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About a generalized Buckley-Leverett equation and lubrication multifluid flow

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In this paper, we analyse the asymptotic system corresponding to a thin film flow with two different fluids, from theoretical and numerical point of view. We also compare this model to the Elrod-Adams one.

1 Introduction

The asymptotic behaviour of a single incompressible flow between two close surfaces in relative motion is described by the well-known Reynolds equation

$$\nabla \cdot \left(\frac{h^3}{6\mu_l} \nabla p \right) = v_0 \frac{\partial}{\partial x_1} (h),$$

in which h is the small gap between the two surfaces, v_0 the relative velocity of the surfaces, μ_l the viscosity and p the pressure. This equation can be rigorously deduced from (Navier) Stokes system by means of an asymptotic analysis [3]. However, in some applications, the lubricant cannot be considered as a single fluid and a multifluid approach has to be introduced. For example, this happens when one of the surfaces has to be particularly protected from contact from the other one and it is covered by a specific fluid; this can be also modelled by the existence of a surface layer with a viscosity which is different from the one of the bulk fluid. Another phenomenon which falls into the scope of the multifluid approach is linked to cavitation, which introduces the existence of air bubbles inside the bulk fluid.

Nevertheless, most of the multifluid problems in the lubrication area assume that the boundary between the two immiscible fluids is known [4, 19, 21]. This assumption allows to obtain a slightly modified Reynolds equation. However, the real problem is a free boundary one and the position of the interface is an additional unknown function. Using a three dimensional multifluid approach introduced by Nouri, Poupaud and Demay [15], a limit system has been derived by Sabil and Paoli [17, 18] with an asymptotic approach. This system describes the behaviour of the pressure and the relative saturation of the two fluids contained in a thin domain. However, this derivation is based upon an assumption on the shape of the interface. Thus, a mathematical study of the obtained asymptotic system has to be made in order to ensure its well-posedness. So far, this kind of result has been given only for the case when the surfaces surrounding the fluid are fixed ($v_0 = 0$),

corresponding for example to the injection of a fluid through a fixed gap. Whether the value of v_0 is zero or not, the system consists of two equations : a generalized Buckley-Leverett equation and a generalized Reynolds lubrication equation.

However, the assumption of zero value for v_0 is not relevant for most of the lubrication problems in which the fluid is sheared, due to the difference of velocities between the surrounding surfaces. Thus, it is the purpose of this paper to give an existence and uniqueness result for the asymptotic system with non zero value for the shear velocity. The main difficulty comes from the study of the generalized Buckley-Leverett equation. More precisely, taking the fact that v_0 is different from 0 prevents us from using the classical results about first order hyperbolic equations: in fact, the flux function is not autonomous and we have to guarantee that the saturation lies in the interval $[0, 1]$, although the maximum principle is not a priori guaranteed anymore.

This paper is organized as follows:

- Section 2 deals with the governing equations of the asymptotic system, obtained from the multifluid Stokes system. Thus, the generalized Buckley-Leverett / Reynolds system is presented, along with the physical assumptions related to realistic modellings.
- Section 3 is devoted to the analysis of the generalized Buckley-Leverett equation. Thus, we present the definition of a weak entropy solution of a scalar conservation law on a bounded domain, and we give some stability results (in particular, the saturation is a function with values in $[0, 1]$), along with an existence and uniqueness theorem, by using the concept of “semi Kružkov entropy-flux pairs”. Moreover, we use a numerical scheme that allows to obtain the unique (physical) solution as the mesh size tends to 0.
- In Section 4, we present the analysis of the generalized Reynolds equation. In particular, we state an existence and uniqueness result for the pressure, and also a priori estimates in the H^1 or L^∞ norm which do not depend on the ratio ε of the viscosities.
- Section 5 deals with numerical computations. In particular, we present some numerical tests showing the importance of the shear effects on the saturation, pressure and velocity profiles but also on the boundary conditions. Then, we focus on cavitation phenomena : we show how our approach allows us to give some comprehensive details on the way the bifluid behaves in thin films. In particular, we compare the solution of the generalized Buckley-Leverett / Reynolds model to the solution of the Elrod-Adams model, which is a frequently used model in mechanical studies.

2 Governing equations

We first recall the set of equations derived by Paoli in [17]. Let be $\Omega =]0, L[$ and let us denote by $\partial\Omega = \{0, L\}$ its boundary, by Q_T the set $]0, T[\times \Omega$ and by Σ_T the set $]0, T[\times \partial\Omega$.

We introduce the ratio $\varepsilon = \mu_g / \mu_l$, μ_l (resp. μ_g) being the viscosity of a fluid in liquid (resp. gaseous) phase. In view of cavitation-related phenomena, the fluid is supposed to be a lubricant: thus, $\mu_l \equiv \mu$, the liquid phase lubricant being considered as a reference fluid, and the gaseous phase lubricant may be considered as air or gas. In that configuration, typically $\varepsilon \sim 10^{-3}$. Now we introduce the main equations:

■ Generalized Buckley-Leverett equation: the saturation s is governed by a scalar conservation law:

$$\frac{\partial}{\partial t} (h(x)s(t, x)) + \frac{\partial}{\partial x} (Q_{in}(t)f(s) + v_0h(x)g(s)) = 0, \quad (t, x) \in Q_T, \quad (2.1)$$

where h is the normalized gap between the surfaces, Q_{in} is the flow input, v_0 is the shear velocity corresponding to the speed of the lower surface, and s denotes the reference (liquid) fluid saturation. The functions f and g , defining the flux, are described later. However, we point out the fact that f represents the classical contribution to the Buckley-Leverett flux, while g is a non-classical contribution induced by shear effects. Equation (2.1) is completed with the following initial and boundary conditions

$$s(0, \cdot) = s_0, \quad \text{on } \Omega \quad (2.2)$$

$$"s = s_1", \quad \text{on } \Sigma_T \quad (2.3)$$

where the sense of the boundary condition (2.3) has to be precised, as it will be discussed later.

This initial boundary value problem is weakly coupled with the following problem:

■ Generalized Reynolds equation: for a given saturation s , the pressure p obeys the following law:

$$\frac{\partial}{\partial x} \left(A(s) \frac{h^3}{6\mu} \frac{\partial p}{\partial x} \right) = v_0 \frac{\partial}{\partial x} (B(s)h), \quad (t, x) \in Q_T, \quad (2.4)$$

with the boundary condition:

$$p = 0, \quad (t, x) \in \Sigma_T. \quad (2.5)$$

Let us mention that expressions of functions f , g , A and B are fully detailed in the Appendix (at the end of this paper). In [17], the derivation of the coupled problem has been done under the assumption that the free boundary, which separates both phases, is a function belonging to $L^\infty((0, T); BV(\Omega))$. Thus, this assumption (on the shape and on the regularity of the free boundary) prevents us from considering multi-layer flows, although they are supposed to be relevant (see, for instance, [17]). The main reason for restricting ourselves to this particular type of free boundaries lies in the difficulty to compute an explicit expression of f and g otherwise. Therefore, we will restrict ourselves to cases (i) and (ii) (in which f , g , A and B can be fully computed):

- (i) The reference (liquid) fluid is stucked to the lower (moving) surface (see FIG.1).
- (ii) The reference (liquid) fluid is stucked to the upper (fixed) surface (see FIG.2).

We will discuss in Section 5 whether the choice of each assumption is relevant or not. Notice that f , g , A and B highly depend on the ratio ε . As it will be pointed out further, the shape of the flux functions and coefficients remains the same, as described by the forthcoming assumptions.

We consider the following assumptions on the data:

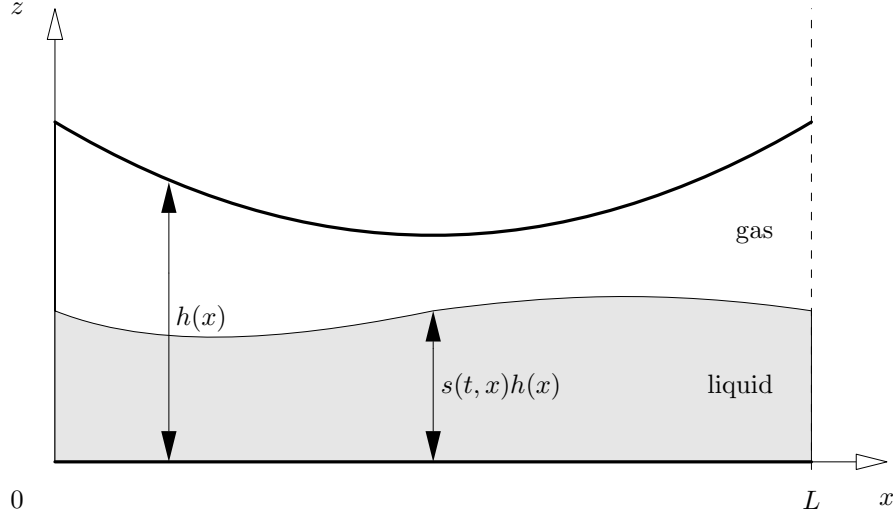


FIGURE 1. Case (i). The liquid phase is stuck to the lower surface

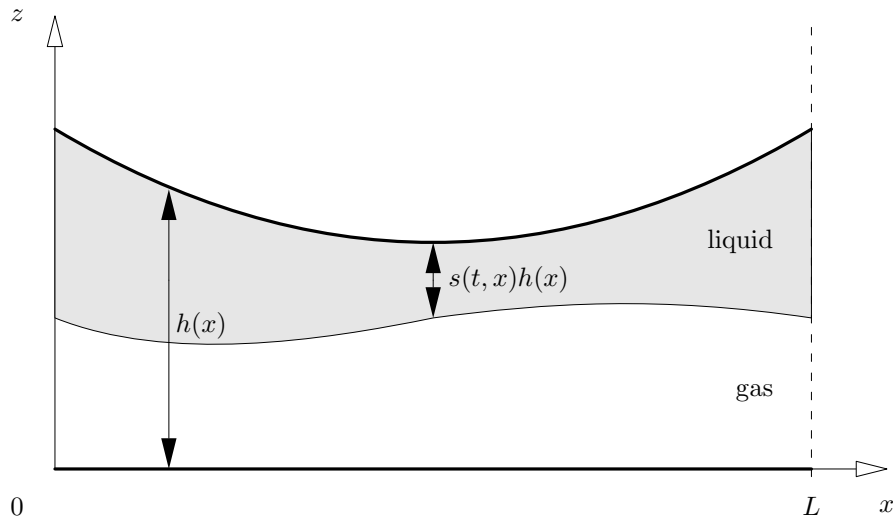


FIGURE 2. Case (ii). The liquid phase is stuck to the upper surface

Assumption 2.1 (initial and boundary functions)

- (i) $s_0 \in L^\infty(\Omega; [0, 1])$,
- (ii) $s_1 \in L^\infty(\mathbb{R}^+; [0, 1])$.

Assumption 2.2 (flow, shear velocity)

- (i) $Q_{in} \in C^0([0, +\infty])$,
- (ii) $\exists Q_{\min}, Q_{\max}, 0 < Q_{\min} \leq Q_{in} \leq Q_{\max}$,

(iii) $v_0 > 0$.

Assumption 2.3 (gap between the surfaces)

- (i) $h \in C^1(\mathbb{R})$,
- (ii) $\exists h_{\min}, h_{\max}, 0 < h_{\min} \leq h \leq h_{\max}$.

