to 0 we get inequality (8).
Then we can extend the inequality (8) for all function g smooth enough such that integrals exist. $\triangleright$

Remark that if $\varphi(x) = \|x\|^2/2 + (n/2)\log(2\pi)$ we obtain the classical logarithmic Sobolev of Gross for the canonical Gaussian measure on $\mathbb{R}^n$.

2.2 Remarks and examples

In the next corollary we give the classical result of perturbation. Of course we lost the optimal constant given in inequality (8).
If Φ is a function on $\mathbb{R}^n$ such that $\int e^{-\Phi} dx < \infty$ we note the probability measure μ_Φ by

$$d\mu_\Phi(x) = \frac{e^{-\Phi(x)}}{Z_\Phi} dx, \tag{11}$$

where $Z_\Phi = \int e^{-\Phi(x)} dx$.

Corollary 2.3 *Assume that φ is a $\mathcal{C}^1$, strictly convex function on $\mathbb{R}^n$ such that $\lim_{|x|\to\infty} \varphi(x)/\|x\| = \infty$. Let $\Phi = \varphi + U$, where U is a bounded function on $\mathbb{R}^n$ and denote by μ_Φ the measure defined by (11).*
Then for all smooth enough function g on $\mathbb{R}^n$. we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq e^{2osc(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\Phi(x), \tag{12}$$

where $osc(U) = \sup(U) - \inf(U)$.

Proof
$\lhd$ First we observe that

$$e^{-\mathrm{osc}(U)} \leq \frac{d\mu_\Phi}{d\mu_\varphi} \leq e^{\mathrm{osc}(U)}. \tag{13}$$

Moreover we have for all probability measure ν on $\mathbb{R}^n$,

$$\mathbf{Ent}_\nu(e^g) = \inf_{a \geqslant 0} \left\{ \int \left(e^g \log \frac{e^g}{a} - e^g + a \right) d\nu \right\},$$

using the fact that $\forall x, a > 0$, $x \log \frac{x}{a} - x + a \geqslant 0$ we get

$$e^{-\mathrm{osc}(U)} \mathbf{Ent}_{\mu_\Phi}(e^g) \leq \mathbf{Ent}_{\mu_\varphi}(e^g) \leq e^{\mathrm{osc}(U)} \mathbf{Ent}_{\mu_\Phi}(e^g).$$

Then if g a smooth enough function g on $\mathbb{R}^n$ we have

$$\begin{aligned} \mathbf{Ent}_{\mu_\Phi}(e^g) &\leq e^{\mathrm{osc}(U)} \mathbf{Ent}_{\mu_\varphi}(e^g) \\ &\leq e^{\mathrm{osc}(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^{g(x)} d\mu_\varphi(x). \end{aligned}$$

Using the fact that φ^* is a convex function on $\mathbb{R}^n$ and $\nabla\varphi^*(\nabla\varphi(x)) = x$ we obtain that

$$\forall x \in \mathbb{R}^n, \quad x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x)) \geqslant 0.$$

Then by (13) we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq e^{2\mathrm{osc}(U)} \int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\} e^g d\mu_\Phi.$$

$\rhd$

Remark 2.4 *It is not necessary to give a tensorisation result because we will obtain exactly the same expression if we compute directly with a product measure.*

Using Theorem 2.1 we find also the same examples given in [BL00] and [BZ05].

Corollary 2.5 *Let $p \geqslant 2$ and let $\Phi(x) = \|x\|^p/p$ where $\|\cdot\|$ is Euclidean norm in $\mathbb{R}^n$. Then we get for all smooth enough function g,*

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq c \int \|\nabla g\|^q e^g d\mu_\Phi, \tag{14}$$

where $1/p + 1/q = 1$ and for some constant $c > 0$.

