Calculation of the Maslov-Type Index Intervening in Gutzwiller's Trace Formula: The Final Chapter?

Maurice A. de Gosson, Serge de Gosson

Calculation of the Maslov-Type Index Intervening in Gutzwiller's Trace Formula: The Final Chapter?

Maurice de Gosson
Universität Potsdam, Inst. f. Mathematik
Am Neuen Palais 10, D-14415 Potsdam
Current address:
Universidade de São Paulo
Departamento de Matemãtica
CEP 05508-900 São Paulo
E-mail: maurice.degosson@gmail.com

Serge de Gosson
Växjö University, MSI,
Växjö, SE-351 95
E-mail: sergedegosson@gmail.com

August 25, 2005

Abstract

The aim of this paper is to give an explicit formula for the calculation of the Gutzwiller–Maslov index of a Hamiltonian periodic orbit. We identify the index appearing in Gutzwiller's trace formula with a non-trivial extension of the Conley–Zehnder index. This index can be related to the usual Maslov index using the theory of the metaplectic group and the formalism of the Arnol'd–Leray–Maslov index developed in previous work, and is extended to symplectic paths with arbitrary endpoint.

Keywords: Gutzwiller trace formula, Conley–Zehnder index, Maslov index.

AMS Classification (2000): 37J05, 53D12, 53Z05, 81S30

1 Introduction

Let us briefly recall the origin of the problem; our argument will be rather heuristic. While the distribution of the spectrum of a quantum Hamiltonian $\widehat{H}$ is well-understood when H is integrable, the situation is far from being fully understood in the general case. Assume that the spectrum of $\widehat{H}$ is discrete and consists of numbers $E_1, E_2, ...$ One wants to find an asymptotic expression, for $\hbar \to 0$, of the "level density"

$$d(E) = \sum_{k=1}^{\infty} \delta(E - E_j) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+} \operatorname{Tr} G(x, x', E + i\varepsilon)$$

(G the Green function for $\widehat{H}$). One of the most popular methods for calculating approximations to $d(E)$ is Gutzwiller's trace formula [11]. Let us recall this formula; "derivations" of which abound in the literature [1, 19] to cite a few. One can write $d(E) = \bar{d}(E) + \widetilde{d}(E)$ where $\bar{d}(E)$ is the Thomas–Fermi or smoothed density of states, and $\widetilde{d}(E)$ the "oscillating term". Gutzwiller's achievement was to propose the following approximation, for small $\hbar$, of $\widetilde{d}(E)$: $\widetilde{d}(E) = \widetilde{d}_{\text{Gutz}}(E) + O(\hbar)$ where

$$\widetilde{d}_{\text{Gutz}}(E) = \frac{1}{\pi\hbar} \operatorname{Re} \sum_{\gamma} \frac{T_\gamma i^{\mu_\gamma}}{\sqrt{|\det(S_\gamma - I)|}} e^{i\mathcal{A}_\gamma/\hbar}. \tag{1}$$

The sum in the right-hand side of the formula above is taken over all *periodic orbits* γ with period T_γ of H (including their repetitions); the exponent μ_γ is an integer, $\mathcal{A}_\gamma = \oint_\gamma p dx$ the action of the orbit, and S_γ the stability matrix of γ. Gutzwiller's formula (1) is plagued, outside a few (interesting!) cases by difficulties due to possible divergences of the series, insufficient error estimates, etc. We will not discuss these delicate problems here; it is well-known that there are formidable roadblocks on the way to a full understanding or justification of Gutzwiller's formula. What we will do is instead to focus on the integers μ_γ, to which much literature has been devoted (see for instance [1, 4, 19] and the references therein). As is well-known, μ_γ is not the usual Maslov index familiar from EBK quantization of Lagrangian manifolds, but rather (up to the sign) the Conley–Zehnder index [3, 12] used in the study of periodic Hamiltonian orbits. This has been

established in the deep paper by Muratore-Ginnaneschi in [17], and by one of the authors of the present paper in [10] using previous constructions in [5, 6]; our argument was based on an ingenious derivation of Gutzwiller's formula by Mehlig and Wilkinson [15] using the Weyl representation of metaplectic operators.

It turns out that the Conley–Zehnder index is not very easy to calculate in practice, and one does not immediately see how it is related to the usual Maslov index. In addition one lacks a general rigorous formula allowing to calculate the index of repeated orbits. (The formulae proposed in [21] are based on a clever but mathematically illegitimate derivation of Gutzwiller's formula).

In this paper we will:

1. Redefine the Conley–Zehnder index in terms of globally defined indices (the Wall–Kashiwara index and the Arnol'd–Leray–Maslov index); we will thus obtain a non-trivial extension which is explicitly computable for all paths, even in the case $\det(S-I)=0$; this is useful in problems where degeneracies arise (for instance the isotropic harmonic oscillator, see [18]);

2. We will prove a formula for the Conley–Zehnder index of the product of two symplectic paths. We will see that in particular the index of an orbit which is repeated r times is

$$\nu_{\gamma_r} = r\nu_\gamma + \tfrac{1}{2}(r-1)\operatorname{sign} M_S$$

where M_S is the symplectic Cayley transform of the monodromy matrix S;

3. We will finally prove that the Conley–Zehnder index ν_γ of a non-degenerate periodic orbit γ is related to its Maslov index m_γ by the simple formula

$$\nu_\gamma = m_\gamma - \operatorname{Inert} W_\gamma$$

where $\operatorname{Inert} W_\gamma$ is Morse's "index of concavity" [16, 17].

We close the article by performing explicit calculations in the case of the two-dimensional anisotropic harmonic oscillator; this allows us to recover a formula obtained by non-rigorous methods in the literature.

Notations

We will denote by σ the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_p$:

$$\sigma(z, z') = \left\langle p, x' \right\rangle - \left\langle p', x \right\rangle \quad \text{if} \quad z = (x, p),\ z' = (x'p')$$

that is, in matrix form

$$\sigma(z, z') = \left\langle Jz, z' \right\rangle \quad , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The real symplectic group $\mathrm{Sp}(n)$ consists of all linear automorphisms S of $\mathbb{R}^{2n}$ such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all z, z'. Equivalently:

$$S \in \mathrm{Sp}(n) \Longleftrightarrow S^T J S = S J S^T = J.$$

$\mathrm{Sp}(n)$ is a connected Lie group and $\pi_1[\mathrm{Sp}(n)] \simeq (\mathbb{Z}, +)$. We denote by $\mathrm{Lag}(n)$ the Lagrangian Grassmannian of $(\mathbb{R}^{2n}, \sigma)$, that is: $\ell \in \mathrm{Lag}(n)$ if and only ℓ is a n-plane in $\mathbb{R}^{2n}$ on which σ vanishes identically. We will write $\ell_X = \mathbb{R}^n_x \times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$ (the "horizontal" and "vertical" polarizations).

