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Out of equilibrium correlations in the XY chain

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Abstract

We study the transversal spin-spin correlations in the non-equilibrium steady state of the XY chain constructed by coupling a finite cutout of the chain to the two infinite parts to its left and right acting as thermal reservoirs at different temperatures. We prove that the spatial decay of these correlations is at least exponentially fast.

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1 Introduction

The XY chain is the one-dimensional integrable spin system, introduced in [11] (see also [9]), whose formal Hamiltonian is specified by

$$H = -\frac{1}{4}\sum_{x\in\mathbb{Z}}\left\{(1+\gamma)\,\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\,\sigma_2^{(x)}\sigma_2^{(x+1)} + 2\lambda\,\sigma_3^{(x)}\right\}, \tag{1.1}$$

where $\sigma_j^{(x)}$ denotes the Pauli matrix at site $x \in \mathbb{Z}$ in the transversal directions, $j = 1,2$, and in the longitudinal direction, $j = 3$. The parameter $\gamma \in (-1,1)$ describes the anisotropy of the spin-spin coupling whereas $\lambda \in \mathbb{R}$ stands for an external magnetic field.

Since the discovery of their ideal thermal conductivity, quasi-one-dimensional Heisenberg spin-$1/2$ systems, made from different materials, have been intensively studied experimentally (see [14, 15]; $SrCuO_2$ and Sr_2CuO_3 are often considered to be the best physical realizations of one-dimensional XYZ Heisenberg models).

Not only this highly unusual transport property motivates the theoretical study of such non-equilibrium models, but the XY chain also represents one of the simplest non-trivial testing grounds for the development of general ideas in rigorous non-equilibrium theory.

Already in [11], for vanishing external magnetic field $\lambda = 0$, the transversal spin-spin correlations in the ground state and at nonzero temperature could be expressed by means of determinants of large truncated Toeplitz matrices. Moreover, bounds on these correlations implied vanishing long-range order, at least in the isotropic case $\gamma = 0$. Later, in [12], this study was continued yielding an asymptotic evaluation of the transversal correlations with the help of Szegö's theorem. Afterwards, in [6], almost the complete phase diagram in γ and λ for the behavior of the correlation functions both in the longitudinal and in the transversal directions were obtained, and so for zero and nonzero temperature: The result of this study for the case of nonzero temperature is that all the correlation functions are asymptotically exponentially decaying with a decay rate which depends on the magnetic field λ and the anisotropy γ.

In this note, we study the large n behavior of the transversal spin-spin correlations,

$$C_j(n) = \omega(\sigma_j^{(0)}\sigma_j^{(n)}), \quad j = 1,2,\ n \in \mathbb{N}, \tag{1.2}$$

in the non-equilibrium steady state (NESS) ω constructed in [5] (and already in [3] for $\gamma = 0$) in a setting which has become to serve as paradigm in non-equilibrium statistical mechanics: a "small" system which is coupled to two infinite reservoirs which are in thermal equilibrium at different inverse temperatures β_L and β_R. We prove that the spatial decay of $C_j(n)$ is at least exponentially fast. This behavior contrasts with the one generally expected by the folklore which predicts quasi-long-range order out of equilibrium.

In the following section, we give a brief informal description of our non-equilibrium setting for the XY chain. We refer to [4, 5, 10] for a precise formulation within the framework of C^* algebraic quantum statistical mechanics.

