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Krein's theory on strings applied to fluctuations of Lévy processes

Sonia Fourati

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LMI de l'INSA de Rouen, place Emile Blondel, 76130 Mont St Aignan. France
LPMA des Universités Paris VI et VII, 4 Place Jussieu, case 188, 75252 Paris, Cedex 05. France

ABSTRACT We give an interpretation of the bilateral exit problem for Lévy processes via the study of an elementary Markov chain. We exhibit a strong connection between this problem and Krein's theory on strings. For instance, for symmetric Lévy processes with bounded variations, the Lévy exponent is the correspondent spectral density and the Wiener-Hopf factorization turns out to be a version of Krein's entropy formula.

1 Wiener-Hopf factorization conditioned by the amplitude

Let (Ω, \mathcal{F}) be the canonical path space of real valued functions with finite or infinite life time ζ , X the canonical process, \check{X} the dual process $-X$. Denote by \mathbf{P} the law of a Lévy process and \mathbf{Q} the "law" of X killed either at an exponential time or at a "Lebesgue" time. In other words \mathbf{Q} is either :

("First case") :

$$\mathbf{Q}(dw) = \int_0^{+\infty} \mathbf{P}((X_s)_{s \in [0, t[} \in dw) e^{-t} dt$$

or ("Second case") :

$$\mathbf{Q}(dw) = \int_0^{+\infty} \mathbf{P}((X_s)_{s \in [0, t[} \in dw) dt$$

Let us define

$$F := X_{\zeta}^- (= \text{the final value of the canonical process})$$

$$M := \sup\{X_t; t \in [0, \zeta]\} \quad (= \text{the maximum value})$$

$$m := \inf\{X_t; t \in [0, \zeta]\} \quad (= \text{the minimum value})$$

Then

$M - m$ is the amplitude.

In the sequel, we exclude compound Poisson processes so that the time t when X_t is at its minimum value is unique and denoted by ρ .

Let \mathbf{Q}^\uparrow be the “law after the minimum”, more precisely, in the first case

$$\mathbf{Q}^\uparrow(dw) = \mathbf{Q}((X_{s+\rho} - m)_{s \in [0, \zeta - \rho]} \in dw)$$

and in the second case,

$$\mathbf{Q}^\uparrow(dw) = \mathbf{P}\left(\int_0^\zeta 1_{(X_t - X_{t-s})_{s \in [0, t]} \in dw} L(dt)\right)$$

where L is a local time at 0 of the reflected process $\sup_{s \in [0, t]} X_s - X_t$, ($\mathbf{Q}^\uparrow(dw)$ is thus defined up to a multiplicative constant):

The following independence property (in both cases) is well known :

$$\mathbf{Q}((X_{\rho-s}^- - m)_{s \in [0, \rho]} \in dw_1), (X_{s+\rho} - m)_{s \in [0, \zeta - \rho]} \in dw_2) = \check{\mathbf{Q}}^\uparrow(dw_1) \mathbf{Q}^\uparrow(dw_2)$$

Proposition 1 *In both cases and for every $x > 0$:*

$$\mathbf{Q}(-m \in du, M - m \leq x, F - m \in dv) = \check{\mathbf{Q}}^\uparrow(M \leq x, F \in du) \mathbf{Q}^\uparrow(M \leq x, F \in dv)$$

Argument : The event $M - m \leq x$ is the intersection of the two events : “The amplitude before the minimum does not exceed x ” and “the amplitude after the minimum does not exceed x ”. The result then follows easily from the independence property given above.

The purpose of the sequel is to give information on the two laws $\mathbf{Q}^\uparrow(M \leq x, F \in dv)$ and $\check{\mathbf{Q}}^\uparrow(M \leq x, F \in du)$ for every $x > 0$. They are obviously a characterization of the bilateral problem or equivalently of the law of the trivariate variable $(-m, M, F)$.

