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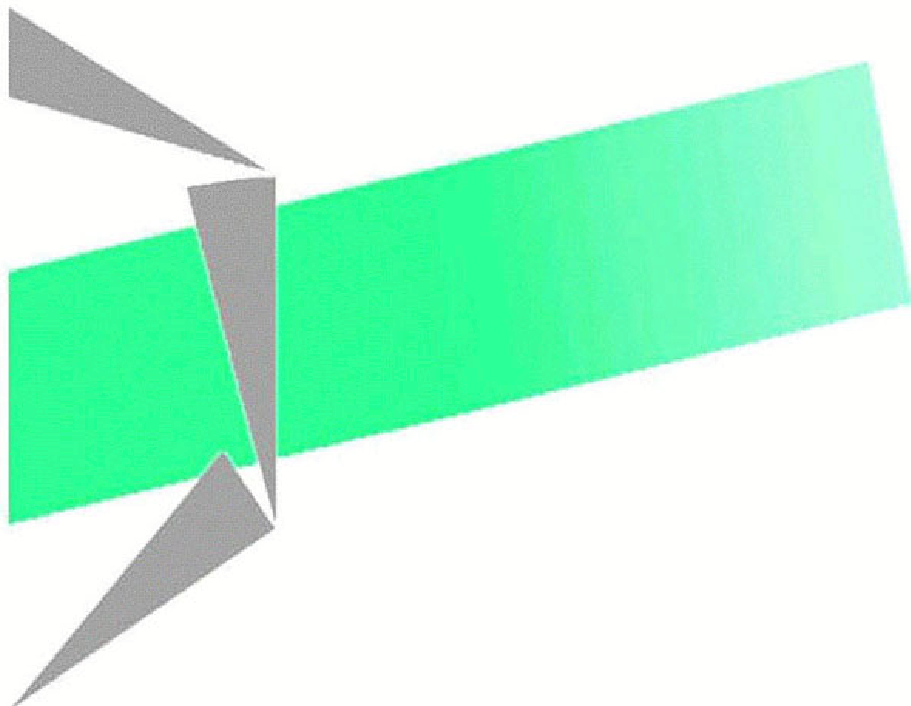
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A  $\mathbb{Z}_n$ -homology version of the  
Borsuk-Ulam theorem

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# A $\mathbb{Z}_n$ -homology version of the Borsuk-Ulam theorem

Frédéric Meunier\*

## 1 Introduction

The aim of our paper is to show that it is possible to generalize the homology version of the Borsuk-Ulam theorem of James W. Walker [2].

A topological space  $X$  is called a *free  $G$ -space*, if  $G$  is a group acting on  $X$  such that for any  $\mu \in G$ ,  $\mu \neq \text{id}$ ,  $\mu$  has no fixed-point. Defining  $\mathcal{S}^d$  as  $\{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ , the  $d$ -sphere  $\mathcal{S}^d$  can be seen as a free  $\mathbb{Z}_2$ -space where the nontrivial  $\mathbb{Z}_2$ -action is the antipodal map  $x \mapsto -x$ .

Let  $G$  be a group acting on two topological spaces  $X$  and  $Y$ . Let  $f : X \rightarrow Y$  be a continuous map such that  $\mu \circ f = f \circ \mu$  for any  $\mu \in G$ , then  $f$  is called a  *$G$ -equivariant map*.

The celebrated Borsuk-Ulam theorem asserts that there is no  $\mathbb{Z}_2$ -equivariant map from the  $d$ -sphere  $\mathcal{S}^d$  to the  $(d-1)$ -sphere  $\mathcal{S}^{d-1}$  with respect to the antipodal map.

