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ADDITIVE FUNCTIONALS OF A NON-DIVERGENCE DIFFUSION IN A RANDOM MEDIUM

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We study the asymptotic properties of a diffusion process associated to a pure non-divergence second order operator with stationary and ergodic coefficients. We first remind the reader of the ergodic properties of the so-called environment seen from the particle as established in the earlier papers of Papanicolaou and Varadhan [27], Yurinskij [35] and Zhikov [37], and then investigate the underlying rate of convergence. We establish more specifically a central limit theorem for additive functionals driven by suitable stationary fields.

Our analysis relies on earlier results for adjoint solutions to second order non-divergence elliptic and parabolic equations.

1. Introduction. Homogenization theory for random second order differential operators goes back to the earlier papers of Papanicolaou and Varadhan [26], [27] and Kozlov [16] and extends the previous analyses for periodic structures (see e.g. Bensoussan, Lions and Papanicolaou [5]). It can be briefly described as follows: the action of a differential operator with highly oscillating coefficients (or, in other words, of a rescaled operator) driven by a stationary and ergodic random medium can be approximated, under certain conditions, by the one of a non-standard Laplacian operator driven by a suitable effective diffusion matrix. From a probabilistic point of view, this amounts to establish the weak convergence of the underlying rescaled diffusion process towards a non-standard Brownian motion. We refer to the monograph of Zhikov, Kozlov and Olejnik [36] for various applications of this theory.

The basic strategy for stochastic homogenization relies on the central limit theorem for martingales and on the ergodic properties of the so-called environment seen from the particle, *i.e.* of the canonical Markov process with values in the random medium and governed by the non-rescaled diffusion kernel. In the case of an operator of pure divergence type, as considered by Papanicolaou and Varadhan [26], Kozlov [16] and Zhikov *et al.* [36],

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Chapter VII, the invariant measure of the environment seen from the particle exactly matches the measure of the random medium itself. Of course, the general setting is far from being so trivial and the long run behaviour of the environment seen from the particle remains a widely open question in most of the cases. However, in the pure non-divergence framework, Papanicolaou and Varadhan [27], Yurinskij [35] and Zhikov [37] proved the existence and uniqueness of an absolutely continuous ergodic invariant measure. The main argument follows from the earlier Aleksandrov and Pucci estimates for the Green functions of a pure non-divergence elliptic operator.

Beside the homogenization property, the ergodic theorem provides for almost every realization of the medium a law of large number for additive functionals of the environment seen from the particle. A natural question then consists in investigating a central limit theorem for such additive functionals. In the reversible case, Kipnis and Varadhan [15] provided a positive answer for a large class of stationary fields driving the additive functional. Several extensions to the non-symmetric case have been investigated for the past twenty years (see for example Osada [25], Oelschläger [23] or the recent review of Olla [24]). The most famous one applies under a suitable assumption that is usually referred as “the strong sector condition” (see again Olla [24], Chapter I, Section 5). Generally speaking, there is no way to apply the strong sector condition or a variant of it to the pure non-divergence framework except several specific cases where the density of the invariant measure is bounded from above and from below by positive constants (see e.g. Osada [25]). In practice, such bounds hold for periodic structures (see e.g. Bensoussan, Lions and Papanicolaou [5], Chapter III) or in the one-dimensional setting (for which the density of the invariant measure can be explicitly written down).

In the current paper, we focus in a deeper way on the pure non-divergence framework. We provide several new estimates for the density of the invariant measure of the environment seen from the particle and then propose a central limit theorem for additive functionals when driven by suitable stationary fields. The strategy takes advantage of crucial estimates for the adjoint solutions to non-divergence elliptic and parabolic equations (see e.g. Bauman [4] and Escauriaza [7]). In short, when normalized by the density of the invariant measure, these solutions satisfy a Harnack inequality that only refers to the ellipticity and *supremum* bounds for the underlying diffusion matrix. In the same way, the normalized kernel associated to the existing operator fulfills a suitable version of the famous Aronson estimates.

Our result just holds when the initial distribution of the environment seen from the particle matches the invariant measure. At this stage of our

research, we are not able to establish a similar extension to the one proposed by Kipnis and Varadhan [15] for Dirac initial distributions. This restriction follows from our specific strategy: for technical reasons, we need to focus on the time reversal of the environment seen from the particle, and thus, to limit our investigation to the equilibrium situation.

Of course, our analysis leaves completely open the non-zero drift case, for which the homogenization problem remains widely unsolved (see however Bricmont and Kupiainen [6] and Sznitman and Zeitouni [33] for deep discussions on the topic). The basic difference between both cases can be explained as follows: a given scaling procedure leaves unchanged the global structure of a pure non-divergence operator but induces an explosive drift in the general case.

The organization of the paper is the following. In Section 2, we remind the reader of the properties of the random medium and introduce the basic objects for our analysis. We expose the main results as well as the strategy in Section 3. The sequel of the paper is then devoted to the proof of the central limit theorem. Crucial properties for the adjoint solutions to non-divergence operators are detailed in Section 4. Section 5 deals with the homogenization property for pure non-divergence operators. In Sections 6 and 7, we aim to apply the strategy due to Kipnis and Varadhan [15] to the time reversal of the environment seen from the particle. We finally complete the proof in Section 8 and then discuss several conceivable extensions.

2. General Notation and Assumption.

2.1. Random Medium. Our standing definition of the random medium follows from the earlier paper of Kozlov [16] and may be found in Zhikov et al. [36], Chapter VII, and in the review of Olla [24], Chapter II, Section 7. The standard set-up of the medium is denoted by $(\Xi, \mathcal{G}, (\mathbb{T}_x)_{x \in \mathbb{R}^d}, \mu)$, where Ξ stands for a Polish space, \mathcal{G} for its Borel σ -algebra, μ for a probability measure on the couple (Ξ, \mathcal{G}) and d for an integer greater than two. The family $(\mathbb{T}_x)_{x \in \mathbb{R}^d}$ denotes a group of transformations acting in a stationary and ergodic way on (Ξ, \mathcal{G}) . This writes:

$$\text{(RM1) (Stationarity)} \quad \forall (A, x) \in \mathcal{G} \times \mathbb{R}^d, \quad \mu(\mathbb{T}_x A) = \mu(A).$$

$$\text{(RM2) (Ergodicity)} \quad \forall A \in \mathcal{G}, \quad (\forall x \in \mathbb{R}^d, \mathbb{T}_x A = A) \Rightarrow \mu(A) \in \{0, 1\}.$$

(RM3) (Measurability) For any measurable function \mathbf{g} on (Ξ, \mathcal{G}, μ) , the function $(x, \xi) \mapsto \mathbf{g}(\mathbb{T}_x \xi)$ is measurable with respect to the product algebra $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$.

The expectation under μ is denoted by \mathcal{E} . For a measurable field \mathbf{g} , we

denote (except very specific cases) by $g : (x, \xi) \mapsto \mathbf{g}(\mathbb{T}_x \xi)$ the realization mapping.

According to the previous references (see in particular Subsection 7.1 in Zhikov et al. [36]), the structure **(RM1-2-3)** induces a strongly continuous unitary group on $L^2(\Xi, \mu)$ given by $\forall x \in \mathbb{R}^d$, $T_x : \mathbf{g} \in L^2(\Xi, \mu) \mapsto \mathbf{g}(\mathbb{T}_x \cdot) \in L^2(\Xi, \mu)$. As a bypass product, the group $(\mathbb{T}_x)_{x \in \mathbb{R}^d}$ is stochastically continuous, *i.e.*

(RM4) (*Continuity*) For every measurable function $\mathbf{g} : \Xi \rightarrow \mathbb{R}$ and for every $\delta > 0$:

$$\lim_{h \rightarrow 0} \mu \{ \xi \in \Xi, |\mathbf{g}(\mathbb{T}_h \xi) - \mathbf{g}(\xi)| \geq \delta \} = 0.$$

For every $i \in \{1, \dots, d\}$, the infinitesimal generator of the one-parameter group $(T_{\lambda e_i})_{\lambda \in \mathbb{R}}$ (e_i denotes the i^{th} vector of the canonical basis) is denoted by D_i . The intersection of the domains D_1, \dots, D_d is denoted by $H(\Xi, \mu)$. For every $\mathbf{g} \in H(\Xi, \mu)$, for μ almost every $\xi \in \Xi$, the function $x \mapsto g(x, \xi) \equiv \mathbf{g}(\mathbb{T}_x \xi)$ belongs to $H_{\text{loc}}^1(\mathbb{R}^d)$ and satisfies, for $i \in \{1, \dots, d\}$, $\partial g / \partial x_i(x, \xi) = D_i \mathbf{g}(\mathbb{T}_x \xi)$ almost everywhere. We denote by $D\mathbf{g}$ the gradient field $(D_1 \mathbf{g}, \dots, D_d \mathbf{g})$.

Recall then the following integration by parts formula:

$$(2.1) \quad \forall \mathbf{f}, \mathbf{g} \in H(\Xi, \mu), \forall i \in \{1, \dots, d\}, \mathcal{E}[(D_i \mathbf{f}) \mathbf{g}] = -\mathcal{E}(\mathbf{f} D_i \mathbf{g}).$$

The group $(T_x)_{x \in \mathbb{R}^d}$ is in fact strongly continuous (and unitary) on every $L^q(\Xi, \mu)$, $q \geq 1$. In particular, the generators D_1, \dots, D_d may be seen as unbounded operators on $L^q(\Xi, \mu)$. The intersection of their domains is denoted by $W^{1,q}(\Xi, \mu)$. Of course, $W^{1,q}(\Xi, \mu) \subset W^{1,q'}(\Xi, \mu)$ for $q' \leq q$, and $D_i \mathbf{h}$, with $i \in \{1, \dots, d\}$ and $\mathbf{h} \in W^{1,q}(\Xi, \mu)$, coincides with $D_i \mathbf{h}$ if \mathbf{h} is seen as an element of $W^{1,q'}(\Xi, \mu)$.

We finally define the space \mathcal{S}_∞ of smooth functions on Ξ :

$$\mathcal{S}_\infty \equiv \text{Span} \left\{ \mathbf{g} \star \varphi : \xi \mapsto \int_{\mathbb{R}^d} \mathbf{g}(\mathbb{T}_x \xi) \varphi(x) dx, \mathbf{g} \in L^\infty(\Xi, \mu), \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \right\},$$

where $\mathcal{C}_c^\infty(\mathbb{R}^d)$ denotes the space of smooth functions on \mathbb{R}^d with compact support (for $\mathbf{g} = \tilde{\mathbf{g}}$ a.s., deduce from Lemma 7.1 in Zhikov et al. [36] that $\mathbf{g} \star \varphi = \tilde{\mathbf{g}} \star \varphi$ a.s.). For $\mathbf{f} \in \mathcal{S}_\infty$, the mapping $x \in \mathbb{R}^d \mapsto f(x, \xi)$ belongs to $C_b^\infty(\mathbb{R}^d)$ (space of bounded smooth functions with bounded derivatives of any order) for every $\xi \in \Xi$ (up to the choice of a representative for \mathbf{f}). In particular, $\mathcal{S}_\infty \subset \cap_{q \geq 1} W^{1,q}(\Xi, \mu)$, and, for \mathbf{g} and φ as above and $i \in \{1, \dots, d\}$, $D_i(\mathbf{g} \star \varphi) = -\mathbf{g} \star (\partial \varphi / \partial x_i)$.

2.2. Non-divergence Operators. For $0 < \lambda \leq \Lambda$, we denote by $\mathcal{A}_d(\lambda, \Lambda)$ the set of symmetric matrices of size d whose lowest eigenvalue is bounded from below by λ and whose largest eigenvalue is bounded from above by Λ .

Assumption (A) We say that a measurable mapping \mathbf{a} from (Ξ, \mathcal{G}) into $\mathbb{R}^{d \times d}$ fulfills Assumption (A) if there exists three constants $0 < \lambda \leq \Lambda$ and $K \geq 0$ such that:

(A1) For every $\xi \in \Xi$, $\mathbf{a}(\xi) \in \mathcal{A}_d(\lambda, \Lambda)$.

(A2) For every $\xi \in \Xi$, the coordinates of the mapping $a(x, \xi) \equiv \mathbf{a}(\mathbb{T}_x \xi)$ belong to $C_b^\infty(\mathbb{R}^d)$ and their first and second order derivatives are bounded by K .

Under Assumption (A), the coordinates of the field \mathbf{a} belong to $W^{2,\infty}(\Xi, \mu) \equiv \{\mathbf{g} \in \cap_{q \geq 1} W^{1,q}(\Xi, \mu), \forall i \in \{1, \dots, d\}, D_i \mathbf{g} \in L^\infty(\Xi, \mu) \cap \cap_{q \geq 1} W^{1,q}(\Xi, \mu), \forall j \in \{1, \dots, d\}, D_{i,j}^2 \mathbf{g} \in L^\infty(\Xi, \mu)\}$. For $(i, j) \in \{1, \dots, d\}^2$, $D \mathbf{a}_{i,j}$ and $D^2 \mathbf{a}_{i,j}$ ($D_{k,\ell}^2(\cdot) = D_k(D_\ell(\cdot))$) are a.s. bounded by K .

Consider now the canonical Wiener space $(\Omega, \mathcal{A}, \mathbb{P}) \equiv (\mathcal{C}([0, +\infty[, \mathbb{R}^d), \mathcal{B}(\mathcal{C}([0, +\infty[, \mathbb{R}^d)), \mathbb{P})$, where \mathbb{P} denotes the Wiener measure. The canonical process is denoted by $(W_t)_{t \geq 0}$. According to Theorems 11.2 and 12.12, Chapter V, in Rogers and Williams [29], for every $\xi \in \Xi$, the SDE:

$$(2.2) \quad X_t = x + \int_0^t a^{1/2}(X_s, \xi) dW_s, \quad x \in \mathbb{R}^d,$$

is strongly solvable. The solution is denoted by $(X_t(x, \xi))_{t \geq 0}$, or just by $(X_t)_{t \geq 0}$ when possible. The operator associated to the process X is given by:

$$\mathcal{L}^\xi \equiv \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(x, \xi) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Since X can be defined through an iteration procedure of Picard type, the mapping $(x, \xi) \in \mathbb{R}^d \times \Xi \mapsto X(x, \xi) \in \Omega$ is jointly measurable with respect to $(x, \xi) \in \mathbb{R}^d \times \Xi$ (due to (A2), it is even continuous in x , see Theorem 13.1, Chapter V, in Rogers and Williams [29]). Moreover, thanks to the strong uniqueness property of (2.2):

$$(2.3) \quad X(x, \xi) = x + X(0, \mathbb{T}_x \xi).$$

Moreover, from Aronson [1] or from Theorem 4.5, Chapter VI, in Friedman [12], for every $\xi \in \Xi$, there exists a fundamental solution $q(t, x, y; \xi)$ to the Markov process $X(\cdot, \xi)$. It satisfies Gaussian lower and upper bounds, but locally in time (see also Stroock [31] and [32]):

PROPOSITION 2.1. *There exists a constant $C_{2.1} > 0$ such that for all $\xi \in \Xi$ and $(t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\begin{aligned} & C_{2.1}^{-1} t^{-d/2} \exp[-C_{2.1} t - C_{2.1} t^{-1} |x - y|^2] \\ & \leq q(t, x, y; \xi) \leq C_{2.1} t^{-d/2} \exp[C_{2.1} t - C_{2.1}^{-1} t^{-1} |x - y|^2]. \end{aligned}$$

Due to **(A2)**, $q(t, x, y; \xi)$ is continuous in (t, x, y) . Due to the measurability of $X(x, \xi)$ with respect to (x, ξ) , it is well-seen that q is jointly measurable with respect to $(t, x, y, \xi) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d \times \Xi$ endowed with the product σ -algebra. Derive also from (2.3):

$$(2.4) \quad \forall z \in \mathbb{R}^d, \quad q(t, x, y; \mathbb{T}_z \xi) = q(t, x + z, y + z; \xi).$$

In the same way, note from Bass [2], Chapter V, Section 5, that, for every $\xi \in \Xi$ and for every $(z, R) \in \mathbb{R}^d \times]0, +\infty[$, \mathcal{L}^ξ admits a Green function with respect to the ball $B(z, R)$ (ball of center z and of radius R). It is denoted by $\gamma(x, y; R, z, \xi)$. Thanks to the Feynman-Kac formula and to the measurability property of $X(x, \xi)$ in (x, ξ) , $\gamma(x, y; R, z, \xi)$ is jointly measurable with respect to (x, y, ξ) . Moreover, standard computations show that for every $u \in \mathbb{R}^d$:

$$(2.5) \quad \gamma(x, y; R, z, \mathbb{T}_u \xi) = \gamma(x + u, y + u; R, z + u, \xi).$$

2.3. *Environment Seen from the Particle.* Following the review of Olla (see [24], Chapter II, Section 7), we define the so-called environment seen from the particle by setting:

$$(2.6) \quad \forall \xi \in \Xi, \quad \forall t \geq 0, \quad \eta_t(\xi) = \mathbb{T}_{X_t(0, \xi)} \xi.$$

Under the measure \mathbb{P} , the process $(\eta_t(\xi))_{t \geq 0}$ is a Markov process with values in Ξ starting from ξ at time 0. The associated semigroup writes:

$$\begin{aligned} \forall \mathbf{h} \in L^\infty(\Xi, \mu), \quad S_t \mathbf{h}(\xi) &= \mathbb{E}[\mathbf{h}(\eta_t(\xi))] \\ &= \int_{\mathbb{R}^d} \mathbf{h}(\mathbb{T}_y \xi) q(t, 0, y; \xi) dy. \end{aligned}$$

From Lemma 7.1 in Zhikov et al. [36], S_t is perfectly defined (*i.e.* $S_t \mathbf{h} = S_t \tilde{\mathbf{h}}$ a.s. for $\mathbf{h} = \tilde{\mathbf{h}}$ a.s.). From Proposition 2.1, we derive:

LEMMA 2.2. *There exists a constant $C_{2.2} > 0$ such that for every $\mathbf{h} \in L^\infty(\Xi, \mu)$, $\mathcal{E}|S_t \mathbf{h}|^2 \leq C_{2.2} \exp(C_{2.2} t) \mathcal{E}|\mathbf{h}|^2$. In particular, $(S_t)_{t \geq 0}$ extends to $L^2(\Xi, \mu)$.*

We now prove that $(S_t)_{t \geq 0}$ is strongly continuous on $L^2(\Xi, \mu)$. For every $\mathbf{h} \in L^2(\Xi, \mu)$:

$$\begin{aligned} & \mathcal{E}|S_t \mathbf{h} - \mathbf{h}|^2 \\ & \leq C_{2.1} \exp(C_{2.1}t) t^{-d/2} \mathcal{E} \int_{\mathbb{R}^d} |\mathbf{h}(\mathbb{T}_y \xi) - \mathbf{h}(\xi)|^2 \exp(-C_{2.1}^{-1} t^{-1} |y|^2) dy \\ & = C_{2.1} \exp(C_{2.1}t) \int_{\mathbb{R}^d} \mathcal{E} |\mathbf{h}(\mathbb{T}_{t^{1/2}y} \xi) - \mathbf{h}(\xi)|^2 \exp(-C_{2.1}^{-1} |y|^2) dy. \end{aligned}$$

Since $\mathcal{E} |\mathbf{h}(\mathbb{T}_{t^{1/2}y} \xi) - \mathbf{h}(\xi)|^2 \leq 4\mathcal{E} |\mathbf{h}|^2$, the continuity of $(T_x)_{x \in \mathbb{R}^d}$ (seen as a group of transformations acting on $L^2(\Xi, \mu)$) and the dominated convergence theorem ensure that $(S_t)_{t \geq 0}$ is strongly continuous on $L^2(\Xi, \mu)$. Its generator is denoted by \mathbf{L} .

For $\mathbf{h} \in W^{2,\infty}(\Xi, \mu)$, the realization mapping $h(\cdot, \xi)$ belongs to the space $W^{2,\infty}(\mathbb{R}^d)$ for a.e. $\xi \in \Xi$. Thanks to Krylov [17], Chapter II, Section 10, Theorem 1, it comes, for every $t \geq 0$, $\mathbb{E}h(X_t(0, \xi), \xi) = h(0, \xi) + \int_0^t \mathbb{E}[\mathcal{L}^\xi h(X_s(0, \xi), \xi)] ds$, so that:

$$S_t \mathbf{h}(\xi) = \mathbf{h}(\xi) + \int_0^t S_s [(1/2) \sum_{i,j=1}^d \mathbf{a}_{i,j} D_{i,j}^2 \mathbf{h}](\xi) ds.$$

Due to the strong continuity of $(S_t)_{t \geq 0}$, $\mathbf{h} \in \mathcal{D}(\mathbf{L})$ and

$$\mathbf{L} \mathbf{h} = \frac{1}{2} \sum_{i,j=1}^d \mathbf{a}_{i,j} D_{i,j}^2 \mathbf{h}.$$

For a given probability measure m on (Ξ, \mathcal{G}) , $(\eta_t)_{t \geq 0}$ is Markov process with m as initial distribution under the probability measure $m \otimes \mathbb{P}$.

3. Statement of the Main Result and Strategy. The basic objective of the paper consists in investigating the asymptotic behaviour (for large r) of additive functionals of the form:

$$(3.1) \quad \int_0^{r^2} \mathbf{f}(\eta_s(\xi)) ds,$$

for a given stationary field \mathbf{f} .

3.1. Ergodic Theorem. Referring to Papanicolaou and Varadhan [27], Yurinskij [35] and Zhikov [37] (see also Chapter X in Zhikov et al. [36]), we claim:

THEOREM 3.1. *Under Assumption **(A)**, the Ξ -valued Markov process $(\eta_t)_{t \geq 0}$ admits an ergodic invariant measure ν , which is absolutely continuous with respect to the measure μ . The density of ν with respect to μ is denoted by \mathbf{p} . It is almost-surely positive, belongs to $L^{d/(d-1)}(\Xi, \mu)$ and is the unique solution in $L^1(\Xi, \mu)$ to $\mathbf{L}^*(\mathbf{p}) = 0$, $\mathcal{E}(\mathbf{p}) = 1$ (i.e. $\forall \mathbf{v} \in W^{2,\infty}(\Xi, \mu)$, $\mathcal{E}(\mathbf{pLv}) = 0$). In particular, the measures μ and ν are equivalent.*

As a bypass product, for every $\mathbf{f} \in L^\infty(\Xi, \mu)$, for μ -almost every $\xi \in \Xi$:

$$\mathbb{P} - \text{a.s.}, \quad \lim_{r \rightarrow +\infty} r^{-2} \int_0^{r^2} \mathbf{f}(\eta_s(\xi)) ds = \mathcal{E}(\mathbf{pf}).$$

3.2. Central Limit Theorem. In this paper, we establish the following central limit theorem:

THEOREM 3.2. *Under Assumption **(A)**, the density \mathbf{p} belongs to the space $W^{1,d/(d-1)}(\Xi, \mu)$, and there exists a constant $C_{3.2}$, depending only on λ, Λ and K such that, μ almost-surely, $\mathbf{p}^{-1} |D\mathbf{p}| \leq C_{3.2}$ (where $D\mathbf{p}$ stands for $(D_1\mathbf{p}, \dots, D_d\mathbf{p})$).*

Moreover, for every stationary field $\mathbf{f} \in L^\infty(\Xi, \mu)$ writing $\mathbf{f} = \mathbf{p}^{-1} D_\ell(\mathbf{pg})$, for $\ell \in \{1, \dots, d\}$ and $\mathbf{g} \in W^{1,\infty}(\Xi, \mu)$ (i.e. $\mathbf{g} \in L^\infty(\Xi, \mu) \cap \cap_{q \geq 1} W^{1,q}(\Xi, \mu)$) and $D\mathbf{g} \in L^\infty(\Xi, \mathcal{G}, \mu; \mathbb{R}^d)$, with $\mathcal{E}(\mathbf{pg}) = 0$, there exists a real $\sigma_{\mathbf{f}} \geq 0$ such that, under $\nu \otimes \mathbb{P}$:

$$(3.2) \quad r^{-1} \int_0^{r^2} \mathbf{f}(\eta_s) ds \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{\mathbf{f}}^2) \text{ as } r \rightarrow +\infty.$$

As in the divergence case (refer for example to Kozlov [16]), the limit diffusion coefficient $\sigma_{\mathbf{f}}^2$ can be characterized through the solution of an auxiliary equation (i.e. $\sigma_{\mathbf{f}}^2 \equiv \mathcal{E}(\mathbf{p}\langle \mathbf{a}\Psi, \Psi \rangle)$ where Ψ satisfies (7.3) for every smooth field Φ).

3.3. Scaling Procedure. The strategy to investigate the asymptotic behaviour of quantities of the form (3.1) highly depends on the scaling properties of the diffusion process $X(0, \xi)$ for $\xi \in \Xi$. After a time change of intensity r^2 , (3.1) writes:

$$(3.3) \quad r^2 \int_0^1 \mathbf{f}(T_{X_{r^2s}(0, \xi)} \xi) ds.$$

Hence, we consider next the rescaled diffusion $(r^{-1} X_{r^2t}(0, \xi))_{t \geq 0}$ or equivalently the rescaled diffusion $(\bar{X}_t(\varepsilon; 0, \xi) \equiv \varepsilon X_{\varepsilon^{-2}t}(0, \xi))_{t \geq 0}$ for a small parameter ε . It is then plain to see that the operator associated to the process

$\bar{X}(\varepsilon; 0, \xi)$ writes again in a non-divergence form:

$$(3.4) \quad \bar{\mathcal{L}}^{\varepsilon, \xi} \equiv \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(\varepsilon^{-1}x, \xi) \frac{\partial^2}{\partial x_i \partial x_j}.$$

The elliptic non-divergence structure of the initial operator is thus preserved by the scaling procedure. In particular, Assumption **(A1)** still holds true for the underlying diffusion matrix $a(\varepsilon^{-1}\cdot, \xi)$. On the opposite, the regularity properties given in Assumption **(A2)** completely fail for small values of ε .

Focus again on (3.3) and note that (3.3) writes as the solution of a linear parabolic PDE associated to $\bar{\mathcal{L}}^{\varepsilon, \xi}$. It is then well-understood that most of the proof of Theorem 3.2 relies on several *a priori* estimates for the solutions of a non-divergence PDE with a uniformly elliptic but discontinuous diffusion matrix. Among them, the famous Aleksandrov and Pucci estimates (see for example Theorem 9.1 in Gilbarg and Trudinger [13] and Krylov [17], Chapter II, Sections 2 and 3) provide an $L^{d/(d-1)}$ estimate for the $\bar{\mathcal{L}}^{\varepsilon, \xi}$ Green functions and the Krylov and Safonov estimates (refer to [19]) provide an *a priori* control for the Hölder continuity of the solutions to $\bar{\mathcal{L}}^{\varepsilon, \xi}$ PDEs.