2 The non-equilibrium setting for the XY chain

Consider the XY chain described by the Hamiltonian (1.1) and remove the two bonds at the sites $\pm M$, $M > 0$. Doing so, the initial chain divides into a compound of three noninteracting subsystems. This configuration is what we call the free system whose Hamiltonian

$$H_0 = H_L + H_S + H_R$$

is built as in (1.1) according to the union $\mathbb{Z}_L \cup \mathbb{Z}_S \cup \mathbb{Z}_R$, where $\mathbb{Z}_L = \{x \in \mathbb{Z} \,|\, x \leq -M-2\}$, $\mathbb{Z}_R = \{x \in \mathbb{Z} \,|\, x \geq M+1\}$, and $\mathbb{Z}_S = \{x \in \mathbb{Z} \,|\, -M \leq x \leq M-1\}$. The infinite pieces $\mathbb{Z}_L$ and $\mathbb{Z}_R$ will play the role of thermal reservoirs to which the finite system on $\mathbb{Z}_S$ is coupled by means of $V = H - H_0$. In contrast, the configuration described by H, i.e. the original XY chain on the whole of $\mathbb{Z}$, is considered to be the perturbed system.
In order to construct a NESS in the sense of [13], we choose the initial state ω_0 to be composed of (τ_j, β_j)-KMS states ω_j on $\mathbb{Z}_j$, $j = L, R$, and of the normalized trace state ω_S on $\mathbb{Z}_S$ (τ_j denotes the time evolution generated by H_j, and β_j is the inverse temperature), i.e. we set

$$\omega_0 = \omega_L \otimes \omega_S \otimes \omega_R.$$

It is well-known that the Jordan-Wigner transformation maps the XY model on a model of free fermions with annihilation and creation operators c_x, c_x^*. Using scattering theory on the 1-particle Hilbert space of these fermions, the NESS ω w.r.t. the initial state ω_0 and the perturbed time evolution τ^t have been constructed in [5],

$$\omega(c_x) = \lim_{t \to +\infty} \omega_0(\tau^t(c_x)).$$

It has been shown in [5] that ω is a quasi-free state with 2-point operator S,

$$\omega(B^*(f)B(g)) = (f, Sg), \tag{2.3}$$

where $f, g \in l^2(\mathbb{Z}) \otimes \mathbb{C}^2 \simeq l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$, and $B(f) = \sum_{x \in \mathbb{Z}} (f_+(x) c_x^* + f_-(x) c_x)$ for $f = (f_+, f_-)$. Moreover, S has been explicitly computed in [5]. In the Fourier picture, $l^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$ (with $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$), it reads

$$S(\xi) = \left(1 + e^{\beta\, h(\xi) + \delta\, k(\xi)}\right)^{-1}, \tag{2.4}$$

with the parameters β and δ given by

$$\beta = \frac{1}{2}(\beta_R + \beta_L), \quad \delta = \frac{1}{2}(\beta_R - \beta_L). \tag{2.5}$$

The 1-particle operators h and k have the form

$$h(\xi) = (\cos\xi - \lambda) \otimes \sigma_3 - \gamma \sin\xi \otimes \sigma_2, \quad k(\xi) = \mathrm{sign}(\kappa(\xi))\, \mu(\xi) \otimes \sigma_0,$$

where the functions $\kappa(\xi)$ and $\mu(\xi)$ are defined by

$$\kappa(\xi) = 2\lambda \sin\xi - (1-\gamma^2)\sin 2\xi, \quad \mu(\xi) = \left((\cos\xi - \lambda)^2 + \gamma^2 \sin^2\xi\right)^{1/2}. \tag{2.6}$$

Furthermore, σ_0 is the identity on $\mathbb{C}^2$, and $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli matrices,

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.7}$$

3 Exponential decay of the correlation function

Let us now state our result.

Theorem *Let ω be the quasi-free NESS characterized by* (2.3)*, and $C_j(n)$, $j = 1,2$, the correlation functions* (1.2)*. For all values of the parameters $0 \le \delta < \beta < \infty$, $\gamma \in (-1,1)$, and $\lambda \in \mathbb{R}$, the transversal spin-spin correlations $C_j(n)$ decay at least exponentially fast, and*

$$\limsup_{n\to\infty} \frac{\log\left|C_j(n)\right|}{n} \le \frac{1}{2}\int_0^{2\pi} \frac{d\xi}{2\pi} \log\left[\mathrm{th}(\beta_L \mu(\xi)/2)\,\mathrm{th}(\beta_R \mu(\xi)/2)\right] < 0.$$