Assumption 2.4 (auxiliary flux functions)

- (i) $f \in C^2([0, 1])$, $f(0) = 0$, $f(1) = 1$, f is a non-decreasing function,
- (ii) $g \in C^2([0, 1])$, $g(0) = g(1) = 0$,

Our purpose is to state an existence and uniqueness result for problem (2.1)–(2.5). In fact, the main difficulty is to state an existence and uniqueness result for the generalized Buckley-Leverett equation (2.1)–(2.3). Moreover, since s denotes a saturation, we have also to state that the (possible) solution takes its values in $[0, 1]$. Indeed, let us recall that the derivation of the generalized Buckley-Leverett equation is not fully rigorous in the sense that a strong assumption on the free boundary shape has been used. Let us recall also that the study of the generalized Buckley-Leverett equation including the shear term has been omitted by Paoli in [17].

3 The generalized Buckley-Leverett equation

In a first subsection, we introduce an auxiliary problem and a corresponding “weak entropy solution”, whose framework lies in the theory of scalar conservation laws on bounded domains. After establishing some results (existence, uniqueness, stability result) on the properties of the auxiliary problem, we will prove, in a second subsection, how it is possible to reduce the generalized Buckley-Leverett problem to the auxiliary one. Finally, in the third subsection, we propose a numerical scheme whose solution converges to the “physical” solution (i.e. the weak entropy solution).

3.1 An auxiliary problem

Let us consider the following assumption:

Assumption 3.1 The “auxiliary boundary / initial data” satisfy :

- (i) $u_0 \in L^\infty(\Omega; [0, 1])$,
- (ii) $u_1 \in L^\infty(\mathbb{R}^+; [0, 1])$.

The “auxiliary gap” satisfies:

- (i) $k \in C^1(\mathbb{R}^2)$,
- (ii) $\exists k_{\min}, k_{\max}, 0 < k_{\min} \leq k \leq k_{\max}$.

We introduce the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (f(u) + k(t, x) g(u)) = 0, \quad (t, x) \in]0, \tilde{T}[\times]0, 1[\quad (3.1)$$

where f, g have been already defined and $\tilde{T} > 0$. Equation (3.1) is completed with the following initial and boundary conditions

$$u(0, \cdot) = u_0, \quad \text{on }]0, 1[, \quad (3.2)$$

$$"u = u_1", \quad \text{on }]0, \tilde{T}[\times \partial]0, 1[. \quad (3.3)$$

Existence and uniqueness of a solution for scalar conservation laws on unbounded domains has been solved in the pioneering work of Kruřkov [11] who introduced the concept of weak entropy solution and related "Kruřkov entropy-flux pairs". When dealing with bounded domains, introducing boundary conditions must be understood in a particular way: in the bounded domain setting, Bardos, Le Roux and Nédélec [1] also proved existence and uniqueness of a weak entropy solution satisfying a "Kruřkov entropy-flux pair" formulation which includes boundary terms, under some regularity assumptions on the data. In particular, the way the boundary condition is satisfied is known as the "BLN" condition. Nevertheless, this formulation would not be sufficient by two reasons:

- 1) boundary and initial conditions lack regularity in comparison to the framework of [1],
- 2) it does not provide a stability result (we recall that we have to state that the possible solution is a function with values in $[0, 1]$).

The notion of weak entropy solution in the L^∞ framework is essentially due to Otto [16] who introduced the so-called "boundary entropy-flux pairs". A more complete exposition appears in [12]. In fact, as it will be pointed out further, it is equivalent to use the "semi Kruřkov entropy-flux pairs" and the "boundary entropy-flux pairs", at least in the case of a scalar conservation law with an autonomous flux. In the case of scalar conservation laws with non-autonomous fluxes, a "boundary entropy-flux pairs" formulation is not obvious anymore, but using the concept of "semi Kruřkov entropy-flux pairs" [6, 20, 22] allows us to generalize the notion of weak entropy solution, state a stability result and prove existence and uniqueness of a weak entropy solution of the auxiliary problem (3.1)–(3.3).

Let us first introduce the definition of a weak entropy solution for the auxiliary problem and state the existence and uniqueness result as a theorem.

Definition 1 *Let us suppose that Assumption 3.1 holds. A function $u \in L^\infty(Q_T)$ is said to be a weak entropy solution of problem (3.1)–(3.3) if it satisfies the inequality*

$$\begin{aligned} \int_0^1 \int_0^{\tilde{T}} & \left((u - \kappa)^\pm \frac{\partial \varphi}{\partial t} + \left(\Phi_{[f]}^\pm(u, \kappa) + k \Phi_{[g]}^\pm(u, \kappa) \right) \frac{\partial \varphi}{\partial x} - \text{sgn}_\pm(u - \kappa) \frac{\partial k}{\partial x} g(\kappa) \varphi \right) dx dt \\ & + \int_\Omega (u_0(x) - \kappa)^\pm \varphi(0, x) dx \\ & + \mathcal{M} \int_0^{\tilde{T}} (u_1(t, 1) - \kappa)^\pm \varphi(t, 1) dt \\ & + \mathcal{M} \int_0^{\tilde{T}} (u_1(t, 0) - \kappa)^\pm \varphi(t, 0) dt \geq 0, \end{aligned} \quad (3.4)$$

for all $\kappa \in [0, 1]$, $\varphi \in \mathcal{D}([0, \tilde{T}] \times [0, 1])$ and $\varphi \geq 0$, the constant \mathcal{M} being defined by

$$\mathcal{M} = \|f'\|_{L^\infty(0,1)} + k_{\max} \|g'\|_{L^\infty(0,1)}. \quad (3.5)$$

The functions $u \mapsto (u - \kappa)^\pm$ are the so-called “semi Kruřkov entropies” (see [6, 20, 22]), defined by

$$(u - \kappa)^+ = \begin{cases} u - \kappa, & \text{if } u \geq \kappa, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad (u - \kappa)^- = (\kappa - u)^+.$$

The functions $\Phi_{[f]}^\pm$ and $\Phi_{[g]}^\pm$ are the corresponding “semi Kruřkov fluxes” defined by

$$\begin{aligned} \Phi_{[f]}^\pm(u, \kappa) &= \operatorname{sgn}_\pm(u - \kappa)(f(u) - f(\kappa)), \\ \Phi_{[g]}^\pm(u, \kappa) &= \operatorname{sgn}_\pm(u - \kappa)(g(u) - g(\kappa)), \end{aligned}$$

where $u \mapsto \operatorname{sgn}_\pm(u)$ is the derivative of the function $u \mapsto u^\pm$ with value 0 at point 0.

The functions $\Phi_{[f]}^\pm(\cdot, \kappa)$ and $\Phi_{[g]}^\pm(\cdot, \kappa)$ are defined on $[0, 1]$ (as f and g). Therefore, both definitions of solutions (weak entropy and strong solutions) make sense only if the function u takes values in $[0, 1]$ a.e., which is not supposed a priori in Definition 1. However, the set is preserved as shown by the following proposition.

Theorem 3.2 (Maximum principle) *In Definition 1 and Equations (3.1)–(3.3), if we replace the functions $f, g, \Phi_{[f]}^\pm, \Phi_{[g]}^\pm$ by the functions \tilde{f} and $\tilde{g}, \tilde{\Phi}_{[f]}^\pm$ and $\tilde{\Phi}_{[g]}^\pm$ defined by:*

$$\begin{aligned} \tilde{f}(v) &= \begin{cases} 0, & \text{if } v < 0 \\ f(v), & \text{if } 0 \leq v \leq 1 \\ 1, & \text{if } v > 1 \end{cases}, & \tilde{\Phi}_{[f]}^\pm(v, \kappa) &= \operatorname{sgn}_\pm(v - \kappa)(\tilde{f}(v) - \tilde{f}(\kappa)), \\ \tilde{g}(v) &= \begin{cases} 0, & \text{if } v < 0 \\ g(v), & \text{if } 0 \leq v \leq 1 \\ 0, & \text{if } v > 1 \end{cases}, & \tilde{\Phi}_{[g]}^\pm(v, \kappa) &= \operatorname{sgn}_\pm(v - \kappa)(\tilde{g}(v) - \tilde{g}(\kappa)), \end{aligned}$$

and if u is a weak entropy solution of problem (3.1)–(3.3), then $0 \leq u \leq 1$ a.e. on $]0, \tilde{T}[\times]0, 1[$.

Proof First let us notice that the following properties hold:

$$\begin{aligned} \tilde{\Phi}_{[f]}^\pm(v, \kappa) &\leq \|f'\|_{L^\infty(0,1)}(v - \kappa)^\pm, \\ \tilde{\Phi}_{[g]}^\pm(v, \kappa) &\leq \|g'\|_{L^\infty(0,1)}(v - \kappa)^\pm. \end{aligned}$$

Now, set $\kappa = 0$ in Inequality (3.4). Since we have $u_0^- = 0, u_1^- = 0$ (see Assumption 3.1), $g(0) = 0$ (see Assumption 2.4), the three last terms in (3.4) vanish. Thus, we have

$$\int_0^1 \int_0^{\tilde{T}} \left(u^- \frac{\partial \varphi}{\partial t} + \left(\tilde{\Phi}_{[f]}^-(u, 0) + k \tilde{\Phi}_{[g]}^-(u, 0) \right) \frac{\partial \varphi}{\partial x} \right) dx dt \geq 0. \quad (3.6)$$

Now, let (τ, R) be such that $0 < \tau < \tilde{T}$, $\delta = \tilde{T} - \tau$ and $0 < R < 1$. Let $r \in \mathcal{D}(\mathbb{R}^+)$ be such that: r is non-increasing, $r \equiv 1$ on $[0, R + \mathcal{M}\tau]$, $r \equiv 0$ on $[R + \mathcal{M}\tau + \delta/2, +\infty)$.