The Krylov and Safonov estimates apply in a very elegant way to the homogenization of the operators $(\bar{\mathcal{L}}^{\varepsilon, \xi})_\varepsilon$. The usual procedure establishes the pointwise convergence of the solutions to $\partial v / \partial t - \bar{\mathcal{L}}^{\varepsilon, \xi}(v) = \chi$, for a suitable function χ from \mathbb{R}^d into \mathbb{R} , towards the solution of a parabolic equation of the same kind, but associated to the Laplacian operator (up to a non-trivial effective diffusion coefficient). Due to the Krylov and Safonov theory, the convergence is also uniform on every compact subset.

This draws the basic background for our strategy. Due to the homogenization property for the family $(\bar{\mathcal{L}}^{\varepsilon, \xi})_\varepsilon$, the fundamental solutions of the operators $(\bar{\mathcal{L}}^{\varepsilon, \xi})_\varepsilon$ weakly converge (in (t, y)) towards the solution of a non-standard Gaussian kernel. These transition densities write:

$$(3.5) \quad \bar{q}(t, x, y; \varepsilon, \xi) \equiv \varepsilon^{-d} q(\varepsilon^{-2}t, \varepsilon^{-1}x, \varepsilon^{-1}y; \xi).$$

As explained in the next Subsection, we then derive that the family $(\bar{q}(t, x, y; \varepsilon, \xi))_\varepsilon$ (up to a normalizing factor) uniformly converges (in (t, x, y)) on every compact subset of $]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$. Thanks to the backward Kolmogorov equation satisfied by \bar{q} in (t, x) , we deduce the (local) L^2 -convergence of the x -gradients of the (normalized) densities towards the x -gradient of the limit Gaussian kernel. This result is the key step towards Theorem 3.2 (see Subsection 3.5).

3.4. *Normalized Solutions.* The strategy to establish the uniform convergence of the transition densities relies on an equicontinuity property for the solutions to the adjoint equations for $(\bar{\mathcal{L}}^{\varepsilon, \xi})_\varepsilon$. Here is a short summary.

Thanks to the Chapman-Kolmogorov equations, the density $\bar{q}(t, x, y; \varepsilon, \xi)$ satisfies in (t, y) the adjoint equation $\partial v / \partial t - (\bar{\mathcal{L}}^{\varepsilon, \xi})^*(v) = 0$. Such adjoint equations to non-divergence operators have been intensely studied in several earlier papers. Mention among others Bauman [4], Escauriaza [7] and Fabes and Stroock [10]. The common idea consists in normalizing the solution v by a nonnegative solution w to the elliptic equation $(\bar{\mathcal{L}}^{\varepsilon, \xi})^*(w) = 0$ and to investigate the Hölder continuity of the product $w^{-1}v$ in terms of its *supremum* norm. The resulting estimate relies on a relevant version of the Harnack inequality for such normalized solutions.

In the stationary setting, the functional $p(\varepsilon^{-1}\cdot; \xi)$, associated to the density \mathbf{p} of the invariant measure, satisfies the adjoint equation $(\bar{\mathcal{L}}^{\varepsilon, \xi})^*(p(\varepsilon^{-1}\cdot, \xi)) = 0$ and thus plays the role of w . We then derive a uniform bound (in (x, ε)) for the Hölder continuity (in (t, y)) of the normalized kernel $p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)$ in terms of its *supremum* norm.

In the same way, the normalized fundamental solution $p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)$ satisfies a suitable version of the well-known Aronson estimates (see Escauriaza [7] for this version and Aronson [1] for the original one in the divergence setting).

Gathering the Harnack inequality for adjoint solutions and the Aronson estimates for the normalized kernel, we establish that the family of (t, y) functions $p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)$ is locally (*i.e.* on every compact subset) equicontinuous with respect to the parameters x and ε .

Thanks to the Krylov and Safonov theory (bounds for the Hölder continuity of the solutions to $\bar{\mathcal{L}}^{\varepsilon, \xi}$), we manage to establish that the family of (t, x, y) (and not (t, y)) functions $p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)$ is locally equicontinuous with respect to the parameter ε .

3.5. *Time Reversal.* Forget for the moment the previous Subsections 3.3 and 3.4 and recall from Olla [24], Chapter I, the basic strategy to establish the central limit theorem for Markov chains. Under the assumptions of Theorem 3.2, solve the family of auxiliary problems:

$$(3.6) \quad -\mathbf{L}(\mathbf{u}(\cdot; \varepsilon)) + \varepsilon^2 \mathbf{u}(\cdot; \varepsilon) = \mathbf{f}.$$

Prove then, as ε tends to zero, that $\varepsilon \mathbf{u}(\cdot; \varepsilon)$ vanishes in $L^2(\Xi, \mathbf{p}, \mu)$ and that $D\mathbf{u}(\cdot; \varepsilon)$ converges (again in $L^2(\Xi, \mathbf{p}, \mu)$). Apply Itô's formula to the process $\varepsilon u(\varepsilon^{-1}\bar{X}(\varepsilon; 0, \xi); \varepsilon, \xi)$, with $u(x; \varepsilon, \xi) \equiv \mathbf{u}(\mathbf{T}_x \xi; \varepsilon)$, to reduce (up to negligible terms) the rescaled additive functional in (3.2) to a martingale. Derive finally Theorem 3.2 from the central limit theorem for martingales.

In our frame (see the writing of \mathbf{f} in the statement of Theorem 3.2), the solution $u(x; \varepsilon, \xi)$ writes:

$$(3.7) \quad u(x; \varepsilon, \xi) = \int_0^{+\infty} \exp(-\varepsilon^2 t) \int_{\mathbb{R}^d} [\mathbf{p}^{-1} D_\ell(\mathbf{p}\mathbf{g})](\mathbb{T}_y \xi) q(t, x, y; \xi) dy dt$$

Thanks to the rescaling procedure (see (3.5)):

$$(3.8) \quad \begin{aligned} & \varepsilon u(\varepsilon^{-1} x; \varepsilon, \xi) \\ &= \varepsilon^{-1} \int_0^{+\infty} \exp(-t) \int_{\mathbb{R}^d} [\mathbf{p}^{-1} D_\ell(\mathbf{p}\mathbf{g})](\mathbb{T}_{\varepsilon^{-1} y} \xi) \bar{q}(t, x, y; \varepsilon, \xi) dy dt \\ &= - \int_0^{+\infty} \exp(-t) \int_{\mathbb{R}^d} (p\mathbf{g})(\varepsilon^{-1} y, \xi) \frac{\partial}{\partial y_\ell} (p^{-1}(\varepsilon^{-1} y, \xi) \bar{q}(t, x, y; \varepsilon, \xi)) dy dt. \end{aligned}$$

From Subsections 3.3 and 3.4, we are able to establish the convergence of the x -gradients (and not y -gradients) of the normalized kernels. Unfortunately, equality (3.8) holds with respect to the y -derivative of the normalized fundamental solutions.

The strategy then consists in reversing in time the process η . This procedure does not change the asymptotic analysis for (3.3) (see Olla [24], Chapter I, and Wu [34] for earlier applications of this technique), and leads to focus on the auxiliary problems (3.6) associated to the normalized adjoint operator $\mathbf{p}^{-1} \mathbf{L}^*(\mathbf{p}\cdot)$. In particular, the time reversal procedure permits to write (3.8) with respect to the x -derivative of \bar{q} instead of the y -derivative. According to the strong convergence property given in Subsections 3.3 and 3.4 and to the ergodic theorem, we derive that the last term in (3.8) vanishes, as expected. We are then able to recover the classical strategy (at least in its main lines), provided $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$ (see Section 7).

3.6. Boundedness of p . The central limit theorem directly follows from the factorizing method due to Osada [25] when \mathbf{p} is bounded from below and from above by positive constants. This situation occurs in the periodic framework (see e.g. Bensoussan et al. [5], Chapter III) and in the one-dimensional setting $d = 1$, in which \mathbf{p} can be explicitly written down: $\mathbf{p} = \mathbf{a}^{-1} / \mathcal{E}(\mathbf{a}^{-1})$.

In the current context, *i.e.* for $d \geq 2$, we are totally unable to predict if such bounds for \mathbf{p} might hold or not.

3.7. Centering Condition. At this early stage of the paper, the reader may be doubtful about the centering condition $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$. Indeed, any asymptotic property for the quantity $\mathbf{p}^{-1} D_\ell(\mathbf{p}\mathbf{g})$ remains unchanged by addition of an extra constant to the product $\mathbf{p}\mathbf{g}$, so that the reader might hope

for an easy extension of Theorem 3.2 to the general case. What we would need to derive a generalized version of Theorem 3.2 from the centered case is in fact slightly stronger: to prove the central limit theorem for a given function \mathbf{f} of the form $\mathbf{f} = \mathbf{p}^{-1}D_\ell(\mathbf{p}\mathbf{g})$ ($\mathcal{E}(\mathbf{p}\mathbf{g})$ being possibly different from zero), we need to write, up to a suitable perturbation term, $\mathbf{p}\mathbf{g}$ as the product of \mathbf{p} times a centered function $\mathbf{h} \in L^\infty(\Xi, \mu)$. For example, $\mathbf{p}\mathbf{g}$ always writes $\mathbf{p}(\mathbf{g} - \mathcal{E}(\mathbf{p}\mathbf{g})) + \mathbf{p}\mathcal{E}(\mathbf{p}\mathbf{g})$ or $\mathbf{p}(\mathbf{g} - \mathbf{p}^{-1}\mathcal{E}(\mathbf{p}\mathbf{g})) + \mathcal{E}(\mathbf{p}\mathbf{g})$. The first transformation (i.e. $\mathbf{g} \mapsto \mathbf{g} - \mathcal{E}(\mathbf{p}\mathbf{g})$) does not preserve the asymptotic behaviour of $\mathbf{p}^{-1}D_\ell(\mathbf{p}\mathbf{g})$ (since $\mathbf{p}\mathcal{E}(\mathbf{p}\mathbf{g})$ induces a non-trivial term), whereas the second one does but maps \mathbf{g} onto $\mathbf{g} - \mathbf{p}^{-1}\mathcal{E}(\mathbf{p}\mathbf{g})$. This latter function does not belong to $L^\infty(\Xi, \mu)$ unless \mathbf{p}^{-1} belongs to $L^\infty(\Xi, \mu)$.

As discussed above, there is no clear reason to ensure the boundedness of \mathbf{p}^{-1} except in the specific case $d = 1$.

3.8. Frequently Used Notation. In the sequel, the generic notations “ C ”, “ C' ” and “ c ” denote constants appearing in the proof of Theorem 3.2. If nothing mentioned, these constants only depend on the parameters λ , Λ and K quoted in Assumption **(A)**. Of course, their values may vary from line to line.

Moreover, for $z \in \mathbb{R}^d$ and $r > 0$, we denote by $B(z, r)$ the Euclidean ball of \mathbb{R}^d of center z and radius r . For $r = 1$, its volume is denoted by V_d . For $t \in \mathbb{R}$, we denote by $Q((t, z), r)$ the cylinder of \mathbb{R}^{d+1} $Q((t, z), r) \equiv]t - r^2, t + r^2[\times B(z, r)$.

4. Normalization with respect to the Density. Follow the strategy described in Section 3 and investigate the properties of the normalized transition densities. To this end, focus first on the density \mathbf{p} of the invariant measure.

4.1. Density of the Invariant Measure. As said in Section 3, the environment seen from the particle admits an invariant measure denoted by $\nu = \mathbf{p}.\mu$. Thanks to Zhikov [37] and to Zhikov et al. [36], Chapter X, or to Theorem 3.1, $p(\cdot, \xi) = \mathbf{p}(\mathbb{T}.\xi)$ satisfies for a.e. $\xi \in \Xi$ the adjoint equation $(\mathcal{L}^\xi)^*(p(\cdot, \xi)) = 0$ in the following sense:

$$(4.1) \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d), \quad \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{i,j}(x, \xi) p(x, \xi) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) dx = 0, \quad p(\cdot, \xi) \in L^1_{\text{loc}}(\mathbb{R}^d).$$

Referring to Sjogren [30], Section 3, Lemma 1, for almost every $\xi \in \Xi$, there exists a continuous version of $x \mapsto p(x, \xi)$. This version belongs to

$\mathcal{C}^2(\mathbb{R}^d)$ and satisfies the adjoint equation in a strong sense (see Theorem 1 and (2.5) in the previous reference). We will always consider this version in the sequel of the paper. In particular, for almost every $\xi \in \Xi$, $\mathbf{p}(\mathbb{T}_x\xi) = p(x, \xi)$ a.e. in x .

From Escoriaza [7], we can provide an explicit writing for the version $p(x, \xi)$. Indeed, from Theorem 1.1 in [7], for every $\xi \in \Xi$, there exists a unique non-negative (continuous) adjoint solution $w(x; \xi)$ to \mathcal{L}^ξ satisfying:

$$\int_{B(0,1)} w(x; \xi) dx = V_d.$$

According to the proof of Theorem 1.1 in [7] and to the measurability of γ , we derive that w is jointly measurable in (x, ξ) .

Since $p(x, \xi)$ satisfies, for a.e. ξ , the adjoint equation for \mathcal{L}^ξ , we claim:

$$p(x, \xi) = V_d^{-1} \left[\int_{B(0,1)} \mathbf{p}(\mathbb{T}_y\xi) dy \right] w(x; \xi).$$

Up to the choice of the version of \mathbf{p} in $L^{d/(d-1)}(\Xi, \mu)$, we can extend $p(x, \xi)$ to the whole set $\mathbb{R}^d \times \Xi$ (due to Lemma 7.1 in Zhikov et al. [36], for two different versions \mathbf{p} and $\tilde{\mathbf{p}}$, $p(\cdot, \xi) = \tilde{p}(\cdot, \xi)$ for a.e. ξ). The resulting mapping $p : \mathbb{R}^d \times \Xi \rightarrow \mathbb{R}$ is jointly measurable in (x, ξ) and satisfies the adjoint equation for every $\xi \in \Xi$.

A first point consists in checking that $p(0, \xi) = \mathbf{p}(\xi)$ for almost every $\xi \in \Xi$:

LEMMA 4.1. *For every $x \in \mathbb{R}^d$, for almost every $\xi \in \Xi$, $\mathbf{p}(\mathbb{T}_x\xi) = p(x, \xi)$.*

PROOF. Assume first without loss of generality that $x = 0$. Due to the continuity of $(T_y)_{y \in \mathbb{R}^d}$ seen as a group acting on $L^{d/(d-1)}(\Xi, \mu)$, $\mathcal{E}|T_y\mathbf{p} - \mathbf{p}| \rightarrow 0$ as $y \rightarrow 0$. Thus,

$$h^{-d} \mathcal{E} \int_{B(0,h)} |\mathbf{p}(\mathbb{T}_y\xi) - \mathbf{p}(\xi)| dy \rightarrow 0 \text{ as } h \rightarrow 0.$$

Deduce that:

$$\mathcal{E} \left| V_d^{-1} h^{-d} \int_{B(0,h)} p(y, \xi) dy - \mathbf{p}(\xi) \right| \rightarrow 0 \text{ as } h \rightarrow 0,$$

so that,

$$V_d^{-1} h^{-d} \int_{B(0,h)} p(y, \xi) dy - \mathbf{p}(\xi) \xrightarrow{\mu\text{-probability}} 0 \text{ as } h \rightarrow 0$$

Since the function $y \mapsto p(y, \xi)$ is continuous for every $\xi \in \Xi$, it comes $p(0, \xi) = \mathbf{p}(\xi)$ for a.e. $\xi \in \Xi$. \square

In the periodic framework, the function $p(\cdot, \xi)$ satisfies a Harnack inequality. As a consequence, it is bounded from below and from above by two nonnegative constants. The story is rather different in the general setting since just a few estimates are available for \mathbf{p} . The $L^{d/(d-1)}$ bound given in Theorem 3.1 follows from the Aleksandrov and Pucci estimates, but is far from being sufficient to establish Theorem 3.2. On the opposite, we know from Osada [25] that Theorem 3.2 holds if \mathbf{p} is bounded from above and from below by positive constants.

It is then well-understood that we need to investigate in a deeper way the density \mathbf{p} to establish the central limit theorem. In this paper, we manage to derive from (4.1) several new local and global properties for \mathbf{p} . For example, we establish the integrability of \mathbf{p}^α for a small exponent α (see Corollary 4.5) as well as a local Harnack inequality (see Proposition 4.10). Since the set of definition of $p(\cdot, \xi)$ (i.e. \mathbb{R}^d) is not compact, there does not seem to be any way to derive a global bound for \mathbf{p} as done in the periodic case.

The strategy relies on the theory of normalized solutions developed first by Bauman [3] and [4] and then by Fabes and Stroock [10] and used more recently by Escauriaza [7]:

DEFINITION 4.2. *Let U be a nonnegative weak solution (in the sense of (4.1)) of $(\mathcal{L}^\xi)^*(U) = 0$. A given function u from \mathbb{R}^d into \mathbb{R} is said to be a normalized solution with respect to $(\mathcal{L}^\xi)^*$ and U if the product function $U \times u$ satisfies $(\mathcal{L}^\xi)^*(U \times u) = 0$.*

This definition extends in a trivial way to a solution $u(t, x)$ of the parabolic adjoint equation.

Here is a first property of ν :

PROPOSITION 4.3. *There exist two constants $\eta_{4.3} > 0$ and $C_{4.3} > 0$ such that $\nu(A) \geq C_{4.3} \mu(A)^{\eta_{4.3}}$ for all $A \in \mathcal{G}$.*

In order to prove Proposition 4.3, derive first the following Proposition from Corollary 2.3 in Fabes and Stroock [10]:

PROPOSITION 4.4. *There exist two constants $c_{4.4} > 0$ and $\eta_{4.4} > 0$ such that for every $\xi \in \Xi$, for every $r > 0$ and for every $E \in \mathcal{B}(\mathbb{R}^d)$, $E \subset B_d(0, r/4)$:*

$$\int_E \gamma(0, y; r, 0, \xi) dy \geq c_{4.4} (r^{-d} |E|)^{\eta_{4.4}} \int_{B(0, r/4)} \gamma(0, y; r, 0, \xi) dy.$$

PROOF OF PROPOSITION 4.3. Fix $r > 0$, drop the index 4.4 in the constants c and η in Proposition 4.4 and deduce that for every $\xi \in \Xi$:

$$(4.2) \quad \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_E(X_t) dt \geq c(r^{-d}|E|)^\eta \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_{B(0,r/4)}(X_t) dt,$$

where E denotes a Borel subset of $B(0, r/4)$, X_t stands for $X_t(0, \xi)$ and $\tau(r)$ for the first exit time of X from the ball $B(0, r)$: $\tau(r) \equiv \inf\{t \geq 0, |X_t| \geq r\}$.

Choose now $A \in \mathcal{G}$ and $\xi \in \Xi$. Apply (4.2) to $E \equiv \{x \in B(0, r/4), \mathbb{T}_x \xi \in A\}$ ($\in \mathcal{B}(\mathbb{R}^d)$ from **(RM3)**):

$$(4.3) \quad \begin{aligned} & r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_A(\eta_t(\xi)) dt \\ &= r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_{\{x \in \mathbb{R}^d, \mathbb{T}_x \xi \in A\}}(X_t) dt \\ &\geq r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_E(X_t) dt \\ &\geq c(r^{-d}|E|)^\eta r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_{B(0,r/4)}(X_t) dt \\ &\geq c \left(r^{-d} \int_{B(0,r/4)} \mathbf{1}_A(\mathbb{T}_x \xi) dx \right)^\eta r^{-2} \mathbb{E}(\tau(r/4)). \end{aligned}$$

Due to the ellipticity assumption **(A1)**, it is plain to see that there exists a constant $C > 0$ such that $C^{-1} \leq r^{-2} \mathbb{E}(\tau(r/4)) \leq C$. This provides a lower bound for the l.h.s. in (4.3) (up to a new constant C):

$$(4.4) \quad r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_A(\eta_t(\xi)) dt \geq C^{-1} \left(r^{-d} \int_{B(0,r/4)} \mathbf{1}_A(\mathbb{T}_x \xi) dx \right)^\eta.$$

Up to a μ -negligible event, the r.h.s. converges as r tends to $+\infty$ towards $C^{-1} 4^{-\eta d} V_d^\eta \mu(A)^\eta$. Turn now to the l.h.s. in (4.4). Since the expectations $(r^{-4} \mathbb{E}(\tau^2(r)))_{r>0}$ are uniformly bounded (in (r, ξ)) (apply Itô's formula to $|X_t|^2$, take the square in each side of the formula and integrate w.r.t. \mathbb{P}), we claim:

$$(4.5) \quad \begin{aligned} & r^{-2} \mathbb{E} \int_0^{\tau(r)} \mathbf{1}_A(\eta_t(\xi)) dt \\ &\leq r^{-2} \mathbb{E}(\tau^2(r))^{1/2} \mathbb{E} \left[\left(\tau^{-1}(r) \int_0^{\tau(r)} \mathbf{1}_A(\eta_t(\xi)) dt \right)^2 \right]^{1/2} \\ &\leq C \mathbb{E} \left[\left(\tau^{-1}(r) \int_0^{\tau(r)} \mathbf{1}_A(\eta_t(\xi)) dt \right)^2 \right]^{1/2}. \end{aligned}$$

The term inside the expectation appearing in the above r.h.s. is bounded by 1. Moreover, thanks to Theorem 3.1, it converges up to a μ -negligible event \mathbb{P} almost-surely towards $(\nu(A))^2$ (note that $\tau(r)$ explodes \mathbb{P} -a.s.). From (4.4) and (4.5), we derive that $\nu(A) \geq C^{-1}\mu(A)^\eta$. \square

Deduce the following corollary:

COROLLARY 4.5. *There exists a constant $\beta_{4.5} > 0$ such that $\mathbf{p}^{-\beta_{4.5}} \in L^1(\Xi, \mu)$.*

PROOF. For every $t \geq 0$, $\mu\{\mathbf{p}^{-1} \geq t\} \leq C_{4.3}t^{-1/\eta}$. Deduce in particular that $\mathbf{p}^{-1/(2\eta)}$ belongs to $L^1(\Xi, \mu)$. \square

4.2. Estimates of the Normalized Densities. The Aronson estimates from Proposition 2.1 do not apply uniformly to the rescaled transition densities, since they are just local in time. To obtain uniform bounds in time for the kernel q , we need to normalize the transition density by $p(\cdot, \xi)$ (see *e.g.* Escauriaza [7], Theorem 1.2):

PROPOSITION 4.6. *There exists a constant $C_{4.6} > 0$ such that for every $\xi \in \Xi$:*

$$\begin{aligned}
& \forall (t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d, \\
& C_{4.6}^{-1} \max \left[\left(\int_{B(x, t^{1/2})} p(z, \xi) dz \right)^{-1}, \left(\int_{B(y, t^{1/2})} p(z, \xi) dz \right)^{-1} \right] \\
(4.6) \quad & \times \exp[-C_{4.6} t^{-1} |x - y|^2] p(y, \xi) \\
& \leq q(t, x, y; \xi) \\
& \leq C_{4.6} \min \left[\left(\int_{B(x, t^{1/2})} p(z, \xi) dz \right)^{-1}, \left(\int_{B(y, t^{1/2})} p(z, \xi) dz \right)^{-1} \right] \\
& \times \exp[-C_{4.6}^{-1} t^{-1} |x - y|^2] p(y, \xi).
\end{aligned}$$

Proposition 4.6 provides a global Gaussian estimate for the normalized kernel $p^{-1}(y, \xi)q(t, x, y; \xi)$ since the ergodic theorem yields for $x \in \mathbb{R}^d$: μ -a.s., $\lim_{t \rightarrow +\infty} t^{d/2} / [\int_{B(x, t^{1/2})} p(z, \xi) dz] = [V_d \mathcal{E}(\mathbf{p})]^{-1} = V_d^{-1}$. According to this global nature, the normalized rescaled kernels $(p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon > 0}$ from (3.5) satisfy a similar version to (4.6), uniformly in ε . Generally speaking, this amounts to say that the constant $C_{4.6}$ only depends on λ and Λ .

4.3. *Harnack Inequality and Hölder Estimate.* We now exhibit a local Harnack inequality for the normalized transition density $p^{-1}(y, \xi)q(t, x, y; \xi)$ as well as a bound for its Hölder norm.

Recall indeed that $q(t, x, y; \xi)$ satisfies in (t, y) the forward Kolmogorov equation and refer again to Escauriaza [7], Theorem 3.7 (with $r = 1/2$ and $\tau = 3/2$):

PROPOSITION 4.7. *There exists a constant $C_{4.7}$ such that for all $\xi \in \Xi$ and $x \in \mathbb{R}^d$:*

$$\begin{aligned} & \sup_{(t,y) \in [1,5/4] \times B(x,1/2)} [p^{-1}(y, \xi)q(t, x, y; \xi)] \\ & \leq C_{4.7} \inf_{(t,y) \in [1/2,3/4] \times B(x,1/2)} [p^{-1}(y, \xi)q(t, x, y; \xi)]. \end{aligned}$$

From Theorem 3.8 in Escauriaza [7], derive the following Hölder property:

PROPOSITION 4.8. *There exist two constants $\alpha_{4.8} > 0$ and $C_{4.8}$ such that for all $\xi \in \Xi$, $r > 0$ and $Q(z, r) \subset]0, +\infty[\times \mathbb{R}^d$, $z \in]0, +\infty[\times \mathbb{R}^d$:*

$$\begin{aligned} & \forall x \in \mathbb{R}^d, (t, y), (t', y') \in Q(z, r/4), \\ & |p^{-1}(y, \xi)q(t, x, y; \xi) - p^{-1}(y', \xi)q(t', x, y'; \xi)| \\ & \leq C_{4.8} [r^{-2}(|y - y'|^2 + |t - t'|)]^{\alpha_{4.8}} 4.8^{1/2} \sup_{Q(z,r)} [p^{-1}(\cdot, \xi)q(\cdot, x, \cdot; \xi)]. \end{aligned}$$

Mention carefully that the point (t', y') in Theorem 3.8 in [7] coincides with the center z of the cylinder. To choose, as above, (t', y') differently from z , we need to pick up (t, y) in the smaller cylinder $Q(z, r/4)$.

In the statement of Proposition 4.8, $p^{-1}(y, \xi)q(t, x, y; \xi)$ is seen as a normalized solution (in (t, y)) to the operator $\partial_t - (\mathcal{L}^\xi)^*$. However, $p^{-1}(y, \xi)q(t, x, y; \xi)$ also satisfies in (t, x) the parabolic equation $[\partial_t - \mathcal{L}^\xi](\cdot) = 0$. Hence, the Krylov and Safonov theory (see e.g. [19] and [20]) yields:

PROPOSITION 4.9. *There exist two constants $\alpha_{4.9} > 0$ and $C_{4.9}$ such that for all $\xi \in \Xi$, $r > 0$ and $Q(z, r) \subset]0, +\infty[\times \mathbb{R}^d$, $z \in]0, +\infty[\times \mathbb{R}^d$:*

$$\begin{aligned} & \forall y \in \mathbb{R}^d, (t, x), (t', x') \in Q(z, r/4), \\ & |p^{-1}(y, \xi)q(t, x, y; \xi) - p^{-1}(y, \xi)q(t', x', y; \xi)| \\ & \leq C_{4.9} [r^{-2}(|x - x'|^2 + |t - t'|)]^{\alpha_{4.9}} 4.9^{1/2} \sup_{Q(z,r)} [p^{-1}(y, \xi)q(\cdot, \cdot, y; \xi)]. \end{aligned}$$

Similarly to Proposition 4.6, Propositions 4.8 and 4.9 uniformly apply to the normalized rescaled kernels $(p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon>0}$ since they hold in a global way. Again, this amounts to say that the underlying constants $\alpha_{4.8}$, $C_{4.8}$, $\alpha_{4.9}$ and $C_{4.9}$ do not refer to the regularity of the matrix $a(x, \xi)$.