Proof
$\lhd$ Using Theorem 2.1, we just have to prove that

$$\forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^n, \quad x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x)) \leq c\|y\|^q.$$

Assume that $y \neq 0$ and let note by

$$\psi(x,y) = \frac{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))}{\|y\|^q}.$$

Then ψ is a bounded function. Indeed an easy calculus prove that $\varphi^*(x) = \|x\|^q/q$. Let take now $z = x\|x\|^{p-2}\|y\|$ and $e = y/\|y\|$ then we obtain

$$\psi(x,y) = \bar{\psi}(z,e) = z \cdot e\|z\|^{q-2} - \frac{1}{q}\|z\|^q + \frac{1}{q}\left\|\frac{z}{\|z\|} - \frac{e}{\|z\|}\right\|^q.$$

We have $\|e\| = 1$, then e is bounded. Using Taylor formula we get $\bar{\psi}(z,e) = O(\|y\|^{q-2})$. But $p \geqslant 2$ implies that $q \leq 2$ and then $\bar{\psi}$ is a bounded function. ψ is then a bounded, if $c = \sup\psi$ we get then inequality (14). $\rhd$

We can remark that Proposition 2.5 is not true when $p \in]1,2[$. As we can see in [GGM05b], when $p \in]1,2[$ we have to change the right hand term of inequality (14) and to add a quadratic term.

In Proposition 2.1 of [BL00], Bobkov and Ledoux prove that Prékopa-Leindler's theorem implies Brascamp-Lieb inequality. In our case we prove that Theorem 2.1 implies also some Brascamp-Lieb inequality as we can see in the next corollary.

Corollary 2.6 *Let φ satisfying conditions of Theorem 2.1 and assume that φ is $\mathcal{C}^2$ on $\mathbb{R}^n$. Then for all smooth enough function g we get*

$$\mathbf{Var}_{\mu_\varphi}(g) \leq \int \nabla g \cdot Hess(\varphi)^{-1}\nabla g d\mu_\varphi,$$

where $Hess(\varphi)^{-1}$ denote the inverse of the Hessian of φ.

Proof
$\lhd$ Assume that g is a $\mathcal{C}^\infty$ function with a compact support and let apply inequality (8) with the function ϵg where $\epsilon > 0$. Using Taylor formula we get

$$\mathbf{Ent}_{\mu_\varphi}(\exp\epsilon f) = 2\epsilon^2\mathbf{Var}_{\mu_\varphi}(f) + o(\epsilon^2),$$

and

$$\int \{x \cdot \nabla g(x) - \varphi^*(\nabla\varphi(x)) + \varphi^*(\nabla\varphi(x) - \nabla g(x))\}e^g(x)d\mu_\varphi(x) = \int \frac{\epsilon^2}{2}\nabla g \cdot \mathrm{Hess}(\varphi^*)(\nabla\varphi)\nabla g d\mu_\varphi + o(\epsilon^2).$$

Using the fact that $\nabla\varphi^*(\nabla\varphi(x)) = x$ we get that $\mathrm{Hess}(\varphi^*)(\nabla\varphi) = \mathrm{Hess}(\varphi)^{-1}$ and the corollary is proved. $\rhd$

Remark 2.7 *Let φ satisfying properties of Theorem 2.1. Note*

$$L(x,y) = \varphi(y) - \varphi(x) + (y-x)\nabla\varphi(x),$$

due to the convexity of φ we get that $L(x,y) \geqslant 0$ for all $x, y \in \mathbb{R}^n$.
Let F be a density of probability with respect to the measure μ_φ, we defined the following Wasserstein distance with the cost function equal to L by

$$W_L(Fd\mu_\varphi, d\mu_\varphi) = \inf\left\{\int L(x,y)d\pi(x,y)\right\},$$

where the infimum is taken for all probabilities measures π on $\mathbb{R}^n \times \mathbb{R}^n$ with marginal distributions $Fd\mu_\varphi$ and $d\mu_\varphi$. Then Bobkov and Ledoux prove again in [BL00] the following transportation inequality

$$W_L(Fd\mu_\varphi, d\mu_\varphi) \leq \mathbf{Ent}_{\mu_\varphi}(F)\,. \tag{15}$$

The main theorem of Otto and Villani in [OV00] is the following: Classical logarithmic Sobolev inequality (when $\varphi(x) = \|x\|^2/2 + (n/2)\log(2\pi)$) implies the transportation inequality (15) for all function F, density of probability with respect to μ_φ (see also [BGL01] for an other proof). By the method developed in [BGL01], one can easily extend the property for $\varphi(x) = \|x\|^p + Z_\varphi$ ($p \geqslant 2$).
In the general case exposed here, we don't know if inequality (8) imply inequality (15).