Then, choosing

$$\varphi(t, x) = \frac{\tau - t}{\tau} \chi_{(0, \tau)}(t) r(x + \mathcal{M}t)$$

in Inequality (3.6) leads to

$$\begin{aligned} & -\frac{1}{\tau} \int_0^1 \int_0^\tau u^- r(x + \mathcal{M}t) dt dx \\ & + \int_0^1 \int_0^\tau \frac{\tau - t}{\tau} r'(x + \mathcal{M}t) \left(\mathcal{M} u^- + \tilde{\Phi}_{[f]}^-(u, 0) + k \tilde{\Phi}_{[g]}^-(u, 0) \right) \geq 0. \end{aligned} \quad (3.7)$$

Now, since $\tilde{\Phi}_{[f]}^-(u, 0) \leq \|f'\|_{L^\infty(0,1)} u^-$, $\tilde{\Phi}_{[g]}^-(u, 0) \leq \|g'\|_{L^\infty(0,1)} u^-$, we use Equality (3.5) to obtain:

$$\mathcal{M} u^- + \tilde{\Phi}_{[f]}^-(u, 0) + k \tilde{\Phi}_{[g]}^-(u, 0) \geq 0.$$

Moreover, since $r'(x + \mathcal{M}t) \leq 0$, the second term of the left-hand side of (3.7) is negative. Now, since $r(x + \mathcal{M}t) = 1$, for all $(t, x) \in (0, \tau) \times (0, R)$ and $r \geq 0$, the left-hand side of (3.7) is upper bounded by

$$-\frac{1}{\tau} \int_0^R \int_0^\tau u^- dx dt$$

which is consequently non-negative. Therefore, we have $u^- = 0$ on $(0, \tau) \times (0, R)$. Now, passing to the limit with $R \rightarrow 1$ and $\tau \rightarrow \tilde{T}$, we have $u \geq 0$ a.e. Similarly, by choosing $\kappa = 1$ in Inequality (3.4) (with the semi entropies $u \mapsto (u - 1)^+$), we prove that $u \leq 1$ a.e. on $]0, \tilde{T}[\times]0, 1[$. \square

Remark 3.3 *Interestingly, the fact that the flux function in the auxiliary problem (3.1)–(3.3) is not autonomous involves a major difference with the autonomous case, concerning the stability intervals. Indeed, the following properties can be easily shown:*

- *if the flux is autonomous (for instance $k \equiv 0$) and if u_0, u_1 are functions with values in an interval $[a, b]$, then, u is a function with values in $[a, b]$;*
- *For the auxiliary problem, if u_0, u_1 are functions with values in an interval $[a, b] \subset [0, 1]$, then, u is a function with values in $[0, 1]$; thus, only the set $[0, 1]$ is preserved. This is due to the properties of g , in particular $g(0) = g(1) = 0$.*

Remark 3.4 *A function u which satisfies Definition 1 is a weak solution in a classical sense. Indeed, for every $\varphi \in H_0^1(Q_T)$, we write $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = -\min(\varphi, 0)$; obviously, $\varphi^\pm \in H_0^1(Q_T)$; thus adding the two inequalities (corresponding to each “semi Kružkov entropy-flux pair”) gives:*

$$\int_{Q_T} \left\{ \left| u - \kappa \right| \frac{\partial \varphi^\pm}{\partial t} + \operatorname{sgn}(u - \kappa) \left((f(u) - f(\kappa)) + k (g(u) - g(\kappa)) \right) \frac{\partial \varphi^\pm}{\partial x} - \frac{\partial k}{\partial x} \operatorname{sgn}(u - \kappa) g(\kappa) \varphi^\pm \right\} dx dt \geq 0.$$

Now, taking $\kappa = \pm \|u\|_{L^\infty(Q_T)}$ gives:

$$\int_{Q_T} \left\{ u \frac{\partial \varphi^\pm}{\partial t} + (f(u) + k g(u)) \frac{\partial \varphi^\pm}{\partial x} \right\} dx dt = 0,$$

Now, by means of subtraction, we immediatly obtain

$$\int_{Q_T} \left\{ u \frac{\partial \varphi}{\partial t} + (f(u) + k g(u)) \frac{\partial \varphi}{\partial x} \right\} dx dt = 0,$$

for every $\varphi \in H_0^1(Q_T)$, so that Equation (3.1) is gained in a weak sense.

Let us explain the way the boundary condition is satisfied:

Definition 2 (Boundary entropy-flux pairs, [16])

Let us define the flux $\hat{f} : (t, x, s) \mapsto f(s) + k(t, x) g(s)$. The pair (H, Q) belonging to $C^1(\mathbb{R}^2) \times C^1([0, \tilde{T}[\times]0, 1[\times \mathbb{R}^2))$ is said to be a “boundary entropy-flux pair” (for the flux \hat{f}) if it satisfies:

1. for all $w \in \mathbb{R}$, $s \mapsto H(s, w)$ is a convex function,
2. $\forall (t, x) \in]0, \tilde{T}[\times]0, 1[$, $\forall w \in \mathbb{R}$, $\partial_3 Q(t, x, s, w) = \partial_1 H(s, w) \partial_3 \hat{f}(t, x, s)$,
3. $\forall w \in \mathbb{R}$, $H(w, w) = 0$, $Q(\cdot, \cdot, w, w) = 0$, $\partial_1 H(w, w) = 0$.

Proposition 3.5 (Boundary condition, [16])

Let $u \in L^\infty(]0, \tilde{T}[\times]0, 1[)$ be a weak entropy solution of problem (3.1)–(3.3). Then,

$$\operatorname{ess\,lim}_{\varrho \rightarrow 0^+} \int_0^{\tilde{T}} \left(Q(t, 1, u(t, 1 - \varrho), u_1(1)) \beta(t, 1) - Q(t, 0, u(t, \varrho), u_1(0)) \beta(t, 0) \right) dt \geq 0,$$

for all “boundary entropy-flux pairs” (H, Q) , $\forall \beta \in L^1(\Sigma_T)$, $\beta \geq 0$ a. (3.8)

Remark 3.6 Now let us give some comprehensive details on the way to understand the boundary condition. This has been given in [12, 16, 22]: in general, the problem should be overdetermined and the boundary equality cannot be required to be assumed at each point of the boundary, even if the solution is a regular function. But, with additional assumptions, the more comprehensive “BLN” condition is recovered:

(i) If u admits a trace, i.e. there exists $u|_b \in L^\infty(]0, \tilde{T}[\times \partial]0, 1[)$ such that

$$\operatorname{ess\,lim}_{\varrho \rightarrow 0^+} \int_0^{\tilde{T}} |u(t, 1 - \varrho) - u|_b(t, 1)| + |u(t, \varrho) - u|_b(t, 0)| dt = 0,$$

then, Inequality (3.8) is equivalent to the following inequality (see [9, 16])

$$Q(\cdot, 1, u|_b(\cdot, 1), u_1(\cdot, 1)) \geq 0, \quad \text{a.e. on }]0, \tilde{T}[, \quad (3.9)$$

$$Q(\cdot, 0, u|_b(\cdot, 0), u_1(\cdot, 0)) \leq 0, \quad \text{a.e. on }]0, \tilde{T}[. \quad (3.10)$$

Moreover, considering the particular “boundary fluxes”

$$H_\delta^+(z, w) = \left((\max(z - w, 0))^2 + \delta^2 \right)^{1/2} - \delta, \quad (3.11)$$

$$Q_\delta^+(t, x, z, w) = \int_w^z \partial_1 H_\delta^+(\lambda, k) \partial_3 \hat{f}(t, x, \lambda) d\lambda, \quad (3.12)$$

and

$$H_\delta^-(z, w) = \left((\min(w - z, 0))^2 + \delta^2 \right)^{1/2} - \delta, \quad (3.13)$$

$$Q_\delta^-(t, x, z, w) = \int_w^z \partial_1 H_\delta^-(\lambda, k) \partial_3 \hat{f}(t, x, \lambda) d\lambda, \quad (3.14)$$

and letting $\delta \rightarrow 0$, we have the following uniform convergences:

$$Q_\delta^\pm(t, x, s, w) \rightarrow \operatorname{sgn}_\pm(s - w) \left(\hat{f}(t, x, s) - \hat{f}(t, x, w) \right).$$

Finally, taking the following “boundary flux” in Inequalities (3.9) and (3.10)

$$\begin{aligned} Q(t, x, s, w) = & \operatorname{sgn}_+(s - \max(w, k)) \left\{ \hat{f}(t, x, s) - \hat{f}(t, x, \max(w, k)) \right\} \\ & + \operatorname{sgn}_-(s - \min(w, k)) \left\{ \hat{f}(t, x, s) - \hat{f}(t, x, \min(w, k)) \right\} \end{aligned}$$

yields to the classical BLN condition given by Bardos, Le Roux and Nédélec [1], that is:

$$\operatorname{sgn}(u|_b(t, 1) - u_1(t, 1))(\hat{f}(t, 1, u|_b(t, 1)) - \hat{f}(t, 1, k)) \geq 0, \quad (3.15)$$

$$\operatorname{sgn}(u|_b(t, 0) - u_1(t, 0))(\hat{f}(t, 0, u|_b(t, 0)) - \hat{f}(t, 0, k)) \leq 0, \quad (3.16)$$

for a.e. $(t, r) \in]0, \tilde{T}[\times \partial]0, 1[, \forall k \in [\min(u|_b, u_1), \max(u|_b, u_1)]$.

(ii) Assume that u admits a trace on the boundary, then Inequalities (3.15) and (3.16) can be simplified in the following cases:

- If $\partial_3 \hat{f}$ is a positive function on $]0, 1[$, then $u(\cdot, 0) = u_1$ (and nothing is imposed at $x = 1$, i.e. the condition $u(\cdot, 1) = u_1$ is not active).
- If $\partial_3 \hat{f}$ is a negative function on $]0, 1[$, then $u(\cdot, 1) = u_1$ (and nothing is imposed at $x = 0$, i.e. the condition $u(\cdot, 0) = u_1$ is not active).

Thus the boundary conditions may be “active” only on a part of the boundary. Unfortunately, in the case of the flux \hat{f} with Assumptions 2.4 and 3.1, monotonicity w.r.t. the third variable lacks, and we have to deal with “relaxed” Dirichlet boundary conditions.

Now, we conclude this subsection with the main result related to the auxiliary problem:

Theorem 3.7 Under Assumption 3.1, problem (3.1)–(3.3) admits a unique weak entropy solution.

Proof Existence is proved using the (classical) parabolic approximation, which consists of adding an artificial diffusive term in the right-hand side of the hyperbolic equation (vanishing viscosity method). Next, passing to the limit on the diffusive parameter gives the existence result. Uniqueness is obtained from the Kruřkov method of doubling vari-

ables. The complete proof is even valid for first order quasilinear equations, and completely detailed in [13]. \square

3.2 Existence and uniqueness of a weak entropy solution for the Buckley-Leverett problem

In this subsection, we show that problem (2.1)–(2.3) can be reduced to an auxiliary problem as the one described in the previous subsection, namely problem (3.1)–(3.3). Then, it suffices to use the results established for the auxiliary problem.

Definition 3 (Direct reduction) *Let us consider the following changes of variables:*

$$Y(x) = L \frac{\int_0^x h(t) dt}{\int_0^L h(x) dx}, \quad T(t) = L \frac{\int_0^t Q_{in}(s) ds}{\int_0^L h(x) dx}.$$

We also define the inverse functions of Y and T , respectively denoted Y^{-1} and T^{-1} .

Definition 4 (Inverse reduction) *Let Y^{-1} and T^{-1} be defined as the respective unique solution of the following Cauchy problems:*

$$\begin{cases} \frac{dY^{-1}}{dy}(y) = \frac{1}{L h(Y^{-1}(y))} \int_0^L h(x) dx, \\ Y^{-1}(0) = 0, \\ \frac{dT^{-1}}{d\tau}(\tau) = \frac{1}{L Q_{in}(T^{-1}(\tau))} \int_0^L h(x) dx, \\ T^{-1}(0) = 0. \end{cases}$$

Remark 3.8 Y is an increasing function which defines an isomorphism from $[0, L]$ to $[0, 1]$. In the same way, T is an increasing function which defines an isomorphism from $[0, T]$ to $[0, \tilde{T}]$, with

$$\tilde{T} = L \frac{\int_0^T Q_{in}(s) ds}{\int_0^L h(x) dx}.$$

Proposition 3.9 *Problem (2.1)–(2.3) can be reduced to an auxiliary problem (3.1)–(3.3).*

Proof First, we define

$$u(T(t), Y(x)) = s(t, x), \tag{3.17}$$

so that Equation (2.1) reduces to

$$\frac{\partial}{\partial \tau} u(\tau, y) + \frac{\partial}{\partial y} (f(u) + k(\tau, y) g(u)) = 0, \quad (\tau, y) \in (0, \tilde{T}) \times (0, 1) \tag{3.18}$$

with $k(\tau, y) = \frac{v_0}{(Q_{in} \circ \mathcal{T}^{-1})(\tau)} (h \circ Y^{-1})(y)$.