The homology version of James W. Walker is the following one (we consider reduced homology):

**Theorem 1** *Suppose  $X$  is a  $\mathbb{Z}_2$ -space and  $g : X \rightarrow \mathcal{S}^d$  is an  $\mathbb{Z}_2$ -equivariant map with respect to the antipodal map. Then there exists an integer  $j \leq d$ , and a homology class  $\beta$  of  $\tilde{H}_j(X; \mathbb{Z}_2)$  such that  $\beta$  is nonzero and  $\nu_*(\beta) = \beta$ , where  $\nu$  is the non trivial  $\mathbb{Z}_2$ -action. Furthermore, if no such  $\beta$  exists for  $j$  less than  $d$ , then  $\beta$  can be chosen such that  $g_*(\beta)$  is the nonzero element of  $\tilde{H}_d(\mathcal{S}^d; \mathbb{Z}_2)$ .*

We show in this note that if we endow the sphere  $\mathcal{S}^d$  with a free  $\mathbb{Z}_n$ -action, we can obtain a direct generalization of the theorem above, and this following step by step Walker's proof.

We recall that if  $n \neq 2$ , the existence of a free  $\mathbb{Z}_n$ -action implies that  $d$  is odd (for instance, see Matousek's book [1], p. 135).

## 2 Tools and Notations

We will denote  $\mathbb{Z}/n\mathbb{Z}$  by  $\mathbb{Z}_n$ .

id will always mean the identity map.

If  $A$  and  $B$  are sets, we write  $A \uplus B$  for the set  $(A \times \{1\}) \cup (B \times \{2\})$ . So  $A \uplus B$  is a disjoint union of  $A$  and  $B$ .

If  $K$  and  $L$  are two simplicial complexes with vertex sets  $V(K)$  and  $V(L)$ , the join of  $K$  and  $L$ , denoted by  $K * L$ , is the simplicial complex with vertex set  $V(K) \uplus V(L)$  and with the set of simplices

$$\{F \uplus G : F \in K, G \in L\}.$$

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The join of a complex  $K$   $n$  times by itself is denoted by  $K^{*n}$ .

Suppose that  $X$  and  $Y$  are two topological spaces which have respectively  $T$  and  $T'$  as triangulations. The join of  $X$  and  $Y$ , denoted by  $X * Y$ , is then the geometric realization of the join  $T * T'$ . The join of  $X$   $n$  times by itself is denoted by  $X^{*n}$ .

Let  $K$  be a simplicial complex and let  $\sigma = [v_0, v_1, \dots, v_r]$ ,  $\sigma' = [v'_0, v'_1, \dots, v'_s]$  be two simplices of  $K$ , then we define the following notation:

$$[\sigma, \sigma'] = \begin{cases} [v_0, v_1, \dots, v_r, v'_0, v'_1, \dots, v'_s] & \text{if } v_i \neq v'_j \text{ for any } i, j \\ 0 & \text{if not.} \end{cases}$$

This allows us to define  $[c, c']$  for two chains  $c = \sum_i n_i \sigma_i$ ,  $c' = \sum_j n'_j \sigma'_j$  as  $\sum_{i,j} n_i n'_j [\sigma_i, \sigma'_j]$ , and to define inductively  $[c, c', \dots, c^{(t)}]$  for  $t$  chains.

An *elementary  $j$ -chain* is a  $j$ -simplex with coefficient 1.

### 3 The Theorem

**Theorem 2** *Let  $X$  be a topological space. Suppose that  $\mathbb{Z}_n$  acts freely on  $X$  and on  $S^d$ , where  $d$  is odd if  $n \neq 2$ . For  $\nu$  the  $\mathbb{Z}_n$ -action corresponding to 1, we define  $\alpha := \sum_{k=0}^{n-1} \nu_{\#}^k$  and  $\bar{\alpha} := \nu_{\#} - id_{\#}$ .*

*If  $f : X \rightarrow S^d$  is a  $\mathbb{Z}_n$ -equivariant map, then there are  $\beta \in \tilde{H}_i(X; \mathbb{Z}_n)$  and  $i$  even  $\leq d$  such that  $\beta \neq 0$  and  $\alpha_*(\beta) = 0$ , or there are  $\beta \in \tilde{H}_i(X; \mathbb{Z}_n)$  and  $i$  odd  $\leq d$  such that  $\beta \neq 0$  and  $\bar{\alpha}_*(\beta) = 0$ . Moreover, if there is no such  $i$  strictly less than  $d$ , then  $\beta$  can be chosen such that  $f_*(\beta)$  is a generator of  $\tilde{H}_d(S^d; \mathbb{Z}_n)$ .*

Remark that for  $n = 2$ , we have  $\alpha = \bar{\alpha}$ , and it is straightforward to check that, in this case, our theorem is exactly the theorem of James Walker.

