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Further examples of explicit Krein representations of certain subordinators

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Abstract : In a previous paper [1], we have shown that the gamma subordinators may be represented as inverse local times of certain diffusions. In the present paper, we give such representations for other subordinators whose Lévy densities are of the form $\frac{\mathcal{C}}{(\sinh(y))^\gamma}$, $0 < \gamma < 2$, and the more general family obtained from those by exponential tilting.

Keywords : Subordinators, Krein correspondence, inverse local times.

1 Aim of the paper and summary of [1]

1.1 In this paper, we continue the program started in [1], that is : to represent as many subordinators $(\mathcal{S}_\ell, \ell \geq 0)$, i.e : increasing Lévy processes, started at 0, as possible as inverse local times $(\tau_\ell, \ell \geq 0)$ of some particular \mathbb{R}_+ -valued diffusion (X_t) , such that 0 is regular for itself (relatively to X). More precisely, assume that :

$$E[\exp(-\lambda \mathcal{S}_\ell)] = \exp(-\ell \Psi(\lambda)),$$

where : $\Psi(\lambda) = \int_0^\infty \nu(dy)(1 - e^{-\lambda y})$, and $\nu(dy)$ - the Lévy measure associated with $(\mathcal{S}_\ell, \ell \geq 0)$ - is of the form :

$$\nu(dy) = h(y)dy, \quad \text{with} \quad h(y) = \int_0^\infty d\lambda(x) e^{-xy},$$

for some positive σ -finite measure $\lambda(dx)$ on \mathbb{R}_+ , then, it is known, as a consequence of Krein's theory (cf : Knight [5], Kotani-Watanabe [6]), that there exists a unique diffusion (X_t) taking values in \mathbb{R}_+ , such that its inverse local time at 0, $(\tau_\ell, \ell \geq 0)$, is distributed as $(\mathcal{S}_\ell, \ell \geq 0)$.

Finding X when ν is given is called (here) *Krein representation problem*.

In our paper [1], we could fill in the following

Table 1

$h(y)$	Generator of (X_t)	Distribution
$\frac{c}{y^{\alpha+1}}$ ($0 < \alpha < 1$)	$\mathcal{L}_{-\alpha} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\delta-1}{2x} \frac{d}{dx}; \quad \delta = 2(1 - \alpha)$	P^δ
$\frac{c}{y^{\alpha+1}} e^{-\mu y}$ ($0 < \alpha < 1; \mu > 0$)	$\mathcal{L}_{-\alpha}^{\mu \downarrow} = \mathcal{L}_{-\alpha} + \sqrt{2\mu} \frac{\widehat{K}'_\alpha(\sqrt{2\mu}x)}{\widehat{K}_\alpha(\sqrt{2\mu}x)} \frac{d}{dx}$ where $\widehat{K}_\alpha(y) = y^\alpha K_\alpha(y), y > 0$	$P^{\delta; \mu \downarrow}$
$\frac{c}{y} e^{-\mu y}$ ($\mu > 0$)	$\mathcal{L}_0^{\mu \downarrow} = \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{1}{2x} + \sqrt{2\mu} \frac{K'_0}{K_0}(\sqrt{2\mu}x) \right) \frac{d}{dx}$	$P^{2; \mu \downarrow}$

1.2 Details of Table 1

In fact, the result for the first row goes back at least to Molchanov-Ostrovski [8], the result for the second row is deduced from that in the first row with the help of the following discussion, which relates Esscher transforms (of subordinators) to Girsanov transforms (of diffusions).

If $(X_t)_{t \geq 0}$ is a \mathbb{R}_+ -valued diffusion, whose inverse local time at 0 :

$$\tau_\ell = \inf\{t : L_t > \ell\}, \ell \geq 0,$$

admits Lévy measure $\nu(dy)$, and Lévy exponent $(\Psi(\theta), \theta \geq 0)$, and if one defines :

$$\varphi_{\theta\downarrow}(x) = E_x[\exp(-\theta T_0(X))],$$

then, there is another diffusion, which we shall denote by $(X_t^{\theta\downarrow}, t \geq 0)$, with laws $(P_x^{\theta\downarrow}, x \geq 0)$, such that :