The initial and boundary conditions are modified as follows:

$$u(0, y) = u_0(0, y) = s_0 \circ Y^{-1}(y), \quad y \in]0, 1[\quad (3.19)$$

$$"u(\tau, y) = u_1(\tau, y) = s_1(\mathcal{T}^{-1}(\tau), y)", \quad (\tau, y) \in (0, \tilde{T}) \times \partial]0, 1[\quad (3.20)$$

□

As a consequence, turning back to the original variables immediatly gives:

Definition 5 *A function $s \in L^\infty(Q_T; [0, 1])$ is said to be a weak entropy solution of problem (2.1)–(2.3) if it satisfies*

$$\begin{aligned} \int_{Q_T} \left\{ h(x) (s - \kappa)^\pm \frac{\partial \varphi}{\partial t} + \left(Q_{in}(t) \Phi_{[f]}^\pm(s, \kappa) + v_0 h(x) \Phi_{[g]}^\pm(s, \kappa) \right) \frac{\partial \varphi}{\partial x} \right. \\ \left. - v_0 \operatorname{sgn}_\pm(s - \kappa) h'(x) g(\kappa) \varphi \right\} dx dt \\ + \int_{\Omega} h(x) (s_0(x) - \kappa)^\pm \varphi(0, x) dx \\ + \mathcal{L} \int_0^T (s_1(t, 1) - \kappa)^\pm \varphi(t, 1) dt \\ + \mathcal{L} \int_0^T (s_1(t, 0) - \kappa)^\pm \varphi(t, 0) dt \geq 0 \end{aligned} \quad (3.21)$$

for all $\kappa \in [0, 1]$, $\varphi \in \mathcal{D}([0, T] \times \mathbb{R})$, $\varphi \geq 0$. Here, we can choose:

$$\mathcal{L} = Q_{\max} \max(\|f'\|_{L^\infty(0,1)}) + v_0 h_{\max} \frac{Q_{\max}}{Q_{\min}} \max(\|g'\|_{L^\infty(0,1)}).$$

Theorem 3.10 *Under Assumptions 2.1–2.4, problem (2.1)–(2.3) admits a unique weak entropy solution.*

3.3 Numerical scheme for the generalized Buckley-Leverett equation

We study the numerical method simulating the solution of problem (3.18)–(3.20) (for convenience) which is equivalent to problem (2.1)–(2.3) up to a change of variables.

The following theorem gives a convergence result which enables us to use some classical finite volume methods (such as the Lax-Friedrichs scheme, for instance) in order to compute the weak entropy solution.

Theorem 3.11 (Vovelle [22]) *Let us consider a finite volume scheme with monotone fluxes associated to problem (3.18)–(3.20) and its corresponding numerical solution $u_{\mathcal{T},k}$. Then $(u_{\mathcal{T},k})$ strongly converges to the weak entropy solution in $L_{loc}^p(\mathbb{R}^+, \Omega)$ for every $p \in [2, +\infty[$.*

From a theoretical point of view, it appears that the simulation of the Buckley-Leverett problem can be easily done thanks to the earlier convergence result. Nevertheless, let us recall that f and g highly depend on the ratio of the viscosities, namely ε (see FIG.14, 15 and 16). In practical situations, ε may be small (10^{-3} for an air-liquid mixture). Thus, let us study the behaviour of the scheme for small values of ε . For this, let us consider a uniform admissible mesh, whose size is denoted Δx , with a time step Δt . We first notice that:

- $\|f'\|_{L^\infty}$ and $\|g'\|_{L^\infty}$ tend to explode as ε tends to 0. More precisely, we have (after omitted computations)

$$\|f'\|_{L^\infty} = \mathcal{O}\left(\varepsilon^{-1/3}\right), \quad \|g'\|_{L^\infty} = \mathcal{O}\left(\varepsilon^{-1/3}\right),$$

so that we get also $\mathcal{M} = \mathcal{O}\left(\varepsilon^{-1/3}\right)$.

- Moreover, Id denoting the identity application, we have that

$$\|f\|_{L^1} = \mathcal{O}\left(\varepsilon^{1/3}\right), \quad \|\text{Id} - g\|_{L^1} = \mathcal{O}\left(\varepsilon^{1/3}\right).$$

Then, Δx needs to be adapted to the value of ε in order to describe phenomena related to the boundary layer, so that the mesh size should satisfy

$$\Delta x = \mathcal{O}(\varepsilon^{1/3}) \tag{3.22}$$

in order to describe boundary layers of f and g .

Now, we recall that the following CFL condition has to be imposed in order to ensure the stability of the numerical scheme:

$$\exists \xi \in]0, 1[, \Delta t \leq (1 - \xi) \frac{\Delta x}{\mathcal{M}}.$$

Thus, we obtain the order of the time step:

$$\Delta t = \mathcal{O}(\varepsilon^{2/3}). \tag{3.23}$$

Now, it clearly appears that it becomes difficult to simulate the Buckley-Leverett equation for too small values of ε . Indeed, for each time step, the number of computations increases with $1/\Delta x$; similarly, the number of times steps increases with $1/\Delta t$ so that it becomes more and more difficult to attain a (possible) stationary solution.

4 The generalized Reynolds equation

4.1 Existence and uniqueness

Definition 6 Let $s \in L^\infty(Q_T; [0, 1])$. A function $p \in L^\infty((0, T); H_0^1(\Omega))$ is a weak solution of problem (2.4)–(2.5) if it satisfies

$$\int_0^L A(s) \frac{h^3}{6\mu_l} \frac{\partial p}{\partial x} \frac{\partial v}{\partial x} dx = v_0 \int_0^L B(s) h \frac{\partial v}{\partial x} dx, \quad \forall v \in H_0^1(\Omega) \tag{4.1}$$

for almost every $t \in (0, T)$.

Next, we establish the existence and uniqueness of solution for the generalized Reynolds equation:

Theorem 4.1 *Under Assumptions 2.1–2.4, problem (2.4)–(2.5) admits a unique solution p in the sense of Definition 6. Moreover, we have the following estimates:*

$$\left\| p \right\|_{L^\infty((0,T);H^1(\Omega))} \leq C_1, \quad \left\| p \right\|_{L^\infty(Q_T)} \leq C_2,$$

uniformly w.r.t. ε .

Proof Obviously, time t plays the role of a parameter. Thus, considering a fixed time t , existence and uniqueness of the pressure follows from Lax-Milgram lemma applied to an elliptic problem. To achieve the proof, it is sufficient to state that the coefficients are bounded and that the A is coercive, which is obvious from the following properties

$$\begin{aligned} 1 &\leq A(s) \leq 1/\varepsilon, & \forall s \in [0, 1], \\ 0 &\leq B(s) \leq 2, & \forall s \in [0, 1], \end{aligned}$$

which are easily obtained from the definitions of A and B (see the Appendix at the end of the paper). Here, ε denotes the ratio of the viscosities μ_g/μ_l . Thus, using $p(t, \cdot)$ ($t \in (0, T)$) as a test-function in the variational formulation (4.1), we have:

$$\begin{aligned} h_{\min}^3 \int_0^L \left| \frac{\partial p}{\partial x}(t, \cdot) \right|^2 &\leq \int_0^L A(s(t, \cdot)) h^3 \left| \frac{\partial p}{\partial x}(t, \cdot) \right|^2 = 6v_0\mu_l \int_0^L B(s(t, \cdot)) h \frac{\partial p}{\partial x}(t, \cdot) \\ &\leq 12v_0\mu_l h_{\max} \int_0^L \frac{\partial p}{\partial x}(t, \cdot). \end{aligned}$$

Next, using the Cauchy-Schwarz inequality, we have:

$$\left\| \frac{\partial p}{\partial x}(t, \cdot) \right\|_{L^2(\Omega)} \leq \frac{12v_0\mu_l h_{\max} L^{1/2}}{h_{\min}^3}.$$

The estimate in the H^1 norm is straightforward from Poincaré-Friedrichs inequality. The estimate in the L^∞ norm comes from the fact that $H^1(\Omega) \subset L^\infty(\Omega)$ with compact injection. Since the earlier inequality holds for almost every $t \in (0, T)$, the proof is concluded. \square

4.2 Simulation of the generalized Reynolds equation

Suppose that the saturation s can be computed for each time step. Then, the pressure p is also obtained at each time step by using any numerical method related to an elliptic problem: finite difference discretization, shooting method with a Runge-Kutta solver (after some linear interpolation procedure on the saturation), finite element method.

5 Numerical simulation

5.1 Influence of the shear effects and boundary conditions

As pointed out in the previous sections, the shear effects play a crucial role on the analysis of the generalized Buckley-Leverett equation. Indeed, we have already mentioned that it

leads to the non-autonomous property of the flux, and also a relaxation of the Dirichlet conditions: without shear velocity, the boundary condition is active only at point $x = 0$ because of the monotonicity of f . But when the shear effects are included, this lacks due to the presence of the partial flux g . We are interested in the contribution of the shear term in the Buckley-Leverett equation. For this, we compare the following regimes:

- $v_0^{(1)} = 0$: injection of a fluid through a fixed gap,
- $v_0^{(2)} = 1$: injection of a fluid through a fixed gap between two surfaces in relative motion (shear effects).

The main difference lies in the properties of the Buckley-Leverett flux function. For this, we choose the following numerical data:

- The lubricant is stucked to the moving surface (Case (i)),
- Geometrical data: $\Omega =]0, 1[$, $h(x) = (2x - 1)^2 + \frac{1}{2}$,
- Reference viscosity: $\mu_l = \mu_g = 1$,
- Flow input: $Q_{in} = v_0^{(2)} \theta_{in} h(0)$ with $\theta_{in} = 0.37$,
- The mesh grid has 600 elements,
- The CFL condition is given by $\frac{\Delta t}{\Delta x} \mathcal{M} = 0.9$.

We start from two different initial conditions:

$$s_0^{(1)}(x) = (\theta_{in} - 0.01)(2x - 1)^6 + 0.01, \quad s_0^{(2)}(x) = (\theta_{in} - 0.99)(2x - 1)^6 + 0.99,$$

($s_0^{(i)}$ is called initial condition (i)) and we observe (see FIG.3-6) that in both regimes ($v_0 = 0$ or $v_0 \neq 0$), the numerical stationary solution (obtained for $T = T_f$) does not depend on the initial condition. Moreover, the shear effects involve a major difference with the autonomous case: the stationary solution is not constant but contains balanced effects due to the non-autonomous flux.

Now, let us focus on the boundary conditions. As it was pointed out, the boundary conditions may be active only on some part of the boundary. Moreover, Theorem 3.11 provides a strong convergence result, up to a possible boundary layer at each point of the boundary. This kind of behaviour is illustrated on FIG.7. The following numerical data have been considered:

- The lubricant is stucked to the moving surface (Case (i)),
- Shear velocity: $v_0 = 1$,
- Geometrical data: $\Omega =]0, 1[$, $h(x) = (2x - 1)^2 + \frac{1}{2}$,
- Viscosities: $\mu_l = 1$, $\mu_g = 10^{-3} \mu_l$,
- Flow input: $Q_{in} = v_0 h(0) g(\theta_{in}) / (1 - f(\theta_{in}))$ with $\theta_{in} = 0.385$,
- The mesh grid has 1400 elements,
- The CFL condition is given by $\frac{\Delta t}{\Delta x} \mathcal{M} = 0.9$.