**Proof of theorem:** In the following, all the chains will be taken with coefficients in  $\mathbb{Z}_n$ . It is easy to see that  $\alpha$  and  $\bar{\alpha}$  commute with  $f_{\#}$  and that  $\alpha \bar{\alpha} = \bar{\alpha} \alpha = 0$ . Furthermore, as  $\nu_{\#} \alpha = \alpha$ , we have  $\alpha \alpha = n \alpha = 0 \pmod n$ .

If there exists a collection  $(h_i)_{i=0, \dots, d}$  of  $i$ -chains of  $S^d$  such that

- $h_0$  is an elementary 0-chain,
- $\partial h_i = \bar{\alpha} h_{i-1}$  for  $i$  odd and  $\geq 1$ ,
- $\partial h_i = \alpha h_{i-1}$  for  $i$  even and  $\geq 2$ ,
- $\alpha h_d$  generates  $\tilde{H}_d(S^d; \mathbb{Z}_n)$ ,

the theorem falls following the scheme used in [2]:

suppose that for  $i < d$ , there is no  $\beta$  as in the terms of the theorem. We have to prove that there is  $\beta \in \tilde{H}_i(X; \mathbb{Z}_n)$  such that  $\bar{\alpha}_* \beta = 0$  and that  $f_*(\beta)$  is a generator of  $\tilde{H}_d(S^d; \mathbb{Z}_n)$ .

We construct singular  $i$ -chains  $(c_i)$  in  $X$  (singular because  $X$  may be not triangulable) such that

- $c_0$  is an elementary 0-chain,
- $\partial c_i = \bar{\alpha} c_{i-1}$  for  $i$  odd and  $\geq 1$ ,
- $\partial c_i = \alpha c_{i-1}$  for  $i$  even and  $\geq 2$ .

Such a construction is easy:  $X$  is not empty because we have assumed that  $\tilde{H}_{-1}(X; \mathbb{Z}_n)$  contains no nonzero  $\beta$  such that  $\bar{\alpha}\beta = 0$  (we work with reduced homology). Pick a vertex in  $X$ , and let  $c_0$  be the corresponding elementary 0-chain.  $\bar{\alpha}c_0$  is a cycle, and, as  $\alpha\bar{\alpha}c_0 = 0$ , there is a 1-chain  $c_1$  such that  $\partial c_1 = \bar{\alpha}c_0$ .

Suppose that  $\partial c_i = \bar{\alpha}c_{i-1}$  for some  $i$  odd  $\leq d$ . We compute  $\partial\alpha c_i = \alpha\partial c_i = \alpha\bar{\alpha}c_{i-1} = 0$ , so  $\alpha c_i$  is a cycle. Since  $\bar{\alpha}\alpha c_i = 0$ , there exists a  $(i+1)$ -chain  $c_{i+1}$  such that  $\partial c_{i+1} = \alpha c_i$ .

Suppose that  $\partial c_i = \alpha c_{i-1}$  for some  $i$  even  $\leq d$ . We compute  $\partial\bar{\alpha}c_i = \bar{\alpha}\partial c_i = \bar{\alpha}\alpha c_{i-1} = 0$ , so  $\bar{\alpha}c_i$  is a cycle. Since  $\alpha\bar{\alpha}c_i = 0$ , there exists a  $(i+1)$ -chain  $c_{i+1}$  such that  $\partial c_{i+1} = \bar{\alpha}c_i$ .

This completes the inductive definition of  $c_0, c_1, \dots, c_d$ .