$$P_{x|\mathcal{F}_t}^{\theta\downarrow} = \frac{\varphi_{\theta\downarrow}(X_t)}{\varphi_{\theta\downarrow}(x)} \exp(\Psi(\theta)L_t - \theta t) \cdot P_{x|\mathcal{F}_t} \quad (1.1)$$

whose inverse local time $(\tau_\ell, \ell \geq 0)$ under $P_0^{\theta\downarrow}$ satisfies :

$$E_0^{\theta\downarrow}(\exp -\lambda \tau_\ell) = \exp(-\ell(\Psi(\lambda + \theta) - \Psi(\theta)))$$

i.e : this inverse local time is the θ -Esscher transform of $(\tau_\ell, \ell \geq 0)$ under P_0 : its Lévy measure (under $P_0^{\theta\downarrow}$) is : $e^{-\theta y} \nu(dy)$.

It is also noteworthy that, under some adequate restriction of their domains, the infinitesimal generators $\mathcal{L}^{\theta\downarrow}$ and \mathcal{L} are related by :

$$\mathcal{L}^{\theta\downarrow} = \mathcal{L} + \frac{d}{dx} (\log(\varphi_{\theta\downarrow}(x))) \cdot \frac{d}{dx}$$

Finally, the result for the third row was deduced by letting $\alpha \rightarrow 0$ in the second row, while taking care of the choice¹ of the local times made for $\mathcal{L}_{-\alpha, \mu\downarrow}$. (A compendium of choices of local times for Bessel-like diffusions is made in [2]).

1.3 In the present paper, we wish to complete the preceding Table 1, by considering the 3 parameter family of Lévy measures on \mathbb{R}_+ :

$$\nu_{\mu, \alpha, k}(dy) = \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp(\mu k y) dy \quad (1.2)$$

¹As is well-known, the local time in a standard Markovian set up, at a given level, is unique up to a multiplicative constant, which for our studies, needs to be chosen carefully.

(The "true" parameters are : $\mu > 0$, k , and α ; as before, \mathcal{C} is simply there to ensure an additional degree of freedom, if necessary).

In order that $\nu_{\mu, \alpha, k}(dy)$ be a Lévy measure, i.e : it must satisfy

$$\int_0^\infty (x \wedge 1) \nu_{\mu, \alpha, k}(dx) < \infty, \quad \text{we need : } 0 \leq \alpha < 1; k < 1 + \alpha.$$

We now recall that, from Pitman-Yor [11] formulae (16), p. 276), if $Q_z^{\delta, \mu}$, for $0 < \delta \equiv 2(1 - \alpha) < 2$, and $\mu > 0$, denotes the distribution of the squared radial Ornstein-Uhlenbeck process, with "dimension" δ , and parameter μ , started from z , i.e : the solution of :

$$dZ_t = 2\sqrt{Z_t}dB_t + (\delta - 2\mu Z_t)dt; \quad Z_t \geq 0, Z_0 = z,$$

then, under $Q_0^{\delta, \mu}$, the inverse local time $(\tau_\ell, \ell \geq 0)$ admits as its Lévy measure :

$$\mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp(\mu \frac{\delta}{2} y) dy \quad (1.3)$$

which is a particular case of (1.2), with $k = \frac{\delta}{2} = (1 - \alpha)$.

In the next section, we shall show, essentially with the help of the recipe (1.1), how to construct a diffusion, indexed by the 3 parameters (α, μ, k) , which solves the Krein representation problem for $\nu_{\mu, \alpha, k}$.

1.4 Some among the new diffusions we are finding as solutions of Krein's problem are related to the diffusions we found in [1] by time changing. We first discovered this relationship by applying the analytical identity :

$$W_{0, \beta}(z) = \sqrt{\frac{z}{\pi}} K_\beta \left(\frac{z}{2} \right) \quad (1.4)$$

between $W_{0, \beta}$, a Whittaker function with parameters $(0, \beta)$, and K_β (see Appendix). Thus, a part of our present discussion may be considered as giving a probabilistic interpretation to (1.4).