Remarks *1) If the temperature difference of the reservoirs vanishes, i.e. $\delta = 0$, the total system is in thermal equilibrium at inverse temperature $\beta_L = \beta_R$, and our bound estimates from above the asymptotically exponentially decaying behavior from [6].*

2) For $\delta \neq 0$, the correlation in the 3*-direction decays like $1/n^2$ at infinity for all $\gamma \in (-1,1)$, $\lambda \in \mathbb{R}$, see [5]. In contrast to this result, our theorem does not confirm the folklore about the change in the type of decay – from short range to long range – when passing from equilibrium to non-equilibrium.*

Before we begin the proof of the theorem, we rewrite the correlation function $C_j(n)$ as the determinant of a $2n \times 2n$ block Toeplitz matrix, in the same way as it has been done at zero and non-zero temperature in [6, 11]. The reader not familiar with Toeplitz theory may consult Appendix A before entering the proof. We restrict our attention to the transversal correlation function $C_1(n) \equiv C(n)$ in the 1-direction (the 2-direction being analogous, see the Jordan-Wigner transformation for σ_2 in (3.8) below),

$$C(n) = \omega(\sigma_1^{(0)}\sigma_1^{(n)}), \quad n \in \mathbb{N}.$$

The well-known Jordan-Wigner transformation expresses the spins $\sigma_1^{(x)}, \sigma_2^{(x)}, \sigma_3^{(x)}$, $x \in \mathbb{Z}$, by means of fermionic creation and annihilation operators $c_x^\sharp$ $(= c_x^*, c_x)$,

$$\sigma_1^{(x)} = TS^{(x)}(c_x + c_x^*), \quad \sigma_2^{(x)} = iTS^{(x)}(c_x - c_x^*), \quad \sigma_3^{(x)} = 2c_x^* c_x - 1, \tag{3.8}$$

where $S^{(x)} = \sigma_3^{(1)} \dots \sigma_3^{(x-1)}$ for $x > 1$, $S^{(1)} = 1$, and $S^{(x)} = \sigma_3^{(x)} \dots \sigma_3^{(0)}$ for $x < 1$, see for example [5]. The element T stems from Araki's C^* crossed product extension of the CAR algebra for the two-sided chain, see [2]. It has the properties $T^2 = 1$, $Tc_0^\sharp = -c_0^\sharp T$, and $Tc_x^\sharp = c_x^\sharp T$ for $x > 0$. Plugging the Jordan-Wigner transformation (3.8) into the product $\sigma_1^{(0)}\sigma_1^{(n)}$, we find

$$\sigma_1^{(0)}\sigma_1^{(n)} = (-1)^n a_-^{(0)} a_+^{(1)} a_-^{(1)} a_+^{(2)} a_-^{(2)} \cdot \ldots \cdot a_+^{(n-1)} a_-^{(n-1)} a_+^{(n)},$$

where $a_\pm^{(x)} = c_x^* \pm c_x$. Moreover, we define $\eta_\pm = (1, \pm 1) \in \mathbb{C}^2$ and $\alpha_\pm^{(x)} = \delta_x \otimes \eta_\pm$ (with $\delta_x \in l^2(\mathbb{Z})$, $\delta_x(y) = 1$ for $y = x$ and zero otherwise). With the linear map B introduced in [1], $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z}) \ni f = (f_+, f_-) \mapsto B(f) = \sum_{x\in\mathbb{Z}} (f_+(x)c_x^* + f_-(x)c_x)$, we have $a_\pm^{(n)} = B(\alpha_\pm^{(n)})$. Hence, we can express the correlation function $C(n)$ as

$$C(n) = (-1)^n \omega(B(\alpha_-^{(0)}) B(\alpha_+^{(1)}) B(\alpha_-^{(1)}) \cdot \ldots \cdot B(\alpha_+^{(n-1)}) B(\alpha_-^{(n-1)}) B(\alpha_+^{(n)})).$$