2 Prerequisites

In this section we review previous results [7, 8, 9] on Lagrangian and symplectic Maslov indices generalizing those of Leray [13]. An excellent comparative study of the indices used here with other indices appearing in the literature can be found in Cappell *et al.* [2].

In what follows (E, ω) is a finite-dimensional symplectic space, $\dim E = 2n$, and $\mathrm{Sp}(E, \omega)$, $\mathrm{Lag}(E, \omega)$ the associated symplectic group and Lagrangian Grassmannian.

2.1 The Wall–Kashiwara index

Let (ℓ, ℓ', ℓ'') be a triple of elements of $\mathrm{Lag}(E, \omega)$; by definition [2, 14, 22] the Wall–Kashiwara index (or: signature) $\tau(\ell, \ell', \ell'')$ is the the signature of the quadratic form

$$Q(z, z', z'') = \sigma(z, z') + \sigma(z', z'') + \sigma(z'', z')$$

on $\ell \oplus \ell' \oplus \ell''$. The index τ is antisymmetric:

$$\tau(\ell, \ell', \ell'') = -\tau(\ell', \ell, \ell'') = -\tau(\ell, \ell'', \ell') = -\tau(\ell'', \ell', \ell);$$

it is a symplectic invariant:

$$\tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'') \ \text{ for } S \in \mathrm{Sp}(n)$$

and it has the following essential cocycle property:

$$\tau(\ell, \ell', \ell'') - \tau(\ell', \ell'', \ell''') + \tau(\ell', \ell'', \ell''') - \tau(\ell', \ell'', \ell''') = 0. \tag{2}$$

Moreover its values modulo 2 are given by the formula:

$$\tau(\ell, \ell', \ell'') \equiv n + \dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \mod 2. \tag{3}$$

Let $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$; identifying $\mathrm{Lag}(E', \omega') \oplus \mathrm{Lag}(E'', \omega'')$ with a subset of $\mathrm{Lag}(E, \omega)$ we have the following additivity formula:

$$\tau(\ell_1 \oplus \ell_2, \ell_1' \oplus \ell_2', \ell_1'' \oplus \ell_2'') = \tau_1(\ell_1, \ell_1', \ell_1'') + \tau_2(\ell_2, \ell_2', \ell_2'')$$

where τ_1 and τ_2 are the signatures on $\mathrm{Lag}(E', \omega')$ and $\mathrm{Lag}(E'', \omega'')$.

The following Lemma will be helpful in our study of the Conley–Zehnder index:

Lemma 1 *(i) If $\ell \cap \ell'' = 0$ then $\tau(\ell, \ell', \ell'')$ is the signature of the quadratic form*

$$Q'(z') = \omega(P_{\ell\ell''} z', z') = \omega(z', P_{\ell''\ell} z')$$

on ℓ', where $P_{\ell\ell''}$ is the projection onto ℓ along ℓ'' and $P_{\ell''\ell} = I - P_{\ell\ell''}$ is the projection on ℓ'' along ℓ. (ii) Let (ℓ, ℓ', ℓ'') be a triple of Lagrangian planes such that an $\ell = \ell \cap \ell' + \ell \cap \ell''$. Then $\tau(\ell, \ell', \ell'') = 0$.

(See *e.g.* [14] for a proof).

The index of inertia of the triple (ℓ, ℓ', ℓ'') is defined by

$$\mathrm{Inert}(\ell, \ell', \ell'') = \frac{1}{2}(\tau(\ell, \ell', \ell'') + n + \dim \ell \cap \ell' - \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell); \tag{4}$$

in view of (3) it is an integer. When the Lagrangian planes ℓ, ℓ', ℓ'' are pairwise transverse it follows from the first part of Lemma 1 that $\mathrm{Inert}(\ell, \ell', \ell'')$ coincides with the index of inertia defined by Leray [13]: see [7, 8].

2.2 The ALM index

Recall [7, 8] (also [9] for a review) that the ALM (=Arnol'd–Leray–Maslov) index on the universal covering $\mathrm{Lag}_\infty(E,\omega)$ of $\mathrm{Lag}(E,\omega)$ is the unique mapping

$$\mu : (\mathrm{Lag}_\infty(E,\omega))^2 \longrightarrow \mathbb{Z}$$

having the two following properties:

- μ is locally constant on each set $\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\}$ $(0 \leq k \leq n)$;
- For all ℓ_∞, ℓ'_∞, ℓ''_∞ in $\mathrm{Lag}_\infty(E,\omega)$ with projections ℓ, ℓ', ℓ'' we have

$$\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'') \tag{5}$$

  where τ is the Wall–Kashiwara index on $\mathrm{Lag}(E,\omega)$.

The ALM index has in addition the following properties:

$$\mu(\ell_\infty, \ell'_\infty) \equiv n + \dim \ell \cap \ell' \mod 2 \tag{6}$$

$(n = \frac{1}{2}\dim E)$ and

$$\mu(\beta^r \ell_\infty, \beta^{r'} \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 2(r - r') \tag{7}$$

for all integers r and r'; here β denotes the generator of $\pi_1[\mathrm{Lag}(E,\omega)] \equiv (\mathbb{Z}, +)$ whose image in $\mathbb{Z}$ is $+1$. From the dimensional additivity property of the signature τ immediately follows that if $\ell_{1,\infty} \oplus \ell_{2,\infty}$ and $\ell'_{1,\infty} \oplus \ell'_{2,\infty}$ are in

$$\mathrm{Lag}_\infty(E',\omega') \oplus \mathrm{Lag}_\infty(E'',\omega'') \subset \mathrm{Lag}_\infty(E,\omega)$$

then

$$\mu(\ell_{1,\infty} \oplus \ell_{2,\infty}, \ell'_{1,\infty} \oplus \ell'_{2,\infty}) = \mu'(\ell_{1,\infty}, \ell'_{1,\infty}) + \mu''(\ell_{2,\infty}, \ell'_{2,\infty}) \tag{8}$$

where μ' and μ'' are the ALM indices on $\mathrm{Lag}_\infty(E',\omega')$ and $\mathrm{Lag}_\infty(E'',\omega'')$, respectively.