2 An elementary Markov Chain

Denote :

$$T_0 = \rho$$

$$Z_1 = \sup\{X_t - m ; t > \rho\} \quad \{T_1\} = \{t ; X_t \vee X_t^- - m = Z_1\}$$

$$Z_2 = \inf\{X_t - X_{T_1} \vee X_{T_1}^-; t > T_1\} \quad \{T_2\} = \{t > T_1; X_t \wedge X_t^- - X_{T_1} \vee X_{T_1}^- = Z_2\}$$

etc. Note that

$$Z_1 > 0, Z_2 < 0, \dots, \quad |Z_n| \geq |Z_{n+1}| \quad \sum_{n=1}^{\infty} Z_i = F - m$$

let

$$H(dx) := \mathbf{Q}^\dagger(M \in dx) = \mathbf{Q}(Z_1 \in dx)$$

$$H(x) := \mathbf{Q}^\dagger(M \leq x) = \mathbf{Q}(Z_1 \leq x) = \phi(x, 0)$$

We obtain easily the next two propositions

Proposition 2 : (Z_n) is a Markov chain whose transition kernel is

$$P(x, dy) = \frac{\check{H}(-dy)}{\check{H}(x)} 1_{y \in [-x, 0]} \quad \text{for } x \geq 0$$

$$P(x, dy) = \frac{H(-dy)}{H(x)} 1_{y \in [0, -x]} \quad \text{for } x \leq 0$$

Put

$$\phi(x, \lambda) := \mathbf{Q}^\dagger(M \leq x, e^{-\lambda F}) = \mathbf{Q}(Z_1 \leq x, e^{-\lambda \sum_{n=1}^{+\infty} Z_n})$$

Proposition 3 : For every $\lambda \in C$,

$$\phi(dx, \lambda) = e^{-\lambda x} \check{\phi}(x, -\lambda) \frac{H(dx)}{\check{H}(x)}$$

$$\check{\phi}(dx, -\lambda) = e^{\lambda x} \phi(x, \lambda) \frac{\check{H}(dx)}{H(x)}$$

For $x \rightarrow 0^+$:

$$\phi(x, \lambda) \sim H(x) \quad \check{\phi}(x, \lambda) \sim \check{H}(x)$$

For $x \rightarrow +\infty$ and $\lambda \in iR$:

$$\phi(x, \lambda) = \mathbf{Q}^\dagger(e^{-\lambda F}) \quad \check{\phi}(x, -\lambda) = \check{\mathbf{Q}}^\dagger(e^{\lambda F})$$

3 An example : Stable processes killed at a Lebesgue time

In this section X is a standard stable process under \mathbf{P} and

$$\mathbf{Q}(dw) := \int_0^{+\infty} \mathbf{P}((X_s)_{s \in [0,t]} \in dw) dt$$

Let α denote the stability index,

$$\gamma := \alpha \mathbf{P}(X_1 > 0) \quad \delta := \alpha \mathbf{P}(X_1 < 0)$$

$$\beta_{a,b}(t) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1} (1-t)^{b-1} 1_{t \in [0,1]} \quad \hat{\beta}_{a,b}(\lambda) := \int_0^{+\infty} e^{-\lambda t} \beta_{a,b}(t) dt$$

Theorem : For some constant $k \in]0, +\infty[$, we have the identity

$$\phi(x, \lambda) = k.x^\gamma \hat{\beta}_{\gamma, \delta+1}(\lambda)$$

Argument : It is clear that in the case involved, the functions H and \check{H} are of the form $H(x) = cx^\gamma$ and $\check{H}(x) = \check{c}x^\delta$ ($c, \check{c} \in]0, +\infty[$). When iterating the differential system given in proposition 3, one gets that the function $x \mapsto \phi(x, \lambda)$ satisfies the confluent hypergeometric equation whose solutions are known.

Consequences (essentially [Rogozin])

1)

$$\frac{\mathbf{P}(\int_0^{+\infty} 1_{X_s - \inf_{s \in [0,t]} X_s \leq 1} dL_s)}{\mathbf{P}(\int_0^{+\infty} 1_{X_s \leq 1} dL_s)} = \frac{\Gamma(\delta + 1)}{\Gamma(\alpha + 1)}$$

where L denote a local time at 0 of the reflected process $\sup_{s \in [0,t]} X_s - X_t$

2) Under \mathbf{Q}^\dagger , $\frac{F}{M}$ and M are independant and $\frac{F}{M} \stackrel{d}{=} \beta_{\gamma+1, \delta}(t) dt$

3)

$$\int_0^{+\infty} \mathbf{P}(S_t \leq x, -I_t \leq y) dt = \frac{\Gamma(\gamma + 1)\Gamma(\delta + 1)}{\Gamma^2(\alpha + 1)} x^\gamma y^\delta$$

4)

$$\mathbf{P}(T^a < T_b) = \int_0^{\frac{b}{a+b}} \beta_{\gamma, \delta}(t) dt$$

($T^a = \inf\{t; X_t > a\}$ $T_b = \inf\{t; X_t < -b\}$).