We also develop a similar discussion for the analytical identity

$$M_{0, \beta}(z) = 4^\beta \Gamma(\beta + 1) \sqrt{z} I_\beta \left(\frac{z}{2} \right).$$

2 Solving Krein's problem for $\nu_{\mu, \alpha, k}$

2.1 We take up the notation in (1.3); in fact, it is more convenient to consider the family of radial Ornstein-Uhlenbeck processes (and not their

squares), which we shall denote as $(R^{\delta,\mu}(t), t \geq 0)$ and their laws $(P_r^{\delta,\mu}, r \geq 0)$. It will be helpful, for the sequel, to have the following formula at hand, for the infinitesimal generator $\mathcal{L}_{-\alpha,\mu}$ of $R^{\delta,\mu}$:

$$\mathcal{L}_{-\alpha,\mu} = \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{\delta - 1}{2x} - \mu x \right) \frac{d}{dx} \quad (2.1)$$

It is well-known (see, e.g., Pitman-Yor [9], p. 454, formula (6.b)) that there is the relationship :

$$R^{\delta,\mu}(t) = e^{-\mu t} R^\delta \left(\frac{e^{2\mu t} - 1}{2\mu} \right), \quad t \geq 0, \quad (2.2)$$

where, on the RHS, $(R^\delta(u), u \geq 0)$ denotes a δ -dimensional Bessel process. Thus, we obtain :

$$\left(\frac{e^{2\mu T_0} - 1}{2\mu}; P_x^{\delta,\mu} \right) \stackrel{(law)}{\underset{(a)}{=}} (T_0; P_x^\delta) \stackrel{(law)}{\underset{(b)}{=}} \frac{x^2}{2\gamma_\alpha} \quad (2.3)$$

where, on the RHS, γ_α denotes a gamma variable with parameter α . [(a) follows from (2.2), while (b) is well-known, and goes back to Gettoor [3], see, e.g., Yor [13], for some variants...].

We thus deduce the following formula from (2.3), with the help of elementary computations :

$$\begin{aligned} E_x^{\delta,\mu}(\exp(-\theta T_0)) &= \frac{1}{\Gamma(\alpha)(\mu x^2)^{\frac{\theta}{2\mu}}} \int_0^\infty \frac{t^{\alpha-1+\frac{\theta}{2\mu}} e^{-t}}{\left(1 + \frac{t}{\mu x^2}\right)^{\frac{\theta}{2\mu}}} dt \\ &= \frac{\Gamma(\alpha + \frac{\theta}{2\mu})}{\Gamma(\alpha)} (\mu x^2)^{\frac{\alpha-1}{2}} e^{\mu \frac{x^2}{2}} W_{\frac{(1-\alpha)-\frac{\theta}{2\mu}}{2}, \frac{\alpha}{2}}(\mu x^2) \quad \text{for } -2\alpha\mu < \theta \end{aligned} \quad (2.4)$$

where $W_{a,b}$ denotes the Whittaker function, with parameters (a, b) .

2.2 We now write :

$$\begin{aligned} \nu_{\mu,\alpha,k}(dy) &= \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp(\mu k y) dy \\ &\equiv \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp\left(\mu \frac{\delta}{2} y\right) \exp(-\theta y) dy, \end{aligned}$$

where : $\theta = \mu \left(\frac{\delta}{2} - k \right)$, $k < 1 + \alpha$.

According to the preceding computation, we now find that the diffusion with infinitesimal generator :

$$\mathcal{L}_{-\alpha,\mu}^{\theta\downarrow} \equiv \mathcal{L}_{-\alpha,\mu} + \frac{d}{dx} \log \left\{ (\mu x^2)^{\frac{\alpha-1}{2}} e^{\mu \frac{x^2}{2}} W_{\frac{(1-\alpha)-\frac{\theta}{2\mu}}{2}, \frac{\alpha}{2}}(\mu x^2) \right\} \cdot \frac{d}{dx} \quad (2.5)$$

solves Krein's representation problem for $\nu_{\mu, \alpha, k}$. (We note in fact that : $\frac{(1-\alpha) - \frac{\theta}{\mu}}{2} = \frac{k}{2}$, so that : $W_{\frac{(1-\alpha) - \frac{\theta}{\mu}}{2}, \frac{\alpha}{2}}(\xi) \equiv W_{\frac{k}{2}, \frac{\alpha}{2}}(\xi)$).