4.4. *Oscillations of $p(\cdot, \xi)$.* To complete the current section, we provide several estimates for the oscillations of the function $p(\cdot, \xi)$. In particular, we establish the gradient estimate given in Theorem 3.2.

Choose first $t = 1$ in Proposition 2.1 and derive from Proposition 4.7 the following local Harnack inequality for $p(\cdot, \xi)$:

PROPOSITION 4.10. *There exists a constant $C_{4.10} > 0$ such that for all $\xi \in \Xi$ and $x \in \mathbb{R}^d$:*

$$C_{4.10}^{-1} \leq \inf_{y \in B(x, 1/2)} [p^{-1}(x, \xi)p(y, \xi)] \leq \sup_{y \in B(x, 1/2)} [p^{-1}(x, \xi)p(y, \xi)] \leq C_{4.10}.$$

In particular, for every $\xi \in \Xi$, the growth and the decay of $p(\cdot, \xi)$ are at most exponential: up to a modification of $C_{4.10}$, for every $x \in \mathbb{R}^d$, $p(0, \xi) \exp(-C_{4.10}|x|) \leq p(x, \xi) \leq p(0, \xi) \exp(C_{4.10}|x|)$.

Derive now the required estimate:

PROPOSITION 4.11. *The density \mathbf{p} belongs to $W^{1, d/(d-1)}(\Xi, \mu)$ and there exists a constant $C_{4.11}$ such that for a.e. $\xi \in \Xi$: $|\mathbf{p}^{-1}(\xi)D\mathbf{p}(\xi)| \leq C_{4.11}$.*

PROOF. Thanks to the smoothness of $a(x, \xi)$ (in x), Equation (4.1) also writes:

$$(4.7) \quad \begin{aligned} & \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(x, \xi) \frac{\partial^2 p}{\partial x_i \partial x_j}(x, \xi) + \sum_{i,j=1}^d \frac{\partial a_{i,j}}{\partial x_j}(x, \xi) \frac{\partial p}{\partial x_i}(x, \xi) \\ & + \frac{1}{2} \left[\sum_{i,j=1}^d \frac{\partial^2 a_{i,j}}{\partial x_i \partial x_j}(x, \xi) \right] p(x, \xi) = 0. \end{aligned}$$

Note that the function $p^{-1}(0, \xi)p(x, \xi)$ still satisfies (4.7). From Proposition 4.10, it is bounded on $B(0, 1/2)$ by a constant C (C does not depend on ξ). Moreover, thanks to Assumption **(A2)**, the coefficients of order zero and one in (4.7) are bounded and Lipschitz continuous. Apply Theorem 6.2 in Gilbarg and Trudinger [13], and deduce that the Hölder norms on $B(0, 1/4)$ of the gradient $p^{-1}(0, \xi)\nabla_x p(x, \xi)$ and of the Hessian matrix

$p^{-1}(0, \xi) \nabla_{x,x}^2 p(x, \xi)$ are bounded by C . By the mean value theorem and by Lemma 4.1, \mathbf{p} belongs to $W^{1,d/(d-1)}(\Xi, \mu)$ and $D\mathbf{p}(\xi) = \nabla_x p(0, \xi)$ a.s..

Due to Lemma 4.1, the bound for $\mathbf{p}^{-1} D\mathbf{p}$ easily follows. \square

5. Homogenization Property. We now investigate the convergence of the family of rescaled kernels $(p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon>0}$. The analysis can be divided in three different steps.

(First Step.)

Thanks to the ergodic properties of η , we establish the homogenization property for the family of operators $(\mathcal{L}^{\varepsilon, \xi})_{\varepsilon>0}$ for a.e. $\xi \in \Xi$. Since these operators are given in a non-divergence form, the effective diffusion coefficient just writes as the expectation of the diffusion matrix under the invariant measure $\mathbf{p} \cdot \mu$. This proves, for every $x \in \mathbb{R}^d$, the weak convergence of $(p^{-1}(\varepsilon^{-1} \cdot, \xi) \bar{q}(\cdot, x, \cdot; \varepsilon, \xi))_{\varepsilon>0}$ towards a Gaussian kernel Γ as $\varepsilon \rightarrow 0$:

PROPOSITION 5.1. *Denote by Γ the Gaussian kernel associated to the diffusion matrix $\bar{\mathbf{a}} = \mathcal{E}(\mathbf{p}\mathbf{a})$. Then, for a.e. $\xi \in \Xi$ and for every $x \in \mathbb{R}^d$, the sequence of functions (in (t, y) , x being fixed) $(p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon>0}$ weakly converges to $\Gamma(t, x, y)$ on $]0, +\infty[\times \mathbb{R}^d$ as ε tends towards zero, i.e.:*

$$(5.1) \quad \begin{aligned} & \forall \varphi \in \mathcal{C}_c^\infty(]0, +\infty[\times \mathbb{R}^d), \\ & \lim_{\varepsilon \rightarrow 0} \int_0^{+\infty} \int_{\mathbb{R}^d} \varphi(t, y) p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi) dy dt \\ & = \int_0^{+\infty} \int_{\mathbb{R}^d} \varphi(t, y) \Gamma(t, x, y) dy dt. \end{aligned}$$

(Second Step.)

Due to Proposition 4.6 (bound for the normalized kernel) and to Propositions 4.8 and 4.9 (Hölder continuity of the normalized kernel), we derive that the fundamental solutions $(p^{-1}(\varepsilon^{-1} \cdot, \xi) \bar{q}(\cdot, \cdot, \cdot; \varepsilon, \xi))_{\varepsilon>0}$ uniformly converge on every compact subset of $]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$ towards the kernel Γ :

PROPOSITION 5.2. *For a.e. $\xi \in \Xi$, the family of functions (in (t, x, y)) $(p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon>0}$ uniformly converges on compact subsets of $]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$ towards $\Gamma(t, x, y)$ as ε tends towards zero.*

Thanks again to Proposition 4.6, we deduce that the convergence also holds in the following L^2 sense:

PROPOSITION 5.3. *For every compact subset $\mathcal{K} \subset \mathbb{R}^d$ and for every $\alpha \in]0, 1[$, the following convergence holds in μ -probability and in \mathbf{p} - μ -probability (thanks to Theorem 3.1 and Corollary 4.5, both are equivalent) as ε vanishes:*

$$\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} \left[\int_{\mathbb{R}^d} p(\varepsilon^{-1}x, \xi) |p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi) - \Gamma(t, x, y)|^2 dx \right] \xrightarrow{(\mathbf{p}\cdot)\mu\text{-probability}} 0.$$

(Third Step.)

We finally obtain the convergence of the gradients:

PROPOSITION 5.4. *For every compact subset $\mathcal{K} \subset \mathbb{R}^d$ and for every $\alpha > 0$, the following convergence holds in μ and \mathbf{p} - μ -probability as ε vanishes:*

$$\sup_{y \in \mathcal{K}} \left[\int_{\alpha}^{+\infty} \int_{\mathbb{R}^d} p(\varepsilon^{-1}x, \xi) |p^{-1}(\varepsilon^{-1}y, \xi) \nabla_x \bar{q}(t, x, y; \varepsilon, \xi) - \nabla_x \Gamma(t, x, y)|^2 dx dt \right] \xrightarrow{(\mathbf{p}\cdot)\mu\text{-probability}} 0.$$

Moreover, the convergence in Proposition 5.3 is uniform over $t \in [\alpha, +\infty[$ and not just over $t \in [\alpha, \alpha^{-1}]$.

At this early stage of the paper, the reader can skip (at least for a first read) the detailed arguments to focus on the sequel of the proof of Theorem 3.2.

5.1. *Proof of Proposition 5.1.* Consider a smooth function φ with a compact support included in $]0, T[\times B(0, R)$, $R, T > 0$, and a bounded stationary field \mathbf{h} . For $\xi \in \Xi$, consider the following backward Cauchy problem over $[0, T[\times \mathbb{R}^d$ (that admits a unique bounded continuous solution in $\cap_{q \geq 1} W_{\text{loc}}^{1,2,q}([0, T] \times \mathbb{R}^d)$, see Theorem 9.1, Chapter IV, in Ladyzhenskaya et al. [21]):

$$\begin{aligned} \frac{\partial}{\partial t} [u(t, x; \varepsilon, \xi)] + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(\varepsilon^{-1}x, \xi) \frac{\partial^2}{\partial x_i \partial x_j} [u(t, x; \varepsilon, \xi)] \\ + \chi(t, x, \varepsilon^{-1}x, \xi) = 0, \quad u(T, x) = 0, \end{aligned}$$

with $\chi(t, x, y, \xi) = \varphi(t, x)h(y, \xi)$.

Referring to Theorem 2, Section 10, Chapter II, in Krylov [17], $u(0, 0; \varepsilon, \xi)$ writes:

$$(5.2) \quad u(0, 0; \varepsilon, \xi) = \mathbb{E} \int_0^T \varphi(t, \bar{X}_t(\varepsilon; 0, \xi)) h(\varepsilon^{-1} \bar{X}_t(\varepsilon; 0, \xi), \xi) dt.$$

According to the homogenization literature (see e.g. the references in Introduction), derive from Theorem 3.1 and from the central limit theorem for martingale processes (see e.g. Jacod and Shiryaev, Chapter VIII, Section 3) that, for a.e. $\xi \in \Xi$, $\bar{X}(\varepsilon; 0, \xi)$ converges in law on $\mathcal{C}([0, +\infty[, \mathbb{R}^d)$ towards a non-standard Brownian motion with values in \mathbb{R}^d and \bar{a} as covariance matrix. Using the tightness of the family $(\bar{X}(\varepsilon; 0, \xi))_{\varepsilon > 0}$ (see e.g. Pardoux [28] for the periodic frame) and referring again to Theorem 3.1, it is then plain to derive from (5.2) that, for a.e. ξ , $u(0, 0; \varepsilon, \xi) \rightarrow \bar{u}(0, 0)$ as ε vanishes, where \bar{u} denotes the (smooth) solution to the following PDE:

$$\frac{\partial \bar{u}}{\partial t}(t, x) + \frac{1}{2} \sum_{i,j=1}^d \bar{a}_{i,j} \frac{\partial^2 \bar{u}}{\partial x_i \partial x_j}(t, x) + \bar{\chi}(t, x) = 0, \quad \bar{u}(T, x) = 0,$$

and $\bar{\chi}(t, x)$ the averaged functional $\mathcal{E}(\mathbf{p}\mathbf{h})\varphi(t, x)$. Of course, the same holds for every $x \in \mathbb{R}^d$: $\forall x \in \mathbb{R}^d$, for a.e. $\xi \in \Xi$, $u(0, x; \varepsilon, \xi) \rightarrow \bar{u}(0, x)$ as ε tends to zero.

Recall now from Krylov and Safonov [19] that the functions $(u(\cdot, \cdot; \varepsilon, \xi))_{\varepsilon, \xi}$ are uniformly Hölder continuous. Deduce that the expressions “ $\forall x \in \mathbb{R}^d$ ” and “for a.e. $\xi \in \Xi$ ” can be exchanged. In other words, for a.e. $\xi \in \Xi$ and all $x \in \mathbb{R}^d$:

$$(5.3) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} \bar{q}(t, x, y; \varepsilon, \xi) \varphi(t, y) h(\varepsilon^{-1} y, \xi) dy dt \\ &= \mathcal{E}(\mathbf{p}\mathbf{h}) \int_0^T \int_{\mathbb{R}^d} \Gamma(t, x, y) \varphi(t, y) dy dt. \end{aligned}$$

We now wish to establish (5.3) with $\mathbf{h} = \mathbf{p}^{-1}$. Assume without loss of generality that φ is \mathbb{R}_+ -valued and choose first $\mathbf{h} = \mathbf{p}^{-1} \wedge n$ for $n \geq 1$. Deduce from (5.3):

$$(5.4) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} \bar{q}(t, x, y; \varepsilon, \xi) p^{-1}(\varepsilon^{-1} y, \xi) \varphi(t, y) dy dt \\ & \geq \mathcal{E}[\mathbf{p}(\mathbf{p}^{-1} \wedge n)] \int_0^T \int_{\mathbb{R}^d} \Gamma(t, x, y) \varphi(t, y) dy dt. \end{aligned}$$

Let $n \rightarrow +\infty$ in (5.3) and derive the lower bound in (5.1).

The upper bound is more difficult. The strategy relies on Proposition 4.6. For every $\xi \in \Xi$, for every $(t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$:

$$p^{-1}(y, \xi)q(t, x, y; \xi) \leq C \exp[-C^{-1}t^{-1}|x - y|^2] \left[\int_{B(x, t^{1/2})} p(z, \xi) dz \right]^{-1}.$$

Hence, from (3.5), for a given $\varepsilon > 0$:

$$\begin{aligned} & p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi) \\ (5.5) \quad & \leq C \exp[-C^{-1}t^{-1}|x - y|^2] \varepsilon^{-d} \left[\int_{B(\varepsilon^{-1}x, \varepsilon^{-1}t^{1/2})} p(z, \xi) dz \right]^{-1} \\ & \leq C \exp[-C^{-1}t^{-1}|x - y|^2] \left[\int_{B(x, t^{1/2})} p(\varepsilon^{-1}z, \xi) dz \right]^{-1}. \end{aligned}$$

Since φ vanishes in the neighbourhood of $t = 0$, there exists $\alpha > 0$ such that $\varphi(t, x) = 0$ for $t < \alpha$. Deduce from (5.5):

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \bar{q}(t, x, y; \varepsilon, \xi) p^{-1}(\varepsilon^{-1}y, \xi) \varphi(t, y) dy dt \\ (5.6) \quad & \leq \int_0^T \int_{\mathbb{R}^d} \bar{q}(t, x, y; \varepsilon, \xi) (p^{-1} \wedge n)(\varepsilon^{-1}y, \xi) \varphi(t, y) dy dt \\ & + C(\varphi, T) \left[\int_{B(x, \alpha^{1/2})} p(\varepsilon^{-1}z, \xi) dz \right]^{-1} \\ & \quad \times \int_{B(0, R)} \mathbf{1}_{]n, +\infty[}(p^{-1}(\varepsilon^{-1}y, \xi)) dy, \end{aligned}$$

where the constant $C(\varphi, T)$ may depend on φ and T . Plug now (5.3) into (5.6) with $\mathbf{h} = \mathbf{p}^{-1} \wedge n$ and apply the ergodic theorem:

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} \bar{q}(t, x, y; \varepsilon, \xi) p^{-1}(\varepsilon^{-1}y, \xi) \varphi(t, y) dy dt \\ (5.7) \quad & \leq \mathcal{E}[\mathbf{p}(\mathbf{p}^{-1} \wedge n)] \int_0^T \int_{\mathbb{R}^d} \Gamma(t, x, y) \varphi(t, y) dy dt \\ & + C(\alpha, \varphi, R, T) \mu\{\mathbf{p}^{-1} \geq n\}. \end{aligned}$$

Let $n \rightarrow +\infty$ in (5.7) and complete the proof (use the separability of $\mathcal{C}_c^\infty(]0, +\infty[\times \mathbb{R}^d)$ to exchange “ $\forall \varphi$ ” and “a.e.” in (5.1)). \square

5.2. Proof of Proposition 5.2. For $0 < \alpha < \beta$ and a compact subset $\mathcal{K} \subset \mathbb{R}^d$, deduce from Proposition 4.6 (see also (5.5)) that for every $\xi \in \Xi$:

$$\sup_{[\alpha, \beta] \times \mathcal{K} \times \mathbb{R}^d} [p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)] \leq C \sup_{x \in \mathcal{K}} \left[\int_{B(x, \alpha^{1/2})} p(\varepsilon^{-1}z, \xi) dz \right]^{-1}.$$

Assume for the moment that there exists $y_0 \in \mathbb{R}^d$ such that $\mathcal{K} \subset B(y_0, \alpha^{1/2}/4)$. Hence, for every $x \in \mathcal{K}$, $B(y_0, \alpha^{1/2}/4) \subset B(x, \alpha^{1/2})$. Thus:

$$\sup_{[\alpha, \beta] \times \mathcal{K} \times \mathbb{R}^d} [p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi)] \leq C \left[\int_{B(y_0, \alpha^{1/2}/4)} p(\varepsilon^{-1}z, \xi) dz \right]^{-1}.$$

Thanks to the ergodic theorem, for a.e. $\xi \in \Xi$:

$$(5.8) \quad \sup_{\varepsilon \in]0, 1]} \sup_{[\alpha, \beta] \times \mathcal{K} \times \mathbb{R}^d} [p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi)] \leq C(\alpha, \xi),$$

where the constant $C(\alpha, \xi)$ may depend on α and ξ . Note now that we can recover a general compact subset \mathcal{K} by a finite number of balls of the form $B(y_0, \alpha^{1/2}/4)$. Hence, (5.8) holds for any compact subset $\mathcal{K} \subset \mathbb{R}^d$.

Thanks to Propositions 4.8 and 4.9, for a.e. $\xi \in \Xi$, the Hölder norms of the functions $(p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi))_{\varepsilon > 0}$ are uniformly bounded on every compact subset of $]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$. Apply Proposition 5.1 and deduce that these functions uniformly converge on compact subsets of $]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$ towards $\Gamma(t, x, y)$. \square

5.3. Proof of Proposition 5.3. To apply Proposition 5.2, write for a cutting function $\eta : \mathbb{R}^d \rightarrow [0, 1]$, matching 1 on $B(0, r)$ and vanishing outside $B(0, r+1)$, $r > 0$ (remove for the sake of simplicity the index ξ):

$$\begin{aligned} & \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) |p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)|^2 dx \\ &= \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) |p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)|^2 \eta(x) dx \\ & \quad + \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) |p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)|^2 (1 - \eta)(x) dx \\ & \equiv T(t, y; 1, \varepsilon) + T(t, y; 2, \varepsilon). \end{aligned}$$

Note that:

$$\begin{aligned} & \sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 1, \varepsilon) \\ & \leq \sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{x \in B(0, r+1)} \sup_{y \in \mathcal{K}} [|p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)|^2] \\ & \quad \times \int_{B(0, r+1)} p(\varepsilon^{-1}z) dz. \end{aligned}$$

Thanks to Proposition 5.2 and to the ergodic theorem, $\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 1, \varepsilon)$ tends to 0 as ε vanishes (a.s. in ξ).

Turn now to $T(t, y; 2, \varepsilon)$. Recall that the normalized kernel $p^{-1}(\varepsilon^{-1}y)\bar{q}(t, x, y; \varepsilon)$ satisfies (5.5) (with respect to the ball $B(y, t^{1/2})$ instead of the ball $B(x, t^{1/2})$, see Proposition 4.6) and that the transition Γ is Gaussian. Hence,

$$(5.9) \quad \begin{aligned} T(t, y; 2, \varepsilon) &\leq C \left[\left(\int_{B(y, t^{1/2})} p(\varepsilon^{-1}z) dz \right)^{-2} + t^{-d} \right] \\ &\quad \times \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) (1 - \eta)(x) \exp[-C^{-1}t^{-1}|x - y|^2] dx \\ &\equiv T(t, y; 2, 1, \varepsilon) \times T(t, y; 2, 2, \varepsilon). \end{aligned}$$

Assume first that there exists $y_0 \in \mathbb{R}^d$ such that $\mathcal{K} \subset B(y_0, \alpha^{1/2}/4)$. Then, for every $y \in \mathcal{K}$, $B(y_0, \alpha^{1/2}/4) \subset B(y, \alpha^{1/2})$. Since $\alpha \leq t \leq \alpha^{-1}$, it comes:

$$T(t, y; 2, 1, \varepsilon) \leq C \left[\left(\int_{B(y_0, \alpha^{1/2}/4)} p(\varepsilon^{-1}z) dz \right)^{-2} + \alpha^{-d} \right].$$

From the ergodic theorem, we derive:

$$(5.10) \quad \begin{aligned} &\mu\text{-a.s.}, \sup_{\varepsilon \in]0, 1]} \sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, 1, \varepsilon) \\ &\leq C \left[\sup_{\varepsilon \in]0, 1]} \left(\int_{B(y_0, \alpha^{1/2}/4)} p(\varepsilon^{-1}z) dz \right)^{-2} + \alpha^{-d} \right] \\ &< +\infty. \end{aligned}$$

Finally, consider in (5.9) the *supremum* of $T(t, y; 2, 2, \varepsilon)$ over $(t, y) \in [\alpha, \alpha^{-1}] \times \mathcal{K}$. Since $|x - y|^2 \geq |x|^2/2 - |y|^2$, we claim:

$$\begin{aligned} &\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, 2, \varepsilon) \\ &\leq \exp(C\alpha^{-1}R^2) \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) (1 - \eta)(x) \exp[-C^{-1}\alpha|x|^2] dx, \end{aligned}$$

with $R \equiv \sup_{y \in \mathcal{K}} |y|$. Taking the expectation with respect to μ , we get:

$$(5.11) \quad \begin{aligned} &\mathcal{E} \left[\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, 2, \varepsilon) \right] \\ &\leq \exp(C\alpha^{-1}R^2) \int_{\mathbb{R}^d} (1 - \eta)(x) \exp[-C^{-1}\alpha|x|^2] dx. \end{aligned}$$

Let us now conclude. Fix to this end $\delta > 0$. From (5.10), we can find $M > 0$ (M being independent of η) such that, for all $\varepsilon \in]0, 1]$, $\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, 1, \varepsilon)$ is bounded by M up to the (μ -)probability $\delta/3$. For this M , we can choose r ($r + 1$ being the radius of the support of η) large enough

in (5.11) to bound, for all $\varepsilon \in]0, 1]$, $\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, 2, \varepsilon)$ by $\delta/(2M)$ up to the (μ_-) probability $\delta/3$. In particular, up to the (μ_-) probability $2\delta/3$, $\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 2, \varepsilon)$ is bounded for all $\varepsilon \in]0, 1]$ by $\delta/2$. Choose now ε small enough to bound $\sup_{\alpha \leq t \leq \alpha^{-1}} \sup_{y \in \mathcal{K}} T(t, y; 1, \varepsilon)$ by $\delta/2$ up to the (μ_-) probability $\delta/3$ and complete the proof when \mathcal{K} is included in $B(y_0, \alpha^{1/2}/4)$. In the general case, the compact set \mathcal{K} can be covered with a finite number of balls of radius $\alpha^{1/2}/4$. \square

5.4. *Proof of Proposition 5.4.* Recall first that for all $\xi \in \Xi$ and $y \in \mathbb{R}^d$, the function $p^{-1}(y, \xi)q(t, x, y; \xi)$ is a solution (in (t, x)) to the operator $\partial_t - \mathcal{L}^\xi$. Apply the scaling procedure (3.5) and derive for $\varepsilon > 0$ and $(t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$:

$$(5.12) \quad \frac{\partial [p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)]}{\partial t} - \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(\varepsilon^{-1}x, \xi) \frac{\partial^2 [p^{-1}(\varepsilon^{-1}y, \xi)\bar{q}(t, x, y; \varepsilon, \xi)]}{\partial x_i \partial x_j} = 0.$$

Apply Theorem 7, Section 6, Chapter IX, in Friedman [11] and note that $\bar{q}(t, x, y; \varepsilon, \xi)$ is a smooth function in (t, x) . In particular, (5.12) holds in a strong sense. Note also that $\bar{q}(t, x, y; \varepsilon, \xi)$, $\nabla_x \bar{q}(t, x, y; \varepsilon, \xi)$ and $\nabla_{x,x}^2 \bar{q}(t, x, y; \varepsilon, \xi)$ are bounded on every set $[\alpha, \beta] \times \mathbb{R}^d \times \mathbb{R}^d$, $\beta > \alpha$, by $C(\alpha, \beta, \varepsilon) \exp(-(C(\alpha, \beta, \varepsilon))^{-1}|x - y|^2)$.