3 Application to Euclidean logarithmic Sobolev inequality

Theorem 3.1 *Assume that the function φ satisfies conditions of Theorem 2.1 then for all $\lambda > 0$ and for all smooth enough function g on $\mathbb{R}^n$ such that integrals exits we get*

$$\mathbf{Ent}_{dx}(e^g) \leq -n\log(\lambda e)\int e^g dx + \int \varphi^*(-\lambda\nabla g)e^g dx. \tag{16}$$

Last inequality is optimal in the sense that if $g = -C(x-\bar{x})$ with $\bar{x} \in \mathbb{R}^n$ and $\lambda = 1$ we get an equality.

Proof
$\lhd$ Using integration by parts on the second term of (8) we obtain for all g smooth enough

$$\int x\cdot\nabla g(x)e^{g(x)}d\mu_\varphi(x) = \int(-n + x\cdot\nabla\varphi(x))e^{g(x)}d\mu_\varphi(x).$$

Then using the equality $\varphi^*(\nabla\varphi) = x\cdot\nabla\varphi(x) - \varphi(x)$ we get for all smooth enough g

$$\mathbf{Ent}_{\mu_\varphi}(e^g) \leq \int(-n + \varphi + \varphi^*(\nabla\varphi - \nabla g))e^g d\mu_\varphi,$$

Let now take $g = f + \varphi$ to obtain

$$\mathbf{Ent}_{dx}\left(e^f\right) \leq \int(-n + \varphi^*(-\nabla g))e^g dx.$$

Let $\lambda > 0$ and take $f(x) = g(\lambda x)$ we get then

$$\mathbf{Ent}_{dx}(e^g) \leq -n\log(\lambda e)\int e^g dx + \int\varphi^*(-\lambda\nabla g)e^g dx,$$

which prove (16).
If now $g = -C(x-\bar{x})$ with $\bar{x} \in \mathbb{R}^n$ an easy calculus prove that if $\lambda = 1$ we get an equality. $\rhd$

In the inequality (16), there exits an optimal $\lambda_0 > 0$. Unfortunately, in the almost case we can't give the expression of the optimal λ_0. It is the unique real satisfying the following equality

$$-n\int e^g dx + \lambda_0 \int \nabla g \cdot \nabla(\varphi^*)(-\lambda_0 \nabla g) e^g dx = 0.$$

But when C is homogeneous, we can give an better expression of the last theorem. We find inequality called Euclidean logarithmic Sobolev inequality which is explained on the next corollary.

Corollary 3.2 *Let C a strictly convex function on $\mathbb{R}^n$ and assume that C is q-homogeneous,*

$$\forall \lambda \geqslant 0 \quad \textit{and} \quad \forall x \in \mathbb{R}^n, \quad C(\lambda x) = \lambda^q C(x).$$

Then for all smooth enough function g in $\mathbb{R}^n$ we get

$$\mathbf{Ent}_{dx}(e^g) \leq \frac{n}{p}\int e^g dx \log\left(\frac{p}{n e^{p-1}\mathcal{L}^{p/n}} \frac{\int C^*(-\nabla g) e^g dx}{\int e^g dx}\right), \tag{17}$$

where $\mathcal{L} = \int e^{-C} dx$ and $1/p + 1/q = 1$.