When (E,ω) is the standard symplectic space $(\mathbb{R}^{2n},\sigma)$ the "Souriau mapping" [20] identifies $\mathrm{Lag}(E,\omega) = \mathrm{Lag}(n)$ with the set

$$\mathrm{W}(n,\mathbb{C}) = \{w \in \mathrm{U}(n,\mathbb{C}) : w = w^T\}$$

of symmetric unitary matrices. This is done by associating to to $\ell = u\ell_P$ $(u \in \mathrm{U}(n,\mathbb{C}))$ the matrix $w = uu^T$; the Maslov bundle $\mathrm{Lag}_\infty(n)$ is then identified with

$$W_\infty(n,\mathbb{C}) = \{(w,\theta) : w \in \mathrm{W}(n,\mathbb{C}),\ \det w = e^{i\theta}\};$$

the projection $\pi^{\rm Lag} : \ell_\infty \longmapsto \ell$ becoming $(w, \theta) \longmapsto w$. The ALM index is then calculated as follows:

- If $\ell \cap \ell' = 0$ then

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi}\left[\theta - \theta' + i\,\mathrm{Tr}\,\mathrm{Log}(-w(w')^{-1}\right] \tag{9}$$

  (the transversality condition $\ell \cap \ell'$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue);

- If $\ell \cap \ell' \neq 0$ one chooses any ℓ'' such that $\ell \cap \ell'' = \ell' \cap \ell'' = 0$ and one then calculates $\mu(\ell_\infty, \ell'_\infty)$ using the formula (5), the values of $\mu(\ell_\infty, \ell''_\infty)$ and $\mu(\ell'_\infty, \ell''_\infty)$ being given by (9). (The cocycle property (2) of τ guarantees that the result does not depend on the choice of ℓ'', see [7, 8]).

2.3 The relative Maslov indices on $\mathrm{Sp}(E, \omega)$

We begin by recalling the definition of the Maslov index for loops in $\mathrm{Sp}(n)$. Let γ be a continuous mapping $[0, 1] \longrightarrow \mathrm{Sp}(n)$ such that $\gamma(0) = \gamma(1)$, and set $\gamma(t) = S_t$. Then $U_t = (S_t S_t)^{-1/2} S_t$ is the orthogonal part in the polar decomposition of S_t:

$$U_t \in \mathrm{Sp}(n) \cap \mathrm{O}(2n, \mathbb{R}).$$

Let us denote by u_t the image $\iota(U_t)$ of U_t in $\mathrm{U}(n, \mathbb{C})$:

$$\iota(U_t) = A + iB \;\text{ if }\; U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

and define $\rho(S_t) = \det u_t$. The Maslov index of γ is by definition the degree of the loop $t \longmapsto \rho(S_t)$ in S^1:

$$m(\gamma) = \deg[t \longmapsto \det(\iota(U_t))]\ ,\ 0 \leq t \leq 1].$$

Let α be the generator of $\pi_1[\mathrm{Sp}(E, \omega)] \equiv (\mathbb{Z}, +)$ whose image in $\mathbb{Z}$ is $+1$; if γ is homotopic to α^r then

$$m(\gamma) = m(\alpha^r) = 2r. \tag{10}$$

The definition of the Maslov index can be extended to arbitrary paths in $\mathrm{Sp}(E, \omega)$ using the properties of the ALM index. This is done as follows: let

$\ell = \pi^{\mathrm{Lag}}(\ell_\infty) \in \mathrm{Lag}(E,\omega)$; we define the Maslov index of $S_\infty \in \mathrm{Sp}_\infty(E,\omega)$ relative to ℓ by

$$\mu_\ell(S_\infty) = \mu(S_\infty \ell_\infty, \ell_\infty); \tag{11}$$

one shows (see [7, 8]) that the right-hand side only depends on the projection ℓ of ℓ_∞, justifying the notation.

Here are three fundamental properties of the relative Maslov index:

- The product formula:

$$\mu_\ell(S_\infty S'_\infty) = \mu_\ell(S_\infty) + \mu_\ell(S'_\infty) + \tau(\ell, S\ell, SS'\ell) \tag{12}$$

  valid for all S_∞, S'_∞ in $\mathrm{Sp}_\infty(E,\omega)$;

- The behavior of μ_ℓ under the action of the fundamental group:

$$\mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 4r \tag{13}$$

  for all $r \in \mathbb{Z}$ (α the generator of $\pi_1[\mathrm{Sp}(E,\omega)]$ with index 1).

- μ_ℓ is locally constant on the sets $\{S_\infty : \dim S\ell \cap \ell = k\}$ for $0 \leq k \leq n$.

The two first properties readily follow from, respectively, (5) and (7). The third follows from the fact that the ALM index is locally constant on the sets $\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\}$. Note that (13) implies that

$$\mu_\ell(\alpha^r) = 4r$$

hence the restriction of any of the μ_ℓ to loops γ in $\mathrm{Sp}(E,\omega)$ is *twice* the Maslov index $m(\gamma)$ defined above; it is therefore sometimes advantageous to use the variant of μ_ℓ defined by

$$m_\ell(S_\infty) = \frac{1}{2}(\mu_\ell(S_\infty) + n + \dim(S\ell \cap \ell))$$

where $n = \frac{1}{2}\dim E$. We will call $m_\ell(S_\infty)$ the reduced (relative) Maslov index. In view of property (6) it is an integer; the properties of m_ℓ are obtained, *mutatis mutandis*, from those of μ_ℓ; for instance property (12) becomes

$$m_\ell(S_\infty S'_\infty) = m_\ell(S_\infty) + m_\ell(S'_\infty) + \mathrm{Inert}(\ell, S\ell, SS'\ell)$$

where Inert is the index of inertia defined by (4).

It follows from the cocycle property of the signature τ that the Maslov indices corresponding to two choices ℓ and ℓ' are related by the formula

$$\mu_\ell(S_\infty) - \mu_{\ell'}(S_\infty) = \tau(S\ell, \ell, \ell') - \tau(S\ell, S\ell', \ell'); \tag{14}$$

similarly

$$m_\ell(S_\infty) - m\mu_{\ell'}(S_\infty) = \mathrm{Inert}(S\ell, \ell, \ell') - \mathrm{Inert}(S\ell, S\ell', \ell').$$

Assume that $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$ and $\ell' \in \mathrm{Lag}(E', \omega')$, $\ell'' \in \mathrm{Lag}(E'', \omega'')$; the additivity property (8) of the ALM index implies that if $S'_\infty \in \mathrm{Sp}_\infty(E', \omega')$, $S''_\infty \in \mathrm{Sp}_\infty(E'', \omega'')$ then

$$\mu_{\ell' \oplus \ell''}(S'_\infty \oplus S''_\infty) = \mu_{\ell'}(S'_\infty) + \mu_{\ell_2}(S''_\infty) \tag{15}$$

where $\mathrm{Sp}_\infty(E', \omega') \oplus \mathrm{Sp}_\infty(E'', \omega'')$ is identified in the obvious way with a subgroup of $\mathrm{Sp}_\infty(E, \omega)$; a similar property holds for the reduced index m_ℓ.