The case where $k = 0$ is particularly interesting, since, on one hand :

$$\nu_{\mu, \alpha, 0}(dy) = \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} dy,$$

and on the other hand (see Appendix) :

$$W_{0, \frac{\alpha}{2}}(\xi) = \sqrt{\frac{\xi}{\pi}} K_{\frac{\alpha}{2}} \left(\frac{\xi}{2} \right) \quad (2.6)$$

so that the diffusion which solves Krein's representation problem for $\nu_{\mu, \alpha, 0}(dy)$ is the solution to :

$$dX_t = dB_t + \left[\frac{\delta - 1}{2X_t} + \mu X_t \left(\frac{\widehat{K}'_{\frac{\alpha}{2}}}{\widehat{K}_{\frac{\alpha}{2}}} \right) \left(\mu \frac{X_t^2}{2} \right) \right] dt \quad (2.7)$$

Here, we need to give some details about this computation :

- a) We deduce from formula (2.5), in the particular case $k = 0$, i.e : $\theta = \frac{\delta\mu}{2}$ with the help of formula (2.6), that :

$$\mathcal{L}_{-\alpha, \mu}^{\theta\downarrow} = \mathcal{L}_{-\alpha, \mu} + \left(\frac{\alpha}{x} + \mu x + \mu x \frac{K'_{\frac{\alpha}{2}} \left(\mu \frac{x^2}{2} \right)}{K_{\frac{\alpha}{2}} \left(\mu \frac{x^2}{2} \right)} \right) \cdot \frac{d}{dx}$$

- b) Now, trivially :

$$\mathcal{L}_{-\alpha, \mu} + \mu x \frac{d}{dx} = \mathcal{L}_{-\alpha},$$

and, equally simply :

$$\frac{\alpha}{x} + \mu x \frac{K'_{\frac{\alpha}{2}} \left(\mu \frac{x^2}{2} \right)}{K_{\frac{\alpha}{2}} \left(\mu \frac{x^2}{2} \right)} = \mu x \frac{\widehat{K}'_{\frac{\alpha}{2}}}{\widehat{K}_{\frac{\alpha}{2}}} \left(\mu \frac{x^2}{2} \right)$$

which translates into the stochastic differential equation form of formula (2.7).

As an introduction to the next discussion, we write down

Table 2

$h(y)$	Generator of (X_t)	Distribution
$\frac{c}{y^{\alpha+1}} e^{-\mu y}$ ($0 < \alpha < 1; \mu > 0$)	$\mathcal{L}_{-\alpha}^{\mu\downarrow} = \mathcal{L}_{-\alpha} + \sqrt{2\mu} \frac{\widehat{K}'_{\alpha}}{\widehat{K}_{\alpha}}(\sqrt{2\mu}x) \frac{d}{dx}$	$P^{\delta; \mu\downarrow}$ ($\delta = 2(1 - \alpha)$)
$\mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} e^{\mu \frac{\delta}{2} y}$ $0 < \delta = 2(1 - \alpha) < 2; \mu > 0$	$\mathcal{L}_{-\alpha, \mu} \equiv \mathcal{L}_{-\alpha} - \mu x \frac{d}{dx}$	$P^{\delta, \mu}$
$\mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1}$	$\mathcal{L}_{-\alpha, \mu}^{\theta\downarrow} \equiv \mathcal{L}_{-\alpha} + \mu x \frac{\widehat{K}'_{\frac{\alpha}{2}}}{\widehat{K}_{\frac{\alpha}{2}}} \left(\mu \frac{x^2}{2} \right) \frac{d}{dx}$ ($\theta = \frac{\delta\mu}{2}$)	$P^{\delta, \mu; \frac{\delta\mu}{2}\downarrow}$

The first row is simply taken from Table 1 (second row there).