The initial condition is $s_0 \equiv \theta_{in}$ and the boundary condition also takes the value $s_1 = \theta_{in}$ (in the sense that has been precised before).

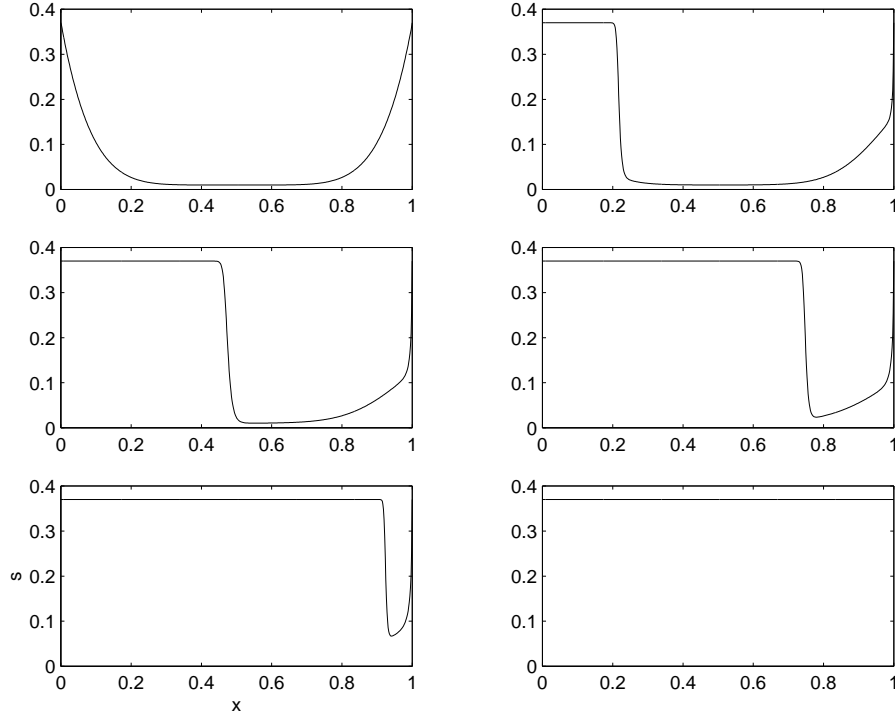


FIGURE 3. Saturation at different time steps $t = (i - 1)T_f/5$ ($i = 1$ to 6), corresponding to initial condition (1), without shear effects

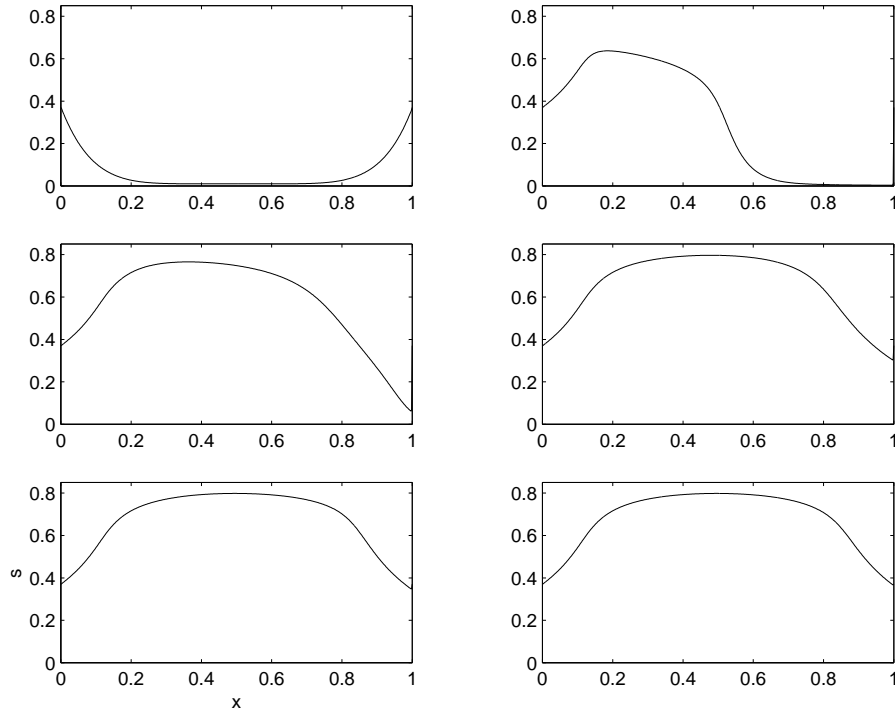


FIGURE 4. Saturation at different time steps $t = (i - 1)T_f/5$ ($i = 1$ to 6), corresponding to initial condition (1), including shear effects

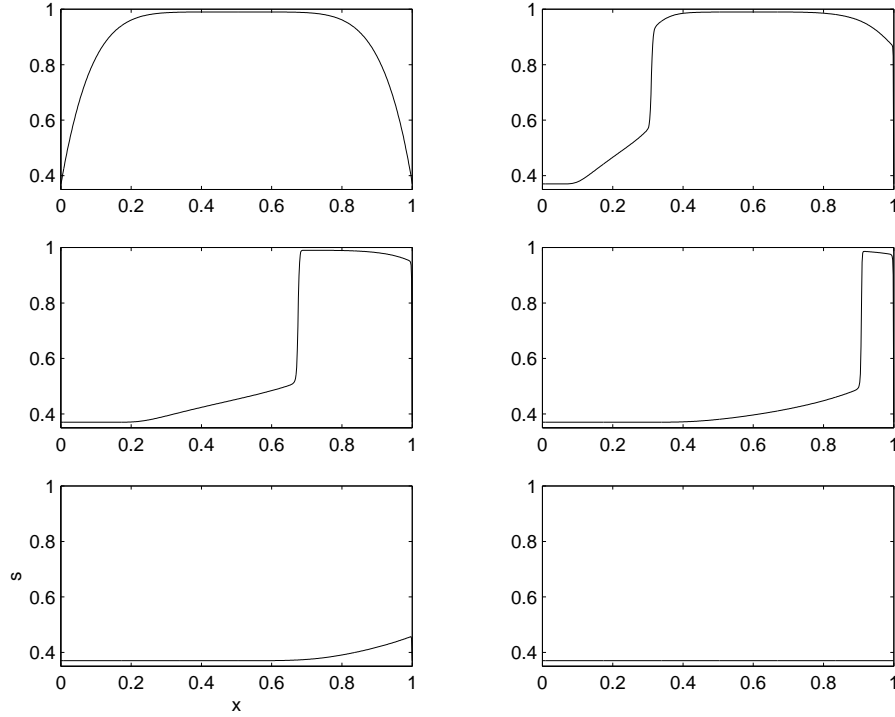


FIGURE 5. Saturation at different time steps $t = (i - 1)T_f/5$ ($i = 1$ to 6), corresponding to initial condition (2), without shear effects

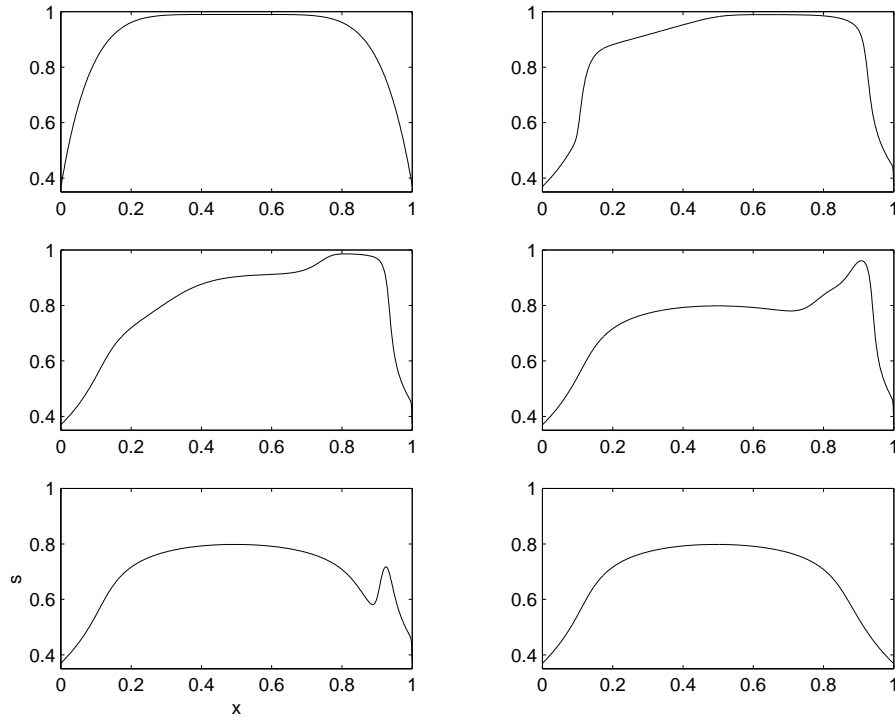


FIGURE 6. Saturation at different time steps $t = (i - 1)T_f/5$ ($i = 1$ to 6), corresponding to initial condition (2), including shear effects

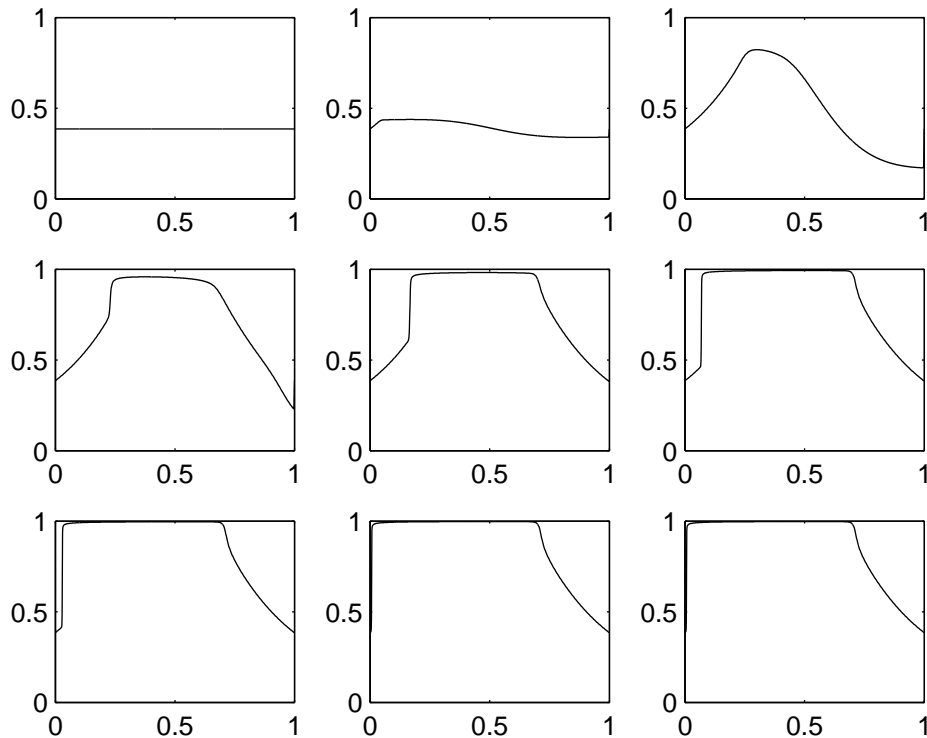


FIGURE 7. Saturation profile with partially active boundary conditions at different time steps: $t = (i - 1)T_f/9$, for i from 1 to 9 (from left to right, up to down)

5.2 About cavitation phenomena in lubrication theory

Lubricated device are generally made of two surfaces which are closely spaced, the annular gap being filled with some lubricant. The radial clearance is very small, typically $\Delta r/r = 10^{-3}$ for infinite oil lubricated bearings. The smallness of this ratio allows for a Cartesian coordinate to be located on the bearing surface. Thus, the Reynolds equation has been used for a long time to describe the behaviour of a viscous flow between two close surfaces in relative motion. The transition of the Stokes equation to the Reynolds equation has been proved by Bayada and Chambat in [3]. It can be written as:

$$\frac{d}{dx} \left(\frac{h^3}{6\mu_l} \frac{dp}{dx} \right) = v_0 \frac{dh}{dx}$$

where μ_l is the lubricant viscosity, p the pressure distribution, h the height between the two surfaces and v_0 the relative speed of the surfaces.