4 A survey of Krein theory on strings

Put

$$\mathcal{M} := \{m : [0, +\infty[\rightarrow [0, +\infty[, m \nearrow, m([0, +\infty[) = [0, +\infty[)\}$$

An element of \mathcal{M} is called a “string”.

$\mathcal{D} := \{\Delta \text{ measure on } [0, +\infty[\text{ such that}$

$$\int_1^{+\infty} \frac{\Delta(du)}{u^2} < +\infty; \int_0^1 \frac{\Delta(du)}{u^2} = +\infty; \int_1^{+\infty} \Delta(du) = +\infty\}$$

An element of \mathcal{D} is called a “spectral measure”.

FACTS

For every string $m \in \mathcal{M}$, we have the following properties,

1) For every $\lambda > 0$, the equation

$$\frac{d^2}{dm dx} X = \lambda^2 X$$

admits a unique solution (called “A-solution”) with conditions

$$A(0) = 1 \text{ and } A'(0) = 0$$

and a unique solution (called “D-solution”) with conditions

$$D'(0) = -1 \text{ and } D(+\infty) = 0$$

2)

$$D(x) = A(x) \int_x^{+\infty} \frac{1}{A^2(t)} dt$$

3) There exists a unique $\Delta \in \mathcal{D}$, (“the spectral measure of the string m ”), such that for every $\lambda > 0$:

$$D(0, \lambda) = \frac{2}{\pi} \int_0^{+\infty} \frac{\Delta(du)}{u^2 + \lambda^2}$$

moreover $m \rightarrow \Delta$ is a bijection from \mathcal{M} onto \mathcal{D}

5 When X is symmetric and $\int_0^1 \frac{1}{H^2(t)} dt < +\infty$

From now on, we suppose that X is symmetric. Let

$$s(x) := \int_0^x \frac{1}{H^2(t)} dt$$

and ψ be the exponent of the Lévy process :

$$\mathbf{P}(e^{-iuX_t}) =: e^{-t\psi(u)}$$

Theorem *We suppose that $s(x) < +\infty$ for every $x > 0$.*

1) *There exists a unique string $m \in \mathcal{M}$ such that*

$$m(s(x)) = \int_0^x \frac{H^4(t)}{4} dt$$

2) *The A-solution associated to m is given by :*

$$A(s(x), \lambda) = \frac{e^{\lambda \frac{x}{2}} \phi(x, \lambda) + e^{-\lambda \frac{x}{2}} \phi(x, -\lambda)}{H(x)}$$

3) *The spectral measure associated to m is $(\psi(u) + 1)du$ in the first case and $\psi(u)du$ in the second case.*

Argument : 1) is trivial.

2) is a consequence of elementary transformations of the differential system given in proposition 3.

In next section, we give a summary of the proof of 3) in the first case. The second case follows when varying the rate of killing (replace the rate 1 of the independant exponential time when the Lévy process X is killed by μ , any positive constant) and then make μ go to 0.

Proposition *The following four assertions are equivalent :*

- (i) $\psi(u)du \in \mathcal{D}$
- (i') $(\psi(u) + 1)du \in \mathcal{D}$
- (ii) *the paths of X have bounded variation*
- (iii) $\int_0^1 \frac{1}{H^2(t)} dt < +\infty$

Argument The equivalence between the properties (i) and (i') is trivial. the equivalence between (i) and (ii) has been proved by Vigon in his thesis.

The equivalence between (i) and (iii) follows easily from the previous theorem.