Still assuming that the collection  $(h_i)$  of  $i$ -chains exists, we define another collection  $(e_i)$  of  $i$ -chains of  $\mathcal{S}^d$  such that  $h_i - f_{\#}c_i - \bar{\alpha}e_i$  is a cycle of  $\mathcal{S}^d$  if  $i$  is odd and  $h_i - f_{\#}c_i - \alpha e_i$  is a cycle of  $\mathcal{S}^d$  if  $i$  is even.

$e_0 := 0$  works:  $\partial(h_0 - f_{\#}c_0) = 1 - 1 = 0$ .

Let  $i$  be even and suppose that  $e_i$  is defined such that  $h_i - g_{\#}c_i - \alpha e_i$  is a cycle. As  $\mathcal{S}^d$  is  $(d-1)$ -connected, there is a  $(i+1)$ -chain  $e_{i+1}$  such that

$$\partial e_{i+1} = h_i - f_{\#}c_i - \alpha e_i.$$

Applying  $\bar{\alpha}$ , we obtain

$$\partial\bar{\alpha}e_{i+1} = \bar{\alpha}h_i - \bar{\alpha}f_{\#}c_i - \bar{\alpha}\alpha e_i.$$

Using the definition of  $h_{i+1}$  and  $c_{i+1}$ , this becomes

$$\partial\bar{\alpha}e_{i+1} = \partial h_{i+1} - \partial f_{\#}c_{i+1}.$$

Thus  $h_{i+1} - f_{\#}c_{i+1} - \bar{\alpha}e_{i+1}$  is a cycle, as desired.

For  $i$  odd, exchanging the role of  $\alpha$  and  $\bar{\alpha}$ , the construction is identical.

Now, we are very close to the conclusion. Thus, we are going to achieve the proof, and after that, we will complete the lack in this proof, that is the definition of the chains  $h_i$ .

$h_d - f_{\#}c_d - \bar{\alpha}e_d$  is a cycle, thus is homologous to  $k(\alpha h_d)$  for a certain integer  $k$ .  $\alpha h_d - f_{\#}\alpha c_d$  is thus in the same class of homology than 0 (remember that  $\alpha^2 = 0 \pmod n$ ). Thus, if we take  $\beta$  in the class of homology of  $\alpha c_d$ ,  $f_*(\beta)$  generates  $\tilde{H}_d(\mathcal{S}^d; \mathbb{Z}_n)$  and  $\bar{\alpha}_*\beta = 0$ . This is the conclusion of the theorem.

It remains to exhibit a construction of the collection  $(h_i)$ . We can assume  $d$  odd (see the last remark of introduction): if  $n = 2$ ,  $h_i$  can be taken as  $i$ -chains corresponding to hemispheres of  $\mathcal{S}^d$  (see [2]).

We write  $d = 2q - 1$ . As between two  $d$ -dimensional  $(d-1)$ -connected free  $\mathbb{Z}_n$ -spaces, it is easy to build a  $\mathbb{Z}_n$ -equivariant map, we can suppose that the  $\mathbb{Z}_n$ -action is defined on  $\mathcal{S}^d$  using a  $\mathbb{Z}_n$ -action on  $\mathcal{S}^1$  (as rotations by  $\frac{2k\pi}{n}$ , for instance) and the well-known homeomorphism  $\mathcal{S}^d \cong (\mathcal{S}^1)^{*q}$ .

So, in the following,  $\mathcal{S}^d$  is seen as the joins of  $q$  copies of  $\mathcal{S}^1$ . For simplicity, each copy of  $\mathcal{S}^1$  is triangulated and becomes a  $n$ -gone. The  $n$  vertices of the  $j$ th copy are in this order:

$$v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(n)},$$

and the  $n$  edges:

$$(v_j^{(1)}, v_j^{(2)}), (v_j^{(2)}, v_j^{(3)}), \dots, (v_j^{(n)}, v_j^{(1)}).$$

On the  $j$ th copy of  $\mathcal{S}^1$ ,  $\nu$  acts by

$$v_j^{(k)} \mapsto v_j^{(k+1)},$$

with the convention  $n + 1 := 1$ .

First define  $h_0$  as the elementary 0-chain corresponding to  $v_1^{(1)}$ .