Nevertheless, this modelling does not take into account cavitation phenomena: cavitation is defined as the rupture of the continuous film due to the formation of air bubbles and makes the Reynolds equation no longer valid in the cavitation area. In order to make it possible, various models have been used, the most popular perhaps being variational

inequalities which have a strong mathematical basis but lack physical evidence. Thus, another model, the Elrod-Adams model, is often used, assuming that the cavitation region is a fluid-air mixture and introducing an additional unknown θ (the saturation of fluid in the mixture) (see [5, 7, 8, 10]). The model includes a modified Reynolds equation: The weak formulation of this problem is the following one:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Find } (p, \theta) \in V \times L^\infty(\Omega) \text{ such that} \\ \int_{\Omega} \frac{h^3}{6\mu_l} \frac{dp}{dx} \frac{d\phi}{dx} = v_0 \int_{\Omega} \theta h \frac{d\phi}{dx} + v_0 \theta_{in} h(0) \phi(0), \quad \forall \phi \in V \\ p \geq 0, \quad 0 \leq \theta \leq 1, \quad p(1 - \theta) = 0, \quad \text{a.e. in } \Omega, \end{array} \right.$$

where the functional space V is defined by

$$V = \{\phi \in H^1(\Omega), \phi(L) = 0\}.$$

Introducing the domains

$$\begin{aligned} \Omega^+ &= \{x \in \Omega, p(x) > 0\} && \text{(lubricated region),} \\ \Omega_0 &= \{x \in \Omega, p(x) = 0\} && \text{(cavitated region),} \\ \Sigma &= \partial\Omega^+ \cup \Omega && \text{(free boundary),} \end{aligned}$$

the free boundary Σ separates a full film area Ω^+ from a cavitated area Ω_0 . Notice that if Q_{in} denotes the flow input, it can be expressed as $Q_{in} = v_0 \theta_{in} h(0)$, with

$$0 \leq \theta_{in} \leq 1.$$

This model has been studied in [2]. In particular, it appears that Neumann conditions are compatible with homogeneous Dirichlet conditions if and only if the normalized flow input θ_{in} belongs to some interval $]\theta_{\min}, \theta_{\max}[$. This corresponds to some ‘‘starvation’’ phenomenon: the domain is divided into three areas: starvation area ($\Omega_0^{(1)}$ in which $p \equiv 0$), then a saturated area (Ω^+ in which $p > 0$) and then another cavitated area ($\Omega_0^{(2)}$ in which $p \equiv 0$).

Now, we want to compare the generalized Buckley-Leverett / Reynolds model to the Elrod-Adams model, which is motivated by the multifluid approach: indeed, the Elrod-Adams model describes partial lubrication of a device composed of a lubricant in two phases (liquid and gas) but strictly focuses on the liquid aspect. The generalized Buckley-Leverett / Reynolds model takes into account the influence of both phases. Thus, recalling that $\varepsilon \sim 10^{-3}$ in the case of a bifluid composed of liquid/gas, we want to numerically compare the two models. More precisely, we aim at comparing the numerical stationary solution of the generalized Buckley-Leverett / Reynolds problem to the solution of the Elrod-Adams problem. For this, let us motivate the choice of the data:

- Geometrical data: $\Omega =]0, 1[$, $h(x) = (2x - 1)^2 + \frac{1}{2}$,
- Shear velocity, reference viscosity (lubricant viscosity) : $v_0 = 1$ and $\mu_l = 1$,
- Flow input : $\theta_{in} = 0.385$. Let us notice that the same flow input Q_{in} has been considered in the Buckley-Leverett / Reynolds model and the Elrod-Adams model.

The choice of such a flow input is designed to compare the Buckley-Leverett / Reynolds model to the Elrod-Adams model with the same data. With the chosen value, the Elrod-Adams solution contains starvation along with a well-known discontinuity in the satu-

ration function corresponding to the rupture of the film thickness; thus, same phenomena observed with the Buckley-Leverett / Reynolds model would lead to some kind of a justification of the Elrod-Adams model. Unfortunately, it is impossible to obtain a Buckley-Leverett saturation with strictly value 1 at one point of the domain with such a choice: indeed we have

Proposition 5.1 *If s is a stationary solution of the generalized Buckley-Leverett equations (2.1)–(2.3), then the following are equivalent:*

- (i) *There exists $x \in \Omega$ such that $s(x) = 1$.*
- (ii) *The relative flow input satisfies the so-called “full saturation condition”:*

$$\theta_{in} = \frac{g(\theta_{in})}{1 - f(\theta_{in})}.$$

Proof Let us assume that s is a stationary solution of Equation (2.1). Then, after integration,

$$Q_{in}f(s(x)) + v_0 h(x) g(s(x)) = C, \quad \forall x \in \Omega, \quad (5.1)$$

C being a constant. This constant is determined by the value of the earlier function at one boundary of Ω (notice that $h(0) = h(1)$), namely

$$C = Q_{in}f(\theta_{in}) + v_0 h(0) g(\theta_{in}). \quad (5.2)$$

Since $f(1) = 1$ and $g(1) = 0$, if there exists $x_0 \in \Omega$, such that $s(x_0) = 1$, one gets

$$Q_{in}f(s(x_0)) + v_0 h(x_0) g(s(x_0)) = Q_{in}. \quad (5.3)$$

Thus, from Equations (5.1)–(5.3),

$$Q_{in} = v_0 h(0) \theta_{in} = v_0 h(0) \frac{g(\theta_{in})}{1 - f(\theta_{in})}.$$

□

Proposition 5.2 *We have the following properties:*

- *For all $\theta_{in} \in]0, 1[$, if s is a stationary solution of the generalized Buckley-Leverett equations (2.1)–(2.3), then s does not reach the value 1.*
- *For all $\theta_{in} \in]0, 1[$, the “full saturation condition” is asymptotically satisfied, i.e.*

$$\lim_{\varepsilon \rightarrow 0} \frac{g(\theta_{in})}{1 - f(\theta_{in})} = \theta_{in}.$$

Proof

- The proof is straightforward from the fact that

$$x > \frac{g(x)}{1 - f(x)}, \quad \forall x \in [0, 1[$$

so that the “full saturation condition” cannot be satisfied.

- Recalling the properties of f and g , we have (see FIG.14 and 15):

$$\lim_{\varepsilon \rightarrow 0} f = 0, \quad \lim_{\varepsilon \rightarrow 0} g = \text{Id},$$

uniformly on every compact set $K \subset [0, 1[$. Thus, we have (for $\theta_{in} \neq 1$):

$$\lim_{\varepsilon \rightarrow 0} \left\{ Q_{in} f(\theta_{in}) + v_0 h(0) g(\theta_{in}) \right\} = Q_{in},$$

which means that the “full saturation condition” is asymptotically satisfied. □

Remark 5.3 *From Propositions 5.1 and 5.2, it is impossible to get a (possible) stationary saturation with value 1 at any location of the domain, because the “full saturation condition” never holds (for non-pathological values of θ_{in}). Nevertheless, since it is asymptotically attained, we infer that considering small values of ε increases our hope to compare the Buckley-Leverett / Reynolds solution to the Elrod-Adams solution. Unfortunately, the behaviour of both fluxes f and g is pathological as ε tends to 0. Indeed, the graph of functions $x \mapsto f(x)$ and $x \mapsto g(x)$ clearly shows the presence of a boundary layer at $x = 1$ when ε tends to 0. Moreover, we have already mentioned that the numerical simulation of the Buckley-Leverett problem for small values of ε becomes difficult.*

Now let us enter into the details of the numerical comparison. Numerical data related to the simulation of the Buckley-Leverett equation are the following ones:

- ▷ The mesh grid has 1400 elements.
- ▷ The CFL condition is given by $\frac{\Delta t}{\Delta x} \mathcal{M} = 0.9$.
- ▷ The numerical stationary solution is attained for a relative error in the discrete L^2 norm with value 10^{-24} .

We study the behaviour of the bifluid in the two mentioned cases (see the Appendix at the end of the paper).

5.2.1 Case (i): the liquid phase is sticked to the lower (moving) surface

FIG.8 (resp. 9) gives the pressure (resp. saturation) distribution obtained with the Buckley-Leverett / Reynolds model along with the solution of the Elrod-Adams model. We can see that when ε decreases, the Buckley-Leverett saturation tends to the Elrod-Adams saturation; nevertheless, it never reaches the value 1, as previously mentioned. For $\varepsilon = 10^{-3}$, which corresponds to physical situations in lubrication theory, the Buckley-Leverett saturation is very near to the Elrod-Adams one, up to the full saturation area. When considering the pressure, we can see that the peak of the Buckley-Leverett pressure never reaches the one of the Elrod-Adams pressure; this is due to the sensibility of the coefficients in the generalized Reynolds equation. Indeed, the fact that the Buckley-Leverett saturation never reaches the value 1 prevents the generalized Reynolds pressure from approaching the Elrod-Adams pressure. Nevertheless, it is observed that non-positive pressures in the Buckley-Leverett / Reynolds model tend to vanish as ε tends to 0. Another

interesting point is to consider that, using the Elrod-Adams saturation in the generalized Reynolds equation allows to exactly obtain the Elrod-Adams pressure, although the main difference between the Elrod-Adams saturation and the Buckley-Leverett saturation lies in the (nearly) saturated region (in the first case, the value 1 is reached although, in the second case, it never reaches this value); this observation shows how the “full saturation condition” is important. However, it is reasonable to say that computations for $\varepsilon = 10^{-3}$ allow to identify cavitated and saturated areas.

FIG.10 describes the horizontal velocity distribution in the domain

$$\{(x, z) \in \mathbb{R}^2, 0 < x < 1, 0 < z < h(x)\}$$

with both models: the Buckley-Leverett / Reynolds model and the Elrod-Adams model.