As said above, the second row follows from Pitman-Yor [11]. In the third row, we have written $\mathcal{L}_{-\alpha, \mu}^{\theta\downarrow}$ for the infinitesimal generator of the process which is defined as : the radial Ornstein-Uhlenbeck process, with dimension $\delta = 2(1 - \alpha)$, and drift parameter $(-\mu)$, pushed downwards with parameter $\theta = \frac{\delta\mu}{2}$. That this infinitesimal generator may be expressed in terms of $\widehat{K}_{\frac{\alpha}{2}}$ will be discussed after (2.7).

2.3 We shall now prove a remarkable relationship between the two families of diffusions whose infinitesimal generators are found on the RHS of Table 2. This relationship explains precisely why (Row 1) may be deduced from (Row 2), and vice-versa.

Proposition.

The following relationship holds with : $\theta = \frac{\delta\mu}{2}$:

$$X^2_{-\alpha, \mu; \theta\downarrow}(t) = X_{-\frac{\alpha}{2}; \left(\frac{\mu^2}{8}\right)\downarrow} \left(4 \int_0^t X^2_{-\alpha, \mu; \theta\downarrow}(u) du \right) \quad (2.8)$$

meaning that : starting from $X \equiv X_{-\alpha, \mu; \theta \downarrow}$ on the LHS, there exists $\left(X_{-\frac{\alpha}{2}; \frac{\mu^2}{8}}(u), u \geq 0\right)$ such that the relationship (2.8) holds.

Comment about our notation

In formula (2.8), and possibly several times below, we have written $X_{i; \theta \downarrow}$, etc... instead of $X_i^{\theta \downarrow}$, for some index i . It seemed more appropriate here, because of the power 2 on the left-side of (2.7).

There should be no confusion between the different diffusions $X_{i, \theta}$ and $X_{i; \theta \downarrow}$.

Proof

We start from the stochastic differential equation satisfied by $(X_{-\alpha, \mu; \theta \downarrow}(t), t \geq 0)$ as described (implicitly) in Row 2 of Table 2.

Then, taking squares, we obtain :

$$X_t^2 = x^2 + 2 \int_0^t X_s dB_s + \delta t + 2 \int_0^t (\mu X_s^2) \frac{\widehat{K}'_{\frac{\alpha}{2}}}{\widehat{K}_{\frac{\alpha}{2}}} \left(\mu \frac{X_s^2}{2} \right) ds$$

We now define $(Y_u \equiv Y(u), u \geq 0)$ via :

$$X_t^2 = Y \left(4 \int_0^t X_s^2 ds \right) , \quad (t \geq 0)$$

and find that Y satisfies :

$$Y_u = x^2 + \beta_u + \left(\frac{\delta}{4} \right) \int_0^u \frac{ds}{Y_s} + \frac{1}{2} \int_0^u \mu \frac{\widehat{K}'_{\frac{\alpha}{2}}}{\widehat{K}_{\frac{\alpha}{2}}} \left(\frac{\mu}{2} Y_s \right) ds$$

since $\frac{\delta}{4} = \frac{\widehat{\delta} - 1}{2}$, with $\widehat{\delta} = 2 - \alpha = 2(1 - \frac{\alpha}{2})$, we find that $(Y_u, u \geq 0)$ is precisely the diffusion with infinitesimal generator $\mathcal{L}_{-\frac{\alpha}{2}}^{\nu \downarrow}$, with $\sqrt{2\nu} = \frac{\mu}{2}$, i.e $\nu = \frac{\mu^2}{8}$. ■

We now remark that the proof we have just given for the Proposition relies upon the identification of the infinitesimal generator of the diffusion $X_{-\alpha, \mu; (\frac{\delta \mu}{2}) \downarrow}$ as given in Table 2; this identification was obtained from an analytical identity between $W_{0, \cdot}$ and $K_{\cdot, \cdot}$ (see formula (2.5)).

We now explain and prove the Proposition without relying on such identities, but rather on absolute continuity relationships between the different laws involved.

We now find it a little more convenient to refer to the laws $\{Q_z^{\delta,\mu}\}$ and the main absolute continuity result we need is :

$$Q_{z|\mathcal{F}_t}^{\delta,\mu} = \exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2} \int_0^t Z_s ds\right) \cdot Q_{z|\mathcal{F}_t}^\delta. \quad (2.9)$$

Here, $(Z_t, t \geq 0)$ denotes the coordinate process on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+)$.