Recall that the Gaussian kernel Γ (see Proposition 5.1) satisfies for $(t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$:

$$\frac{\partial \Gamma(t, x, y)}{\partial t} - \frac{1}{2} \sum_{i,j=1}^d \bar{a}_{i,j} \frac{\partial^2 \Gamma(t, x, y)}{\partial x_i \partial x_j} = 0.$$

We then deduce for $(t, x, y) \in]0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$ (remove the index ξ):

$$(5.13) \quad \frac{\partial [p^{-1}(\varepsilon^{-1}y)\bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)]}{\partial t} - \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(\varepsilon^{-1}x) \frac{\partial^2 [p^{-1}(\varepsilon^{-1}y)\bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)]}{\partial x_i \partial x_j} = \frac{1}{2} \sum_{i,j=1}^d [a_{i,j}(\varepsilon^{-1}x) - \bar{a}_{i,j}] \frac{\partial^2 \Gamma(t, x, y)}{\partial x_i \partial x_j}.$$

Thanks to the previous Gaussian bounds and to Proposition 4.10 (growth of p), we can multiply both sides in (5.13) by $p(\varepsilon^{-1}x)[p^{-1}(\varepsilon^{-1}y)\bar{q}(t, x, y; \varepsilon) -$

$\Gamma(t, x, y)$] and integrate with respect to $x \in \mathbb{R}^d$. Since $p(\varepsilon^{-1}x, \xi)$ satisfies in x the adjoint equation $(\bar{\mathcal{L}}^{\varepsilon, \xi})^* (p(\varepsilon^{-1}x, \xi)) = 0$, an integration by parts yields for a suitable constant $C > 0$ and for every $t \geq \alpha$:

$$\begin{aligned}
(5.14) \quad & \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) [p^{-1}(\varepsilon^{-1}y)\bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y)]^2 dx \\
& + C^{-1} \int_{\alpha}^t \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) |\nabla_x [p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon)] - \nabla_x \Gamma(s, x, y)|^2 dx ds \\
& \leq \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) [p^{-1}(\varepsilon^{-1}y)\bar{q}(\alpha, x, y; \varepsilon) - \Gamma(\alpha, x, y)]^2 dx \\
& + \frac{1}{2} \sum_{i,j=1}^d \int_{\alpha}^t \int_{\mathbb{R}^d} \left[p(\varepsilon^{-1}x) [a_{i,j}(\varepsilon^{-1}x) - \bar{a}_{i,j}] \right. \\
& \quad \left. \times \frac{\partial^2 \Gamma(s, x, y)}{\partial x_i \partial x_j} [p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon) - \Gamma(s, x, y)] \right] dx ds.
\end{aligned}$$

The proof of Proposition 5.4 then follows from Proposition 5.3 and from the following Lemma:

LEMMA 5.5. *Under Assumption of Proposition 5.4,*

$$\begin{aligned}
& \sup_{t \geq \alpha} \sup_{y \in \mathcal{K}} \left| \int_{\alpha}^t \left[\sum_{i,j=1}^d \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) [a_{i,j}(\varepsilon^{-1}x) - \bar{a}_{i,j}] \frac{\partial^2 \Gamma(s, x, y)}{\partial x_i \partial x_j} \right. \right. \\
& \quad \left. \left. \times [p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon) - \Gamma(s, x, y)] dx \right] ds \right| \xrightarrow{\mu\text{-probability}} 0.
\end{aligned}$$

PROOF. Recall first that the second order derivatives of Γ satisfy for all $(t, x, y) \in]0, +\infty[\times \mathbb{R} \times \mathcal{K}$ (denote $R \equiv \sup_{y \in \mathcal{K}} |y|$):

$$\begin{aligned}
(5.15) \quad & |\nabla_{x,x}^2 \Gamma(t, x, y)| \\
& \leq Ct^{-d/2-1} [1 + t^{-1}|x - y|^2] \exp(-C^{-1}t^{-1}|x - y|^2) \\
& \leq Ct^{-d/2-1} [1 + t^{-1}(R^2 + |x|^2)] \exp(Ct^{-1}R^2 - C^{-1}t^{-1}|x|^2).
\end{aligned}$$

Fix now $T > \alpha$ and deduce from the Cauchy-Schwartz inequality and from Proposition 5.3:

$$\begin{aligned}
(5.16) \quad & \sup_{y \in \mathcal{K}} \left[\int_{\alpha}^T \left| \sum_{i,j=1}^d \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) [a_{i,j}(\varepsilon^{-1}x) - \bar{a}_{i,j}] \frac{\partial^2 \Gamma(s, x, y)}{\partial x_i \partial x_j} \right. \right. \\
& \quad \left. \left. \times [p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon) - \Gamma(s, x, y)] dx \right] ds \right] \xrightarrow{\mu\text{-probability}} 0.
\end{aligned}$$

It thus remains to establish the convergence of the above integral over the interval $[T, +\infty[$. Follow the proof of Proposition 5.3 (and in particular (5.9)) and assume for the moment that $\mathcal{K} \subset B(y_0, T^{1/2}/4)$ for a given $y_0 \in \mathbb{R}^d$. Then, for $t \geq T$,

$$\begin{aligned}
& \sup_{y \in \mathcal{K}} \int_{\mathbb{R}^d} \left[p(\varepsilon^{-1}x) |a_{i,j}(\varepsilon^{-1}x) - \bar{a}_{i,j}| \left| \frac{\partial^2 \Gamma(t, x, y)}{\partial x_i \partial x_j} \right| \right. \\
& \quad \left. \times (p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) + \Gamma(t, x, y)) \right] dx \\
(5.17) \quad & \leq C \exp(Ct^{-1}R^2) \left[\sup_{y \in \mathcal{K}} \left(\int_{B(y, t^{1/2})} p(\varepsilon^{-1}z) dz \right)^{-1} + t^{-d/2} \right] \\
& \quad \times \left[\int_{\mathbb{R}^d} p(\varepsilon^{-1}x) t^{-d/2-1} [1 + t^{-1}(R^2 + |x|^2)] \exp[-C^{-1}t^{-1}|x|^2] dx \right] \\
& \leq C \exp(CT^{-1}R^2) \left[t^{d/2} \left(\int_{B(y_0, t^{1/2}/4)} p(\varepsilon^{-1}z) dz \right)^{-1} + 1 \right] \\
& \quad \times \left[\int_{\mathbb{R}^d} p(\varepsilon^{-1}x) t^{-d-1} [1 + t^{-1}(R^2 + |x|^2)] \exp[-C^{-1}t^{-1}|x|^2] dx \right] \\
& \equiv C \exp(CT^{-1}R^2) \times S(t; 1, \varepsilon) \times S(t; 2, \varepsilon).
\end{aligned}$$

Note that $S(t; 1, \varepsilon) = \left(\int_{B(t^{-1/2}y_0, 1/4)} p(\varepsilon^{-1}t^{1/2}z) dz \right)^{-1} + 1$.

In particular, following (5.10), we can prove from the ergodic theorem that $\sup_{t \geq T} \sup_{\varepsilon \in]0, 1]} S(t; 1, \varepsilon)$ is a.s. finite. Similarly to (5.11), the expectation $\mathcal{E}[\int_T^{+\infty} |S(t; 2, \varepsilon)| dt]$ can be made, uniformly in ε , as small as necessary by choosing T large. \square

6. Resolvent Equation. In this new section, we investigate the resolvent equation $-\mathbf{L}^*(\mathbf{p}\mathbf{u}(\cdot; \varepsilon)) + \varepsilon^2 \mathbf{p}\mathbf{u}(\cdot; \varepsilon) = \mathbf{p}\mathbf{f}$, $\varepsilon > 0$.

(First Step.)

In this frame, we first prove:

PROPOSITION 6.1. *The family $(T_x)_{x \in \mathbb{R}^d}$ defines a strongly continuous group on $L^2(\Xi, \mathbf{p}, \mu)$ (so that the realization of a field in $L^2(\Xi, \mathbf{p}, \mu)$ belongs to $L^2_{\text{loc}}(\mathbb{R}^d)$). The intersection of the domains of the generators D_1, \dots, D_d is denoted by $H(\Xi, \mathbf{p}, \mu)$ (the definitions of $(D_i \mathbf{h})_{1 \leq i \leq d}$ coincide if $\mathbf{h} \in H(\Xi, \mu) \cap H(\Xi, \mathbf{p}, \mu)$ since the convergences in μ and \mathbf{p}, μ -probability are equivalent).*

Moreover, for every $\mathbf{k} \in H(\Xi, \mathbf{p}, \mu)$, $\mathbf{p}\mathbf{k} \in W^{1,1}(\Xi, \mu)$ with $D(\mathbf{p}\mathbf{k}) \in L^2(\Xi, \mathbf{p}^{-1}, \mu; \mathbb{R}^d)$, and the following integration by parts holds for all $\mathbf{h} \in H(\Xi, \mathbf{p}, \mu)$ and $i \in \{1, \dots, d\}$:

$$(6.1) \quad \mathcal{E}[\mathbf{p}(D_i \mathbf{h}) \mathbf{k}] = -\mathcal{E}[\mathbf{h} D_i(\mathbf{p}\mathbf{k})].$$

Finally, for every $\mathbf{h} \in H(\Xi, \mathbf{p}, \mu)$, $h(\cdot, \xi) \in H_{\text{loc}}^1(\mathbb{R}^d)$ for a.e. $\xi \in \Xi$ and $\nabla_x h(x, \xi) = D\mathbf{h}(\mathbb{T}_x \xi)$ almost everywhere in x .

We then investigate the resolvent equation:

THEOREM 6.2. *For a given $\mathbf{h} \in L^2(\Xi, \mathbf{p}, \mu)$ and for every $\varepsilon > 0$, there exists a unique stationary field $\mathbf{v}(\cdot; \varepsilon)$ in the domain of $\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot)$ (adjoint operator of \mathbf{L} with respect to the scalar product induced by \mathbf{p}) such that:*

$$-\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{v}(\cdot; \varepsilon)) + \varepsilon^2\mathbf{v}(\cdot; \varepsilon) = \mathbf{h}.$$

It writes:

$$\begin{aligned} \mathbf{v}(\xi; \varepsilon) &= \int_0^{+\infty} \left[\exp(-\varepsilon^2 t) \int_{\mathbb{R}^d} \mathbf{h}(\mathbb{T}_y \xi) p(y, \xi) p^{-1}(0, \xi) q(t, y, 0; \xi) dy \right] dt. \\ (6.2) \quad &= \varepsilon^{-2} \int_0^{+\infty} \left[\exp(-t) \right. \\ &\quad \left. \times \int_{\mathbb{R}^d} \mathbf{h}(\mathbb{T}_{\varepsilon^{-1}y} \xi) p(\varepsilon^{-1}y, \xi) p^{-1}(0, \xi) \bar{q}(t, y, 0; \varepsilon, \xi) dy \right] dt. \end{aligned}$$

Moreover, for every $\mathbf{h} \in L^2(\Xi, \mathbf{p}, \mu)$, the solution $\mathbf{v}(\cdot; \varepsilon)$ belongs to the space $H(\Xi, \mathbf{p}, \mu)$. In this frame:

$$(6.3) \quad \frac{1}{2} \mathcal{E}(\mathbf{p}(\mathbf{a}D\mathbf{v}(\cdot; \varepsilon), D\mathbf{v}(\cdot; \varepsilon))) + \varepsilon^2 \mathcal{E}(\mathbf{p}|\mathbf{v}(\cdot; \varepsilon)|^2) = \mathcal{E}(\mathbf{p}\mathbf{v}(\cdot; \varepsilon)\mathbf{h}).$$

(Second Step.)

For $\mathbf{h} = \mathbf{f}$ in Theorem 6.2, we write $\mathbf{u}(\cdot; \varepsilon)$ for the solution to the resolvent equation. Since $\mathbf{f} = \mathbf{p}^{-1}D_\ell(\mathbf{p}\mathbf{g})$, derive from (6.1) (since $D\mathbf{g} \in L^\infty(\Xi, \mu; \mathbb{R}^d)$, $\mathbf{g} \in H(\Xi, \mathbf{p}, \mu)$) that $\mathcal{E}(\mathbf{p}\mathbf{u}(\cdot; \varepsilon)\mathbf{f}) = -\mathcal{E}(\mathbf{p}\mathbf{g}D_\ell\mathbf{u}(\cdot; \varepsilon))$. Due to Assumption **(A.1)**, (6.3) yields:

$$(6.4) \quad \sup_{\varepsilon > 0} [\mathcal{E}(\mathbf{p}|D\mathbf{u}(\cdot; \varepsilon)|^2) + \varepsilon^2 \mathcal{E}(\mathbf{p}\mathbf{u}^2(\cdot; \varepsilon))] < +\infty.$$

We then prove the following asymptotic property for the family $(\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$:

THEOREM 6.3. *For every compact subset $\mathcal{K} \subset \mathbb{R}^d$, the following convergence holds in μ -probability and \mathbf{p}, μ -probability as ε vanishes:*

$$\varepsilon^2 \int_{\mathcal{K}} p(\varepsilon^{-1}x, \xi) u^2(\varepsilon^{-1}x; \varepsilon, \xi) dx \xrightarrow{(\mathbf{p}, \mu)\text{-probability}} 0.$$

(Third Step.)

We finally improve Theorem 6.3:

THEOREM 6.4. *The family $(\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$ satisfies: $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathcal{E}(\mathbf{p}\mathbf{u}^2(\cdot; \varepsilon)) = 0$.*

This result is well-known in homogenization theory and usually derives from the properties of the weak limits of the sequence $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$. Indeed, if we manage to prove that every weak limit $\Psi \in L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$ of the sequence $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$ satisfies $\mathcal{E}(\mathbf{p}\langle \mathbf{a}\Psi, \Psi \rangle) = -2\mathcal{E}(\mathbf{p}\mathbf{g}\Psi_\ell)$, we easily deduce that Theorem 6.4 holds true (see e.g. Olla [24], Chapter I). Unfortunately, we are not able to directly prove such an equality. Too little properties for the density \mathbf{p} are available. The strategy then follows from Theorem 6.3.

6.1. *Proof of Proposition 6.1.* Consider first a function $\mathbf{h} \in L^2(\Xi, \mathbf{p}, \mu)$. Due to Lemma 4.1 and to Proposition 4.10, $T_x\mathbf{h}$ still belongs to $L^2(\Xi, \mathbf{p}, \mu)$ for every $x \in \mathbb{R}^d$. We then aim to prove that $\lim_{x \rightarrow 0} \mathcal{E}(\mathbf{p}(T_x\mathbf{h} - \mathbf{h})^2) = 0$. Since the function \mathbf{h} is μ -a.s. finite, \mathbf{h} satisfies **(RM4)** (stochastic continuity of the group $(T_x)_{x \in \mathbb{R}^d}$). It is then sufficient to prove that the family $(\mathbf{p}(\mathbf{h}(T_x\cdot) - \mathbf{h})^2)_{x \in B(0, 1/4)}$ is uniformly integrable with respect to μ . For $A \in \mathcal{G}$:

$$\mathcal{E}[\mathbf{1}_A \mathbf{p}(\mathbf{h}(T_x\cdot) - \mathbf{h})^2] \leq 2\mathcal{E}[\mathbf{1}_A \mathbf{p}\mathbf{h}^2(T_x\cdot)] + 2\mathcal{E}[\mathbf{1}_A \mathbf{p}\mathbf{h}^2].$$

There is no difficulty to handle the second term in the r.h.s.. Focus then on the first one. Thanks to **(RM1)** (stationarity of the medium), to Lemma 4.1 and to Proposition 4.10, it writes for $x \in B(0, 1/4)$:

$$(6.5) \quad \mathcal{E}[\mathbf{1}_A \mathbf{p}\mathbf{h}^2(T_x\cdot)] = \mathcal{E}[\mathbf{1}_A(T_{-x}\cdot)\mathbf{p}(T_{-x}\cdot)\mathbf{h}^2] \leq C\mathcal{E}[\mathbf{1}_A(T_{-x}\cdot)\mathbf{p}\mathbf{h}^2].$$

Since $\mu(T_x A) = \mu(A)$, we derive that the family $(\mathbf{p}(\mathbf{h}(T_x\cdot) - \mathbf{h})^2)_{x \in B(0, 1/4)}$ is U.I. and thus establish the announced strong continuity property.

Turn now to the proof of the integration by parts (6.1). Following the previous construction of $H(\Xi, \mathbf{p}, \mu)$, we can define in a similar way the space $H(\Xi, \mathbf{p}^{-1}, \mu)$ (mention carefully that \mathbf{p}^{-1}, μ is just σ -finite). Since $\mathbf{p} \in L^1(\Xi, \mu)$, it is well seen that $H(\Xi, \mathbf{p}^{-1}, \mu) \subset W^{1,1}(\Xi, \mu)$. In particular, the notation $D_i\Phi$, for $\Phi \in H(\Xi, \mathbf{p}^{-1}, \mu)$ and $i \in \{1, \dots, d\}$, still makes sense.

From standard computations, we deduce that the product $\mathbf{h}\Phi$ belongs to $W^{1,1}(\Xi, \mu)$ if $\mathbf{h} \in H(\Xi, \mathbf{p}, \mu)$ and $\Phi \in H(\Xi, \mathbf{p}^{-1}, \mu)$. For all $i \in \{1, \dots, d\}$, $D_i(\mathbf{h}\Phi)$ writes $\mathbf{h}D_i\Phi + \Phi D_i\mathbf{h}$.

Thanks to Propositions 4.10 and 4.11, it is plain to prove that $\mathbf{p}\mathbf{k} \in H(\Xi, \mathbf{p}^{-1}, \mu)$ for $\mathbf{k} \in H(\Xi, \mathbf{p}, \mu)$. The integration by parts (6.1) follows easily.

The last assertion follows from Proposition 4.10 and from (7.18) in Zhikov et al. [36]. \square

6.2. *Proof of Theorem 6.2.* Since $\mathbf{p} \cdot \mu$ is invariant for η , $(S_t)_{t \geq 0}$ defines a contraction semigroup on $L^2(\Xi, \mathbf{p} \cdot \mu)$ (see Subsection 2.3 for the definition of $(S_t)_{t \geq 0}$). Following Subsection 2.3, we derive from Proposition 6.1 (strong continuity of $(T_x)_{x \in \mathbb{R}^d}$ on $L^2(\Xi, \mathbf{p} \cdot \mu)$) that $(S_t)_{t \geq 0}$ is strongly continuous. The domain of \mathbf{L} , seen as an operator on $L^2(\Xi, \mathbf{p} \cdot \mu)$, is denoted by $\mathcal{D}_{\mathbf{p}}(\mathbf{L})$. It contains $W^{2,\infty}(\Xi, \mu)$.

We are now in position to construct the semigroup for $\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot)$:

PROPOSITION 6.5. *Define for a function $\mathbf{h} \in L^1(\Xi, \mathbf{p} \cdot \mu)$ and a real $t \geq 0$ the following mapping:*

$$R_t \mathbf{h} : \xi \in \Xi \mapsto \int_{\mathbb{R}^d} \mathbf{h}(\mathbb{T}_x \xi) \mathbf{p}(\mathbb{T}_x \xi) \mathbf{p}^{-1}(\xi) q(t, x, 0; \xi) dx \quad (= \mathbf{h}(\xi) \text{ if } t = 0).$$

Then, $(R_t)_{t \geq 0}$ is a family of contractions on $L^1(\Xi, \mathbf{p} \cdot \mu)$ and induces a family of contractions on $L^\infty(\Xi, \mu)$. Moreover, it defines a strongly continuous contraction semigroup on $L^2(\Xi, \mathbf{p} \cdot \mu)$ whose generator is the adjoint of \mathbf{L} seen as an operator on $L^2(\Xi, \mathbf{p} \cdot \mu)$. This generator writes formally $\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot)$, where \mathbf{L}^* denotes the adjoint of \mathbf{L} seen as an operator on $L^2(\Xi, \mu)$. Its domain is given by:

$$\begin{aligned} \mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot)) \\ = \{ \mathbf{h} \in L^2(\Xi, \mathbf{p} \cdot \mu), \exists c \geq 0, \forall \mathbf{l} \in W^{2,\infty}(\Xi, \mu), \mathcal{E}[\mathbf{p}\mathbf{h}\mathbf{L}(\mathbf{l})]^2 \leq c\mathcal{E}[\mathbf{p}|\mathbf{l}|^2] \}, \end{aligned}$$

and for $\mathbf{h} \in \mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot))$ and $\mathbf{l} \in \mathcal{D}_{\mathbf{p}}(\mathbf{L})$, $\mathcal{E}[\mathbf{p}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{h}))\mathbf{l}] = \mathcal{E}[\mathbf{p}\mathbf{h}\mathbf{L}(\mathbf{l})]$. In particular, $W^{2,\infty}(\Xi, \mu)$ is included in $\mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot))$.

PROOF. Note first from Lemma 7.1 in Zhikoy et al. [36] that $R_t \mathbf{h}$ is well defined for $\mathbf{h} \geq 0$ (i.e. $R_t \mathbf{h} = R_t \hat{\mathbf{h}}$ a.s. for $\mathbf{h} = \hat{\mathbf{h}}$ a.s.).

From (2.4), for all $\mathbf{h} \geq 0$ and $t > 0$:

$$\begin{aligned} \mathcal{E}[\mathbf{p}(R_t \mathbf{h})] &= \int_{\mathbb{R}^d} \left[\int_{\Xi} \mathbf{h}(\mathbb{T}_x \xi) \mathbf{p}(\mathbb{T}_x \xi) q(t, x, 0; \xi) d\mu(\xi) \right] dx \\ (6.6) \quad &= \int_{\mathbb{R}^d} \left[\int_{\Xi} \mathbf{h}(\mathbb{T}_x \xi) \mathbf{p}(\mathbb{T}_x \xi) q(t, 0, -x; \mathbb{T}_x \xi) d\mu(\xi) \right] dx \\ &= \int_{\mathbb{R}^d} \left[\int_{\Xi} \mathbf{h}(\xi) \mathbf{p}(\xi) q(t, 0, -x; \xi) d\mu(\xi) \right] dx = \mathcal{E}(\mathbf{h}\mathbf{p}). \end{aligned}$$

Recall that $q(t, 0, -x; \xi)$ is a density (in x) over \mathbb{R}^d to derive the last equality.

Deduce in particular that $R_t \mathbf{h}$ defines an element of $L^1(\Xi, \mathbf{p} \cdot \mu)$ for $\mathbf{h} \in L^1(\Xi, \mathbf{p} \cdot \mu)$. It is then readily seen that R_t is a contraction on $L^1(\Xi, \mathbf{p} \cdot \mu)$. We now prove that it defines a contraction on $L^2(\Xi, \mathbf{p} \cdot \mu)$ and $L^\infty(\Xi, \mu)$. We need to this end the following lemma:

LEMMA 6.6. *For every $\xi \in \Xi$, the function $x \in \mathbb{R}^d \mapsto p^{-1}(0, \xi)p(x, \xi)q(t, x, 0; \xi)$ is a density.*

PROOF OF LEMMA 6.6. Since $p(x, \xi)$ is at most of exponential growth (see Proposition 4.10) and is continuous with respect to the first argument, derive from Theorem 15, Chapter I, in Friedman [11], that for every $\xi \in \Xi$:

$$(6.7) \quad \lim_{t \rightarrow 0} \int_{\mathbb{R}^d} p^{-1}(0, \xi)p(x, \xi)q(t, x, 0; \xi)dx = 1.$$

Recall now that $q(t, x, 0; \xi)$ is a solution to the operator $\partial_t - \mathcal{L}^\xi$ and satisfies the bounds given in the proof of Proposition 5.4. Hence,

$$\begin{aligned} & \frac{d}{dt} \left[\int_{\mathbb{R}^d} p^{-1}(0, \xi)p(x, \xi)q(t, x, 0; \xi)dx \right] \\ &= \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} p^{-1}(0, \xi)p(x, \xi)a_{i,j}(x, \xi) \frac{\partial^2 q}{\partial x_i \partial x_j}(t, x, 0; \xi)dx. \end{aligned}$$

Since $p(x, \xi)$ is a solution to the adjoint equation $(\mathcal{L}^\xi)^*(p(\cdot, \xi)) = 0$, the above l.h.s. is equal to zero. From (6.7), we complete the proof. \square

END OF THE PROOF OF PROPOSITION 6.5. Thanks to Lemmas 4.1 and 6.6, it is readily seen that R_t is a contraction on $L^\infty(\Xi, \mu)$ (recall by the way that $L^\infty(\Xi, \mu)$ is also $L^\infty(\Xi, \mathbf{p} \cdot \mu)$ since μ and $\mathbf{p} \cdot \mu$ are equivalent). From the Cauchy-Schwarz inequality, it is also a contraction on $L^2(\Xi, \mathbf{p} \cdot \mu)$.

Thanks to the Chapman-Kolmogorov equation (applied to the kernel $q(t, x, y; \xi)$), it is plain to see that $(R_t)_{t \geq 0}$ defines a semigroup on $L^2(\Xi, \mathbf{p} \cdot \mu)$. It thus remains to prove that the semigroup is strongly continuous. For $t > 0$ and $\mathbf{h} \in L^2(\Xi, \mathbf{p} \cdot \mu)$, deduce from Lemma 6.6 and Proposition 2.1:

$$\begin{aligned} & \mathcal{E}[\mathbf{p}(R_t \mathbf{h} - \mathbf{h})^2] \\ & \leq \mathcal{E} \left[\int_{\mathbb{R}^d} [\mathbf{h}(\mathbb{T}_x \xi) - \mathbf{h}(\xi)]^2 \mathbf{p}(\mathbb{T}_x \xi) q(t, x, 0; \xi) dx \right] \\ & \leq C t^{-d/2} \exp(Ct) \mathcal{E} \left[\int_{\mathbb{R}^d} \mathbf{p}(\mathbb{T}_x \xi) [\mathbf{h}(\mathbb{T}_x \xi) - \mathbf{h}(\xi)]^2 \exp(-C^{-1} t^{-1} |x|^2) dx \right] \\ & = C t^{-d/2} \exp(Ct) \int_{\mathbb{R}^d} \mathcal{E}[\mathbf{p}(\xi) [\mathbf{h}(\mathbb{T}_x \xi) - \mathbf{h}(\xi)]^2] \exp(-C^{-1} t^{-1} |x|^2) dx \\ & = C \exp(Ct) \int_{\mathbb{R}^d} \mathcal{E}[\mathbf{p}(\xi) [\mathbf{h}(\mathbb{T}_{t^{1/2} x} \xi) - \mathbf{h}(\xi)]^2] \exp(-C^{-1} |x|^2) dx. \end{aligned}$$

Following Subsection 2.3, we derive the strong continuity property (apply Proposition 4.10 to establish the domination condition $\mathcal{E}[\mathbf{p}(\xi) \mathbf{h}^2(\mathbb{T}_{t^{1/2} x} \xi)] \leq \exp(C t^{1/2} |x|) \mathcal{E}(\mathbf{p} \mathbf{h}^2)$).

Compute finally the generator A of $(R_t)_{t \geq 0}$. For $\mathbf{h}, \mathbf{l} \in L^2(\Xi, \mathbf{p}, \mu)$, it is well-seen that $\mathcal{E}[\mathbf{p}(R_t \mathbf{h}) \mathbf{l}] = \mathcal{E}[\mathbf{p} \mathbf{h}(S_t \mathbf{l})]$, where $(S_t)_{t \geq 0}$ denotes the semigroup associated to \mathbf{L} . Hence, for \mathbf{h} in the domain of A and \mathbf{l} in the domain of \mathbf{L} , $\mathcal{E}[\mathbf{p}(R_t \mathbf{h} - \mathbf{h}) \mathbf{l}] = \mathcal{E}[\mathbf{p} \mathbf{h}(S_t \mathbf{l} - \mathbf{l})]$. In particular, $\mathcal{E}[\mathbf{p} A(\mathbf{h}) \mathbf{l}] = \mathcal{E}[\mathbf{p} \mathbf{h} \mathbf{L}(\mathbf{l})]$. We easily derive that A matches $\mathbf{p}^{-1} \mathbf{L}^*(\mathbf{p} \cdot)$.

The representation for $\mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1} \mathbf{L}(\mathbf{p} \cdot))$ given in the statement of Proposition 6.5 is well-known when $W^{2,\infty}(\Xi, \mu)$ is replaced by $\mathcal{D}_{\mathbf{p}}(\mathbf{L})$. To obtain the announced representation, we need to prove that for every $\mathbf{l} \in \mathcal{D}_{\mathbf{p}}(\mathbf{L})$, there exists $(\mathbf{l}_n)_{n \geq 0} \in (W^{2,\infty}(\Xi, \mu))^{\mathbb{N}}$, such that $(\mathbf{l}_n, \mathbf{L} \mathbf{l}_n) \rightarrow (\mathbf{l}, \mathbf{L} \mathbf{l})$ in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^2)$. This can be done as follows. Referring to the next paragraph (end of the proof of Theorem 6.2), we can prove that $(I - \mathbf{L})^{-1}$ (I denotes the identity) maps \mathcal{S}_{∞} into $W^{2,\infty}(\Xi, \mu)$. Considering a sequence $(\mathbf{h}_n)_{n \geq 0} \in \mathcal{S}_{\infty}^{\mathbb{N}}$ converging towards $(I - \mathbf{L}) \mathbf{l}$ in $L^2(\Xi, \mathbf{p}, \mu)$ (obviously, \mathcal{S}_{∞} is dense in $L^2(\Xi, \mathbf{p}, \mu)$), we can choose $\mathbf{l}_n \equiv (I - \mathbf{L})^{-1}(\mathbf{h}_n)$, so that \mathbf{l}_n tends towards $(I - \mathbf{L})^{-1}((I - \mathbf{L}) \mathbf{l}) = \mathbf{l}$ in $L^2(\Xi, \mathbf{p}, \mu)$. Since $(I - \mathbf{L}) \mathbf{l}_n = \mathbf{h}_n \rightarrow (I - \mathbf{L}) \mathbf{l}$, we derive that $\mathbf{L} \mathbf{l}_n \rightarrow \mathbf{L} \mathbf{l}$ in $L^2(\Xi, \mathbf{p}, \mu)$.