3 The index ν on $\mathrm{Sp}_\infty(n)$

In this section we define and study a function $\nu : \mathrm{Sp}_\infty(n) \longrightarrow \mathbb{Z}$ generalizing the Conley–Zehnder index [3]. We begin by recalling the definition and main properties of the latter.

3.1 Review of the Conley–Zehnder index

Let Σ be a continuous path $[0, 1] \longrightarrow \mathrm{Sp}(n)$ such that $\Sigma(0) = I$ and $\det(\Sigma(1) - I) \neq 0$. Loosely speaking, the Conley–Zehnder index [3] counts algebraically the number of points in the open interval $]0, 1[$ for which $\Sigma(t)$ has 1 as an eigenvalue. To give a more precise definition we need some notations. Let us define three subsets of $\mathrm{Sp}(n)$ by

$$\begin{aligned} \mathrm{Sp}_0(n) &= \{S : \det(S - I) = 0\} \\ \mathrm{Sp}^+(n) &= \{S : \det(S - I) > 0\} \\ \mathrm{Sp}^-(n) &= \{S : \det(S - I) < 0\}. \end{aligned}$$

These sets partition $\mathrm{Sp}(n)$, and $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$ are moreover arcwise connected; the symplectic matrices $S^+ = -I$ and

$$S^- = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \quad , \quad L = \mathrm{diag}[2, -1, ..., -1]$$

belong to $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$, respectively.

Let us now denote by $C_\pm(2n,\mathbb{R})$ the space of all paths $\Sigma : [0,1] \longrightarrow \mathrm{Sp}(n)$ with $\Sigma(0) = I$ and $\Sigma(1) \in \mathrm{Sp}^\pm(n)$. Any such path can be extended into a path $\widetilde{\Sigma} : [0,2] \longrightarrow \mathrm{Sp}(n)$ such that $\widetilde{\Sigma}(t) \in \mathrm{Sp}^\pm(n)$ for $1 \leq t \leq 2$ and $\widetilde{\Sigma}(2) = S^+$ or $\widetilde{\Sigma}(2) = S^-$. Let ρ be the mapping $\mathrm{Sp}(n) \longrightarrow S^1$, $\rho(S_t) = \det u_t$, used in the definition of the Maslov index for symplectic loops. The Conley–Zehnder index of Σ is, by definition, the winding number of the loop $(\rho \circ \tilde{\Sigma})^2$ in S^1:

$$i_{\mathrm{CZ}}(\Sigma) = \deg[t \longmapsto (\rho(\tilde{\Sigma}(t)))^2,\, 0 \leq t \leq 2].$$

It turns out that $i_{\mathrm{CZ}}(\Sigma)$ is invariant under homotopy as long as the endpoint $S = \Sigma(1)$ remains in $\mathrm{Sp}^\pm(n)$; in particular it does not change under homotopies with fixed endpoints so we may view i_{CZ} as defined on the subset

$$\mathrm{Sp}^*_\infty(n) = \{S_\infty : \det(S - I) \neq 0\}$$

of the universal covering group $\mathrm{Sp}_\infty(n)$. With this convention one proves [12] that the Conley-Zehnder index is the unique mapping $i_{\mathrm{CZ}} : \mathrm{Sp}^*_\infty(n) \longrightarrow \mathbb{Z}$ having the following properties:

(CZ$_1$) *Antisymmetry*: For every S_∞ we have

$$i_{\mathrm{CZ}}(S_\infty^{-1}) = -i_{\mathrm{CZ}}(S_\infty)$$

where S_∞^{-1} is the homotopy class of the path $t \longmapsto S_t^{-1}$;

(CZ$_2$) *Continuity*: Let Σ be a symplectic path representing S_∞ and Σ' a path joining S to an element S' belonging to the same component $\mathrm{Sp}^\pm(n)$ as S. Let S'_∞ be the homotopy class of $\Sigma * \Sigma'$. We have

$$i_{\mathrm{CZ}}(S_\infty) = i_{\mathrm{CZ}}(S'_\infty);$$

(CZ$_3$) *Action of* $\pi_1[\mathrm{Sp}(n)]$:

$$i_{\mathrm{CZ}}(\alpha^r S_\infty) = i_{\mathrm{CZ}}(S_\infty) + 2r$$

for every $r \in \mathbb{Z}$.

We observe that these three properties are characteristic of the Conley–Zehnder index in the sense that any other function $i'_{\mathrm{CZ}} : \mathrm{Sp}^*_\infty(n) \longrightarrow \mathbb{Z}$ satisfying then must be identical to i_{CZ}. Set in fact $\delta = i_{\mathrm{CZ}} - i'_{\mathrm{CZ}}$. In view of (CZ$_3$) we have $\delta(\alpha^r S_\infty) = \delta(S_\infty)$ for all $r \in \mathbb{Z}$ hence δ is defined

on $\mathrm{Sp}^*(n) = \mathrm{Sp}^+(n) \cup \mathrm{Sp}^-(n)$ so that $\delta(S_\infty) = \delta(S)$ where $S = S_1$, the endpoint of the path $t \longmapsto S_t$. Property (CZ_2) implies that this function $\mathrm{Sp}^*(n) \longrightarrow \mathbb{Z}$ is constant on both $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$. We next observe that since $\det S = 1$ we have $\det(S^{-1} - I) = \det(S - I)$ so that S and S^{-1} always belong to the same set $\mathrm{Sp}^+(n)$ or $\mathrm{Sp}^-(n)$ if $\det(S - I) \neq 0$. Property (CZ_1) then implies that δ must be zero on both $\mathrm{Sp}^+(n)$ or $\mathrm{Sp}^-(n)$.

Two other noteworthy properties of the Conley–Zehnder are:

($\mathbf{CZ}_4$) *Normalization*: Let J_1 be the standard symplectic matrix in $Sp(1)$. If S_1 is the path $t \longrightarrow e^{\pi t J_1}$ $(0 \leq t \leq 1)$ joining I to $-I$ in $\mathrm{Sp}(1)$ then $i_{\mathrm{CZ},1}(S_{1,\infty}) = 1$ ($i_{\mathrm{CZ},1}$ the Conley–Zehnder index on $\mathrm{Sp}(1)$);

($\mathbf{CZ}_5$) *Dimensional additivity*: if $S_{1,\infty} \in \mathrm{Sp}^*_\infty(n_1)$, $S_{2,\infty} \in \mathrm{Sp}^*_\infty(n_2)$, $n_1 + n_2 = n$, then

$$i_{\mathrm{CZ}}(S_{1,\infty} \oplus S_{2,\infty}) = i_{\mathrm{CZ},1}(S_{1,\infty}) + i_{\mathrm{CZ},2}(S_{2,\infty})$$

where $i_{\mathrm{CZ},j}$ is the Conley–Zehnder index on $\mathrm{Sp}(n_j)$, $j = 1, 2$.