Since ω is quasi-free [5], the correlation function $C(n)$ is a Pfaffian,

$$C(n) = \mathrm{Pf}\,\Omega(n),$$

where the matrix $(\Omega(n)_{jk})_{j,k=1}^{2n} \in \mathbb{C}^{2n\times 2n}$ is defined to be skew-symmetric, and, for $j < k$,

$$\Omega(n)_{jk} = \omega(B(f_j)B(f_k)) \quad \text{with} \quad f_{2i-1} = \alpha_-^{(i-1)},\ f_{2i} = \alpha_+^{(i)},\ i = 1, 2, \ldots, n.$$

The Pfaffian $\mathrm{Pf}A$ of a matrix $A \in \mathbb{C}^{2n\times 2n}$ is defined by $\mathrm{Pf}A = \sum_\pi \mathrm{sign}(\pi) \prod_{k=1}^n A_{\pi_{2k-1},\pi_{2k}}$ with the sum running over all π in the permutation group S_{2n} which satisfy $\pi_{2k}, \pi_{2k+1} > \pi_{2k-1}$. If A is skew-symmetric, $A^T = -A$ (A^T denotes the transpose of A), the Pfaffian of A is related to the determinant of A through $(\mathrm{Pf}A)^2 = \det A$. Thus, we are led to study the large n asymptotics of the determinant of $\Omega(n)$,

$$C(n)^2 = \det \Omega(n). \tag{3.9}$$

In order to cast $\Omega(n)$ into the form of a truncated block Toeplitz matrix, we compute the matrix entries

$$A_{\pm\pm}^{jk} = \omega(B(\alpha_\pm^{(j)})B(\alpha_\pm^{(k)})),\ 1 \le j < k \le n. \tag{3.10}$$

For this purpose, we rewrite the 2-point operator $S(\xi)$ on $L^2(\mathbb{T}) \otimes \mathbb{C}^2$ from (2.4) as

$$S(\xi) = s_0(\xi) \otimes \sigma_0 + \sum_{k=1}^3 s_k(\xi) \otimes \sigma_k, \tag{3.11}$$

using the Pauli matrices from (2.7). The first component $s_0(\xi)$ looks like

$$s_0(\xi) \;=\; \frac{1}{2} + \frac{1}{2}\,\mathrm{sign}(\kappa(\xi))\,\varphi_\delta(\xi), \tag{3.12}$$

and $s(\xi) = (s_1(\xi), s_2(\xi), s_3(\xi))$ has the form

$$s(\xi) \;=\; \frac{1}{2}\,\varphi_\beta(\xi)\,r(\xi), \quad r(\xi) = \frac{1}{\mu(\xi)}\,(0, -\gamma\sin\xi, \cos\xi - \lambda), \tag{3.13}$$

with the functions $\kappa(\xi)$ and $\mu(\xi)$ from (2.6). Moreover, we used the definition

$$\varphi_\alpha(\xi) \;=\; \frac{\mathrm{sh}(\alpha\mu(\xi))}{\mathrm{ch}(\beta\mu(\xi)) + \mathrm{ch}(\delta\mu(\xi))}, \quad \alpha \in \mathbb{R}. \tag{3.14}$$

Let us start with the computation of A^{jk}_{++} in (3.10). Using the property $B^*(f) = B(Jf)$ for $J : f = (f_+, f_-) \mapsto (\bar{f}_-, \bar{f}_+)$ from [5], and the expressions in (3.11), (3.12), and (3.13), we find, for $1 \le j < k \le n$, and with the definition $e_j(\xi) = e^{-ij\xi}$, that

$$\begin{aligned} A^{jk}_{++} &= (e_j \otimes \eta_+, S e_k \otimes \eta_+)_{L^2\otimes\mathbb{C}^2} \\ &= 2\int_0^{2\pi} \frac{d\xi}{2\pi}\, s_0(\xi) e^{-i(k-j)\xi} \\ &= \int_0^{2\pi} \frac{d\xi}{2\pi}\, \mathrm{sign}(\kappa(\xi))\, \varphi_\delta(\xi)\, e^{-i(k-j)\xi}. \end{aligned}$$