6 Proof of part 3 of the theorem

Lemma 1 *In the first case and for any Lévy process, we have for $\lambda > 0$*

$$-\log \mathbf{Q}^\uparrow(e^{-\lambda F}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \log(\psi(u) + 1) \left(\frac{1}{-iu + \lambda} - \frac{1}{-iu} \right) du$$

and if X is symmetric :

$$-\log \mathbf{Q}^\uparrow(e^{-\lambda F}) = \frac{2\lambda}{\pi} \int_0^{+\infty} \frac{\log(\psi(u) + 1)}{u^2 + \lambda^2} du$$

Argument for $\lambda \in iR$, the independence property of trajectories before and after the minimum (see first section) gives the equality :

$$-\log \mathbf{Q}(e^{-\lambda F}) = -\log \mathbf{Q}^\uparrow(e^{-\lambda F}) - \log \check{\mathbf{Q}}^\uparrow(e^{\lambda F})$$

One can prove that $\lambda \mapsto -\log \mathbf{Q}(e^{-\lambda F})$ is the exponent of a Lévy process with bounded variation and no drift and that the above equality is its decomposition into a sum of the exponent of a subordinator and the exponent of the opposite of a subordinator. Then extract $-\log \mathbf{Q}^\uparrow(e^{-\lambda F})$ from $-\log \mathbf{Q}(e^{-\lambda F})$ by a (sort of) Stieljes transform.

Lemma 2 [*Krein's Entropy Formula*] *When $\Delta(du) = \Delta'(u)du$, one gets*

$$\frac{2\lambda}{\pi} \int_0^{+\infty} \frac{\log[\Delta'(u)]}{u^2 + \lambda^2} du = \lim_{x \rightarrow +\infty} \log(-e^{2\lambda \cdot s^{-1}(x)} D'(x, \lambda) D(x, \lambda)) + \log \lambda$$

Final Argument We have noticed previously that

$$D(x) = A(x) \int_x^{+\infty} \frac{1}{A^2(t)} dt \quad \text{and} \quad A(s(x)) \sim e^{\lambda \frac{x}{2}} \mathbf{Q}^\uparrow(e^{-\lambda F}) \quad (x \rightarrow +\infty)$$

With the help of these two properties, one can easily find out that the second member of lemma 2 is equal to $-\log \mathbf{Q}^\uparrow(e^{-\lambda F})$. Lemma 1 then gives the equality $\Delta'(u) = \psi(u) + 1$ by identification.

7 When X is symmetric and has unbounded variations

In that case, we have $s(x) = \int_0^x \frac{1}{H^2(t)} dt = +\infty$. But one still can define :

$$\tilde{A}(x) := \frac{e^{\lambda \frac{x}{2}} \phi(x, \lambda) + e^{-\lambda \frac{x}{2}} \phi(x, -\lambda)}{H(x)}$$

$$\tilde{D}(x) := \tilde{A}(x) \int_x^{+\infty} \frac{1}{\tilde{A}^2(t)} \frac{1}{H^2(t)} dt$$

Notice that $\tilde{D}(x) \in]0, +\infty[$ for every $x > 0$ but $\tilde{D}(0) = +\infty$.

Still we have that the measure $\frac{\psi(u)+1}{u^2+1} du$ is in \mathcal{D} (this is true for any symmetric Lévy process). Denote by D_1 the D -solution associated to this spectral measure $\frac{\psi(u)+1}{u^2+1} du$.

And define

$$t(x) := x + \frac{1}{H^2(x)} \cdot \frac{\tilde{D}(x, 1)}{-\tilde{D}'(x, 1)}$$

Theorem [*Adaptation of “ Rule 4” of Dym-Mac Kean*] :
t is increasing, $t(0^+) = 0$ and $t(+\infty) = +\infty$ and,

$$\frac{D_1(t(x), \lambda)}{-\tilde{D}'(x, \lambda)} = \frac{1}{\lambda^2 - 1} \left(\frac{\tilde{D}(x, \lambda)}{-\tilde{D}'(x, \lambda)} - \frac{\tilde{D}(x, 1)}{-\tilde{D}'(x, 1)} \right)$$

Remark

The Brownian case can be computed through the solution corresponding to the spectral measure $\Delta(du) = du \in \mathcal{D}!!!$

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