3.2 Symplectic Cayley transform

Our extension of the index i_{CZ} requires a notion of Cayley transform for symplectic matrices. If $S \in \mathrm{Sp}(n)$, $\det(S - I) \neq 0$, we call the matrix

$$M_S = \frac{1}{2} J(S + I)(S - I)^{-1} \tag{16}$$

the "symplectic Cayley transform of S". Equivalently:

$$M_S = \frac{1}{2} J + J(S - I)^{-1}. \tag{17}$$

It is straightforward to check that M_S always is a symmetric matrix: $M_S = M_S^T$ (it suffices for this to use the fact that $S^T J S = S J S^T = J$).

The symplectic Cayley transform has in addition the following properties:

Lemma 2 *(i) The symplectic Cayley transform of the product SS' is (when defined) given by the formula*

$$M_{SS'} = M_S + (S^T - I)^{-1} J (M_S + M_{S'})^{-1} J (S - I)^{-1}. \tag{18}$$

(ii) The symplectic Cayley transform of S and S^{-1} are related by

$$M_{S^{-1}} = -M_S. \tag{19}$$

Proof. *(i)* We begin by noting that (17) implies that

$$M_S + M_{S'} = J(I + (S-I)^{-1} + (S'-I)^{-1}) \tag{20}$$

hence the identity

$$(M_S + M_{S'})^{-1} = -(S'-I)(SS'-I)^{-1}(S-I)J. \tag{21}$$

In fact, writing $SS'-I = S(S'-I)+S-I$, we have

$$\begin{aligned}(S'-I)(SS'-I)^{-1}(S-I) &= (S'-I)(S(S'-I)+S-I)^{-1}(S-I)\\ &= ((S-I)^{-1}S(S'-I)(S'-I)^{-1} + (S'-I)^{-1})^{-1}\\ &= ((S-I)^{-1}S + (S'-I)^{-1})\\ &= I + (S-I)^{-1} + (S'-I)^{-1};\end{aligned}$$

the equality (21) follows in view of (20). We have to prove that

$$M_S + M = M_{SS'} \tag{22}$$

where M is the matrix defined by

$$M = (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S-I)^{-1}$$

that is, in view of (21),

$$M = (S^T - I)^{-1}J(S'-I)(SS'-I)^{-1}.$$

Using the obvious relations $S^T = -JS^{-1}J$ and $(-S^{-1}+I)^{-1} = S(S-I)^{-1}$ we have

$$\begin{aligned}M &= (S^T - I)^{-1}J(S'-I)(SS'-I)^{-1}\\ &= -J(-S^{-1}+I)^{-1}(S'-I)(SS'-I)^{-1}\\ &= -JS(S-I)^{-1}(S'-I)(SS'-I)^{-1}\end{aligned}$$

that is, writing $S = S - I + I$,

$$M = -J(S'-I)(SS'-I)^{-1} - J(S-I)^{-1}(S'-I)(SS'-I)^{-1}.$$

Replacing M_S by its value (17) we have

$$M_S + M = J(\frac{1}{2}I + (S-I)^{-1} - (S'-I)(SS'-I)^{-1} - (S-I)^{-1}(S'-I)(SS'-I)^{-1});$$

noting that

$$(S-I)^{-1}-(S-I)^{-1}(S'-I)(SS'-I)^{-1} = (S-I)^{-1}(SS'-I-S'+I)(SS'-I)^{-1})$$

that is

$$\begin{aligned}(S-I)^{-1}-(S-I)^{-1}(S'-I)(SS'-I)^{-1} &= (S-I)^{-1}(SS'-S')(SS'-I)^{-1}) \\ &= S'(SS'-I)^{-1})\end{aligned}$$

we get

$$\begin{aligned}M_S + M &= J(\tfrac{1}{2}I-(S'-I)(SS'-I)^{-1}+S'(SS'-I)^{-1}) \\ &= J(\tfrac{1}{2}I+(SS'-I)^{-1}) \\ &= M_{SS'}\end{aligned}$$

which we set out to prove. *(ii).* Formula (19) follows from the sequence of trivial equalities

$$\begin{aligned}M_{S^{-1}} &= \tfrac{1}{2}J+J(S^{-1}-I)^{-1} \\ &= \tfrac{1}{2}J-JS(S-I)^{-1} \\ &= \tfrac{1}{2}J-J(S-I+I)(S-I)^{-1} \\ &= -\tfrac{1}{2}J-J(S-I)^{-1} \\ &= -M_S.\end{aligned}$$

■

3.3 Definition of $\nu(S_\infty)$ and first properties

We define on $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ a symplectic form $\sigma^\ominus$ by

$$\sigma^\ominus(z_1,z_2;z_1',z_2') = \sigma(z_1,z_1')-\sigma(z_2,z_2')$$

and denote by $\mathrm{Sp}^\ominus(2n)$ and $\mathrm{Lag}^\ominus(2n)$ the corresponding symplectic group and Lagrangian Grassmannian. Let $\mu^\ominus$ be the ALM index on $\mathrm{Lag}^\ominus_\infty(2n)$ and $\mu^\ominus_L$ the Maslov index on $\mathrm{Sp}^\ominus_\infty(2n)$ relative to $L\in\mathrm{Lag}^\ominus(2n)$.

For $S_\infty\in\mathrm{Sp}_\infty(n)$ we define

$$\nu(S_\infty) = \frac{1}{2}\mu^\ominus((I\oplus S)_\infty\Delta_\infty,\Delta_\infty) \tag{23}$$

where $(I\oplus S)_\infty$ is the homotopy class in $\mathrm{Sp}^\ominus(2n)$ of the path

$$t\longmapsto\{(z,S_tz):z\in\mathbb{R}^{2n}\}\ \ ,\ \ 0\leq t\leq 1$$

and $\Delta = \{(z,z) : z \in \mathbb{R}^{2n}\}$ the diagonal of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Setting $S_t^\ominus = I \oplus S_t$ we have $S_t^\ominus \in \mathrm{Sp}^\ominus(2n)$ hence formulae (23) is equivalent to

$$\nu(S_\infty) = \frac{1}{2}\mu_\Delta^\ominus(S_\infty^\ominus) \tag{24}$$

where $\mu_\Delta^\ominus$ is the relative Maslov index on $\mathrm{Sp}_\infty^\ominus(2n)$ corresponding to the choice $\Delta \in \mathrm{Lag}^\ominus(2n)$.