We now combine this relation (2.9) with that of the "push downwards" with parameter θ , so that, with notations which we shall explain after writing the formula :

$$\begin{aligned} & Q_{z|\mathcal{F}_t}^{\delta,\mu;\theta\downarrow} \quad (2.10) \\ = & \frac{\varphi_{\theta\downarrow}(Z_t)}{\varphi_{\theta\downarrow}(z)} \exp(\Psi(\theta)L_t - \theta t) \exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2} \int_0^t ds Z_s\right) \cdot Q_{z|\mathcal{F}_t}^\delta \end{aligned}$$

and we note that for precisely : $\theta = \frac{\delta\mu}{2}$, this relation simplifies as :

$$Q_{z|\mathcal{F}_t}^{\delta,\mu;(\frac{\delta\mu}{2})\downarrow} = \frac{\exp(-\frac{\mu}{2}Z_t) \varphi_{\theta\downarrow}(Z_t)}{\exp(-\frac{\mu}{2}z) \varphi_{\theta\downarrow}(z)} \exp\left(\Psi(\theta)L_t - \frac{\mu^2}{2} \int_0^t ds Z_s\right) \cdot Q_{z|\mathcal{F}_t}^\delta \quad (2.11)$$

(The due explanation of the formula (2.10) is that we have combined the "push-downwards" formula (1.1), relative to $\{Q_z^{\delta,\mu}\}$, - i.e. the function $\varphi_{\theta\downarrow}$ and $\Psi(\theta)$ are relative to that diffusion - with the preceding formula (2.9).

From now on, we keep : $\theta = \frac{\delta\mu}{2}$.

2.4 We now consider what becomes of formula (2.11), once we time change both sides with the inverse of $\left(4 \int_0^t Z_u du, t \geq 0\right)$, so that, by a slight abuse of notation, the process of reference is now $(\widehat{Z}(h), h \geq 0)$, with \widehat{Z} defined by :

$$\boxed{Z_t = \widehat{Z}\left(4 \int_0^t Z_u du\right)} \quad (2.12)$$

Thus, we obtain :

$$\widehat{Q}_{z|\widehat{\mathcal{F}}_u}^{\delta,\mu;(\frac{\delta\mu}{2})\downarrow} = \frac{\exp\left(-\frac{\mu}{2}\widehat{Z}_u\right) \varphi_{\theta\downarrow}(\widehat{Z}_u)}{\left(e^{-\frac{\mu}{2}z} \varphi_{\theta\downarrow}(z)\right)} \exp\left(\Psi(\theta)\widehat{L}_u - \frac{\mu^2}{8}u\right) \cdot \widehat{Q}_{z|\widehat{\mathcal{F}}_u}^\delta \quad (2.13)$$

From the well-known property of time change for Bessel processes (see [12], Chap.XI, Prop.1.11), \widehat{Q}_z^δ is the distribution of a Bessel process of index $\alpha/2$,

i.e. of dimension $\hat{\delta} = 2 - \alpha$, that is $\widehat{Q}_z^\delta = P_z^\delta$. Again, with obvious notation, the right-hand side of (2.13) may be written :

$$\frac{\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(\widehat{Z}_u)}{\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z)} \exp\left(\widehat{\Psi}\left(\frac{\mu^2}{8}\right)\widehat{L}_u - \frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_{z|\widehat{F}_u}^\delta,$$

and we discover that :

$$\begin{cases} \widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) = e^{-\frac{\mu}{2}z}\varphi_{\theta\downarrow}(z) \\ \widehat{\Psi}\left(\frac{\mu^2}{8}\right) = \Psi(\theta) \end{cases} \quad (2.14)$$

and $\widehat{Q}_z^{\delta, \mu; (\frac{\delta\mu}{2})\downarrow} = P_z^{\delta; \frac{\mu^2}{8}\downarrow}$.

Again, let us explain, very much in the same spirit, e.g : the first relation : $\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) = e^{-\frac{\mu}{2}z}\varphi_{\theta\downarrow}(z)$ in (2.14).