Proof
$\lhd$ Let apply Theorem 3.1 with $\varphi = C + \log \mathcal{L}$. Then φ satisfies conditions of Theorem 3.1 and we get then

$$\mathbf{Ent}_{dx}(e^g) \leq -n \log\left(\lambda e \mathcal{L}^{1/n}\right) \int e^g dx + \int C^*(-\lambda \nabla g) e^g dx.$$

Due to the fact that C is q-homogeneous an easy calculus prove that C^* is p-homogeneous where $1/p + 1/q = 1$. An optimization over $\lambda > 0$ gives inequality (17). $\rhd$

Inequality (17) is called Euclidean logarithmic Sobolev inequality. This inequality with $p = 2$ appears in the work of Weissler in [Wei78]. It was discussed and extended to this last version in many articles see [Car91, Led96, Bec99, DPD03, Gen03, AGK04].

Remark 3.3 *Of course as it is explained in the introduction, calculus used in Corollary 3.2 prove that inequality* (17) *is equivalent to inequality* (16)*. Agueh, Ghoussoub and Kang, in [AGK04], used Monge-Kantorovich theory for mass transport to prove inequalities* (16) *and* (17)*. Then it gives an other way to establish Theorem 2.1.*

Note also that inequality (17) *is optimal, extremal functions is given by $g(x) = -bC(x - \bar{x})$, with $\bar{x} \in \mathbb{R}^n$ and $b > 0$. But we don't know if it's only extremal functions.*

4 Application to logarithmic Sobolev inequality for large entropy

In [GGM05b, GGM05a] is given a convex logarithmic Sobolev inequality for measure μ_φ between $e^{-|x|}$ and e^{-x^2}. More precisely let Φ a function on the real line and assume that Φ is even and satisfies the following property, there exists $M \geqslant 0$ and $0 < \varepsilon \leq 1/2$ such that

$$\forall x \geqslant M, \quad (1+\varepsilon)\Phi(x) \leq x\Phi'(x) \leq (2-\varepsilon)\Phi(x). \tag{H}$$

Then there exists $A, B, D > 0$ such that for all smooth functions g we have

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq A \int H_\Phi\left(g'\right) e^g d\mu_\Phi, \tag{18}$$

where

$$H_\Phi(x) = \begin{cases} \Phi^*(Bx) & \text{if } |x| \geqslant D, \\ x^2 & \text{if } |x| \leq D, \end{cases}$$

and μ_Φ is defined on (11).
The proof of inequality (18) is technical and it divided between two parts: the large and the small entropy. We give in the next theorem a n-dimensional version of this inequality but only for large entropy.

Theorem 4.1 *Let Φ be a $\mathcal{C}^1$, strictly convex and even function on $\mathbb{R}^n$, such that $\lim_{|x|\to\infty} \Phi(x)/\|x\| = \infty$. Assume that $\Phi \geqslant 0$ and $\Phi(0) = 0$ (it implies that 0 is the unique minimum of Φ).*
Assume that

$$\lim_{\alpha\to 0,\, \alpha\in[0,1]} \sup_{x\in\mathbb{R}^n} \left\{ (1-\alpha)\frac{\Phi^*\left(\frac{x}{1-\alpha}\right)}{\Phi^*(x)} \right\} = 1, \tag{19}$$

assume also that there exists $A > 0$ such that

$$\forall x \in \mathbb{R}^n, \quad x \cdot \nabla\Phi(x) \leq (A+1)\Phi(x). \tag{20}$$

Then there exists $C_1, C_2 \geqslant 0$ such that for all smooth enough function g such that $\int e^g d\mu_\Phi = 1$ and $\mathbf{Ent}_{\mu_\Phi}(e^g) \geqslant 1$ we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq C_1 \int \Phi^*(C_2\nabla g) e^g d\mu_\Phi. \tag{21}$$

Proof
$\lhd$ Let apply Theorem 2.1 with $\varphi = \Phi + \log Z_\Phi$ we get then

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq \int \{x \cdot \nabla g(x) - \Phi^*(\nabla\Phi(x)) + \Phi^*(\nabla\Phi(x) - \nabla g(x))\} e^g d\mu_\Phi.$$