Note that replacing n by $2n$ in the congruence (6) we have

$$\begin{aligned}\mu^\ominus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty) &\equiv \dim((I \oplus S)\Delta \cap \Delta) \mod 2 \\ &\equiv \dim \mathrm{Ker}(S - I) \mod 2\end{aligned}$$

and hence

$$\nu(S_\infty) \equiv \frac{1}{2}\dim \mathrm{Ker}(S - I) \mod 1$$

so that $\nu(S_\infty)$ is always an integer (the eigenvalue 1 of S has even multiplicity).

The index ν has the following three important properties; the third is essential for the calculation of the index of repeated periodic orbits (it clearly shows that this index is not in general additive):

Proposition 3 *(i) For all $S_\infty \in \mathrm{Sp}_\infty(n)$ we have*

$$\nu(S_\infty^{-1}) = -\nu(S_\infty) \quad , \quad \nu(I_\infty) = 0 \tag{25}$$

(I_∞ the identity of $\mathrm{Sp}_\infty(n)$). (ii) For all $r \in \mathbb{Z}$ we have

$$\nu(\alpha^r S_\infty) = \nu(S_\infty) + 2r \quad , \quad \nu(\alpha^r) = 2r \tag{26}$$

(iii) Let S_∞ be the homotopy class of a path Σ in $\mathrm{Sp}(n)$ joining the identity to $S \in \mathrm{Sp}_0(n)$, and let $S' \in \mathrm{Sp}(n)$ be in the same connected component $\mathrm{Sp}^\pm(n)$ as S. Then $\nu(S'_\infty) = \nu(S_\infty)$ where S'_∞ is the homotopy class in $\mathrm{Sp}(n)$ of the concatenation of Σ and a path joining S to S' in $\mathrm{Sp}_0(n)$.

Proof. (i) Formulae (25) immediately follows from the equality $(S_\infty^\ominus)^{-1} = (I \oplus S^{-1})_\infty$ and the antisymmetry of $\mu_\Delta^\ominus$. (ii) The second formula (26) follows from the first using (25). To prove the first formula (26) it suffices to observe that to the generator α of $\pi_1[\mathrm{Sp}(n)]$ corresponds the generator $I_\infty \oplus \alpha$ of $\pi_1[\mathrm{Sp}^\ominus(2n)]$; in view of property (13) of the Maslov index it follows that

$$\begin{aligned}\nu(\alpha^r S_\infty) &= \frac{1}{2}\mu_\Delta^\ominus((I_\infty \oplus \alpha)^r S_\infty^\ominus) \\ &= \frac{1}{2}(\mu_\Delta^\ominus(S_\infty^\ominus) + 4r) \\ &= \nu(S_\infty) + 2r.\end{aligned}$$

(iii) Assume in fact that S and S' belong to, say, $\mathrm{Sp}^+(n)$and let Σ be a symplectic path representing S_∞ and $t \longmapsto \Sigma'(t)$ $0 \leq t \leq 1$, a path joining S to S'. Let $S_\infty(t)$ be the homotopy class of $\Sigma * \Sigma'(t)$. We have $\det(S(t)-I) > 0$ for all $t \in [0,1]$ hence $S_\infty^\ominus(t)\Delta \cap \Delta \neq 0$ as t varies from 0 to 1. It follows from the fact that the $\mu_\Delta^\ominus$ is locally constant on $\{S_\infty^\ominus : S_\infty^\ominus \Delta \cap \Delta = 0\}$ (see §2.3) that the function $t \longmapsto \mu_\Delta^\ominus(S_\infty^\ominus(t))$ is constant,and hence

$$\mu_\Delta^\ominus(S_\infty^\ominus) = \mu_\Delta^\ominus(S_\infty^\ominus(0)) = \mu_\Delta^\ominus(S_\infty^\ominus(1)) = \mu_\Delta^\ominus(S_\infty'^\ominus)$$

which was to be proven. ■

The following consequence of this result shows that the indices ν and i_{CZ} coincide on their common domain of definition:

Corollary 4 *The restriction of the index ν to $\mathrm{Sp}^*(n)$ is the Conley–Zehnder index:*

$$\nu(S_\infty) = i_{CZ}(S_\infty) \quad \textit{if} \quad \det(S-I) \neq 0.$$

Proof. The restriction of ν to $\mathrm{Sp}^*(n)$ satisfies the properties (CZ_1), (CZ_2), and (CZ_3) of the Conley–Zehnder index listed in §3.1; we showed that these properties uniquely characterize i_{CZ}. ■

Using the properties of the ALM index we can prove a formula for the product of

Proposition 5 *If S_∞, S'_∞, and $S_\infty S'_\infty$ are such that $\det(S-I) \neq 0$, $\det(S'-I) \neq 0$, and $\det(SS'-I) \neq 0$ then*

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \tfrac{1}{2}\operatorname{sign} M_S \tag{27}$$

where M_S is the symplectic Cayley transform of S; in particular

$$\nu(S_\infty^r) = r\nu(S_\infty) + \tfrac{1}{2}(r-1)\operatorname{sign} M_S \tag{28}$$

for every integer r.

Proof. In view of (17) we have

$$\langle M_S z, z\rangle = \sigma((S-I)^{-1}z, z);$$

since the quadratic forms $z \longmapsto \sigma((S-I)^{-1}z,z)$ and $z \longmapsto \sigma(z,(S-I)z) = \sigma(z,Sz)$ are equivalent they have same signature, and formula (27) is therefore equivalent to

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \tfrac{1}{2}\operatorname{sign}\sigma(Sz,z) \tag{29}$$

where $\operatorname{sign}\sigma(Sz,z)$ is the signature of the quadratic form $z \longmapsto \sigma(Sz,z)$. Let us prove (29). In view of (24) and the product property (12) of the Maslov index we have

$$\begin{aligned}\nu(S_\infty S'_\infty) &= \nu(S_\infty)+\nu(S'_\infty)+\tfrac{1}{2}\tau^\ominus(\Delta, S^\ominus\Delta, S^\ominus S'^\ominus\Delta)\\ &= \nu(S_\infty)+\nu(S'_\infty)-\tfrac{1}{2}\tau^\ominus(S^\ominus S'^\ominus\Delta, S^\ominus\Delta, \Delta)\end{aligned}$$