It now remains to prove that $W^{2,\infty}(\Xi, \mu) \subset \mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1} \mathbf{L}^*(\mathbf{p} \cdot))$. Choose \mathbf{h} and \mathbf{l} in $W^{2,\infty}(\Xi, \mu)$ and investigate $\mathcal{E}(\mathbf{p} \mathbf{h} \mathbf{L} \mathbf{l}) = (1/2) \sum_{i,j=1}^d \mathcal{E}(\mathbf{p} a_{i,j} \mathbf{h} D_{i,j}^2 \mathbf{l})$. It is clear that $\mathbf{h} \mathbf{l} \in W^{2,\infty}(\Xi, \mu)$ and that $D_{i,j}^2(\mathbf{h} \mathbf{l}) = \mathbf{h} D_{i,j}^2 \mathbf{l} + \mathbf{l} D_{i,j}^2 \mathbf{h} + D_i \mathbf{h} D_j \mathbf{l} + D_j \mathbf{h} D_i \mathbf{l}$. Since $\mathbf{L}^*(\mathbf{p}) = 0$, it comes $\mathcal{E}(\mathbf{p} \mathbf{h} \mathbf{L} \mathbf{l}) = -\mathcal{E}(\mathbf{p}(\mathbf{L} \mathbf{h}) \mathbf{l}) - \mathcal{E}(\mathbf{p} \langle a D \mathbf{h}, D \mathbf{l} \rangle)$. Apply the integration by parts (6.1) to complete the proof. \square

END OF THE PROOF OF THEOREM 6.2. The first integral writing of $\mathbf{v}(\cdot; \varepsilon)$ (see (6.2)) directly follows from Proposition 6.5 and Lemma 4.1 (to identify $\mathbf{p}(\xi)$ with $p(0, \xi)$ in $R_t \mathbf{h}$). The second writing follows from an obvious change of variable.

We now prove, for a given $\varepsilon > 0$, that $\mathbf{v}(\cdot; \varepsilon)$ belongs to $H(\Xi, \mathbf{p}, \mu)$. Assume first that \mathbf{h} belongs to the space \mathcal{S}_{∞} and show that $v(x; \varepsilon, \xi) \equiv \mathbf{v}(\mathbb{T}_x \xi; \varepsilon)$ is a strong solution to the realization of the resolvent equation.

Referring to (4.1) and to the proof of Proposition 4.11, for every $\xi \in \Xi$, the operator $p^{-1}(\cdot, \xi)(\mathcal{L}^{\xi})^*(p(\cdot, \xi) \cdot)$ develops in a non-divergence form, with a as diffusion matrix, with a bounded and Hölder continuous drift and with a null zero order term. Therefore, there exists a bounded classical solution w to the equation $-p^{-1}(x, \xi)(\mathcal{L}^{\xi})^*(p(x, \xi)w(x)) + \varepsilon^2 w(x) = h(x, \xi)$ (see Theorems 4.3.1 and 4.3.2 in Krylov [18]). Since the first order term of the PDE admits Hölder continuous derivatives, w admits bounded derivatives of order one, two and three (refer to the Schauder theory, Friedman [11], Chapter III, Section 3) and the common bound for w , $\nabla_x w$, $\nabla_{x,x}^2 w$ and $\nabla_{x,x,x}^3 w$ does not depend on ξ . Moreover, due to Theorem 15, Chapter I, in Friedman [11], w

writes:

$$(6.8) \quad w(x) = \int_0^{+\infty} \left[\exp(-\varepsilon^2 t) \int_{\mathbb{R}^d} h(y, \xi) p(y, \xi) p^{-1}(x, \xi) q(t, y, x; \xi) dy \right] dt.$$

Comparing (6.8) to (6.2), $v(\cdot; \varepsilon, \xi)$ admits for a.e. $\xi \in \Xi$ a smooth version that satisfies the realization of the resolvent equation in a strong sense. In the sequel, we identify $v(\cdot; \varepsilon, \xi)$ with this version, so that, for every $\xi \in \Xi$, $v(\cdot; \varepsilon, \xi)$ is three times differentiable with bounded derivatives. The bound for the derivatives does not refer to ξ . Following Lemma 4.1, for every $x \in \mathbb{R}^d$, $v(x; \varepsilon, \xi)$ and $\mathbf{v}(\mathbb{T}_x \xi; \varepsilon)$ coincide for a.e. $\xi \in \Xi$.

Thanks to the mean value theorem, it is then well seen that \mathbf{v} (and $D\mathbf{v}$) belongs to $\cap_{q>1} W^{1,q}(\Xi, \mu)$ and to $H(\Xi, \mathbf{p}, \mu)$. Of course, $D\mathbf{v}$ and $D^2\mathbf{v}$ are bounded, so that $\mathbf{v} \in \mathcal{D}_{\mathbf{p}}(\mathbf{L}) \cap \mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot))$. Proposition 6.5 applies:

$$\mathcal{E}(\mathbf{p}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{v}))\mathbf{v}) = \mathcal{E}(\mathbf{p}\mathbf{v}\mathbf{L}(\mathbf{v})).$$

Standard computations show that $\mathbf{L}(\mathbf{v}^2) = 2\mathbf{v}\mathbf{L}(\mathbf{v}) + \langle \mathbf{a}D\mathbf{v}, D\mathbf{v} \rangle$. Since \mathbf{p} satisfies $\mathbf{L}^*(\mathbf{p}) = 0$, this proves that:

$$(6.9) \quad \mathcal{E}(\mathbf{p}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{v}))\mathbf{v}) = \mathcal{E}(\mathbf{p}\mathbf{v}\mathbf{L}(\mathbf{v})) = -(1/2)\mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{v}, D\mathbf{v} \rangle).$$

Plug (6.9) into the resolvent equation in Theorem 6.2 to get (6.3).

It now remains to conclude when $\mathbf{h} \notin \mathcal{S}_\infty$. In this case, there exists a sequence $(\mathbf{h}_n)_{n \geq 0} \in \mathcal{S}_\infty^{\mathbb{N}}$ converging to \mathbf{h} in $L^2(\Xi, \mathbf{p}, \mu)$. Since $(\varepsilon^2 I - \mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot))^{-1}$ is a bounded operator (I denotes the identity), the regularized solutions $\mathbf{v}_n(\cdot; \varepsilon)$ to the resolvent equation converge in the same space towards $\mathbf{v}(\cdot; \varepsilon)$.

Thanks to (6.3) applied to the sequence $(\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$, deduce that the sequence $(D\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$ is bounded in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$. Thanks to the Mazur theorem, there exists a sequence of convex combinations of $(D\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$, still denoted by $(D\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$, that converges in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$. The limit is denoted by $(\mathbf{w}_1, \dots, \mathbf{w}_d)$. Of course, the re-indexed sequences $(\mathbf{h}_n)_{n \geq 0}$ and $(\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$ still converge towards \mathbf{h} and $\mathbf{v}(\cdot; \varepsilon)$ (and still satisfy the resolvent equation).

The strategy now consists in identifying \mathbf{w} with $D\mathbf{v}$. We seek first for uniform controls for $D\mathbf{v}_n(\cdot; \varepsilon)$ and $D^2\mathbf{v}_n(\cdot; \varepsilon)$. Apply to this end Theorem 9.11 in Gilbarg and Trudinger [13] and deduce from Proposition 4.10 (to obtain the second term in the r.h.s. below):

$$(6.10) \quad \begin{aligned} & \int_{B(0,1/2)} |\nabla_{x,x} v_n(x; \varepsilon, \xi)|^2 dx \\ & \leq Cp^{-1}(0, \xi) \left[\int_{B(0,1)} (v_n^2(x; \varepsilon, \xi) + h_n^2(x, \xi)) p(x, \xi) dx \right]. \end{aligned}$$

Complete now the proof. For $i \in \{1, \dots, d\}$ and $\delta \in \mathbb{R} \setminus \{0\}$:

$$\begin{aligned}
& \mathcal{E}[\mathbf{p} |\delta^{-1}(\mathbf{v}(\mathbb{T}_{\delta e_i} \cdot; \varepsilon) - \mathbf{v}(\cdot; \varepsilon)) - \mathbf{w}_i|^2] \\
&= \lim_{n \rightarrow +\infty} \mathcal{E}[\mathbf{p} |\delta^{-1}(\mathbf{v}_n(\mathbb{T}_{\delta e_i} \cdot; \varepsilon) - \mathbf{v}_n(\cdot; \varepsilon)) - D_i \mathbf{v}_n(\cdot; \varepsilon)|^2] \\
&= 4^d V_d^{-1} \lim_{n \rightarrow +\infty} \mathcal{E} \left[\int_{B(0,1/4)} p(x, \xi) \right. \\
&\quad \left. \times |\delta^{-1}(v_n(x + \delta e_i; \varepsilon, \xi) - v_n(x; \varepsilon, \xi)) - \frac{\partial v_n}{\partial x_i}(x; \varepsilon, \xi)|^2 dx \right] \\
&\leq 4^d V_d^{-1} \limsup_{n \rightarrow +\infty} \mathcal{E} \left[\int_{B(0,1/4)} p(x, \xi) \right. \\
&\quad \left. \times \left[\int_0^1 \left[\frac{\partial v_n}{\partial x_i}(x + \lambda \delta e_i; \varepsilon, \xi) - \frac{\partial v_n}{\partial x_i}(x; \varepsilon, \xi) \right]^2 d\lambda \right] dx \right] \\
&\leq 4^d V_d^{-1} \delta \limsup_{n \rightarrow +\infty} \mathcal{E} \left[\int_{B(0,1/4)} p(x, \xi) \right. \\
&\quad \left. \times \left[\int_0^1 \int_0^1 |\nabla_{x,x}^2 v_n|^2(x + \lambda \mu \delta e_i; \varepsilon, \xi) d\lambda d\mu \right] dx \right].
\end{aligned}$$

Apply again Proposition 4.10 to bound $p(x, \xi)$ by $Cp(0, \xi)$ (at least for δ small) and (6.10) to bound the partial derivatives of order two:

$$\begin{aligned}
(6.11) \quad & \mathcal{E}[\mathbf{p} |h^{-1}(\mathbf{v}(\mathbb{T}_{\delta e_i} \cdot; \varepsilon) - \mathbf{v}(\cdot; \varepsilon)) - \mathbf{w}_i|^2] \\
& \leq C\delta \limsup_{n \rightarrow +\infty} \mathcal{E}[\mathbf{p} (|\mathbf{v}_n|^2(\cdot; \xi) + |\mathbf{h}_n|^2)].
\end{aligned}$$

From (6.3), bound the r.h.s. in (6.11) and deduce that $\mathbf{v}(\cdot; \varepsilon) \in H(\Xi, \mathbf{p} \cdot \mu)$. Apply again (6.3) to the (re-indexed sequences) $(\mathbf{h}_n)_{n \geq 0}$, $(\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$ and $(D\mathbf{v}_n(\cdot; \varepsilon))_{n \geq 0}$ and deduce that (6.3) holds for $\mathbf{v}(\cdot; \varepsilon)$. \square

6.3. Strategy to Prove Theorem 6.3. Consider first, for a given $\varepsilon > 0$, the realization of the resolvent equation $-p^{-1}(x, \xi)(\mathcal{L}^\xi)^*(p(x, \xi)w(x)) + \varepsilon^2 w(x) = f(x, \xi)$. Since $f(x, \xi)$ is not smooth in x , we cannot solve this PDE in a strong sense as in the previous discussion. Considering a regularizing sequence for the bounded function $f(x, \xi)$, we can still build a sequence of regularized solutions with uniform bounds in $\cap_{q \geq 1} W_{\text{loc}}^{2,q}(\mathbb{R}^d)$ (apply again Theorem 9.11 in Gilbarg and Trudinger [13]). Extracting a converging subsequence, this provides a bounded continuous solution in $\cap_{q \geq 1} W_{\text{loc}}^{2,q}(\mathbb{R}^d)$ to the realization of the resolvent equation (due to the Sobolev imbedding theorems, its gradient is also bounded and continuous). Passing to the limit in (6.8), the resulting solution writes:

$$(6.12) \quad w(x) = \int_0^{+\infty} \left[\exp(-\varepsilon^2 t) \int_{\mathbb{R}^d} f(y, \xi) p(y, \xi) p^{-1}(x, \xi) q(t, y, x; \xi) dy \right] dt.$$

Up the choice of a version for the stationary field \mathbf{f} , (6.2) and (6.12) permit to define, for every $\varepsilon > 0$, a smooth version (*i.e.* bounded and continuous with a bounded and continuous gradient and with second order derivatives in $\cap_{q \geq 1} L^q_{\text{loc}}(\mathbb{R}^d)$) for the realization of the field $\mathbf{u}(\cdot; \varepsilon)$. The resulting mapping $u(\cdot; \varepsilon, \cdot) : \mathbb{R}^d \times \Xi \rightarrow \mathbb{R}$ is jointly measurable in (x, ξ) and satisfies the realization of the resolvent equation for every $\xi \in \Xi$. Following Lemma 4.1, for every $x \in \mathbb{R}^d$, $u(x; \varepsilon, \xi)$ and $\mathbf{u}(\mathbb{T}_x \xi; \varepsilon)$ coincide for almost every ξ .

Consider now a given smooth cut-off function η from \mathbb{R}^d into $[0, 1]$, matching 1 on $B(0, r)$ and vanishing outside $B(0, r + 1)$, $r > 0$, and focus on the product $u(y; \varepsilon, \xi)\eta(\varepsilon y)$, the environment ξ being fixed. From standard computations, we claim that the cut-off solution satisfies:

$$\begin{aligned}
(6.13) \quad & -p^{-1}(x, \xi)(\mathcal{L}^\xi)^*[p(x, \xi)u(x; \varepsilon, \xi)\eta(\varepsilon x)] + \varepsilon^2 u(x; \varepsilon, \xi)\eta(\varepsilon x) \\
& = f(x, \xi)\eta(\varepsilon x) \\
& \quad - \varepsilon \langle a(x, \xi) \nabla_x u(x; \varepsilon, \xi), \nabla_x \eta(\varepsilon x) \rangle \\
& \quad - \varepsilon \langle B(x, \xi), \nabla_x \eta(\varepsilon x) \rangle u(x; \varepsilon, \xi) \\
& \quad - \frac{1}{2} \varepsilon^2 \langle a(x, \xi), \nabla_{x,x}^2 \eta(\varepsilon x) \rangle u(x; \varepsilon, \xi) \\
& \equiv f(x, \xi)\eta(\varepsilon x) + m(x; \varepsilon, \xi),
\end{aligned}$$

with $B_j(x, \xi) \equiv p^{-1}(x, \xi) \sum_{i=1}^d \partial[a_{i,j} p] / \partial x_i(x, \xi)$, $1 \leq j \leq d$.

As above, the product $u(x; \varepsilon, \xi)\eta(\varepsilon x)$ writes:

$$\begin{aligned}
(6.14) \quad u(x; \varepsilon, \xi)\eta(\varepsilon x) &= \int_0^{+\infty} \left[\exp(-\varepsilon^2 t) \int_{\mathbb{R}^d} (f(y, \xi)\eta(\varepsilon y) + m(y; \varepsilon, \xi)) \right. \\
& \quad \left. \times p(y, \xi)p^{-1}(x, \xi)q(t, y, x; \xi)dy \right] dt.
\end{aligned}$$

Deduce from an obvious change of variable in (t, x, y) :

$$\begin{aligned}
(6.15) \quad & u(\varepsilon^{-1}x; \varepsilon, \xi)\eta(x) \\
& = \varepsilon^{-2} \int_0^{+\infty} \left[\exp(-t) \int_{\mathbb{R}^d} (f(\varepsilon^{-1}y, \xi)\eta(y) + m(\varepsilon^{-1}y; \varepsilon, \xi)) \right. \\
& \quad \left. \times p(\varepsilon^{-1}y, \xi)p^{-1}(\varepsilon^{-1}x, \xi)\bar{q}(t, y, x; \varepsilon, \xi)dy \right] dt.
\end{aligned}$$

Assume for the moment the following results:

PROPOSITION 6.7. *Define for $\xi \in \Xi$ and $(t, x) \in [0, +\infty[\times \mathbb{R}^d$:*

$$\begin{aligned}
\bar{m}(t, x; \varepsilon, \xi) &\equiv \varepsilon^{-1} \int_0^t \exp(-s) \int_{\mathbb{R}^d} [f(\varepsilon^{-1}y, \xi)\eta(y) + m(\varepsilon^{-1}y; \varepsilon, \xi)] \\
& \quad \times p(\varepsilon^{-1}y, \xi)p^{-1}(\varepsilon^{-1}x, \xi)\bar{q}(s, y, x; \varepsilon, \xi)dy ds.
\end{aligned}$$

Then, for every $r > 0$ (recall that $r + 1$ denotes the radius of the support of η) and for every $\delta > 0$, there exists $\alpha(\delta, r) > 0$ such that:

$$\forall \varepsilon > 0, \forall t \leq \alpha(\delta, r), \mu \left\{ \int_{\mathbb{R}^d} \bar{m}^2(t, x; \varepsilon, \xi) p(\varepsilon^{-1}x, \xi) dx > \delta \right\} \leq \delta.$$

PROPOSITION 6.8. For every compact subset $\mathcal{K} \subset \mathbb{R}^d$ and for every $\delta > 0$, there exists $r(\delta) > 0$, such that:

$$\begin{aligned} & \forall \alpha > 0, \forall r > r(\delta), \exists \varepsilon(\delta, \alpha, r) > 0, \forall \varepsilon < \varepsilon(\delta, \alpha, r), \\ & \mu \left\{ \varepsilon^{-1} \sup_{x \in \mathcal{K}} \left| \int_{\alpha}^{+\infty} \exp(-t) \int_{\mathbb{R}^d} [f(\varepsilon^{-1}y, \xi) \eta(y) + m(\varepsilon^{-1}y; \varepsilon, \xi)] \right. \right. \\ & \quad \left. \left. \times p(\varepsilon^{-1}y, \xi) p^{-1}(\varepsilon^{-1}x, \xi) \bar{q}(t, y, x; \varepsilon, \xi) dy dt \right| > \delta \right\} \leq \delta. \end{aligned}$$

PROOF OF THEOREM 6.3. Fix a compact set $\mathcal{K} \subset \mathbb{R}^d$ and a real $\delta > 0$ and choose r large enough such that $B(0, r) \supset \mathcal{K}$ and $r > r(\delta)$ in Proposition 6.8. Choose then $\alpha < \alpha(\delta, r)$ in Proposition 6.7. Choose finally $\varepsilon < \varepsilon(\delta, \alpha, r)$ in Proposition 6.8. To conclude, note from (6.15) that, for every $x \in \mathcal{K}$, $\varepsilon u(\varepsilon^{-1}x; \varepsilon, \xi)$ writes as the sum of the terms considered in Propositions 6.7 and 6.8. \square

6.4. *Proof of Proposition 6.7.* Consider $\xi \in \Xi$, $\varepsilon > 0$ and $\alpha \in]0, 1]$. Remove for the sake of simplicity the index ξ in the notations \bar{m} and m . It is then well seen that \bar{m} satisfies in $\cap_{q \geq 1} W_{\text{loc}}^{1,2,q}([0, \alpha] \times \mathbb{R}^d)$ the PDE (see Theorem 9.1, Chapter IV, in Ladyzhenskaya et al. [21] and Theorem 15, Chapter I, in Friedman [11]):

$$(6.16) \quad \begin{aligned} & \frac{\partial \bar{m}(t, x; \varepsilon)}{\partial t} - \frac{1}{2} p^{-1}(\varepsilon^{-1}x) \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} [(a_{i,j} p)(\varepsilon^{-1}x) \bar{m}(t, x; \varepsilon)] \\ & + \bar{m}(t, x; \varepsilon) = \varepsilon^{-1} [f(\varepsilon^{-1}x) \eta(x) + m(\varepsilon^{-1}x; \varepsilon)], \quad (t, x) \in]0, \alpha] \times \mathbb{R}^d. \end{aligned}$$

Referring to (6.13), the function $f\eta(\varepsilon \cdot) + m(\cdot; \varepsilon)$ is bounded by a constant $C_\varepsilon > 0$ depending on ε (indeed, f is bounded and m is continuous with a compact support) and vanishes outside $B(0, r + 1)$. From Propositions 2.1 and 4.10, the quantity $|\bar{m}(t, x; \varepsilon)|$ can be bounded by $C_\varepsilon \exp(C_\varepsilon(r + |x|)) \int_0^t \int_{B(0, r+1)} s^{-d/2} \exp(-C_\varepsilon^{-1} s^{-1} |x - y|^2) dy ds$. For $|x| > 2(r + 1)$, every $y \in B(0, r + 1)$ satisfies $|x - y| > |x|/2$. Hence,

$$\begin{aligned} |\bar{m}(t, x; \varepsilon)| & \leq C_\varepsilon \exp(C_\varepsilon(r + |x|) - C_\varepsilon^{-1} |x|^2/8) \\ & \quad \times \int_0^t s^{-d/2} \int_{B(0, r+1)} \exp(-C_\varepsilon^{-1} s^{-1} |x - y|^2/2) dy \\ & \leq C_\varepsilon \exp(C_\varepsilon(r + |x|) - C_\varepsilon^{-1} |x|^2/8). \end{aligned}$$

From the gradient estimate in Friedman [11], Chapter VII, Section 2, Theorem 4, similar bounds hold for large x for the first x -derivatives of $\bar{m}(\cdot, \cdot; \varepsilon)$ (recall from **(A2)** that $a(x, \xi)$ is smooth in x). Due to the exponential growth of p , we can multiply both sides in (6.16) by $\bar{m}(t, x; \varepsilon)p(\varepsilon^{-1}x)$ and integrate with respect to (t, x) . An integration by parts yields:

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^d} \bar{m}^2(t, x; \varepsilon)p(\varepsilon^{-1}x)dx - \frac{1}{2} \int_{\mathbb{R}^d} \bar{m}^2(0, x; \varepsilon)p(\varepsilon^{-1}x)dx \\
& + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} [(a_{i,j}p)(\varepsilon^{-1}x)\bar{m}(s, x; \varepsilon)] \frac{\partial \bar{m}}{\partial x_j}(s, x; \varepsilon)dx ds \\
(6.17) \quad & + \int_0^t \int_{\mathbb{R}^d} \bar{m}^2(s, x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& = \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^d} \bar{m}(s, x; \varepsilon)m(\varepsilon^{-1}x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& + \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^d} \bar{m}(s, x; \varepsilon)(fp)(\varepsilon^{-1}x)\eta(x)dx ds
\end{aligned}$$

Since $\mathcal{L}^*(p) = 0$, the second order term in (6.17) writes $\frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} (a_{i,j}p)(\varepsilon^{-1}x)(\partial \bar{m} / \partial x_i)(s, x; \varepsilon)(\partial \bar{m} / \partial x_j)(s, x; \varepsilon)dx ds$. Thanks to **(A.1)** (ellipticity of a), deduce from the boundary condition $\bar{m}(0, x) = 0$ that:

$$\begin{aligned}
T(t; \xi) & \equiv \frac{1}{2} \int_{\mathbb{R}^d} \bar{m}^2(t, x; \varepsilon)p(\varepsilon^{-1}x)dx \\
& + C^{-1} \int_0^t \int_{\mathbb{R}^d} p(\varepsilon^{-1}x)|\nabla_x \bar{m}|^2(s, x; \varepsilon)dx ds \\
(6.18) \quad & + \int_0^t \int_{\mathbb{R}^d} \bar{m}^2(s, x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& \leq \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^d} \bar{m}(s, x; \varepsilon)m(\varepsilon^{-1}x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& + \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^d} \bar{m}(s, x; \varepsilon)(fp)(\varepsilon^{-1}x)\eta(x)dx ds.
\end{aligned}$$

Recall that m is given by (6.13). Plug (6.13) into (6.18) and deduce:

$$\begin{aligned}
T(t; \xi) & \leq C \int_0^t \int_{\mathbb{R}^d} |\nabla_x u|(\varepsilon^{-1}x; \varepsilon)|\nabla_x \eta|(x)|\bar{m}|(s, x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& + C\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|(\varepsilon^{-1}x; \varepsilon)|\nabla_{x,x}^2 \eta|(x)|\bar{m}|(s, x; \varepsilon)p(\varepsilon^{-1}x)dx ds \\
& - \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (a_{i,j}p)(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_j}(x)u(\varepsilon^{-1}x; \varepsilon)\bar{m}(s, x; \varepsilon)dx ds \\
& + \varepsilon^{-1} \int_0^t \int_{\mathbb{R}^d} \bar{m}(s, x; \varepsilon)(fp)(\varepsilon^{-1}x)\eta(x)dx ds.
\end{aligned}$$

Due to the statement of Theorem 3.2, for a.e. $\xi \in \Xi$, $pg \in W_{\text{loc}}^{1,d/(d-1)}(\mathbb{R}^d)$ and $(pf)(x, \xi) = \partial(pg)/\partial x_\ell(x, \xi)$ almost everywhere in x . Deduce from two integrations by parts:

$$\begin{aligned}
(6.19) \quad T(t; \xi) &\leq C \int_0^t \int_{\mathbb{R}^d} |\nabla_x u|(\varepsilon^{-1}x; \varepsilon) |\nabla_x \eta|(x) |\bar{m}|(s, x; \varepsilon) p(\varepsilon^{-1}x) dx ds \\
&\quad + C\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|(\varepsilon^{-1}x; \varepsilon) |\nabla_{x,x}^2 \eta|(x) |\bar{m}|(s, x; \varepsilon) p(\varepsilon^{-1}x) dx ds \\
&\quad + C\varepsilon \int_0^t \int_{\mathbb{R}^d} |u|(\varepsilon^{-1}x; \varepsilon) |\nabla_x \eta|(x) |\nabla_x \bar{m}|(s, x; \varepsilon) p(\varepsilon^{-1}x) dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} |\nabla_x \bar{m}|(s, x; \varepsilon) (|g|p)(\varepsilon^{-1}x) \eta(x) dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} |\bar{m}|(s, x; \varepsilon) (|g|p)(\varepsilon^{-1}x) |\nabla_x \eta|(x) dx ds.
\end{aligned}$$