This translates as :

$$\widehat{E}_z\left(\exp -\frac{\mu^2}{8}T_0(\widehat{Z})\right) = e^{-\frac{\mu}{2}z}E_z\left(e^{-\frac{\delta\mu}{2}T_0(Z)}\right) \quad (2.15)$$

where \widehat{Z} simply denotes a BES process with dimension $\widehat{\delta}$, and Z a process with law $Q_z^{\delta, \mu}$. This may be well understood by considering the absolute continuity relationship (2.9), when we replace t by $T_0(Z)$. Then, it follows from that relationship that :

$$\begin{aligned} Q_z^{\delta, -\mu}\left(\exp\left(-\frac{\delta\mu}{2}T_0(Z)\right)\right) &= e^{\frac{\mu z}{2}}Q_z^\delta\left(e^{-\frac{\mu^2}{2}\int_0^{T_0} ds Z_s}\right) \\ &= e^{\frac{\mu z}{2}}Q_z^{\widehat{\delta}}\left(e^{-\frac{\mu^2}{8}T_0(\widehat{Z})}\right), \end{aligned}$$

which is precisely (2.15).

Now, it is well known (see [3], [4], [10]) that the Laplace transform of T_0 , under the distribution P_z^δ of a Bessel process, is given by:

$$\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) := E_z^\delta\left(\exp\left(-\frac{\mu^2}{8}T_0\right)\right) = 2^{1-\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\left(\frac{\mu z}{2}\right)^{\frac{\alpha}{2}}K_{\frac{\alpha}{2}}\left(\frac{\mu z}{2}\right). \quad (2.16)$$

Using (2.14), we can recover the expression of $\varphi_{\theta\downarrow}$ for $\theta = \frac{\delta\mu}{2}$ obtained in (2.4) using the identity (2.6).

2.5 We now develop a discussion similar to that made in (2.8), but with the downwards arrows \downarrow now changed into upwards arrows \uparrow (for the definition of these pushed upwards and downwards processes obtained from a diffusion, see Pitman-Yor [10]).

The analogue of formula (2.10) is now :

$$\begin{aligned} & Q_{z|\mathcal{F}_t}^{\delta, \mu; \theta \uparrow} \\ &= \frac{\varphi_{\theta \uparrow}(Z_t)}{\varphi_{\theta \uparrow}(z)} \exp(-\theta t) \exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2} \int_0^t ds Z_s\right) \cdot Q_{z|\mathcal{F}_t}^\delta \end{aligned} \quad (2.17)$$

and we note again that, precisely for : $\theta = \frac{\delta\mu}{2}$, this relation simplifies as :

$$Q_{z|\mathcal{F}_t}^{\delta, \mu; \frac{\delta\mu}{2} \uparrow} = \frac{\exp\left(-\frac{\mu}{2} Z_t\right) \varphi_{\theta \uparrow}(Z_t)}{\exp\left(-\frac{\mu}{2} z\right) \varphi_{\theta \uparrow}(z)} \exp\left(-\frac{\mu^2}{2} \int_0^t ds Z_s\right) \cdot Q_{z|\mathcal{F}_t}^\delta \quad (2.18)$$

(We note that this formula is even simpler than (2.11) since here there is no local time contribution).

We now continue to develop an analogous discussion to that made in subsection (2.8).

Thus, we time-change both sides of the absolute continuity relation (2.18) with the inverse of $4 \int_0^t Z_u du$, $t \geq 0$, with \widehat{Z} , as defined from Z in (2.12).

We obtain :

$$\widehat{Q}_{z|\widehat{\mathcal{F}}_u}^{\delta, \mu; (\frac{\delta\mu}{2}) \uparrow} = \frac{\exp\left(-\frac{\mu}{2} \widehat{Z}_u\right) \varphi_{\theta \uparrow}(\widehat{Z}_u)}{\exp\left(-\frac{\mu}{2} z\right) \varphi_{\theta \uparrow}(z)} \exp\left(-\frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_{z|\widehat{\mathcal{F}}_u}^\delta$$