where $S^\ominus = I\oplus S$, $S'^\ominus = I\oplus S'$ and $\tau^\ominus$ is the Kashiwara signature on the symplectic space $(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n},\sigma^\ominus)$. The condition $\det(SS'-I)\neq 0$ is equivalent to $S^\ominus S'^\ominus\Delta\cap\Delta = 0$ hence we can apply property *(i)* in Lemma 1 with $\ell = S^\ominus S'^\ominus\Delta$ and $\ell'' = \Delta$. The projection operator onto $S^\ominus S'^\ominus\Delta$ along Δ is

$$\operatorname{Pr}(S^\ominus S'^\ominus\Delta,\Delta) = \begin{bmatrix}(I-SS')^{-1} & -(I-SS')^{-1}\\ SS'(I-SS')^{-1} & -SS'(I-SS')^{-1}\end{bmatrix}$$

hence $\tau^\ominus(S^\ominus S'^\ominus\Delta, S^\ominus\Delta,\Delta)$ is the signature of the quadratic form

$$Q(z) = \sigma^\ominus(\operatorname{Pr}(S^\ominus S'^\ominus\Delta,\Delta)(z,Sz);(z,Sz))$$

that is, since $\sigma^\ominus = \sigma\ominus\sigma$:

$$\begin{aligned}Q(z) &= \sigma((I-SS')^{-1}(I-S)z,Sz)) - \sigma(SS'(I-SS')^{-1}(I-S)z,Sz))\\ &= \sigma((SS'-I)(I-SS')^{-1}(I-S)z,Sz))\\ &= \sigma(Sz,z).\end{aligned}$$

Formula (28) follows from (27) by induction on r. ■

3.4 Relation between ν and μ_{ℓ_P}

The index ν can be expressed in simple – and useful – way in terms of the Maslov index μ_{ℓ_P} on $\operatorname{Sp}_\infty(n)$. The following technical result will be helpful in establishing this relation. Recall that $S\in\operatorname{Sp}(n)$ is said to be "free" if we have $S\ell_P\cap\ell_P = 0$; this condition is equivalent to $\det B\neq 0$ when S is identified with the matrix

$$S = \begin{bmatrix}A & B\\ C & D\end{bmatrix} \tag{30}$$

in the canonical basis. The set of all free symplectic matrices is dense in $\operatorname{Sp}(n)$. The quadratic form W on $\mathbb{R}^n_x\times\mathbb{R}^n_x$ defined by

$$W(x,x') = \frac{1}{2}\left\langle Px,x\right\rangle - \left\langle Lx,x'\right\rangle + \frac{1}{2}\left\langle Qx',x'\right\rangle$$

where $P = DB^{-1}$, $L = B^{-1}$, $Q = B^{-1}A$ then generates S in the sense that

$$(x,p) = S(x',p') \Longleftrightarrow p = \partial_x W(x,x')\ ,\ p' = -\partial_{x'} W(x,x').$$

We have:

Lemma 6 *Let $S = S_W \in \mathrm{Sp}(n)$ be given by (30). We have*

$$\det(S_W - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}) \qquad (31)$$

that is:

$$\det(S_W - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T).$$

In particular the symmetric matrix

$$P + Q - L - L^T = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1}$$

is invertible.

Proof. Since B is invertible we can factorize $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D-I)B^{-1}(A-I) & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

and hence

$$\begin{aligned} \det(S_W - I) &= \det(-B) \det(C - (D-I)B^{-1}(A-I)) \\ &= (-1)^n \det B \det(C - (D-I)B^{-1}(A-I)). \end{aligned}$$

Since S is symplectic we have $C - DB^{-1}A = -(B^T)^{-1}$ and hence

$$C - (D-I)B^{-1}(A-I)) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows. ■

Let us now introduce the notion of index of concavity of a Hamiltonian periodic orbit γ, defined for $0 \leq t \leq T$, with $\gamma(0) = \gamma(T) = z_0$. As t goes from 0 to T the linearized part $D\gamma(t) = S_t(z_0)$ goes from the identity to $S_T(z_0)$ (the monodromy matrix) in $\mathrm{Sp}(n)$. We assume that $S_T(z_0)$ is free and that $\det(S_T(z_0) - I) \neq 0$. Writing

$$S_t(z_0) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$$

we thus have $\det B(t) \neq 0$ in a neighborhood $[T-\varepsilon, T+\varepsilon]$ of the time T. The generating function

$$W(x,x',t) = \frac{1}{2}\left\langle P(t)x, x\right\rangle - \left\langle L(t)x, x'\right\rangle + \frac{1}{2}\left\langle Q(t)x', x'\right\rangle$$

(with $P(t)$, $Q(t)$, $L = L(t)$ defined as above) thus exists for $T-\varepsilon \leq t \leq T+\varepsilon$. By definition the index of concavity of the periodic orbit γ is the index of inertia of matrix of second derivatives of the function $W(x,x;T)$, that is

$$W'' = P + Q - L - L^T$$

where we have set $P = P(T)$, $Q = Q(T)$, $L = L(T)$.

Let us now prove the following essential result:

Proposition 7 *Let $t \longmapsto S_t$ be a symplectic path, $0 \leq t \leq 1$. Let $S_\infty \in \mathrm{Sp}_\infty(n)$ be the homotopy class of that path and set $S = S_1$. If $\det(S-I) \neq 0$ and $S\ell_P \cap \ell_P = 0$ then*

$$\nu(S_\infty) = \frac{1}{2}(\mu_{\ell_P}(S_\infty) + \operatorname{sign} W_S) = m_{\ell_P}(S_\infty) - \operatorname{Inert} W'' \tag{32}$$

where $\operatorname{Inert} W''$ is the index of concavity corresponding to the endpoint S of the path $t \longmapsto S_t$.

Proof. We will divide the proof in three steps. *Step 1.* Let $L \in \mathrm{Lag}^\ominus(4n)$. Using successively formulae (24) and (14) we have

$$\nu(S_\infty) = \frac{1}{2}(\mu^\ominus_L(S^\ominus_\infty) + \tau^\ominus(S^\ominus\Delta, \Delta, L) - \tau^\ominus(S^\ominus\Delta, S^\ominus L, L)). \tag{33}$$

Choosing in particular $L = L_0 = \ell_P \oplus \ell_P$ we get

$$\begin{aligned}
\mu^\ominus_{L_0}(S^\ominus_\infty) &= \mu^\ominus((I\oplus S)_\infty(\ell_P\oplus\ell_P), (\ell_P\oplus\ell_P)) \\
&= \mu(\ell_{P,\infty}, \ell_{P,\infty}) - \mu(\ell_{P,\infty}, S_\infty\ell_{P,\infty}) \\
&= -\mu(\ell_{P,\infty}, S_\infty\ell_{P,\infty}) \\
&= \mu_{\ell_P}(S_\infty)
\end{aligned}$$

so that there remains to prove that

$$\tau^\ominus(S^\ominus\Delta, \Delta, L_0) - \tau^\ominus(S^\ominus\Delta, S^\ominus L_0, L_0) = -\operatorname{sign} W_S.$$