- *Computation of the velocity for the Buckley-Leverett / Reynolds model:* Let us denote (U, V) the velocity in the bifluid. The following relationship has been derived in [17]:

$$\eta(x, z) \frac{\partial^2 U}{\partial z^2}(x, z) = \frac{dp}{dx}(x), \quad (5.4)$$

$$U(x, 0) = v_0, \quad (5.5)$$

$$U(x, h(x)) = 0, \quad (5.6)$$

with

$$\eta(x, z) = \begin{cases} \mu_l, & \text{if } 0 < z < s(x)h(x), \\ \mu_g, & \text{if } s(x)h(x) < z < h(x). \end{cases}$$

Moreover, $z \mapsto U(x, z)$ is continuous at the interface $z = s(x)h(x)$. Thus, integrating twice Equation (5.4) and taking account of boundary conditions (5.5) and (5.6) allow to get the velocity profile, up to the velocity at the free boundary, denoted U^* . Nevertheless, U^* is obtained by taking account of

$$\lim_{z \rightarrow (s(x)h(x))^-} \left\{ \mu_l \frac{\partial U}{\partial z}(x, z) \right\} = \lim_{z \rightarrow (s(x)h(x))^+} \left\{ \mu_g \frac{\partial U}{\partial z}(x, z) \right\}.$$

- *Computation of the velocity for the Elrod-Adams model:* The velocity profile for the Elrod-Adams model is computed under the additional assumption that the liquid phase of the lubricant is stucked to the moving surface (we recall that, actually, the Elrod-Adams model does not provide any information on the position of the liquid phase). Thus, the velocity at the free boundary (which separates the lubricant and the “empty” phase) is equal to 0. The velocity in the “empty” phase is reduced to 0. In details, we obtain:

$$\mu_l \frac{\partial^2 U}{\partial z^2}(x, z) = \frac{dp}{dx}(x), \text{ for } z \in]0, s(x)h(x)[$$

with the boundary conditions $U(x, 0) = v_0$ and $U(x, s(x)h(x)) = 0$. In the empty phase, we have $U(x, z) = 0$, for $z \in]s(x)h(x), h(x)[$.

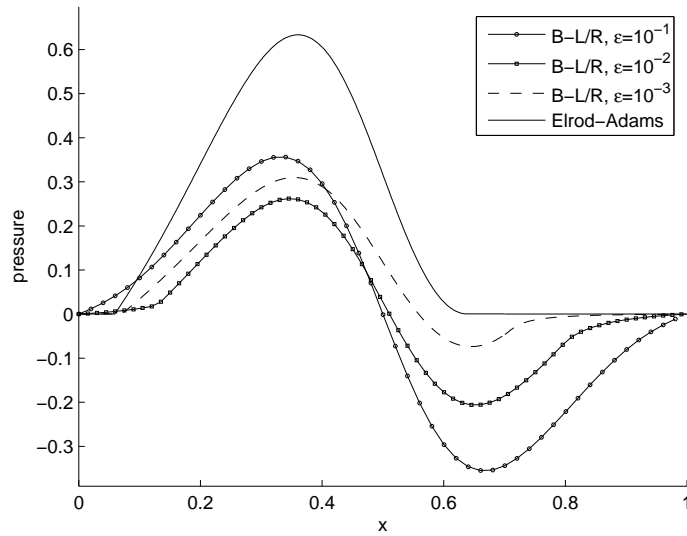


FIGURE 8. Pressure distribution for the Buckley-Leverett / Reynolds and Elrod-Adams models (Case (i))

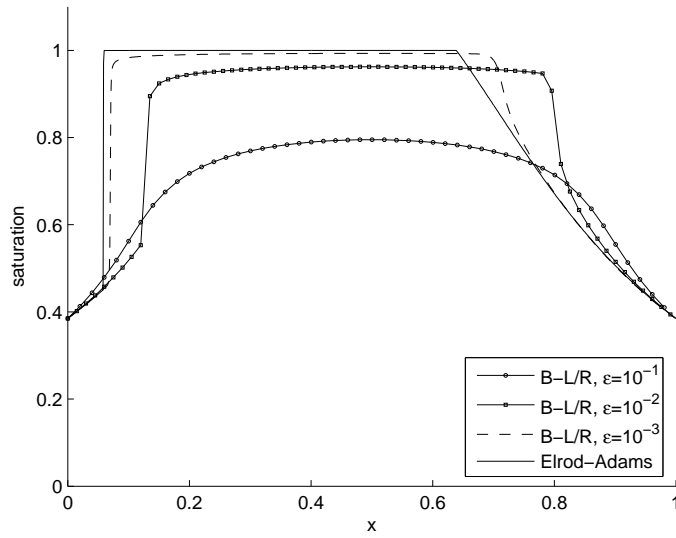


FIGURE 9. Saturation distribution for the Buckley-Leverett / Reynolds and Elrod-Adams models (Case (i))

5.2.2 Case (ii): the liquid phase is sticked to the upper (fixed) surface

FIG.11 (resp. 12) gives the pressure (resp. saturation) distribution obtained with the Buckley-Leverett / Reynolds model for various values of ε . We can see that when ε decreases, the Buckley-Leverett saturation tends to the uniform distribution $s \equiv \theta_{in}$.

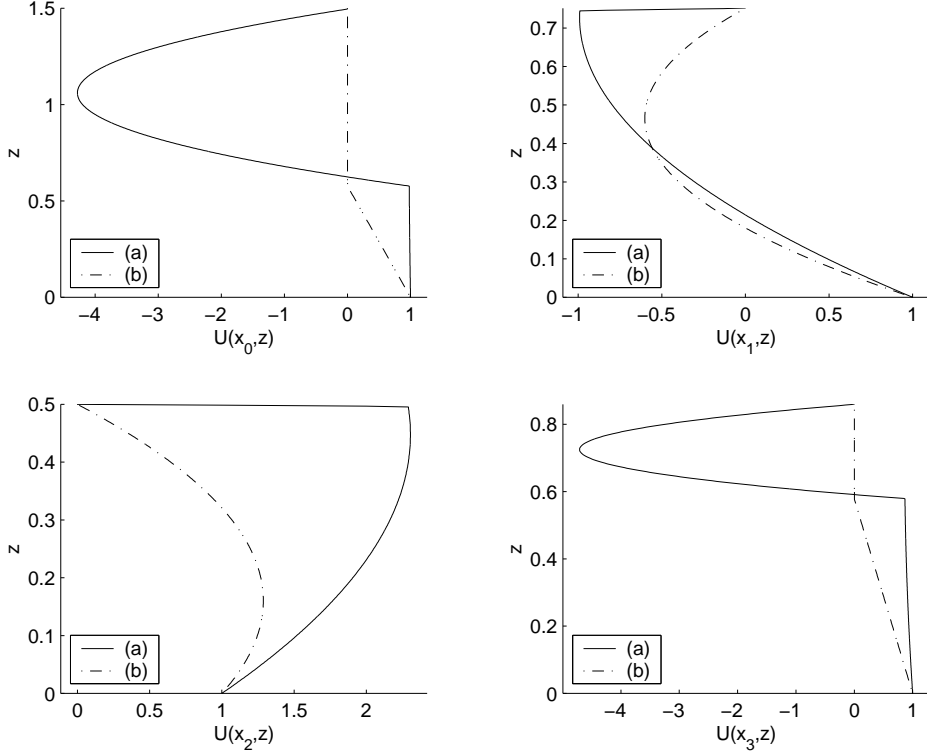


FIGURE 10. Horizontal velocity U at $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 4/5$, for the Buckley-Leverett model with $\varepsilon = 10^{-3}$ (a) and the Elrod-Adams model (b) (Case (i))

The pressure tends to 0 uniformly. Therefore, the Buckley-Leverett / Reynolds model does not approach the Elrod-Adams model in that configuration: assuming that the lubricant is stucked to the upper (fixed) surface is not relevant. In fact, we can observe that, in this configuration, the shear effects tend to vanish, as it is pointed out further. FIG.13 describes the horizontal velocity distribution in the whole domain with the two models (Buckley-Leverett / Reynolds and Elrod-Adams).

- *Computation of the velocity for the Buckley-Leverett / Reynolds model:* Equations (5.4)–(5.6) still hold, up to this adapted definition (which takes into account the position of each phase):

$$\eta(x, z) = \begin{cases} \mu_g, & \text{if } 0 < z < (1 - s(x))h(x), \\ \mu_l, & \text{if } (1 - s(x))h(x) < z < h(x). \end{cases}$$

Now, as in Case (i), this is completed with the following equality at the interface:

$$\lim_{z \rightarrow ((1-s(x))h(x))^-} \left\{ \mu_g \frac{\partial U}{\partial z}(x, z) \right\} = \lim_{z \rightarrow ((1-s(x))h(x))^+} \left\{ \mu_l \frac{\partial U}{\partial z}(x, z) \right\}.$$

- *Computation of the velocity for the Elrod-Adams model:* The velocity profile for the Elrod-Adams model is computed under the additional assumption that the liquid phase of the lubricant is stucked to the upper surface (which is not supposed by the model).

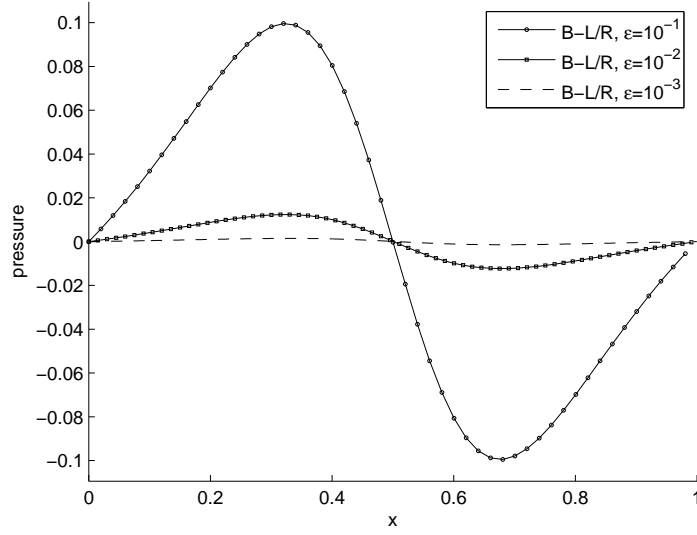


FIGURE 11. Pressure distribution for the Buckley-Leverett model (Case (ii))

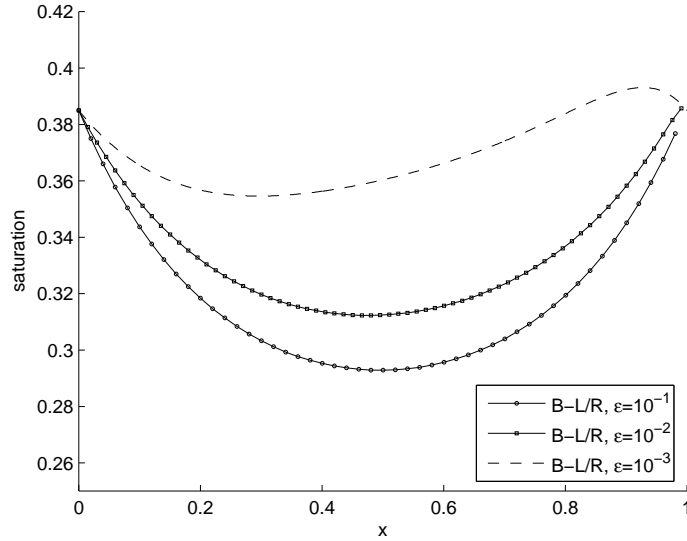


FIGURE 12. Saturation distribution for the Buckley-Leverett model (Case (ii))

We obtain:

$$\mu_l \frac{\partial^2 U}{\partial z^2}(x, z) = \frac{dp}{dx}(x), \text{ for } z \in](1 - s(x)) h(x), h(x)[,$$

with the boundary conditions $U(x, h(x)) = v_0$ and $U(x, (1 - s(x)) h(x)) = 0$. In the empty phase, we have $U(x, z) = 0$, for $z \in]0, (1 - s(x)) h(x)[$.