Apply the standard inequality $2yz \leq y^2 + z^2$ to every term in the above r.h.s. and remove as a bypass product the gradient terms in \bar{m} in (6.19) (compare with $T(t; \xi)$ in (6.18)). Apply then the Gronwall lemma to remove the \bar{m} term in the r.h.s. and take finally the expectation w.r.t $\xi \in \Xi$. Derive that there exists a constant C such that:

$$\begin{aligned}
&\mathcal{E} \left[\int_{\mathbb{R}^d} \bar{m}^2(t, x; \varepsilon, \xi) p(\varepsilon^{-1}x, \xi) dx \right] \\
&\leq C\alpha \exp(C\alpha) [\mathcal{E}[\mathbf{p}|\mathbf{g}|^2] + \mathcal{E}[\mathbf{p}|D\mathbf{u}(\cdot; \varepsilon)|^2] + \varepsilon^2 \mathcal{E}[\mathbf{p}|\mathbf{u}(\cdot; \varepsilon)|^2]] \\
&\quad \times \int_{\mathbb{R}^d} [|\eta(x)|^2 + |\nabla_x \eta(x)|^2 + |\nabla_{x,x}^2 \eta(x)|^2] dx.
\end{aligned}$$

From (6.4), this clearly completes the proof. \square

6.5. *Proof of Proposition 6.8.* Start first with the following term:

LEMMA 6.9. *The term :*

$$\begin{aligned}
S(y; \varepsilon, \xi) &\equiv \varepsilon^{-1} \int_{\alpha}^{+\infty} \left[\exp(-t) \int_{\mathbb{R}^d} f(\varepsilon^{-1}x, \xi) \eta(x) \right. \\
&\quad \left. \times p(\varepsilon^{-1}x, \xi) p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi) dx \right] dt
\end{aligned}$$

writes $S(y; \varepsilon, \xi) \equiv S(y; 1, \varepsilon, \xi) + S(y; 2, \varepsilon, \xi)$, with

- a) $\forall r > 0, \forall \alpha > 0, \sup_{y \in \mathcal{K}} |S(y; 1, \varepsilon, \xi)| \xrightarrow{(\mathbf{p}.)\mu\text{-probability}} 0,$
- b) $\forall r > 0, \forall \alpha > 0, \mu\text{-a.s.}, \limsup_{\varepsilon \rightarrow 0} \sup_{y \in \mathcal{K}} |S(y; 2, \varepsilon, \xi)| = 0.$

PROOF. Since $(pf)(x, \xi) = \partial(pg)/\partial x_\ell(x, \xi)$ almost everywhere in x , derive from an integration by parts that (remove the index ξ):

$$\begin{aligned} S(y; \varepsilon) &= - \int_\alpha^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \frac{\partial}{\partial x_\ell} (p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon)) dx dt \\ &\quad - \int_\alpha^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \frac{\partial \eta}{\partial x_\ell}(x)(gp)(\varepsilon^{-1}x) p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon) dx dt \\ &\equiv S(y; 0, 1, \varepsilon) + S(y; 0, 2, \varepsilon). \end{aligned}$$

Start with $S(y; 0, 1, \varepsilon)$

$$\begin{aligned} &- S(y; 0, 1, \varepsilon) \\ &= \int_\alpha^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \left[\frac{\partial}{\partial x_\ell} (p^{-1}(\varepsilon^{-1}y)\bar{q}(s, x, y; \varepsilon)) \right. \\ &\quad \left. - \frac{\partial \Gamma}{\partial x_\ell}(t, x, y) \right] dx dt \\ &\quad + \int_\alpha^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \frac{\partial \Gamma}{\partial x_\ell}(t, x, y) dx dt \\ &\equiv S(y; 0, 1, 1, \varepsilon) + S(y; 0, 1, 2, \varepsilon). \end{aligned}$$

Due to Proposition 5.4 (convergence of the gradients), note from the Cauchy-Schwarz inequality that:

$$(6.20) \quad \forall r > 0, \forall \alpha > 0, \sup_{y \in \mathcal{K}} |S(y; 0, 1, 1, \varepsilon)| \xrightarrow{\mu\text{-probability}} 0.$$

Turn now to $S(y; 0, 1, 2, \varepsilon)$. Since the kernel Γ is Gaussian, derive a bound for the first derivatives of Γ similar to (5.15). Hence, the ergodic theorem yields μ -a.s., for all $t \geq \alpha$ and $y \in \mathcal{K}$ (with $R \equiv \sup_{y \in \mathcal{K}} |y|$):

$$\begin{aligned} (6.21) \quad &\sup_{\varepsilon \in]0, 1]} \left| \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \frac{\partial \Gamma}{\partial x_\ell}(t, x, y) dx \right| \\ &\leq Ct^{-d/2-1} \sup_{\varepsilon \in]0, 1]} \int_{\mathbb{R}^d} \eta(x)(|x| + R)p(\varepsilon^{-1}x) dx \\ &\leq C\alpha^{-d/2-1}(R + r + 1) \sup_{\varepsilon \in]0, 1]} \int_{B(0, r+1)} p(\varepsilon^{-1}x) dx < +\infty. \end{aligned}$$

Hence, the ergodic theorem (applied to a sequence of simple functions approximating the compact support function $\eta \partial \Gamma / \partial x_\ell$), the dominated con-

vergence theorem and (6.21) yield for every $y \in \mathcal{K}$ and μ -a.s.:

$$(6.22) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\alpha}^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \frac{\partial \Gamma}{\partial x_{\ell}}(t, x, y) dx dt \\ &= \mathcal{E}(\mathbf{g}\mathbf{p}) \int_{\alpha}^{+\infty} \exp(-t) \int_{\mathbb{R}^d} \eta(x) \frac{\partial \Gamma}{\partial x_{\ell}}(t, x, y) dx dt = 0. \end{aligned}$$

Note in fact that the convergence is uniform in $y \in \mathcal{K}$, since the derivatives of $S(y; 0, 1, 2, \varepsilon)$ are controlled in the following way (recall that $\partial^2 \Gamma / \partial y_i \partial x_{\ell} = -\partial^2 \Gamma / \partial x_i \partial x_{\ell}$ and apply (5.15)): μ -a.s., for all $t \geq \alpha$ and $y \in \mathcal{K}$:

$$(6.23) \quad \begin{aligned} & \sup_{\varepsilon \in]0,1]} \left| \int_{\mathbb{R}^d} \eta(x)(gp)(\varepsilon^{-1}x) \frac{\partial^2 \Gamma}{\partial x_i \partial x_{\ell}}(t, x, y) dx \right| \\ & \leq Ct^{-d/2-1} \sup_{\varepsilon \in]0,1]} \int_{\mathbb{R}^d} \eta(x) [t^{-1}(|x|^2 + R^2) + 1] p(\varepsilon^{-1}x) dx \\ & \leq C\alpha^{-d/2-2} \sup_{\varepsilon \in]0,1]} \int_{B(0, r+1)} (r^2 + R^2 + 1 + \alpha) p(\varepsilon^{-1}x) dx < +\infty. \end{aligned}$$

Hence, $S(y; 0, 1, 2, \varepsilon)$ satisfies $b)$ in the statement of Lemma 6.9. Note that $S(y; 0, 1, 2, \varepsilon)$ satisfies a weaker version of $b)$ if the centering condition $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$ fails (since the integral of the gradient of Γ over \mathbb{R}^d vanishes, $S(y; 0, 1, 2, \varepsilon)$ is small for large values of r). This weaker form is still sufficient to establish Theorem 6.3 so that the centering condition is not crucial at this stage of the proof.

The term $S(y; 0, 2, \varepsilon)$ is treated in a similar way (use the last assertion in Proposition 5.4). \square

LEMMA 6.10. *The term*

$$\begin{aligned} T(y; \varepsilon, \xi) \equiv & \varepsilon^{-1} \int_{\alpha}^{+\infty} \left[\exp(-t) \int_{\mathbb{R}^d} m(\varepsilon^{-1}x; \varepsilon, \xi) \right. \\ & \left. \times p(\varepsilon^{-1}x, \xi) p^{-1}(\varepsilon^{-1}y, \xi) \bar{q}(t, x, y; \varepsilon, \xi) dx \right] dt, \end{aligned}$$

writes $T(y; \varepsilon, \xi) \equiv T(y; 1, \varepsilon, \xi) + T(y; 2, \varepsilon, \xi)$, with:

- a) $\forall r > 0, \forall \alpha > 0, \sup_{y \in \mathcal{K}} |T(y; 1, \varepsilon, \xi)| \xrightarrow{(\mathbf{p}.)\mu\text{-probability}} 0,$
b) $\forall \delta > 0, \exists r(\delta) > 0, \forall r > r(\delta), \forall \alpha > 0, \forall \varepsilon > 0, \mathcal{E} \left[\sup_{y \in \mathcal{K}} |T(y; 2, \varepsilon, \xi)| \right] \leq \delta.$

PROOF. Remove the index ξ . According to (6.13), write:

$$T(y; \varepsilon) \equiv - \int_{\alpha}^{+\infty} \exp(-t) [T(t, y; 1, \varepsilon) + T(t, y; 2, \varepsilon) + T(t, y; 3, \varepsilon)] dt,$$

with:

$$\begin{aligned}
T(t, y; 1, \varepsilon) &\equiv \int_{\mathbb{R}^d} [\langle a(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle \\
&\quad \times p(\varepsilon^{-1}x) p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon)] dx, \\
T(t, y; 2, \varepsilon) &\equiv \int_{\mathbb{R}^d} [\langle B(\varepsilon^{-1}x), \nabla_x \eta(x) \rangle \\
&\quad \times u(\varepsilon^{-1}x; \varepsilon) p(\varepsilon^{-1}x) p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon)] dx, \\
T(t, y; 3, \varepsilon) &\equiv \frac{1}{2} \varepsilon \int_{\mathbb{R}^d} [\langle a(\varepsilon^{-1}x), \nabla_{x,x}^2 \eta(x) \rangle \\
&\quad \times u(\varepsilon^{-1}x; \varepsilon) p(\varepsilon^{-1}x) p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon)] dx.
\end{aligned}$$

(First Step: $T(t, y; 1, \varepsilon)$.)

$$\begin{aligned}
&T(t, y; 1, \varepsilon) \\
&= \int_{\mathbb{R}^d} [\langle a(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle p(\varepsilon^{-1}x) \\
(6.24) \quad &\quad \times (p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) - \Gamma(t, x, y))] dx \\
&+ \int_{\mathbb{R}^d} \langle a(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle p(\varepsilon^{-1}x) \Gamma(t, x, y) dx \\
&\equiv T(t, y; 1, 1, \varepsilon) + T(t, y; 1, 2, \varepsilon).
\end{aligned}$$

Deal first with the second term in (6.24). Due to the Gaussian decay of Γ , we claim (with $R \equiv \sup_{y \in \mathcal{K}} |y|$):

$$\begin{aligned}
&|T(t, y; 1, 2, \varepsilon)| \\
&\leq C t^{-d/2} \\
(6.25) \quad &\times \int_{\mathbb{R}^d} |\nabla_x u(\varepsilon^{-1}x; \varepsilon)| |\nabla_x \eta(x)| p(\varepsilon^{-1}x) \exp(-C^{-1} t^{-1} |x - y|^2) dx \\
&\leq C t^{-d/2} \exp(C t^{-1} R^2) \\
&\times \int_{\mathbb{R}^d} |\nabla_x u(\varepsilon^{-1}x; \varepsilon)| |\nabla_x \eta(x)| p(\varepsilon^{-1}x) \exp(-C^{-1} t^{-1} |x|^2) dx.
\end{aligned}$$

Take the supremum over \mathcal{K} in the l.h.s. in (6.25) and then integrate with respect to $\xi \in \Xi$:

$$\begin{aligned}
\mathcal{E}[\sup_{y \in \mathcal{K}} |T(t, y; 1, 2, \varepsilon)|] &\leq C \exp(C t^{-1} R^2) \mathcal{E}[|D\mathbf{u}(\cdot; \varepsilon)|_{\mathbf{p}}] \\
&\quad \times \int_{|x| > r} t^{-d/2} \exp(-C^{-1} t^{-1} |x|^2) dx.
\end{aligned}$$

Apply now the Bernstein inequality to estimate the Gaussian kernel:

$$\begin{aligned}
(6.26) \quad & \mathcal{E} \left[\sup_{y \in \mathcal{K}} |T(t, y; 1, 2, \varepsilon)| \right] \\
& \leq C \exp(Ct^{-1}R^2) \mathcal{E} [|D\mathbf{u}(\cdot; \varepsilon)|_{\mathbf{p}}] \exp(-C^{-1}t^{-1}r^2) \\
& = C \exp(t^{-1}(CR^2 - C^{-1}r^2)) \mathcal{E} [|D\mathbf{u}(\cdot; \varepsilon)|_{\mathbf{p}}].
\end{aligned}$$

Deduce from (6.26) that:

$$\begin{aligned}
(6.27) \quad & \mathcal{E} \left[\sup_{y \in \mathcal{K}} \left| \int_{\alpha}^{+\infty} \exp(-t) T(t, y; 1, 2, \varepsilon) dt \right| \right] \\
& \leq C \mathcal{E} [|D\mathbf{u}(\cdot; \varepsilon)|_{\mathbf{p}}] \int_0^{+\infty} \exp(-t + t^{-1}(CR^2 - C^{-1}r^2)) dt.
\end{aligned}$$

Due to (6.4), the last term in (6.27) vanishes as $r \rightarrow +\infty$, uniformly in ε , so that the term $\int_{\alpha}^{+\infty} \exp(-t) T(t, y; 1, 2, \varepsilon) dt$ satisfies *b*) in the statement of Lemma 6.10.

Focus then on $T(t, y; 1, 1, \varepsilon)$ in (6.24). Thanks to the Cauchy-Schwarz inequality:

$$\begin{aligned}
& \sup_{y \in \mathcal{K}} \left[\int_{\alpha}^{+\infty} \exp(-t) |T(t, y; 1, 1, \varepsilon)| dt \right] \\
& \leq C \left[\int_{\mathbb{R}^d} |\nabla_x u(\varepsilon^{-1}x; \varepsilon)|^2 |\nabla_x \eta(x)|^2 p(\varepsilon^{-1}x) dx \right]^{1/2} \\
& \quad \times \sup_{y \in \mathcal{K}} \left[\int_{\alpha}^{+\infty} \int_{\mathbb{R}^d} \exp(-t) [p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) \right. \\
& \quad \left. - \Gamma(t, x, y)]^2 p(\varepsilon^{-1}x) dx dt \right]^{1/2}.
\end{aligned}$$

Apply now Proposition 5.4. Thanks to (6.4), deduce that:

$$(6.28) \quad \sup_{y \in \mathcal{K}} \int_{\alpha}^{+\infty} \exp(-t) |T(t, y; 1, 1, \varepsilon)| dt \xrightarrow{\mu\text{-probability}} 0.$$

Deduce that $\int_{\alpha}^{+\infty} \exp(-t) T(t, y; 1, 1, \varepsilon) dt$ satisfies *a*) in the statement of Lemma 6.10.

(*Second Step: $T(t, y; 2, \varepsilon)$.)*

Deduce with an integration by parts that (refer to (6.13) for the definition

of B):

$$\begin{aligned}
& T(t, y; 2, \varepsilon) \\
&= -\varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} (a_{i,j}p)(\varepsilon^{-1}x) \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) u(\varepsilon^{-1}x; \varepsilon) p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) dx \\
&\quad - \sum_{i,j=1}^d \int_{\mathbb{R}^d} (a_{i,j}p)(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(\varepsilon^{-1}x; \varepsilon) p^{-1}(\varepsilon^{-1}y) \bar{q}(t, x, y; \varepsilon) dx \\
&\quad - \varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} (a_{i,j}p)(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_i}(x) u(\varepsilon^{-1}x; \varepsilon) p^{-1}(\varepsilon^{-1}y) \frac{\partial \bar{q}}{\partial x_j}(t, x, y; \varepsilon) dx \\
&\equiv T(t, y; 2, 1, \varepsilon) + T(t, y; 2, 2, \varepsilon) + T(t, y; 2, 3, \varepsilon).
\end{aligned}$$

Up to a multiplicative constant, $T(t, y; 2, 1, \varepsilon)$ matches $T(t, y; 3, \varepsilon)$. It is thus treated in the sequel. In the same way, $T(t, y; 2, 2, \varepsilon)$ is equal, up to a suitable factor, to $T(t, y; 1, \varepsilon)$. It thus remains to deal with $T(t, y; 2, 3, \varepsilon)$. Write:

$$\begin{aligned}
& -T(t, y; 2, 3, \varepsilon) \\
&= \varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} \left[(pa_{i,j})(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_i}(x) u(\varepsilon^{-1}x; \varepsilon) \right. \\
(6.29) \quad & \quad \left. \times \left[p^{-1}(\varepsilon^{-1}y) \frac{\partial \bar{q}}{\partial x_j}(t, x, y; \varepsilon) - \frac{\partial \Gamma}{\partial x_j}(t, x, y) \right] \right] dx \\
& \quad + \varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} (pa_{i,j})(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_i}(x) u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \Gamma}{\partial x_j}(t, x, y) dx \\
& \equiv T(t, y; 2, 3, 1, \varepsilon) + T(t, y; 2, 3, 2, \varepsilon).
\end{aligned}$$

Deal first with $T(t, y; 2, 3, 2, \varepsilon)$. Follow (6.25):

$$\begin{aligned}
& \sup_{y \in \mathcal{K}} |T(t, y; 2, 3, 2, \varepsilon)| \\
&\leq C\varepsilon \int_{\mathbb{R}^d} [p(\varepsilon^{-1}x) |\nabla_x \eta(x)| |u(\varepsilon^{-1}x; \varepsilon)| |x - y| t^{-d/2-1} \\
&\quad \times \exp(-C^{-1}t^{-1}|x - y|^2)] dx \\
&\leq C\varepsilon \exp(Ct^{-1}R^2) t^{-d/2-1} \int_{\mathbb{R}^d} [p(\varepsilon^{-1}x) |\nabla_x \eta(x)| |u(\varepsilon^{-1}x; \varepsilon)| (|x| + R) \\
&\quad \times \exp(-C^{-1}t^{-1}|x|^2)] dx \\
&\leq C\varepsilon \exp(Ct^{-1}R^2) (r + R + 1) t^{-d/2-1} \int_{\mathbb{R}^d} [p(\varepsilon^{-1}x) |\nabla_x \eta(x)| |u(\varepsilon^{-1}x; \varepsilon)| \\
&\quad \times \exp(-C^{-1}t^{-1}|x|^2)] dx.
\end{aligned}$$

Following (6.26),

$$\begin{aligned}
& \mathcal{E} \left[\sup_{y \in \mathcal{K}} |T(t, y; 2, 3, 2, \varepsilon)| \right] \\
& \leq C\varepsilon \mathcal{E}(\mathbf{p}|\mathbf{u}|(\cdot; \varepsilon)) \exp(Ct^{-1}R^2)(r + R + 1)t^{-d/2-1} \\
& \quad \times \int_{|x|>r} \exp(-C^{-1}t^{-1}|x|^2) dx \\
& \leq C\varepsilon \mathcal{E}(\mathbf{p}|\mathbf{u}|(\cdot; \varepsilon))(r + R + 1)t^{-1} \exp(t^{-1}(CR^2 - C^{-1}r^2)).
\end{aligned}$$

Deduce finally:

$$\begin{aligned}
(6.30) \quad & \mathcal{E} \left[\sup_{y \in \mathcal{K}} \int_{\alpha}^{+\infty} \exp(-t) |T(t, y; 2, 3, 2, \varepsilon)| dt \right] \\
& \leq C\varepsilon \mathcal{E}(\mathbf{p}|\mathbf{u}|(\cdot; \varepsilon))(r + R + 1) \\
& \quad \times \int_0^{+\infty} \exp(-t) t^{-1} \exp(t^{-1}(CR^2 - C^{-1}r^2)) dt.
\end{aligned}$$

Thanks to (6.4), the r.h.s. in (6.30) vanishes as $r \rightarrow +\infty$: the term $\int_{\alpha}^{+\infty} \exp(-t) T(t, y; 2, 3, 2, \varepsilon) dt$ satisfies *b* in the statement of Lemma 6.10.

Turn now to $T(t, y; 2, 3, 1, \varepsilon)$ in (6.29). Follow the strategy to handle $T(t, y; 1, 1, \varepsilon)$ in (6.24) (see (6.28)) and deduce that $\int_{\alpha}^{+\infty} \exp(-t) T(t, y; 2, 3, 1, \varepsilon) dt$ satisfies *a* in Lemma 6.10.

(Third Step: $T(t, y; 3)$.)

Note that the final term $T(t, y; 3, \varepsilon)$ is treated as $T(t, y; 1, \varepsilon)$ with the following difference: $\varepsilon \mathbf{u}(\cdot; \varepsilon)$ plays the role of $D\mathbf{u}(\cdot; \varepsilon)$ (and $\nabla_{x,x}^2 \eta$ the role of $\nabla_x \eta$). This completes the proof of Lemma 6.10 \square

(Conclusion.)

Complete now the proof of Proposition 6.8. Fix $\delta > 0$ and deduce from Lemma 6.9 that, for all $\alpha > 0$ and $r > 0$, $P(1) \equiv \mu\{\sup_{y \in \mathcal{K}} |S(y; \varepsilon, \xi)| > \delta\}$ vanishes as ε tends to zero. Choose now $r \geq r(\delta^2/2)$ in Lemma 6.10. We claim from the Markov inequality:

$$\begin{aligned}
(6.31) \quad & P(2) \equiv \mu\{\sup_{y \in \mathcal{K}} |T(y; \varepsilon, \xi)| > \delta\} \\
& \leq \mu\{\sup_{y \in \mathcal{K}} |T(y; 1, \varepsilon, \xi)| > \delta/4\} + \mu\{\sup_{y \in \mathcal{K}} |T(y; 2, \varepsilon, \xi)| > 3\delta/4\} \\
& \leq \mu\{\sup_{y \in \mathcal{K}} |T(y; 1, \varepsilon, \xi)| > \delta/4\} + (2/3)\delta.
\end{aligned}$$

Deduce, that for every $\alpha > 0$, $P(2)$ vanishes as ε tends to zero. \square

6.6. *Proof of Theorem 6.4.* The following notation is frequently used in this subsection: every term labelled with the symbol (\spadesuit) denotes a sequence that tends to zero in μ or $\mathbf{p}.\mu$ probability (recall that both are equivalent) and that is uniformly integrable in $L^1(\Xi, \mu)$. In particular, every term labelled with the symbol (\spadesuit) vanishes in $L^1(\Xi, \mu)$.

(*First Step.*)

The proof of Theorem 6.4 relies on the following lemma:

LEMMA 6.11. *Let θ be a convex even function from \mathbb{R} into $[0, +\infty[$ such that $\theta(0) = 0$, $\theta'(0) = 0$, $\theta''(z) = 1$ for $|z| \leq 1$, $0 \leq \theta''(z) \leq 1$ for $1 \leq |z| \leq 2$ and $\theta''(z) = 0$ for $|z| \geq 2$ (of course such a function does exist: integrate twice θ'') and let η be the smooth function from \mathbb{R}^d into $[0, 1]$ given by $\eta(x) = \exp(-1/(1 - |x|^2))$, $|x| < 1$ and $\eta(x) = 0$ for $|x| \geq 1$. Then, as $\varepsilon \rightarrow 0$,*

$$(6.32) \quad \begin{aligned} & \int_{\mathbb{R}^d} \langle a(\varepsilon^{-1}x, \xi) \nabla_x u(\varepsilon^{-1}x; \varepsilon, \xi), \nabla_x u(\varepsilon^{-1}x; \varepsilon, \xi) \rangle \\ & \times \eta^2(x) \theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon, \xi) \eta(x)) p(\varepsilon^{-1}x, \xi) dx \\ & + 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x, \xi) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon, \xi) \eta^2(x) dx \xrightarrow{L^1(\Xi, \mu)} 0. \end{aligned}$$

Assume for the moment that Lemma 6.11 holds. According to (6.3):

$$(6.33) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{u}(\cdot; \varepsilon), D\mathbf{u}(\cdot; \varepsilon) \rangle) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{u}(\cdot; \varepsilon), D\mathbf{u}(\cdot; \varepsilon) \rangle) \\ & \leq \limsup_{\varepsilon \rightarrow 0} [\mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{u}(\cdot; \varepsilon), D\mathbf{u}(\cdot; \varepsilon) \rangle) + 2\varepsilon^2 \mathcal{E}(\mathbf{p}\mathbf{u}^2(\cdot; \varepsilon))] \\ & = -2 \liminf_{\varepsilon \rightarrow 0} \mathcal{E}(\mathbf{p}\mathbf{g}D_\ell \mathbf{u}(\cdot; \varepsilon)), \end{aligned}$$

where the last equality follows from (6.1).

Since the family $D\mathbf{u}(\cdot; \varepsilon)$ is bounded in $L^2(\Xi, \mathbf{p}.\mu; \mathbb{R}^d)$, there exists $(\Psi_1, \dots, \Psi_d) \in L^2(\Xi, \mathbf{p}.\mu; \mathbb{R}^d)$ and $(\varepsilon_n)_{n \geq 0} \in]0, 1]^{\mathbb{N}}$, $\varepsilon_n \rightarrow 0$, such that $D\mathbf{u}(\cdot; \varepsilon_n) \rightharpoonup \Psi$ in the weak sense as $n \rightarrow +\infty$. In this frame, (6.33) still holds true with respect to $(\varepsilon_n)_{n \geq 0}$ and the last \liminf in (6.33) reduces to $\mathcal{E}(\mathbf{p}\mathbf{g}\Psi_\ell)$.

Take now the expectation \mathcal{E} in (6.32). Since θ'' is bounded by 1, it comes:

$$(6.34) \quad \begin{aligned} & \liminf_{n \rightarrow +\infty} \left[\mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{u}(\cdot; \varepsilon_n), D\mathbf{u}(\cdot; \varepsilon_n) \rangle) \int_{\mathbb{R}^d} \eta^2(x) dx \right] \\ & \geq -2\mathcal{E}(\mathbf{p}\mathbf{g}\Psi_\ell) \int_{\mathbb{R}^d} \eta^2(x) dx. \end{aligned}$$

Plug (6.34) into (6.33) and derive that $(\varepsilon_n^2 \mathcal{E}(\mathbf{p}u^2(\cdot; \varepsilon_n)))_{n \geq 0}$ vanishes. This completes the proof of Theorem 6.4. \square

(Second Step.)