The computation of A^{jk}_{--} yields $A^{jk}_{--} = -A^{jk}_{++}$. Next, the entry A^{jk}_{+-} looks like

$$\begin{aligned} A^{jk}_{+-} &= (e_j \otimes \eta_+, S e_k \otimes \eta_-)_{L^2\otimes\mathbb{C}^2} \\ &= 2\int_0^{2\pi} \frac{d\xi}{2\pi}\, (s_3(\xi) + i s_2(\xi)) e^{-i(k-j)\xi} \\ &= \int_0^{2\pi} \frac{d\xi}{2\pi}\, \frac{\cos\xi - \lambda - i\gamma\sin\xi}{\mu(\xi)}\, \varphi_\beta(\xi)\, e^{-i(k-j)\xi}. \end{aligned}$$

Finally, $A^{jk}_{-+} = -A^{kj}_{+-}$. Due to translation invariance, we can define $A^x_{\pm\pm} = A^{jj+x}_{\pm\pm}$ with $x \in \mathbb{Z}$. Using $A^x_{--} = -A^x_{++}$, $A^x_{-+} = -A^{-x}_{+-}$, and the symmetry property $A^{-x}_{++} = -A^x_{++}$, we can write $\Omega(n)$ in (3.9) in the form of a $2n \times 2n$ truncated block Toeplitz matrix with 2×2 blocks a_x,

$$\Omega(n) = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-(n-1)} \\ a_1 & a_0 & \dots & a_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix}, \tag{3.15}$$

where the blocks are given by

$$a_x = \begin{bmatrix} A^x_{++} & -A^{x-1}_{+-} \\ A^{-x-1}_{+-} & -A^x_{++} \end{bmatrix}, \quad x \in \mathbb{Z}. \tag{3.16}$$

Using the relations $a_x = \int_0^{2\pi} d\xi/(2\pi)\, a(\xi) e^{-ix\xi}$ from (A.20), we can extract from (3.16) the symbol $a(\xi)$

$$a(\xi) = \begin{bmatrix} \mathrm{sign}(\kappa(\xi))\, \varphi_\delta(\xi) & -q(\xi)\, \varphi_\beta(\xi) \\ \overline{q}(\xi)\, \varphi_\beta(\xi) & -\mathrm{sign}(\kappa(\xi))\, \varphi_\delta(\xi) \end{bmatrix}. \tag{3.17}$$

Here, $\varphi_\alpha(\xi)$ is defined in (3.14), $\kappa(\xi)$, $\mu(\xi)$ in (2.6), and $\overline{q}(\xi)$ is the complex conjugate of

$$q(\xi) = \frac{\cos\xi - \lambda - i\gamma \sin\xi}{\mu(\xi)}\, e^{i\xi}.$$

Therefore, denoting by $T[a]$ the block Toeplitz matrix associated with the symbol a given by (3.17), we can write $\Omega(n)$ as a truncated block Toeplitz matrix (see (A.23)),

$$\Omega(n) = T_n[a] = P_n T[a] P_n.$$

Proof of the Theorem For $n \in \mathbb{N}$, let $s_1^{(n)} \leq s_2^{(n)} \leq \ldots \leq s_{2n}^{(n)}$ be the singular values of $T_n[a]$, and denote by $N_{\varepsilon,n}$ the number of singular values in $[0,\varepsilon]$, $0 < \varepsilon < 1$, including multiplicity. From the above computations, we have

$$|C(n)|^2 = |\det \Omega(n)| = |\det T_n[a]| = \prod_{j=1}^{2n} s_j^{(n)} \leq \varepsilon^{N_{\varepsilon,n}} \prod_{j=N_{\varepsilon,n}+1}^{2n} s_j^{(n)}$$