Let $\alpha \in [0,1[$, Φ^* is convex then

$$\Phi^*(\nabla\Phi(x) - \nabla g(x)) \leq (1-\alpha)\Phi^*\left(\frac{\nabla\Phi(x)}{1-\alpha}\right) + \alpha\Phi^*\left(\frac{-\nabla g(x)}{\alpha}\right), \tag{22}$$

recall that Φ^* is also a even function. Young's inequality implies that

$$x \cdot \frac{\nabla g(x)}{\alpha} \leq \Phi(x) + \Phi^*\left(\frac{\nabla g(x)}{\alpha}\right). \tag{23}$$

Using (22) and (23) we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq 2\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \alpha \int \Phi(x) e^g d\mu_\Phi + \int \left((1-\alpha)\Phi^*\left(\frac{\nabla\Phi(x)}{1-\alpha}\right) - \Phi^*(\nabla\Phi(x)) \right) e^g d\mu_\Phi.$$

We have $\Phi^*(\nabla\Phi(x)) = x \cdot \nabla\Phi(x) - \Phi(x)$, then inequality (20) implies that $\Phi^*(\nabla\Phi(x)) \leq A\Phi(x)$. Due to the fact that $\Phi(0) = 0$ we have $\Phi^* \geqslant 0$ we get

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq \alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + (\alpha + A|\psi(\alpha) - 1|) \int \Phi e^g d\mu_\Phi,$$

where

$$\psi(\alpha) = \sup_{x\in\mathbb{R}^n} \left\{ (1-\alpha)\frac{\Phi^*\left(\frac{x}{1-\alpha}\right)}{\Phi^*(x)} \right\}. \tag{24}$$

Let $\lambda > 0$ then due to the fact that $\int e^g d\mu_\Phi = 1$ we get

$$\int \Phi e^g d\mu_\Phi \leq \lambda \left(\mathbf{Ent}_{\mu_\Phi}(e^g) + \log \int e^{\Phi/\lambda} d\mu_\Phi \right).$$

We have $\lim_{\lambda\to\infty} \log \int e^{\Phi/\lambda} d\mu_\Phi = 0$, then let now choose λ large enough such that $\log \int e^{\Phi/\lambda} d\mu_\Phi \leq 1$. Using the property (19), take α such that $(\alpha + A|\psi(\alpha) - 1|)\lambda \leq 1/4$ we obtain

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq 2\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi + \frac{1}{4}(\mathbf{Ent}_{\mu_\Phi}(e^g) + 1).$$

Then using $\mathbf{Ent}_{\mu_\Phi}(e^g) \geqslant 1$ we obtain

$$\mathbf{Ent}_{\mu_\Phi}(e^g) \leq 4\alpha \int \Phi^*\left(\frac{\nabla g}{\alpha}\right) e^g d\mu_\Phi.$$

$\triangleright$

We need a lemma to give non-trivial examples. This lemma explains how property (19) is a infinity property.

Lemma 4.2 *Let Φ_1 and Φ_2 be two strictly convex and even functions such that $\Phi_1, \Phi_2 \geqslant 0$, $\Phi_1(0) = \Phi_2(0) = 0$ and $\lim_{|x|\to\infty} \Phi_1(x)/\|x\| = \lim_{|x|\to\infty} \Phi_2(x)/\|x\| = \infty$. Assume also that $\Phi_1(x) \overset{\pm\infty}{\sim} \Phi_2(x)$.*

If Φ_2 satisfies the property (19) then Φ_1 satisfies also the same property.

Proof

$\triangleleft$ First we prove that $\Phi_1^*(x) \overset{\pm\infty}{\sim} \Phi_2^*(x)$. Let $\epsilon > 0$, then there exists $A > 0$ such that

$$\forall y \in \mathbb{R}^n, \quad \|y\| \geqslant A, \quad (1-\epsilon)\Phi_2(y) \leq \Phi_1(y) \leq (1+\epsilon)\Phi_2(y),$$

then

$$\forall x \in \mathbb{R}^n, \quad \sup_{\|y\|\geqslant A} \{x\cdot y - (1+\epsilon)\Phi_2(y)\} \leq \sup_{\|y\|\geqslant A} \{x\cdot y - \Phi_1(y)\} \leq \sup_{\|y\|\geqslant A} \{x\cdot y - (1-\epsilon)\Phi_2(y)\}.$$