It now remains to establish Lemma 6.11. Recall from (6.13) that, for all $\varepsilon > 0$ and $\xi \in \Xi$, $u(\varepsilon^{-1}x; \varepsilon, \xi)\eta(x)$ satisfies in $\cap_{q \geq 1} W^{2,q}(\mathbb{R}^d)$ a truncated version of the resolvent equation. Removing the index ξ , it comes:

$$\begin{aligned} & p^{-1}(\varepsilon^{-1}x)(\mathcal{L}^\varepsilon)^*(p(\varepsilon^{-1}x)\theta(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))) \\ &= \varepsilon\theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))p^{-1}(\varepsilon^{-1}x)(\mathcal{L}^\varepsilon)^*(p(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\ &+ \frac{\varepsilon^2}{2}\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\ &\quad \times \langle a(\varepsilon^{-1}x)\nabla_x(u(\varepsilon^{-1}x; \varepsilon)\eta(x)), \nabla_x(u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \rangle. \end{aligned}$$

Apply (6.15) to develop the first term in the r.h.s.:

$$\begin{aligned} & p^{-1}(\varepsilon^{-1}x)(\mathcal{L}^\varepsilon)^*(p(\varepsilon^{-1}x)\theta(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))) \\ &= \varepsilon^{-1}\theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))(-f(\varepsilon^{-1}x)\eta(x) - m(\varepsilon^{-1}x; \varepsilon) + \varepsilon^2 u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\ &+ \frac{1}{2}\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))\langle a(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x u(\varepsilon^{-1}x; \varepsilon) \rangle \eta^2(x) \\ &+ \frac{\varepsilon^2}{2}\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))u^2(\varepsilon^{-1}x; \varepsilon)\langle a(\varepsilon^{-1}x)\nabla_x \eta(x), \nabla_x \eta(x) \rangle \\ &+ \varepsilon\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))u(\varepsilon^{-1}x; \varepsilon)\eta(x)\langle a(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle. \end{aligned}$$

Integrate both sides with respect to $p(\varepsilon^{-1}x)$. Since the l.h.s. reduces to zero, it comes:

$$\begin{aligned} & \int_{\mathbb{R}^d} [\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\ & \quad \times \langle a(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x u(\varepsilon^{-1}x; \varepsilon) \rangle \eta^2(x)p(\varepsilon^{-1}x)] dx \\ &+ \varepsilon^2 \int_{\mathbb{R}^d} [\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))u^2(\varepsilon^{-1}x; \varepsilon) \\ & \quad \times \langle a(\varepsilon^{-1}x)\nabla_x \eta(x), \nabla_x \eta(x) \rangle p(\varepsilon^{-1}x)] dx \quad (\spadesuit) \\ (6.35) \quad &+ 2\varepsilon \int_{\mathbb{R}^d} [\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))u(\varepsilon^{-1}x; \varepsilon)\eta(x) \\ & \quad \times \langle a(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle p(\varepsilon^{-1}x)] dx \quad (\spadesuit) \\ &+ 2\varepsilon \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))u(\varepsilon^{-1}x; \varepsilon)\eta(x)p(\varepsilon^{-1}x) dx \quad (\spadesuit) \\ &= 2\varepsilon^{-1} \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))f(\varepsilon^{-1}x)\eta(x)p(\varepsilon^{-1}x) dx \quad (T(1, \varepsilon)) \\ &+ 2\varepsilon^{-1} \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x))m(\varepsilon^{-1}x; \varepsilon)p(\varepsilon^{-1}x) dx \quad (T(2, \varepsilon)). \end{aligned}$$

All the terms in (6.35) labelled with the symbol (\spadesuit) tend to 0 in probability as ε vanishes: recall that θ' and θ'' are bounded and apply Theorem 6.3 and (6.4). From **(RM1)** (see e.g. (6.5)) and again from (6.4), the labelled term containing θ' is clearly U.I. in $L^1(\Xi, \mu)$. To establish that the remaining terms (*i.e.* those containing θ'') are also U.I., note that, in each of these integrals, θ'' vanishes for $\varepsilon|u|(\varepsilon^{-1}x; \varepsilon)\eta(x) \geq 2$. Hence, $\varepsilon|u|(\varepsilon^{-1}x; \varepsilon)$ can be bounded by $(2\varepsilon)^{1/2}|u|^{1/2}(\varepsilon^{-1}x; \varepsilon)\eta^{-1/2}(x)$. It is then well seen that the product $\nabla_x \eta(x)\eta^{-1/2}(x)$ is a bounded function with compact support. Again, **(RM1)** and (6.4) permit to derive the required U.I. property.

Focus now on the term $T(2, \cdot)$ in (6.35). Recall from (6.13):

$$\begin{aligned}
(6.36) \quad & -T(2, \varepsilon) \\
& = 2 \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle dx \quad (\spadesuit) \\
& + \varepsilon \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) u(\varepsilon^{-1}x; \varepsilon) \langle (pa)(\varepsilon^{-1}x), \nabla_{x,x}^2 \eta(x) \rangle dx \quad (\spadesuit) \\
& + 2 \int_{\mathbb{R}^d} [\theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) u(\varepsilon^{-1}x; \varepsilon) \\
& \quad \times \langle B(\varepsilon^{-1}x), \nabla_x \eta(x) \rangle p(\varepsilon^{-1}x)] dx \quad (T(3, \varepsilon)).
\end{aligned}$$

Recall that $|\theta'(z)| \leq \min(2, |z|)$ for $z \in \mathbb{R}$ and apply again Theorem 6.3 and (6.4) to derive that the labelled terms tend to 0 in probability as ε vanishes. Follow the previous argument to handle (6.35) to establish the U.I. property. Focus then on the last term in (6.36) and refer again to (6.13):

$$\begin{aligned}
(6.37) \quad & (1/2)T(3, \varepsilon) \\
& = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) u(\varepsilon^{-1}x; \varepsilon) \frac{\partial (pa_{i,j})}{\partial x_i}(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_j}(x) dx \\
& = - \int_{\mathbb{R}^d} \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle dx \\
& - \varepsilon \int_{\mathbb{R}^d} [u(\varepsilon^{-1}x; \varepsilon)\eta(x)\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\
& \quad \times \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle] dx \\
& - \varepsilon^2 \int_{\mathbb{R}^d} [u^2(\varepsilon^{-1}x; \varepsilon)\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \\
& \quad \times \langle (pa)(\varepsilon^{-1}x) \nabla_x \eta(x), \nabla_x \eta(x) \rangle] dx \\
& - \varepsilon \int_{\mathbb{R}^d} u(\varepsilon^{-1}x; \varepsilon)\theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \langle (pa)(\varepsilon^{-1}x), \nabla_{x,x}^2 \eta(x) \rangle dx \\
& = (\spadesuit).
\end{aligned}$$

Again, all the labelled terms in (6.37) vanish in probability as ε tends to 0: apply Theorem 6.3 and (6.4). Follow the previous argument to handle (6.35) to establish the U.I. property. From (6.35), (6.36) and (6.37), we derive the following:

$$(6.38) \quad \int_{\mathbb{R}^d} [\theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) \langle a(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x u(\varepsilon^{-1}x; \varepsilon) \rangle \\ \times \eta^2(x) p(\varepsilon^{-1}x)] dx - T(1, \varepsilon) \xrightarrow{L^1(\Xi, \mu)} 0.$$

(Fourth Step.)

We thus investigate $T(1, \varepsilon)$ in (6.35). Due to the choice of \mathbf{f} in Theorem 3.2:

$$(6.39) \quad - (1/2)T(1, \varepsilon) \\ = -\varepsilon^{-1} \int_{\mathbb{R}^d} \frac{\partial(pg)}{\partial x_\ell}(\varepsilon^{-1}x) \eta(x) \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) dx \\ = \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial \eta}{\partial x_\ell}(x) \theta'(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) dx \quad (\spadesuit) \\ + \varepsilon \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \eta(x) u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) \theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) dx \quad (\spadesuit) \\ + \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \eta^2(x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) \theta''(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) dx \quad (T(4, \varepsilon)).$$

Again, the labelled terms in (6.39) vanish in probability as ε tends to zero: apply again Theorem 6.3 and (6.4) and use the bounds $|\theta'(z)| \leq |z|$, $|\theta''| \leq 1$. Recall also that \mathbf{g} is bounded and derive the U.I. property. Moreover,

$$(6.40) \quad T(4, \varepsilon) \\ = \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \eta^2(x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) (\theta'' - 1)(\varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)) dx \quad (\spadesuit) \\ + \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \eta^2(x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) dx \quad (T(5, \varepsilon)).$$

To prove the above convergence in probability, note that $|\theta'' - 1|(z) \leq \mathbf{1}_{|z| \geq 1} \leq |z|$. Apply then Theorem 6.3 and (6.4). The U.I. follows from the boundedness of \mathbf{g} and (6.4).

Plug finally (6.39) and (6.40) into (6.38) and complete the proof of (6.32).

□

7. Strong Convergence of the Gradient Fields. We now establish:

THEOREM 7.1. *The family $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$ converges in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$ as ε tends towards 0. The limit field is denoted by (Ψ_1, \dots, Ψ_d) .*

The proof holds in two parts. We first assume that Ψ denotes a weak limit of the family $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$ (recall that it is bounded in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$) and prove that the convergence holds in the strong sense. We finally prove that the limit is unique.

7.1. Proof of the Strong Convergence. The following lemma will be very useful. It derives from the maximum principle and from **(A1)**:

LEMMA 7.2. *There exists a constant $C_{\gamma, \varrho}$ such that for all $\xi \in \Xi$, $R > 0$ and $(x, z) \in (\mathbb{R}^d)^2$:*

$$\int_{B(z, R)} \gamma(x, y; R, z, \xi) dy \leq C_{\gamma, \varrho} R^2.$$

The Green function $\bar{\gamma}(x, y; R, z, \varepsilon, \xi)$ of the rescaled operator $\bar{\mathcal{L}}^{\varepsilon, \xi}$ (given by $\bar{\gamma}(x, y; R, z, \varepsilon, \xi) \equiv \varepsilon^{2-d} \gamma(\varepsilon^{-1}x, \varepsilon^{-1}y; \varepsilon^{-1}R, \varepsilon^{-1}z, \xi)$) satisfies the same bound.

From (6.4), we can extract a subsequence, still denoted by $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$ that weakly converges in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$. The weak limit is denoted by Ψ . Note that Ψ satisfies several crucial properties. Recall indeed that for a smooth function $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$, null outside a compact set, for a bounded stationary field $\varphi \in L^\infty(\Xi, \mu)$, and for $\varepsilon > 0$ and $(i, j) \in \{1, \dots, d\}^2$:

$$(7.1) \quad \mathcal{E} \left[\mathbf{p}\varphi \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x; \varepsilon, \xi) \frac{\partial \eta}{\partial x_j}(x) dx \right] = \mathcal{E} \left[\mathbf{p}\varphi \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_j}(x; \varepsilon, \xi) \frac{\partial \eta}{\partial x_i}(x) dx \right].$$

Letting ε tend to zero, the same holds for $\psi(x, \xi) = \Psi(\Gamma_x \xi)$:

$$\mathcal{E} \left[\mathbf{p}\varphi \int_{\mathbb{R}^d} \psi_i(x, \xi) \frac{\partial \eta}{\partial x_j}(x) dx \right] = \mathcal{E} \left[\mathbf{p}\varphi \int_{\mathbb{R}^d} \psi_j(x, \xi) \frac{\partial \eta}{\partial x_i}(x) dx \right],$$

so that, μ -a.s., for every smooth function η with compact support:

$$(7.2) \quad \int_{\mathbb{R}^d} \psi_i(x, \xi) \frac{\partial \eta}{\partial x_j}(x) dx = \int_{\mathbb{R}^d} \psi_j(x, \xi) \frac{\partial \eta}{\partial x_i}(x) dx.$$

Equality (7.2) permits to define, for almost every $\xi \in \Xi$, a primitive functional to ψ (see e.g. Kozlov [16] for the original argument). Choose for

example the primitive functional vanishing at zero:

$$z(x; \xi) \equiv \sum_{j=1}^d \int_0^1 x_j \psi_j(tx, \xi) dt.$$

Due to Lemma 7.1 in Zhikov et al. [36], $\Psi = \tilde{\Psi}$ a.s. implies $\mu\{\xi, \{z(x; \xi) = \tilde{z}(x; \xi) \text{ a.e.}\}\} = 1$, with \tilde{z} given by the above formula with ψ replaced by $\tilde{\psi}$. For a.e. $\xi \in \Xi$, the function $\psi(x, \xi)$ is then the $L^2_{\text{loc}}(\mathbb{R}^d)$ Sobolev derivative (in x) of $z(x; \xi)$. Thanks to **(RM3)**, $z(x; \xi)$ is jointly measurable in (x, ξ) .

Contrary to Kozlov [16], the classical relationship, $\mathcal{E}(\Psi_i) = 0$, $i \in \{1, \dots, d\}$, fails in our specific framework, since Ψ does not belong to $L^2(\mu; \mathbb{R}^d)$ but to $L^2(\mathbf{p}.\mu; \mathbb{R}^d)$. A possible strategy would consist in imbedding $L^2(\mathbf{p}.\mu; \mathbb{R}^d)$ into $L^1(\mu; \mathbb{R}^d)$ to define the expectation under μ of a given square integrable field under $\mathbf{p}.\mu$: this leads to assume that \mathbf{p}^{-1} belongs to $L^1(\mu; \mathbb{R}^d)$. Except trivial cases, there is no way, to the best of our knowledge, to obtain such a bound.

In the sequel of the paper, we aim to apply the strategy developed by Kozlov [16] for the divergence case to prove that $\varepsilon z(\varepsilon^{-1}x, \xi)$ tends to 0 (in a suitable sense), but we really miss the natural condition $\mathcal{E}(\Psi_i) = 0$, $i \in \{1, \dots, d\}$. This is the reason why the specific centering condition $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$ appears in the statement of Theorem 3.2.

(First Step.)

Anyway, the starting point of our strategy remains the same as in the divergence case: we focus on the equation satisfied by the weak limit Ψ . From an integration by parts (see (6.1)), we derive from Theorem 6.2 and Proposition 6.5 that, for all fields $\mathbf{h}, \Phi \in \mathcal{S}_\infty$ and all $\varepsilon > 0$, $\mathcal{E}(\mathbf{p}\langle \mathbf{a}D\mathbf{v}(\cdot; \varepsilon), D\Phi \rangle) - \sum_{i,j=1}^d \mathcal{E}(D_j(\mathbf{p}\mathbf{a}_{i,j})D_i\mathbf{v}(\cdot; \varepsilon)\Phi) + 2\varepsilon^2 \mathcal{E}(\mathbf{p}\mathbf{v}(\cdot; \varepsilon)\Phi) = 2\mathcal{E}(\mathbf{p}\mathbf{h}\Phi)$, with $\mathbf{v}(\cdot; \varepsilon) \equiv (\varepsilon^2 I - \mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\cdot))^{-1}\mathbf{h} \in W^{2,\infty}(\Xi, \mu)$. From a density argument, the same holds for $\mathbf{h} = \mathbf{f}$ and $\mathbf{v}(\cdot; \varepsilon) = \mathbf{u}(\cdot; \varepsilon)$ (see Subsection 6.2, Paragraph ‘‘End of the Proof of Theorem 6.2’’). Letting ε tend to zero, it comes from (6.1):

$$(7.3) \quad \mathcal{E} \left[\sum_{i,j=1}^d (\mathbf{p}\mathbf{a}_{i,j}) \Psi_i D_j \Phi \right] - \mathcal{E} \left[\sum_{i,j=1}^d D_j (\mathbf{p}\mathbf{a}_{i,j}) \Psi_i \Phi \right] = -2\mathcal{E}[\mathbf{p}\mathbf{g} D_\ell \Phi].$$

Choosing $\Phi = \mathbf{k} \star \eta$, with $\mathbf{k} \in L^\infty(\Xi, \mu)$, we claim that for a.e. $\xi \in \Xi$ and for every smooth function η with compact support:

$$(7.4) \quad \int_{\mathbb{R}^d} \langle (p\mathbf{a}\psi)(x, \xi), \nabla_x \eta(x) \rangle dx - \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial (p\mathbf{a}_{i,j})}{\partial x_j}(x, \xi) \psi_i(x, \xi) \eta(x) dx \\ = -2 \int_{\mathbb{R}^d} (p\mathbf{g})(x, \xi) \frac{\partial \eta}{\partial x_\ell}(x) dx.$$

We deduce that there exists an event $A \in \mathcal{G}$, with full measure, such that for every $\xi \in A$, $z(x; \xi)$ satisfies the following PDE in $H_{\text{loc}}^1(\mathbb{R}^d)$:

$$(7.5) \quad \begin{aligned} & - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} ((pa_{i,j})(x, \xi)) \frac{\partial}{\partial x_j} (z(x; \xi)) \\ & - \sum_{i,j=1}^d \frac{\partial (pa_{i,j})}{\partial x_j} (x, \xi) \frac{\partial}{\partial x_i} (z(x; \xi)) = 2 \frac{\partial}{\partial x_\ell} (pg)(x, \xi). \end{aligned}$$

Due to Assumption **(A)**, to the boundedness of $D\mathbf{g}$, to the positivity of \mathbf{p} and to Proposition 4.11, $z(x; \xi)$ satisfies, for every $\xi \in A$, the De Giorgi and Nash estimates and thus admits a continuous version (see Theorem 8.22 in Gilbarg and Trudinger [13]). This continuous version is given by the pointwise limit of $V_d^{-1} \delta^{-d} \int_{B(0, \delta)} z(x+y; \xi) dy$ as $\delta \rightarrow 0$. For the sake of simplicity, we then assume $z(x; \xi)$ to match this continuous version for $\xi \in A$ and to vanish for $\xi \in A^c$. The resulting mapping $z(x; \xi)$ is still jointly measurable with respect to (x, ξ) and satisfies (7.5) for $\xi \in A$. Due to Theorem 9.11 and to Corollary 9.18 in Gilbarg and Trudinger [13] (Calderón and Zygmund inequalities for PDEs of non-divergence type), $z(x; \xi)$ belongs, for every $\xi \in A$, to the space $\cap_{q \geq 1} W_{\text{loc}}^{2,q}(\mathbb{R}^d)$.

Set now, for every $\varepsilon > 0$, $z(x; \varepsilon, \xi) \equiv \varepsilon z(\varepsilon^{-1}x; \varepsilon, \xi)$ and derive from (7.5) that, for all $\xi \in A$ and $\varepsilon > 0$, $z(x; \varepsilon, \xi)$ satisfies the following equation in $\cap_{q \geq 1} W_{\text{loc}}^{2,q}(\mathbb{R}^d)$:

$$(7.6) \quad \begin{aligned} & - p^{-1}(\varepsilon^{-1}x, \xi) \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} ((pa_{i,j})(\varepsilon^{-1}x, \xi)) z(x; \varepsilon, \xi) \\ & = 2p^{-1}(\varepsilon^{-1}x, \xi) \frac{\partial}{\partial x_\ell} ((pg)(\varepsilon^{-1}x, \xi)). \end{aligned}$$

Here is the equivalent of Lemma 2 in Kozlov [16]:

PROPOSITION 7.3. *For almost every $\xi \in \Xi$, for all $\varepsilon > 0$, the function $z(\cdot; \varepsilon, \xi)$ writes $z(\cdot; \varepsilon, \xi) = z_1(\cdot; \varepsilon, \xi) + z_2(\cdot; \varepsilon, \xi) + \varepsilon u(\varepsilon^{-1}x; \varepsilon, \xi) + \Gamma(\varepsilon, \xi)$ with:*

$$\begin{aligned} a) \quad & \lim_{\varepsilon \rightarrow 0} \mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |z_1(x; \varepsilon, \xi)|^2 dx \right] = 0, \\ b) \quad & \mathcal{E} \left[\left(\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) dx \right) \left(\sup_{x \in B(0,1)} |z_2(x; \varepsilon, \xi)|^2 \right. \right. \\ & \left. \left. + \sup_{(x,y) \in \bar{B}(0,1/4), x \neq y} [|x-y|^{-\alpha} \gamma \cdot \mathcal{J} |z_2(x; \varepsilon, \xi) - z_2(y; \varepsilon, \xi)|^2] \right) \right] \leq C \gamma \cdot \mathcal{J}, \end{aligned}$$

for two constants $\alpha \gamma \cdot \mathcal{J} > 0$ and $C \gamma \cdot \mathcal{J}$. Moreover, $\Gamma(\varepsilon, \xi)$ is independent of x .

PROOF. Consider $\bar{z}(x; \varepsilon, \xi) \equiv z(x; \varepsilon, \xi) - \varepsilon u(\varepsilon^{-1}x; \varepsilon, \xi)$. For $\xi \in A$ (see (7.4) and (7.5) for the definition of A), $\bar{z}(x; \varepsilon, \xi)$ satisfies the PDE (remove the index ξ for notational convenience):

$$(7.7) \quad -p^{-1}(\varepsilon^{-1}x) \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} ((pa_{i,j})(\varepsilon^{-1}x) \bar{z}(x; \varepsilon)) = 2\varepsilon u(\varepsilon^{-1}x; \varepsilon).$$

Denote by $z_1(x; \varepsilon)$ the solution to the PDE (see Theorem 9.15 in Gilbarg and Trudinger [13] since the operator writes in a non-divergence form):

$$(7.8) \quad \begin{aligned} & -p^{-1}(\varepsilon^{-1}x) \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} ((pa_{i,j})(\varepsilon^{-1}x) z_1(x; \varepsilon)) \\ & = 2\varepsilon u(\varepsilon^{-1}x; \varepsilon), \quad |x| < 2, \quad z_1(x; \varepsilon) = 0 \text{ for } |x| = 2. \end{aligned}$$

The solution $z_1(x; \varepsilon)$ is continuous and belongs to $\cap_{q \geq 1} (W_0^{1,q}(B(0,2)) \cap W^{2,q}(B(0,2)))$. We can multiply both sides in (7.8) by $p(\varepsilon^{-1}x) z_1(x; \varepsilon)$ and integrate over \mathbb{R}^d . An integration by parts yields:

$$(7.9) \quad \begin{aligned} & \int_{B(0,2)} p(\varepsilon^{-1}x) |\nabla_x z_1(x; \varepsilon)|^2 dx \\ & \leq C \left[\varepsilon^2 \int_{B(0,2)} p(\varepsilon^{-1}x) u^2(\varepsilon^{-1}x; \varepsilon) dx \right]^{1/2} \\ & \quad \times \left[\int_{B(0,2)} p(\varepsilon^{-1}x) |z_1(x; \varepsilon)|^2 dx \right]^{1/2}. \end{aligned}$$

Write z_1 with respect to the Green function of $p^{-1}(\varepsilon^{-1}\cdot)(\mathcal{L}^\varepsilon)^*(p(\varepsilon^{-1}\cdot))$:

$$(7.10) \quad z_1(x; \varepsilon) = 2\varepsilon \int_{B(0,2)} u(\varepsilon^{-1}y; \varepsilon) p^{-1}(\varepsilon^{-1}x) p(\varepsilon^{-1}y) \bar{\gamma}(y, x; 2, 0, \varepsilon) dy.$$

Deduce from Young's inequality that for every $\delta > 0$:

$$\begin{aligned} |z_1(x; \varepsilon)|^2 & \leq \delta^{-2} \varepsilon^2 \int_{B(0,2)} u^2(\varepsilon^{-1}y; \varepsilon) p^{-1}(\varepsilon^{-1}x) p(\varepsilon^{-1}y) \bar{\gamma}(y, x; 2, 0, \varepsilon) dy \\ & \quad + \delta^2 \int_{B(0,2)} p^{-1}(\varepsilon^{-1}x) p(\varepsilon^{-1}y) \bar{\gamma}(y, x; 2, 0, \varepsilon) dy. \end{aligned}$$

Precise now the index ξ and apply (2.5): $\bar{\gamma}(y, x; 2, 0, \varepsilon, \xi) = \varepsilon^{2-d} \gamma(\varepsilon^{-1}y, \varepsilon^{-1}x; 2\varepsilon^{-1}, 0, \xi) = \varepsilon^{2-d} \gamma(0, \varepsilon^{-1}(x - y); 2\varepsilon^{-1}, -\varepsilon^{-1}y, \mathbb{T}_{\varepsilon^{-1}y}\xi) = \bar{\gamma}(0, x -$

$y; 2, -y, \varepsilon, \mathbb{T}_{\varepsilon^{-1}y}\xi$). Thus,

$$\begin{aligned} & p(\varepsilon^{-1}x, \xi) |z_1(x; \varepsilon, \xi)|^2 \\ & \leq \delta^{-2} \varepsilon^2 \int_{B(0,2)} u^2(\varepsilon^{-1}y; \varepsilon, \xi) p(\varepsilon^{-1}y, \xi) \bar{\gamma}(0, x - y; 2, -y, \varepsilon, \mathbb{T}_{\varepsilon^{-1}y}\xi) dy \\ & \quad + \delta^2 \int_{B(0,2)} p(\varepsilon^{-1}y, \xi) \bar{\gamma}(0, x - y; 2, -y, \varepsilon, \mathbb{T}_{\varepsilon^{-1}y}\xi) dy. \end{aligned}$$

Integrate now with respect to $x \in B(0, 2)$ and then with respect to μ (thanks to (7.10), $z_1(x; \varepsilon, \xi)$ is jointly measurable in (x, ξ)):

$$\begin{aligned} & \mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |z_1(x; \varepsilon, \xi)|^2 dx \right] \\ & \leq \delta^{-2} \varepsilon^2 \int_{B(0,2)} \int_{\Xi} \mathbf{u}^2(\xi; \varepsilon) \mathbf{p}(\xi) \left[\int_{B(0,2)} \bar{\gamma}(0, x - y; 2, -y, \varepsilon, \xi) dx \right] d\mu(\xi) dy \\ & \quad + \delta^2 \int_{B(0,2)} \int_{\Xi} \mathbf{p}(\xi) \left[\int_{B(0,2)} \bar{\gamma}(0, x - y; 2, -y, \varepsilon, \xi) dx \right] dy d\mu(\xi). \end{aligned}$$

Recall from Lemma 7.2 that the integral of the Green function is bounded by C . Hence:

$$\mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |z_1(x; \varepsilon, \xi)|^2 dx \right] \leq C \delta^{-2} \varepsilon^2 \mathcal{E}(\mathbf{u}^2(\cdot; \varepsilon) \mathbf{p}) + C \delta^2.$$

Thanks to Theorem 6.4, this proves that:

$$\mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |z_1(x; \varepsilon, \xi)|^2 dx \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Hence, thanks to (7.9) and again to Theorem 6.4:

$$(7.11) \quad \mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |\nabla_x z_1(x; \varepsilon, \xi)|^2 dx \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

This gives *a*) in the statement of Proposition 7.3.