Thus,

$$\log|\det T_n[a]| \quad \leq \quad \sum_{j=N_{\varepsilon,n}+1}^{2n} \log s_j^{(n)} \leq \sum_{j=1}^{2n} \chi_\varepsilon(s_j^{(n)}) \log s_j^{(n)},$$

where $0 \leq \chi_\varepsilon \leq 1$ is a smooth characteristic function with support in $(\varepsilon, \|T[a]\|+1)$ and value 1 on $[\varepsilon+\varepsilon^2, \|T[a]\|]$. Since the symbol a of $T_n[a]$ has the two singular values $\varphi_\beta - \varphi_\delta$ and $\varphi_\beta + \varphi_\delta$, the Avram-Parter theorem [8, p.203] (see also (A.26)) yields

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log|\det T_n[a]| \quad &\leq \quad \int_0^{2\pi} \frac{d\xi}{2\pi} \mathrm{tr}\,[(\chi_\varepsilon \log)(|a(\xi)|)] \\ &= \quad \int_0^{2\pi} \frac{d\xi}{2\pi} (\chi_\varepsilon \log)(\varphi_\beta(\xi) - \varphi_\delta(\xi)) + (\chi_\varepsilon \log)(\varphi_\beta(\xi) + \varphi_\delta(\xi)). \end{aligned}$$

Since the inequality holds for all $\varepsilon > 0$, and since for all $\xi \in [0, 2\pi]$ we have $0 < \varphi_\beta(\xi) - \varphi_\delta(\xi) \leq \varphi_\beta(\xi) + \varphi_\delta(\xi) < 1$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log|\det T_n[a]| \quad \leq \quad \int_0^{2\pi} \frac{d\xi}{2\pi} \log[\mathrm{th}(\beta_L \mu(\xi)/2)\, \mathrm{th}(\beta_R \mu(\xi)/2)] \tag{3.18}$$

which concludes the proof. □

Remarks *3) We do not provide a lower bound in our proof (but see remark 1 for $\delta = 0$). Due to the lack of results on the invertibility of block Toeplitz matrices $T[a]$, the behavior of the singular values of $T_n[a]$ close to zero is, in general, rather difficult to control.*

4) In the critical regime of the XY chain, $\gamma = 0$, $\lambda \in [-1,1]$, and $\gamma \neq 0$, $\lambda = \pm 1$, where the nonnegative function $\mu(\xi)$ from (2.6) *can have zeroes on $[0,2\pi]$, the bound* (3.18) *remains finite due to the integrability of $\log x$ at the origin.*

5) A straightforward proof using the inequalities $|\det T_n[a]| \leq \|T_n[a]\|^{2n}$ and

$$\|T_n[a]\| \leq \|T[a]\| = \operatorname*{ess\,sup}_{\xi\in\mathbb{T}} \|a(\xi)\|_{\mathscr{L}(\mathbb{C}^N)} = \operatorname*{ess\,sup}_{\xi\in\mathbb{T}} s_2(a(\xi))$$

(see [8] and (A.24)-(A.25)*) yields the weaker bound $|\det T_n[a]| \leq \mathrm{th}^{2n}(\beta_R\|\mu\|_\infty/2)$ which, in contrast to* (3.18)*, is valid for all n. On the other hand, in the limit $\beta_R \to \infty$, this bound is void whereas* (3.18) *still implies exponential decay as long as $\beta_L < \infty$.*