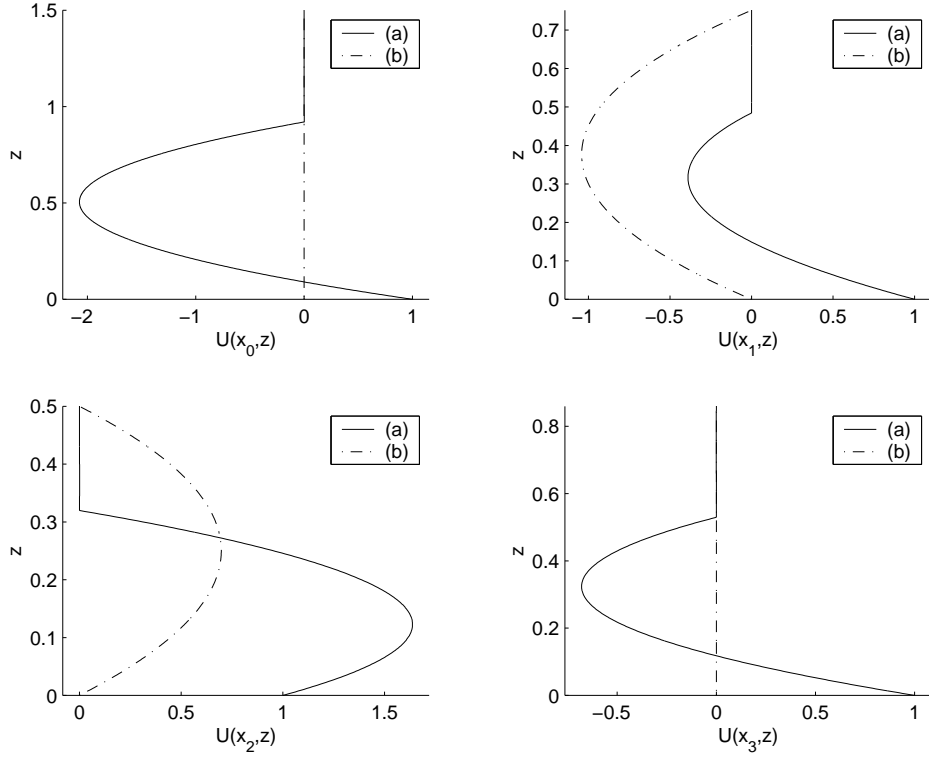


FIGURE 13. Horizontal velocity U at $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 4/5$, for the Buckley-Leverett model with $\varepsilon = 10^{-3}$ (a) and the Elrod-Adams model (b) (Case (ii))

Let us recall that the lubricant in liquid phase is considered as the reference fluid. Thus, velocity profiles show that the lubricant does not support the shear effects at all. This can be viewed also through the expression of the Buckley-Leverett flux, $Q_{inf}(s) + v_0 h(x)g(s)$, in which it is observed that when ε tends to 0, the non-classical contribution to the Buckley-Leverett flux, i.e. the shear contribution $v_0 h(x)g(s)$, tends to vanish.

Appendix. Expression of the flux functions

In [14, 17], the flow of two Newtonian, incompressible, nonmiscible fluids in a 2D thin domain is considered. Starting from the Stokes equations, a generalized Buckley-Leverett equation for the reference fluid saturation is derived. Studying the asymptotic behavior of the flow when the thickness of the gap tends to zero and assuming that the fluids interface, which is a free boundary, is described by curves of uniformly bounded variation, it is proved that the limit problem obeys a generalized Reynolds law.

For the sake of simplicity, let us introduce the following function:

$$\alpha_\varepsilon^{(i)}(s) = 1 - (1 - \varepsilon)s^i, \quad i = 1, 2, 3.$$

Case (i): the lubricant is stucked to the lower (moving) surface

This corresponds to FIG.1. The functions A , B are given by:

$$\begin{aligned} A(s) &:= A^{(1)}(s) = \frac{4\alpha_\varepsilon^{(1)}(s)\alpha_\varepsilon^{(3)}(s) - 3(\alpha_\varepsilon^{(2)}(s))^2}{\varepsilon \alpha_\varepsilon^{(1)}(s)}, \\ B(s) &:= B^{(1)}(s) = \frac{\alpha_\varepsilon^{(2)}(s)}{\alpha_\varepsilon^{(1)}(s)}, \end{aligned}$$

and the flux functions f and g are defined by:

$$\begin{aligned} f(s) &:= f^{(1)}(s) = \varepsilon s^2 \frac{3(\alpha_\varepsilon^{(2)}(s))^2 - 2s\alpha_\varepsilon^{(1)}(s)}{4\alpha_\varepsilon^{(1)}(s)\alpha_\varepsilon^{(3)}(s) - 3(\alpha_\varepsilon^{(2)}(s))^2}, \\ g(s) &:= g^{(1)}(s) = -f^{(1)}(s) \frac{\alpha_\varepsilon^{(2)}(s)}{2\alpha_\varepsilon^{(1)}(s)} + s \left(1 - \varepsilon \frac{s}{2\alpha_\varepsilon^{(1)}(s)} \right). \end{aligned}$$

Case (ii): the lubricant is stucked to the upper (fixed) surface

This corresponds to FIG.2. We emphasize that the flux functions or Reynolds coefficients can be written, in a simple way, as a perturbation of the ones described in the first case. Indeed, the functions A , B are given by:

$$\begin{aligned} A(s) &:= A^{(2)}(s) = A^{(1)}(s), \\ B(s) &:= B^{(2)}(s) = B^{(1)}(s) - \frac{2s(1-s)(1-\varepsilon)}{\alpha_\varepsilon^{(1)}(s)}, \end{aligned}$$

and the flux functions f and g are defined by:

$$\begin{aligned} f(s) &:= f_\varepsilon^{(2)}(s) = f^{(1)}(s), \\ g(s) &:= g_\varepsilon^{(2)}(s) = g^{(1)}(s) - \frac{s(1-s)}{\alpha_\varepsilon^{(1)}(s)} \left(1 - (1-\varepsilon)f^{(2)}(s) \right). \end{aligned}$$

From the previous formulas, it can be deduced that only the shear terms are modified by the assumption on the geometrical assumption: indeed, in the Buckley-Leverett (resp. Reynolds) equation, only the flux function g (resp. right-hand side B) is modified.

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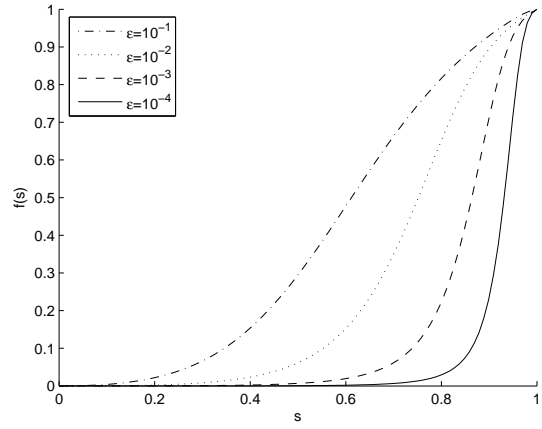


FIGURE 14. Classical contribution to the Buckley-Leverett flux function f (Cases (i) – (ii))

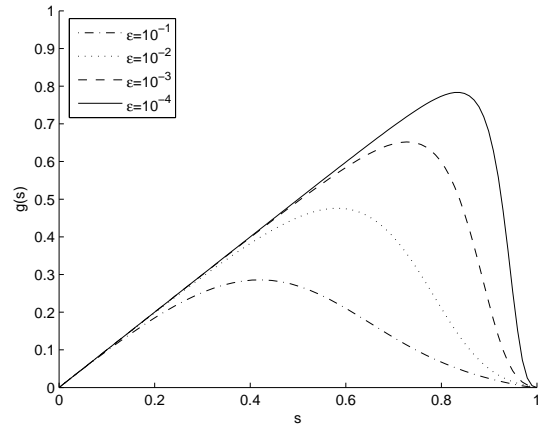


FIGURE 15. Shear contribution to the Buckley-Leverett flux function g (Case (i))

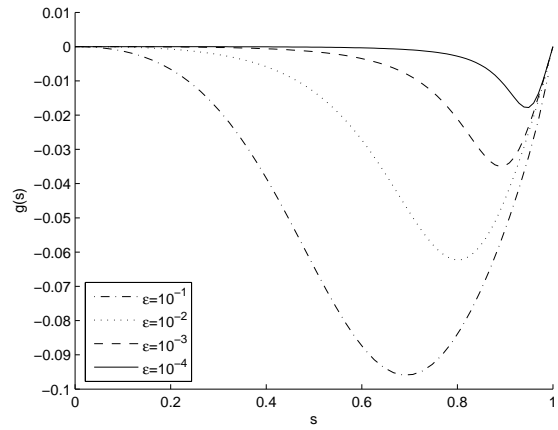


FIGURE 16. Shear contribution to the Buckley-Leverett flux function g (Case (ii))

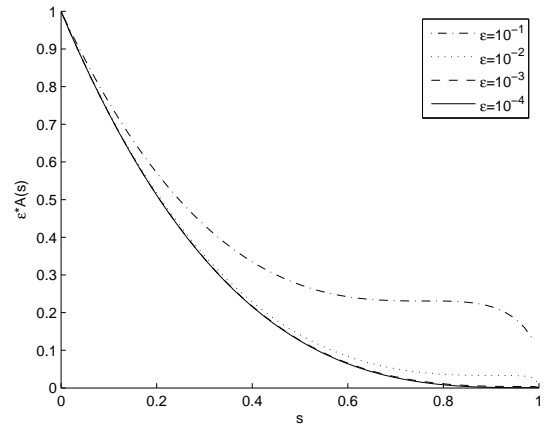


FIGURE 17. Left-hand side weight function in the Reynolds equation A (Cases $(i) - (ii)$)

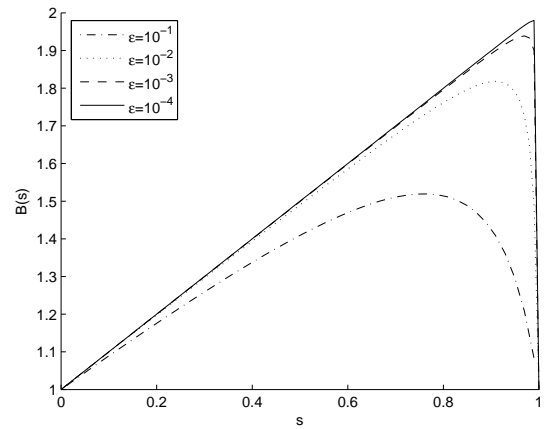


FIGURE 18. Right-hand side weight function in the Reynolds equation B (Case (i))

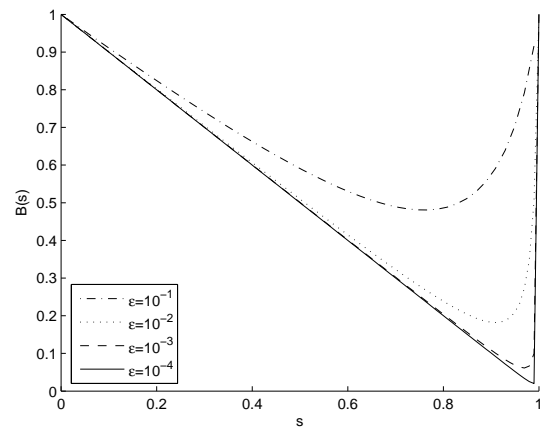


FIGURE 19. Right-hand side weight function in the Reynolds equation B (Case (ii))

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