Set now $z_2(x; \varepsilon, \xi) \equiv \bar{z}(x; \varepsilon, \xi) - z_1(x; \varepsilon, \xi)$, $x \in B(0, 2)$. For $\xi \in A$, deduce from (7.7) and (7.8) that $z_2(x; \varepsilon, \xi)$ satisfies the following PDE in $\cap_{q \geq 1} W^{2,q}(B(0, 2))$:

$$(7.12) \quad p^{-1}(\varepsilon^{-1}x, \xi) \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} ((pa_{i,j})(\varepsilon^{-1}x, \xi) z_2(x; \varepsilon, \xi)) = 0.$$

Note that $\nabla_x z_2(x; \varepsilon, \xi) = \nabla_x \bar{z}(x; \varepsilon, \xi) - \nabla_x z_1(x; \varepsilon, \xi) = \psi(\varepsilon^{-1}x, \xi) - \nabla_x u(\varepsilon^{-1}x; \varepsilon, \xi) - \nabla_x z_1(x; \varepsilon, \xi)$. Derive from (6.4) and (7.11) that:

$$(7.13) \quad \mathcal{E} \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |\nabla_x z_2(x; \varepsilon, \xi)|^2 dx \right] \leq C.$$

We now aim to apply the Poincaré inequality for normalized solutions established in Escauriaza and Kenig [8], Lemma 10. The proof given in the previous reference applies to a slightly different normalization procedure: a solution $v(x)$ to the adjoint equation $(\bar{\mathcal{L}}^{\varepsilon, \xi})^*(v) = 0$ is not divided by $p(\varepsilon^{-1}x, \xi)$, but by $\bar{\gamma}(x_0, x; 20, 0, \varepsilon, \xi)$ with $|x_0| = 9$ (see page 277 in [8]). According to Theorems 2.3, 2.4, 2.5 and 2.7 in Escauriaza [7] and to Lemma 2.0 in Fabes and Stroock [10], the reader can verify that the following still holds in our frame:

$$(7.14) \quad \begin{aligned} & \sup_{x \in B(0,1)} |z_2(x; \varepsilon, \xi) - z_2(0; \varepsilon, \xi)|^2 \\ & \leq C \left[\int_{B(0,2)} p(\varepsilon^{-1}x, \xi) dx \right]^{-1} \int_{B(0,2)} p(\varepsilon^{-1}x, \xi) |\nabla_x z_2(x; \varepsilon, \xi)|^2 dx. \end{aligned}$$

Thanks to the Harnack inequality for normalized adjoint solutions (see Theorem 2.5 in [7]), there exists a real $\alpha > 0$ such that (7.14) holds with $\sup_{x, y \in \bar{B}(0, 1/4), x \neq y} [|x - y|^{-\alpha} |z_2(x; \varepsilon, \xi) - z_2(y; \varepsilon, \xi)|^2]$ in addition to the l.h.s..

Plug finally (7.13) into (7.14) and complete the proof by setting $\Gamma(\varepsilon, \xi) = z_2(0; \varepsilon, \xi)$ and by subtracting $z_2(0; \varepsilon, \xi)$ to $z_2(\cdot; \varepsilon, \xi)$. \square

(Second Step.)

Considering $z(\cdot; \varepsilon, \xi) - \Gamma(\varepsilon, \xi)$ instead of $z(\cdot; \varepsilon, \xi)$ itself, we can assume without loss of generality that $\Gamma(\varepsilon, \xi)$ vanishes (of course $z(0; \varepsilon, \xi)$ may then differ from zero). Note in particular that $z(\cdot; \varepsilon, \xi)$ still satisfies (7.6) for $\xi \in A$. Of course, the gradient of $z(x; \xi)$ is still given by $\psi(x, \xi)$.

Consider now a smooth function φ with compact support. Removing the index ξ , derive from (7.6) (see also (7.5) for the divergence form):

$$(7.15) \quad \begin{aligned} & \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \psi(\varepsilon^{-1}x), \nabla_x \varphi(x) \rangle dx \\ & - \varepsilon^{-1} \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial (pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x) \psi_i(\varepsilon^{-1}x) \varphi(x) dx \\ & = -2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial \varphi}{\partial x_\ell}(x) dx. \end{aligned}$$

Choose now $\varphi(x) = \varepsilon u(\varepsilon^{-1}x; \varepsilon)\eta(x)$ in (7.15), where η denotes a smooth function with compact support included in $B(0, 1/4)$. We claim:

$$\begin{aligned}
(7.16) \quad & \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x)\psi(\varepsilon^{-1}x), \nabla_x u(\varepsilon^{-1}x; \varepsilon) \rangle \eta(x) dx \\
& + \varepsilon \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x)\psi(\varepsilon^{-1}x), \nabla_x \eta(x) \rangle u(\varepsilon^{-1}x; \varepsilon) dx \\
& - \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial(pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x)\psi_i(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon)\eta(x) dx \\
& = -2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) \eta(x) dx \\
& - 2\varepsilon \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx.
\end{aligned}$$

We now exchange the role between $z(x; \varepsilon)$ and $\varepsilon u(\varepsilon^{-1}x; \varepsilon)$. Recall to this end that $\varepsilon u(\varepsilon^{-1}x; \varepsilon)$ satisfies (7.6), but with $-2\varepsilon u(\varepsilon^{-1}x; \varepsilon)$ in addition to the second member. Following (7.15) (up to an integration by parts for the second term in (7.15)), write for a smooth function φ with compact support:

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \varphi(x) \rangle dx \\
& + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial(pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \varphi}{\partial x_i}(x) dx \\
& = -2\varepsilon \int_{\mathbb{R}^d} p(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon)\varphi(x) dx - 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial \varphi}{\partial x_\ell}(x) dx.
\end{aligned}$$

Choose now $\varphi = z(x; \varepsilon)\eta(x)$ with the same η as in (7.16):

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \psi(\varepsilon^{-1}x) \rangle \eta(x) dx \\
& + \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x)\nabla_x u(\varepsilon^{-1}x; \varepsilon), \nabla_x \eta(x) \rangle z(x; \varepsilon) dx \quad (\clubsuit) \\
& + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial(pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon)\psi_i(\varepsilon^{-1}x)\eta(x) dx \\
& + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial(pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \eta}{\partial x_i}(x) z(x; \varepsilon) dx \quad (\diamond) \\
& = -2\varepsilon \int_{\mathbb{R}^d} p(\varepsilon^{-1}x)u(\varepsilon^{-1}x; \varepsilon)z(x; \varepsilon)\eta(x) dx \\
& - 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x)\psi_\ell(\varepsilon^{-1}x)\eta(x) dx - 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x)z(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx.
\end{aligned}$$

Apply an integration by parts to the term (\diamond) to get rid of the term (\clubsuit):

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \psi(\varepsilon^{-1}x) \rangle \eta(x) dx \\
& + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial(pa_{i,j})}{\partial x_j}(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) \psi_i(\varepsilon^{-1}x) \eta(x) dx \\
& - \varepsilon \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \psi(\varepsilon^{-1}x), \nabla_x \eta(x) \rangle u(\varepsilon^{-1}x; \varepsilon) dx \\
(7.17) \quad & - \varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} (pa_{i,j})(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) z(x; \varepsilon) \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) dx \\
& = -2\varepsilon \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) z(x; \varepsilon) \eta(x) dx \\
& - 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \psi_\ell(\varepsilon^{-1}x) \eta(x) dx \\
& - 2 \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) z(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx.
\end{aligned}$$

Add together equalities (7.16) and (7.17):

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \psi(\varepsilon^{-1}x) \rangle \eta(x) dx \\
& - \frac{1}{2} \varepsilon \sum_{i,j=1}^d \int_{\mathbb{R}^d} (pa_{i,j})(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) z(x; \varepsilon) \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) dx \quad (T(1; \varepsilon)) \\
& = -\varepsilon \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) z(x; \varepsilon) \eta(x) dx \quad (T(2; \varepsilon)) \\
(7.18) \quad & - \varepsilon \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) u(\varepsilon^{-1}x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx \quad (T(3; \varepsilon)) \\
& - \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) \eta(x) dx \\
& - \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \psi_\ell(\varepsilon^{-1}x) \eta(x) dx \\
& - \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) z(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx \quad (T(4; \varepsilon)).
\end{aligned}$$

(Third Step.)

From Theorem 6.3, $T(3; \varepsilon)$ in (7.18) tends to 0 in probability. Focus on $T(1; \varepsilon)$ and $T(2; \varepsilon)$. From the Young inequality and the boundedness of the

support of η , there exists a constant C such that for all $\delta > 0$:

$$\begin{aligned}
(7.19) \quad & T(1; \varepsilon) + T(2; \varepsilon) \\
& \leq C\delta^{-1}\varepsilon^2 \int_{B(0,1/4)} p(\varepsilon^{-1}x)u^2(\varepsilon^{-1}x; \varepsilon)dx \\
& \quad + C\delta \int_{B(0,1/4)} p(\varepsilon^{-1}x)z^2(x; \varepsilon)dx \\
& \equiv T(5; \varepsilon) + T(6; \varepsilon).
\end{aligned}$$

Note from Theorem 6.4 that $T(5; \varepsilon)$ converges towards zero in $L^1(\Xi, \mu)$. Focus then on $T(6; \varepsilon)$. Referring to Proposition 7.3,

$$\begin{aligned}
(7.20) \quad & T(6; \varepsilon) \leq 3C\delta \left[\int_{B(0,1/4)} p(\varepsilon^{-1}x)|z_1|^2(x; \varepsilon)dx \right. \\
& \quad + \sup_{x \in B(0,1/4)} [|z_2|^2(x; \varepsilon)] \int_{B(0,1/4)} p(\varepsilon^{-1}x)dx \\
& \quad \left. + \varepsilon^2 \int_{B(0,1/4)} p(\varepsilon^{-1}x)u^2(\varepsilon^{-1}x; \varepsilon)dx \right]. \\
& \equiv T(7; \varepsilon) + T(8; \varepsilon) + T(9; \varepsilon).
\end{aligned}$$

Again from Proposition 7.3, $T(7; \varepsilon)$ tends to zero in $L^1(\Xi, \mu)$. From Theorem 6.4, the same holds for $T(9; \varepsilon)$. Finally, again from Proposition 7.3, the quantity $\mu\{\xi, T(8; \varepsilon) \geq \delta^{1/2}\}$ is bounded (up to a multiplicative constant) by $\delta^{1/2}$. Hence, from (7.19) and (7.20), we deduce that $T(1, \varepsilon)$ and $T(2, \varepsilon)$ in (7.18) tend to 0 in probability.

Focus now on $T(4; \varepsilon)$. Refer again to Proposition 7.3 and note that:

$$\begin{aligned}
(7.21) \quad & |T(4; \varepsilon)| \leq \int_{B(0,1/4)} |pg|(\varepsilon^{-1}x) [|z_1|(x; \varepsilon) + \varepsilon|u|(\varepsilon^{-1}x; \varepsilon)] dx \\
& \quad + \left| \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x)z_2(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx \right| \\
& \equiv T(10; \varepsilon) + T(11; \varepsilon).
\end{aligned}$$

As done for $T(1; \varepsilon)$, $T(2; \varepsilon)$ and $T(3; \varepsilon)$, the term $T(10; \varepsilon)$ tends to zero in probability. It thus remains to deal with $T(11; \varepsilon)$.

Note from Proposition 7.3 and from the ergodic theorem that the family of measures on the space of continuous real-valued functions on $\bar{B}(0, 1/4)$ induced by the family of processes $(z_2(\cdot; \varepsilon, \cdot))_{0 < \varepsilon \leq 1}$ is tight: for every $\delta > 0$, there exists a compact subset $\mathcal{K} \subset \mathcal{C}(\bar{B}(0, 1/4), \mathbb{R})$ such that, for all $\varepsilon \in]0, 1]$,

$\mu\{\xi \in \Xi, z_2(\cdot; \varepsilon, \xi) \notin \mathcal{K}\} \leq \delta$. Since the set \mathcal{K} is compact, there exists a family $(\ell_i)_{i \in \{1, \dots, N\}}$, $N \geq 1$, of continuous functions on $\bar{B}(0, 1/4)$ such that:

$$(7.22) \quad \mathcal{K} = \bigcup_{i=1}^N \left\{ \ell \in \mathcal{K}, \sup_{x \in \bar{B}(0, 1/4)} |\ell(x) - \ell_i(x)| \leq \delta \right\} \equiv \bigcup_{i=1}^N A_i,$$

with $A_i \equiv \{ \ell \in \mathcal{K}, \sup_{x \in \bar{B}(0, 1/4)} |\ell(x) - \ell_i(x)| \leq \delta \} \setminus (A_0 \cup \dots \cup A_{i-1})$, for $i \in \{1, \dots, N\}$ and $A_0 = \emptyset$.

Focus then on $T(11; \varepsilon)$ for $z_2(\cdot; \varepsilon, \xi) \in \mathcal{K}$. Thanks to (7.22) (remove again the index ξ):

$$\begin{aligned} & \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) z_2(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx \\ &= \sum_{i=1}^N \mathbf{1}_{A_i}(z_2(\cdot; \varepsilon)) \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) (z_2(x; \varepsilon) - \ell_i(x)) \frac{\partial \eta}{\partial x_\ell}(x) dx \\ & \quad + \sum_{i=1}^N \mathbf{1}_{A_i}(z_2(\cdot; \varepsilon)) \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \ell_i(x) \frac{\partial \eta}{\partial x_\ell}(x) dx. \end{aligned}$$

Hence,

$$(7.23) \quad \left| \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) z_2(x; \varepsilon) \frac{\partial \eta}{\partial x_\ell}(x) dx \right| \leq C\delta \int_{\mathbb{R}^d} p(\varepsilon^{-1}x) |\nabla_x \eta|(x) dx + \sum_{i=1}^N \left| \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \ell_i(x) \frac{\partial \eta}{\partial x_\ell}(x) dx \right|.$$

Since $\mathcal{E}(\mathbf{pg}) = 0$, derive now from the ergodic theorem (applied to a sequence of simple functions approximating the compact support function $\ell_i \partial \eta / \partial x_\ell$) that for every $i \in \{1, \dots, N\}$:

$$(7.24) \quad \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \ell_i(x) \frac{\partial \eta}{\partial x_\ell}(x) dx \rightarrow 0 \text{ in } L^1(\Xi, \mu).$$

From (7.23) and (7.24), deduce that $T(11; \varepsilon)$ tends to zero in probability.

Return back to (7.18): the terms $T(1; \varepsilon)$, $T(2; \varepsilon)$, $T(3; \varepsilon)$ and $T(4; \varepsilon)$ tend to zero in probability as ε vanishes. Deduce in the end that:

$$(7.25) \quad \begin{aligned} & \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \psi(\varepsilon^{-1}x) \rangle \eta(x) dx \\ & + \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \frac{\partial u}{\partial x_\ell}(\varepsilon^{-1}x; \varepsilon) \eta(x) dx \\ & + \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x) \psi_\ell(\varepsilon^{-1}x) \eta(x) dx, \end{aligned}$$

tends to zero in probability.

(Fourth Step.)

We now aim to prove that the term in (7.25) also tends to zero in $L^1(\Xi, \mu)$. To this end, we need to establish the uniform integrability of (7.25) with respect to the parameter $\varepsilon \in]0, 1]$. The U.I. of the last term in (7.25) follows from **(RM1)** (see (6.5)).

Turn now to the first term. Due to (6.4), we claim for $A \in \mathcal{G}$:

$$\begin{aligned} & \mathcal{E} \left[\mathbf{1}_A \left| \int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x) \nabla_x u(\varepsilon^{-1}x; \varepsilon), \psi(\varepsilon^{-1}x) \rangle \eta(x) dx \right| \right] \\ & \leq C \left[\int_{B(0, 1/4)} \mathcal{E}(\mathbf{1}_A(\xi) (\mathbf{p}|\Psi|^2)(\mathbb{T}_{\varepsilon^{-1}x}\xi)) dx \right]^{1/2}. \end{aligned}$$

Again from **(RM1)**, the first term in (7.25) is also U.I.. The second term in (7.25) is treated in a similar way.

Hence, the quantity in (7.25) tends to zero in $L^1(\Xi, \mu)$. In particular, the expectation of (7.25) vanishes as ε tends to zero. Due to the weak convergence (up to a subsequence) of the gradient, this proves that $\mathcal{E}[\mathbf{p}\langle \mathbf{a}\Psi, \Psi \rangle] = -2\mathcal{E}[\mathbf{p}\mathbf{g}\Psi_\ell]$.

Since $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}[\mathbf{p}\langle \mathbf{a}D\mathbf{u}(\cdot; \varepsilon), D\mathbf{u}(\cdot; \varepsilon) \rangle] \leq -2\mathcal{E}[\mathbf{p}\mathbf{g}\Psi_\ell]$ (see (6.33)), this completes the proof of the strong convergence. \square

7.2. Uniqueness of the Limit. We finally prove the uniqueness of the limit Ψ . Assume to this end that $\tilde{\Psi} \in L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$ denotes another weak limit (and thus strong limit from Subsection 7.1) of the family $(D\mathbf{u}(\cdot; \varepsilon))_{\varepsilon > 0}$. Noting that the $L^1(\Xi, \mu)$ convergence in (7.25) holds for $\varepsilon \rightarrow 0$ and not just for a subsequence tending to 0, and making the difference between both (7.25) associated to Ψ and $\tilde{\Psi}$, derive that $\Delta\Psi \equiv \Psi - \tilde{\Psi}$ satisfies:

$$(7.26) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathcal{E} \left[\int_{\mathbb{R}^d} \langle (pa)(\varepsilon^{-1}x, \xi) \nabla_x u(\varepsilon^{-1}x; \varepsilon, \xi), \Delta\Psi(\mathbb{T}_{\varepsilon^{-1}x}\xi) \rangle \eta(x) dx \right. \\ & \left. + \int_{\mathbb{R}^d} (pg)(\varepsilon^{-1}x, \xi) \Delta\Psi_\ell(\mathbb{T}_{\varepsilon^{-1}x}\xi) \eta(x) dx \right] = 0. \end{aligned}$$

Hence, for any weak limit $\bar{\Psi}$ of the sequence $(D\mathbf{u}(\cdot; \varepsilon))_{0 < \varepsilon \leq 1}$,

$$(7.27) \quad \mathcal{E}[\langle (\mathbf{p}\mathbf{a})\bar{\Psi}, \Delta\Psi \rangle] = -\mathcal{E}[\mathbf{p}\mathbf{g}(\Delta\Psi)_\ell].$$

Both Ψ and $\tilde{\Psi}$ satisfy (7.27). In particular, the difference $\Delta\Psi$ satisfies $\mathcal{E}[\langle (\mathbf{p}\mathbf{a})\Delta\Psi, \Delta\Psi \rangle] = 0$. This completes the proof of uniqueness. \square

8. Conclusion.

8.1. *Proof of Theorem 3.2.* We now complete the proof of Theorem 3.2. Under $(\mathbf{p}, \mu) \otimes \mathbb{P}$, for every $r > 0$, the process $(\eta_{r^2-t})_{0 \leq t \leq r^2}$ is a Markov process with $\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p})$ as generator (see e.g. Nagasawa [22]). Hence, from Proposition 1.7, Chapter IV, in Ethier and Kurtz [9], for every $r > 0$, $(\mathbf{p}, \mu) \otimes \mathbb{P}$ a.s.:

$$\begin{aligned} \mathbf{u}(\eta_0; r^{-1}) &= \mathbf{u}(\eta_{r^2}; r^{-1}) \\ &\quad + r^{-2} \int_0^{r^2} \mathbf{u}(\eta_{r^2-s}; r^{-1}) ds - \int_0^{r^2} \mathbf{f}(\eta_{r^2-s}) ds + M_{r^2}^{(r)}, \end{aligned}$$

where $(M_t^{(r)})_{0 \leq t \leq r^2}$ is a continuous square integrable martingale with respect to the filtration $(\sigma\{\eta_{r^2-s}, 0 \leq s \leq t\})_{0 \leq t \leq r^2}$. From Lemma VIII.3.68 in Jacod and Shiryaev [14], the quadratic variation writes $(\int_0^t \langle D\mathbf{u}(\cdot; r^{-1}), \mathbf{a}D\mathbf{u}(\cdot; r^{-1}) \rangle (\eta_{r^2-s}) ds)_{0 \leq t \leq r^2}$ (following the regularization argument in Subsection 6.2, Paragraph “End of the Proof of Theorem 6.2”, we can prove that $\mathbf{u}^2(\cdot; r^{-1}) \in \mathcal{D}_{\mathbf{p}}(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}))$ and that $\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{u}^2(\cdot; r^{-1})) = 2\mathbf{u}(\cdot; r^{-1})(\mathbf{p}^{-1}\mathbf{L}^*(\mathbf{p}\mathbf{u}(\cdot; r^{-1}))) + \langle \mathbf{a}D\mathbf{u}(\cdot; r^{-1}), D\mathbf{u}(\cdot; r^{-1}) \rangle \in L^2(\Xi, \mathbf{p}, \mu)$).

Thanks to Theorem 6.4, deduce that:

$$\lim_{r \rightarrow +\infty} \mathcal{E} \otimes \mathbb{E} \left[\mathbf{p} \left| r^{-1} \int_0^{r^2} \mathbf{f}(\eta_{r^2-s}) ds - r^{-1} M_{r^2}^{(r)} \right| \right] = 0.$$

Note to conclude that $r^{-1}(M_{r^2 t}^{(r)})_{0 \leq t \leq 1}$ is a martingale (with respect to a rescaled filtration) whose quadratic variation writes $(r^{-2} \int_0^{r^2 t} \langle D\mathbf{u}(\cdot; r^{-1}), \mathbf{a}D\mathbf{u}(\cdot; r^{-1}) \rangle (\eta_{r^2-s}) ds)_{0 \leq t \leq 1}$. Apply Theorem 3.1 (ergodic theorem), Theorem 7.1 and Theorem VIII.3.11 in Jacod and Shiryaev [14] and deduce that $r^{-1}(M_{r^2 t}^{(r)})_{0 \leq t \leq 1}$ converges in law towards a non-standard real valued Brownian motion with $\sigma_{\mathbf{f}}^2 = \mathcal{E}(\mathbf{p}\langle \Psi, \mathbf{a}\Psi \rangle)$ as variance. \square

8.2. *Further Extensions and Investigations.* Note first that Theorem 3.2 still holds true if \mathbf{f} writes as the linear combination of terms of the form $(\mathbf{p}^{-1}D_{\ell_i}(\mathbf{p}\mathbf{g}_i))_{1 \leq i \leq N}$, with $N \geq 1$, $(\ell_1, \dots, \ell_N) \in \{1, \dots, d\}^N$ and $\mathbf{g}_1, \dots, \mathbf{g}_d$ as in the statement of Theorem 3.2. The arguments to establish such an extension are just the same as those used to prove Theorem 3.2.

A natural question then consists in investigating Theorem 3.2 when \mathbf{g} (or more generally $\mathbf{g}_1, \dots, \mathbf{g}_d$ as above) belongs to $H(\Xi, \mathbf{p}, \mu)$ and satisfies $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$. In this case, the field \mathbf{g} can be approximated in $L^2(\Xi, \mathbf{p}, \mu)$ by a sequence of the form $(\mathbf{g}_n \star k_n)_{n \geq 0}$, for a sequence $(\mathbf{g}_n)_{n \geq 0} \in (L^\infty(\Xi, \mu))^{\mathbb{N}}$ converging towards \mathbf{g} in $L^2(\Xi, \mathbf{p}, \mu)$ and a sequence $(k_n)_{n \geq 0}$ of mollifiers on

\mathbb{R}^d . Choose for example $\mathbf{g}_n = \varphi_n(\mathbf{g})$, with $\varphi_n(t) = t$ for $|t| \leq r_n$, $\varphi_n(t) = r_n \text{sign}(t)$ for $|t| > r_n$, $r_n \rightarrow +\infty$, and $k_n(x) = n^d k(nx)$ for a smooth non-negative function k with compact support. Considering $\mathbf{g}_n \star k_n - \mathcal{E}[\mathbf{p}(\mathbf{g}_n \star k_n)]$ instead of $\mathbf{g}_n \star k_n$, we then recover the centering condition for the regularizing sequence. Thanks to Proposition 4.10, we then prove that, for every $n \geq 0$, $\mathbf{g}_n \star k_n$ belongs to $H(\Xi, \mathbf{p}, \mu)$ and that $D(\mathbf{g}_n \star k_n) = -\mathbf{g}_n \star \nabla_x k_n \in L^\infty(\Xi, \mu; \mathbb{R}^d)$, so that the regularizing sequence fulfills the assumption in Theorem 3.2. For a suitable sequence $(r_n)_{n \geq 0}$, we can prove that $D(\mathbf{g}_n \star k_n)$ converges in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$ towards $D\mathbf{g}$. In the end, $\mathbf{p}^{-1}D[\mathbf{p}(\mathbf{g}_n \star k_n - \mathcal{E}[\mathbf{p}(\mathbf{g}_n \star k_n)])]$ converges in $L^2(\Xi, \mathbf{p}, \mu; \mathbb{R}^d)$ towards $\mathbf{p}^{-1}D(\mathbf{p}\mathbf{g})$. The central limit theorem for the field $\mathbf{p}^{-1}D_\ell(\mathbf{p}\mathbf{g})$, $1 \leq \ell \leq d$, then follows from Theorem 3.2 applied to the regularizing sequence and from the crucial Proposition 2.2, Chapter I, in Olla [24] which provides an estimate for the $L^2(\Xi, \mathbf{p}, \mu)$ norm of the additive functional driven by $\mathbf{p}^{-1}D_\ell(\mathbf{p}\mathbf{g})$ in terms of the $L^2(\Xi, \mathbf{p}, \mu)$ norm of \mathbf{g} .

More generally, the same proposition permits to extend the result to $L^2(\Xi, \mathbf{p}, \mu)$ converging infinite combinations of the form $\sum_{n \geq 0} \mathbf{p}^{-1}D_{\ell_n}(\mathbf{p}\mathbf{g}_n)$, $(\ell_n)_{n \geq 0} \in \{1, \dots, d\}^{\mathbb{N}}$, $(\mathbf{g}_n)_{n \geq 0} \in (H(\Xi, \mathbf{p}, \mu))^{\mathbb{N}}$, $\forall n \geq 0$, $\mathcal{E}(\mathbf{p}\mathbf{g}_n) = 0$, such that:

$$(8.1) \quad \Sigma \equiv \sum_{n \geq 0} \mathcal{E}(\mathbf{p}\mathbf{g}_n^2)^{1/2} < +\infty.$$

Indeed, due to (8.1), for every smooth function $\mathbf{h} \in \mathcal{S}_\infty$,

$$\begin{aligned} \mathcal{E}\left(\mathbf{p}\mathbf{h} \sum_{n \geq 0} \mathbf{p}^{-1}D_{\ell_n}(\mathbf{p}\mathbf{g}_n)\right) &= \sum_{n \geq 0} \mathcal{E}(\mathbf{h}D_{\ell_n}(\mathbf{p}\mathbf{g}_n)) \\ &= - \sum_{n \geq 0} \mathcal{E}(\mathbf{p}\mathbf{g}_n D_{\ell_n} \mathbf{h}) \\ &\leq \Sigma \mathcal{E}(\mathbf{p}|D\mathbf{h}|^2)^{1/2}, \end{aligned}$$

so that Proposition 2.2, Chapter I, in Olla [24] applies.

A more challenging problem would consist in discussing the centering condition $\mathcal{E}(\mathbf{p}\mathbf{g}) = 0$, from which we derived the strong convergence of the gradient fields of the resolvent. This condition appears as a trick to overcome the lack of centering properties for the gradient fields in $L^2(\Xi, \mathbf{p}, \mu)$, since this latter space is not imbedded into $L^1(\Xi, \mu)$. At this stage of our research, we have no clues for such a possible extension.

Another interesting open question concerns the choice of the initial condition. Due to the specific time reversal procedure for the process $(\eta_t)_{t \geq 0}$, our central limit theorem just holds under the invariant measure, and we are

not able to follow the discussion of Kipnis and Varadhan [15] about initial distributions of Dirac type. Again, right now, we have no ideas to improve our result in this direction.

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