A Toeplitz operators

Toeplitz operators [8, p.185] Let $N \in \mathbb{N}$. We define the space l^2_N of all $\mathbb{C}^N$-valued sequences $f = \{f_i\}_{i=1}^\infty$, $f_i \in \mathbb{C}^N$, by

$$l^2_N = \{f : \mathbb{N} \to \mathbb{C}^N \,|\, \|f\| < \infty\}, \quad \|f\| = \left(\sum_{i=1}^\infty \|f_i\|^2_{\mathbb{C}^N}\right)^{1/2}, \tag{A.19}$$

where $\|\cdot\|_{\mathbb{C}^N}$ denotes the l^2 norm on $\mathbb{C}^N$. We write $l^2 \equiv l^2_1$. Let $\{a_x\}_{x\in\mathbb{Z}}$ be a sequence of $N\times N$ matrices, $a_x \in \mathbb{C}^{N\times N}$. The Toeplitz operator defined through its action on elements of l^2_N by $f \mapsto \{\sum_{j=1}^\infty a_{i-j} f_j\}_{i=1}^\infty$ is a bounded operator on l^2_N, if and only if

$$a_x = \int_0^{2\pi} \frac{d\xi}{2\pi} a(\xi) e^{-ix\xi} \tag{A.20}$$

for some $a \in L^\infty_{N\times N}$ (see [8, p.186]), where we define (with $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$)

$$L^\infty_{N\times N} = \{\phi : \mathbb{T} \to \mathbb{C}^{N\times N} \,|\, \phi_{ij} \in L^\infty(\mathbb{T}), i,j = 1,...,N\}. \tag{A.21}$$

In this case, we write the Toeplitz operator as

$$T[a] = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{A.22}$$

The function $a \in L^\infty_{N\times N}$ is called the symbol of $T[a]$. If $N = 1$ the symbol $a \in L^\infty \equiv L^\infty_{1\times 1}$ and the Toeplitz operator $T[a]$ are called scalar, whereas for $N > 1$ they are called block.

For $n \in \mathbb{N}$, let P_n be the projections on l^2_N,

$$P_n(\{x_1, ..., x_n, x_{n+1}, ...\}) = \{x_1, ..., x_n, 0, 0, ...\}.$$

With the help of these P_n, we define the truncated $Nn \times Nn$ Toeplitz matrices as

$$T_n[a] = P_n T[a] P_n|_{\mathrm{Im}\, P_n}. \tag{A.23}$$

Norm of Toeplitz operators [8, p.186] The norm of a Toeplitz operator is related to its symbol,

$$\|T[a]\| = \|a\|_\infty, \tag{A.24}$$

where $\|a\|_\infty$ is defined to be the operator norm of the multiplication operator acting on $\oplus_1^N L^2(\mathbb{T})$ by multiplication with the matrix function $a \in L^\infty_{N\times N}$. From [7, p. 95], we have

$$\|a\|_\infty = \operatorname*{ess\,sup}_{\xi\in\mathbb{T}} \|a(\xi)\|_{\mathscr{L}(\mathbb{C}^N)}, \tag{A.25}$$

where $\|\cdot\|_{\mathscr{L}(\mathbb{C}^N)}$ is the operator norm induced by the l^2 norm on $\mathbb{C}^N$.

Avram-Parter theorem [8, p.203] Let $s_1^{(n)}, ..., s_{Nn}^{(n)}$ be the singular values of the truncated Toeplitz operator $T_n[a]$ with block symbol $a \in L^\infty_{N\times N}$, and g a continuous function with compact support, $g \in C_0(\mathbb{R})$. Then,

$$\lim_{n\to\infty} \frac{1}{Nn} \sum_{j=1}^{Nn} g(s_j^{(n)}) = \frac{1}{N} \int_0^{2\pi} \frac{d\xi}{2\pi} \mathrm{tr}\, g(|a(\xi)|), \tag{A.26}$$

where tr$A$ is the trace of the matrix $A \in \mathbb{C}^{m\times m}$, $m \in \mathbb{N}$, and $|A| = \sqrt{A^*A}$ its modulus.

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