Φ_1 and Φ_2 are strictly convex then there exists $B > 0$ such that

$$\forall x \in \mathbb{R}^n, \quad \|x\| \geqslant B, \quad \Phi_1^*(x) = \sup_{\|y\|\geqslant A} \{x\cdot y - \Phi_1(y)\},$$

and the same for Φ_2, then

$$\forall x \in \mathbb{R}^n,\ \|x\| \geqslant B, \quad (1+\epsilon)\Phi_2^*\left(\frac{x}{1+\epsilon}\right) \leq \Phi_1^*(x) \leq (1-\epsilon)\Phi_2^*\left(\frac{x}{1-\epsilon}\right).$$

Using now property (19) for Φ_2 we get

$$\forall x \in \mathbb{R}^n, \quad \Phi_2^*(x) \leq \psi\left(\frac{\epsilon}{1+\epsilon}\right)(1+\epsilon)\Phi_2^*\left(\frac{x}{1+\epsilon}\right) \text{ and } \Phi_2^*\left(\frac{x}{1-\epsilon}\right) \leq \frac{\psi(\epsilon)}{1-\epsilon}\Phi_2^*(x),$$

where ψ is defined on (24). We get then

$$\forall x \in \mathbb{R}^n,\ \|x\| \geqslant B, \quad \psi\left(\frac{\epsilon}{1+\epsilon}\right)^{-1} \Phi_2^*(x) \leq \Phi_1^*(x) \leq \psi(\epsilon)\Phi_2^*(x).$$

The function Φ_2 satisfies (19) then $\lim_{\alpha\to 0} \psi(\alpha) = 1$ then $\Phi_1^*(x) \overset{\pm\infty}{\sim} \Phi_2^*(x)$.
The end of the proof is elementary, we just have to remark that using a compact argument we get

$$\forall A > 0, \quad \lim_{\alpha\to 0,\, \alpha\in[0,1]} \sup_{\|x\|\leq A} \left\{ (1-\alpha)\frac{\Phi_1^*\left(\frac{x}{1-\alpha}\right)}{\Phi_1^*(x)} \right\} = 1.$$

Then, when $\|x\|$ is large Φ_1^* is equivalent to Φ_2^*. $\triangleright$

Example 4.3 • *Let Φ be a $\mathcal{C}^1$, strictly convex function on $\mathbb{R}^1$. Assume that $\Phi \geqslant 0$ and $\Phi(0) = 0$. Assume that*

$$\forall x \in \mathbb{R},\ |x| \geqslant 2, \quad \Phi(x) = x^a \log^b x$$

with $a > 1$ and $b \in \mathbb{R}$. Then Φ satisfies property (19). Remark that if $a \in]1, 2[$ and $b = 0$ then the measure μ_Φ doesn't satisfies (14) for small entropy.

- *Here is now an example of measure on $\mathbb{R}^n$ with interactions. Let h be a $\mathcal{C}^1$, strictly convex function on $\mathbb{R}^1$. Assume that $h \geqslant 0$, $h(0) = 0$ and that h satisfies assumptions (19) and (20). Assume also that*

$$\lim_{|x| \to \infty} \frac{h(x)}{x^2} = +\infty. \tag{25}$$

Note

$$\Phi(x) = \sum_{i=1}^{n} (x_i x_{i+1} + h(x_i)),$$

where $x = (x_1, \cdots, x_n)$ and $x_{n+1} = x_1$. Then it's easy to prove that Φ is convex, even with $\Phi(0) = 0$ and satisfies inequality (20). Then using (25) we get that

$$\Phi(x) \overset{\pm\infty}{\sim} \sum_{i=1}^{n} h(x_i).$$

By Lemma 4.2 we prove that Φ satisfy (19).

This example in interesting because it gives an measure on $\mathbb{R}^n$ which is not a product measure on $\mathbb{R}^n$ and satisfies inequality (21) for large entropy.

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