For  $h_1$ , we take the elementary 1-chain corresponding to the simplex  $[v_1^{(1)}, v_1^{(2)}]$ . We have as required  $\partial h_1 = \nu_{\#} h_0 - h_0 = \bar{\alpha} h_0$ .

We define  $h_2 := [v_2^{(1)}, \alpha h_1]$ ,

$\partial h_2 = \alpha h_1 - [v_2^{(1)}, \partial \alpha h_1] = \alpha h_1 - [v_2^{(1)}, \alpha \bar{\alpha} h_0] = \alpha h_1$ , as required.

We go on: for  $q - 1 \geq j \geq 1$ , we define

$$h_{2j} := [v_{j+1}^{(1)}, \alpha h_{2j-1}]$$

$$h_{2j+1} := [v_{j+1}^{(1)}, \nu_{\#} h_{2j}].$$

We have  $\partial h_{2j} = \alpha h_{2j-1} - [v_{j+1}^{(1)}, \alpha \partial h_{2j-1}] = \alpha h_{2j-1} - [v_{j+1}^{(1)}, \alpha \bar{\alpha} h_{2j-2}] = \alpha h_{2j-1}$ .

And  $\partial h_{2j+1} = \nu_{\#} h_{2j} - [v_{j+1}^{(1)}, \nu_{\#} \partial h_{2j}] = \nu_{\#} h_{2j} - [v_{j+1}^{(1)}, \nu_{\#} \alpha h_{2j-1}] = \nu_{\#} h_{2j} - [v_{j+1}^{(1)}, \alpha h_{2j-1}] = \bar{\alpha} h_{2j}$ .

So it remains to check that the homology class of  $\alpha h_{2q-1}$  generates  $\tilde{H}_{2q-1}(\mathcal{S}^d; \mathbb{Z}_n)$ .

This is straightforward:  $h_1 = [v_1^{(1)}, v_2^{(2)}]$ ,  $h_3 = [v_2^{(1)}, v_2^{(2)}, \alpha[v_1^{(1)}, v_1^{(2)}]]$ ,

and, by direct induction for  $j \geq 2$ ,

$$h_{2j+1} = [v_{j+1}^{(1)}, v_{j+1}^{(2)}, \alpha[v_j^{(1)}, v_j^{(2)}], \alpha[v_{j-1}^{(1)}, v_{j-1}^{(2)}], \dots, \alpha[v_1^{(1)}, v_1^{(2)}].$$

Hence

$$\alpha h_{2q-1} = [\alpha[v_q^{(1)}, v_q^{(2)}], \dots, \alpha[v_1^{(1)}, v_1^{(2)}]]$$

This is the formal sum of all  $d$ -simplices of the triangulation of  $\mathcal{S}^d$  induced by the triangulations of the  $\mathcal{S}^1$ , and thus generates  $\tilde{H}_{2q-1}(\mathcal{S}^d; \mathbb{Z}_n)$ . ■

**Remark 1** The Dold's theorem is the following generalization of Borsuk-Ulam theorem: there is no  $\mathbb{Z}_n$ -equivariant map  $f : X \rightarrow Y$  between free  $\mathbb{Z}_n$ -spaces such that the dimension of  $Y$  is not larger than the connectivity of  $X$ . Our theorem implies Dold's theorem in the case when the dimension of  $Y$  is odd.

**Remark 2** Even in the case when we know nothing about the homology groups of  $X$ , the conclusion of the theorem may be true if we can exhibit a collection of  $i$ -chains  $(c_i)$ . As we saw, this is sufficient, the definition of the homology groups is used only to build the collection  $(e_i)$ , which are chains of  $\mathcal{S}^d$ , whose homology groups are perfectly known.

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- [1] Jirí Matousek, *Using the Borsuk-Ulam Theorem*, Springer-Verlag, Berlin, Heidelberg, 2003.
- [2] James W. Walker, A homology version of the Borsuk-Ulam theorem, *Amer. Math. Monthly*, 90 (1983), 466-468.

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