With obvious notation, this right-hand side may be written :

$$\frac{\widehat{\varphi}_{\frac{\mu^2}{8} \uparrow}(\widehat{Z}_u)}{\widehat{\varphi}_{\frac{\mu^2}{8} \uparrow}(z)} \exp\left(-\frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_{z|\widehat{\mathcal{F}}_u}^\delta$$

with :

$$\widehat{\varphi}_{\frac{\mu^2}{8} \uparrow}(z) = e^{-\frac{\mu}{2} z} \varphi_{\theta \uparrow}(z) \quad (2.19)$$

$\widehat{Q}_z^{\delta, \mu; (\frac{\delta\mu}{2}) \uparrow}$ is the distribution of a Bessel process of dimension $\widehat{\delta} = 2 - \alpha$ with drift $\frac{\mu^2}{8}$, i.e.

$$\widehat{Q}_z^{\delta, \mu; (\frac{\delta\mu}{2}) \uparrow} = P_z^{\widehat{\delta}; \frac{\mu^2}{8} \uparrow}.$$

The analytical counterpart of (2.19) is the companion formula of (2.6) (see Appendix):

$$M_{0, -\frac{\alpha}{2}}(\xi) = 4^{-\frac{\alpha}{2}} \Gamma(1 - \frac{\alpha}{2}) \sqrt{\xi} I_{-\frac{\alpha}{2}} \left(\frac{\xi}{2} \right);$$

while the companion formula of (2.16) is:

$$\widehat{\varphi}_{\frac{\mu^2}{8}\uparrow}(z) := \frac{1}{E_0^\delta \left(\exp(-\frac{\mu^2}{8} T_z) \right)} = 2^{-\frac{\alpha}{2}} \Gamma(1 - \frac{\alpha}{2}) \left(\frac{\mu z}{2} \right)^{\frac{\alpha}{2}} I_{-\frac{\alpha}{2}} \left(\frac{\mu z}{2} \right)$$

a well-known formula which goes back to Kent [4], Pitman-Yor [10].

Appendix : On the Whittaker and Bessel - Mc Donald functions

The following formulae involving these classical special functions are found in Lebedev [7], to which we refer with numberings such as : $(N)_*$...

- a) The Whittaker functions $M_{k, \mu}(z)$ and $W_{k, \mu}(z)$ are a pair of solutions of Whittaker's equation :

$$u'' + \left(-\frac{1}{4} + \frac{k}{z} + \frac{(\frac{1}{4} - \mu^2)}{z^2} \right) u = 0$$

(p. 279*).

- b) $W_{k, \mu}$ admits the integral representation :

$$W_{k, \mu}(z) = \frac{z^k e^{-\frac{z}{2}}}{\Gamma(\mu - k + \frac{1}{2})} \int_0^\infty e^{-t} t^{\mu - k - \frac{1}{2}} \left(1 + \frac{t}{z} \right)^{\mu - k + \frac{1}{2}} dt$$

(see Problem 17*, p. 279*).

- c)

$$W_{0, \mu}(z) = \sqrt{\frac{z}{\pi}} K_\mu \left(\frac{z}{2} \right)$$

(see Problem 19*, p. 279*).

- d) In terms of the confluent hypergeometric function Ψ , there are the relations :

$$W_{k, \mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Psi \left(\frac{1}{2} - k + \mu, 2\mu + 1; z \right)$$

(see (9.13.16)*, p. 274*).

$$K_\mu(z) = \sqrt{\pi}(2z)^\mu e^{-z} \Psi \left(\mu + \frac{1}{2}, 2\mu + 1; 2z \right)$$

(see (9.13.15)*, p. 274*).

Taking $k = 0$ in the above formula for $W_{k,\mu}$, one recovers c).

- e) In terms of the confluent hypergeometric function Φ , there are the relations :

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \Phi \left(\frac{1}{2} - k + \mu, 2\mu + 1; z \right)$$

(see (9.13.16)*, p. 274*).

$$I_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu + 1)} e^{-z} \Phi \left(\mu + \frac{1}{2}, 2\mu + 1; 2z \right)$$

(see (9.13.14)*, p. 274*).

Taking $k = 0$ in the above formula for $M_{k,\mu}$, we obtain:

$$M_{0,\mu}(z) = 4^\mu \Gamma(\mu + 1) \sqrt{z} I_\mu \left(\frac{z}{2} \right)$$

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