Step 2. We are going to show that

$$\tau^\ominus(S^\ominus\Delta, S^\ominus L_0, L_0) = 0;$$

in view of the symplectic invariance and the antisymmetry of $\tau^\ominus$ this is equivalent to

$$\tau^\ominus(L_0, \Delta, L_0, (S^\ominus)^{-1}L_0) = 0. \tag{34}$$

We have

$$\Delta \cap L_0 = \{(0, p; 0, p) : p \in \mathbb{R}^n\}$$

and $(S^\ominus)^{-1}L_0 \cap L_0$ consists of all $(0, p', S^{-1}(0, p''))$ with $S^{-1}(0, p'') = (0, p')$; since S (and hence S^{-1}) is free we must have $p' = p'' = 0$ so that

$$(S^\ominus)^{-1}L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.$$

It follows that we have

$$L_0 = \Delta \cap L_0 + (S^\ominus)^{-1}L_0 \cap L_0$$

hence (34) in view of property *(ii)* in Lemma 1. *Step 3.* Let us finally prove that.

$$\tau^\ominus(S^\ominus\Delta, \Delta, L_0) = -\operatorname{sign} W_S;$$

this will complete the proof of the proposition. The condition $\det(S-I) \neq 0$ is equivalent to $S^\ominus\Delta \cap \Delta = 0$ hence, using property *(i)* in Lemma 1:

$$\tau^\ominus(S^\ominus\Delta, \Delta, L_0) = -\tau^\ominus(S^\ominus\Delta, L_0, \Delta)$$

is the signature of the quadratic form Q on L_0 defined by

$$Q(0, p, 0, p') = -\sigma^\ominus(P_\Delta(0, p, 0, p'); 0, p, 0, p')$$

where

$$P_\Delta = \begin{bmatrix} (S-I)^{-1} & -(S-I)^{-1} \\ S(S-I)^{-1} & -S(S-I)^{-1} \end{bmatrix}$$

is the projection on $S^\ominus\Delta$ along Δ in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. It follows that the quadratic form Q is given by

$$Q(0, p, 0, p') = -\sigma^\ominus((I-S)^{-1}(0, p''), S(I-S)^{-1}(0, p''); 0, p, 0, p')$$

where we have set $p'' = p - p'$; by definition of $\sigma^\ominus$ this is

$$Q(0, p, 0, p') = -\sigma((I-S)^{-1}(0, p''), (0, p)) + \sigma(S(I-S)^{-1}(0, p''), (0, p')).$$

Let now M_S be the symplectic Cayley transform (16) of S; we have

$$(I-S)^{-1} = JM_S + \tfrac{1}{2}I \quad , \quad S(I-S)^{-1} = JM_S - \tfrac{1}{2}I$$

and hence

$$\begin{aligned}
Q(0,p,0,p') &= -\sigma((JM_S+\tfrac{1}{2}I)(0,p''),(0,p)) + \sigma((JM_S-\tfrac{1}{2}I)(0,p''),(0,p')) \\
&= -\sigma(JM_S(0,p''),(0,p)) + \sigma(JM_S(0,p''),(0,p')) \\
&= \sigma(JM_S(0,p''),(0,p'')) \\
&= -\left\langle M_S(0,p''),(0,p'')\right\rangle .
\end{aligned}$$

Let us calculate explicitly M_S. Writing S in usual block-form we have

$$S-I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} C-(D-I)B^{-1}(A-I) & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

that is

$$S-I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} W_S & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

where we have used the identity

$$C-(D-I)B^{-1}(A-I)) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}$$

which follows from the relation $C - DB^{-1}A = -(B^T)^{-1}$ (the latter is a rephrasing of the equalities $D^TA - B^TC = I$ and $D^TB = B^TD$, which follow from the fact that $S^TJS = S^TJS$ since $S \in \mathrm{Sp}(n)$). It follows that

$$\begin{aligned}
(S-I)^{-1} &= \begin{bmatrix} W_S^{-1} & 0 \\ B^{-1}(I-A)W_S^{-1} & I \end{bmatrix} \begin{bmatrix} (I-D)B^{-1} & I \\ B^{-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} W_S^{-1}(I-D)B^{-1} & W_S^{-1} \\ B^{-1}(I-A)W_S^{-1}(I-D)B^{-1} + B^{-1} & B^{-1}(I-A)W_S^{-1} \end{bmatrix}
\end{aligned}$$

and hence

$$M_S = \begin{bmatrix} B^{-1}(I-A)W_S^{-1}(I-D)B^{-1} + B^{-1} & \frac{1}{2}I + B^{-1}(I-A)W_S^{-1} \\ -\frac{1}{2}I - W_S^{-1}(I-D)B^{-1} & -W_S^{-1} \end{bmatrix}$$

from which follows that

$$Q(0,p,0,p') = \left\langle W_S^{-1}p'',p''\right\rangle = \left\langle W_S^{-1}(p-p'),(p-p')\right\rangle .$$

The matrix of the quadratic form Q is thus

$$2\begin{bmatrix} W_S^{-1} & -W_S^{-1} \\ -W_S^{-1} & W_S^{-1} \end{bmatrix}$$

and this matrix has signature $\mathrm{sign}(W_S^{-1}) = \mathrm{sign}\, W_S$, concluding the proof.

■

Remark 8 *Lemma 6 above shows that if S is free then we have*

$$\begin{aligned}\frac{1}{\pi}\arg\det(S-I) &\equiv n+\arg\det B+\arg\det W'' \mod 2\\ &\equiv n-\arg\det B+\arg\det W'' \mod 2\end{aligned}$$

In [5, 6] we have shown that the reduced Maslov index $m_{\ell_P}(S_\infty)$ corresponds to a choice of $\arg\det B$ *modulo* 4*; Proposition 7 thus justifies the following definition of the argument of* $\det(S-I)$*:*

$$\frac{1}{\pi}\arg\det(S-I)\equiv n-\nu(S_\infty) \mod 4.$$

That this is indeed the correct choice modulo 4 *has been proven by other means (the theory of the metaplectic group) by one of us in a recent publication [10].*

4 An Example

Consider a two-dimensional harmonic oscillator with Hamiltonian function

$$H=\frac{\omega_x}{2}(p_x^2+x^2)+\frac{\omega_y}{2}(p_y^2+y^2);$$

we assume that the frequencies ω_y, ω_x are incommensurate, so that the only periodic orbits are librations along the x and y axes. Let us focus on the orbit γ_x along the x axis; its prime period is $T=2\pi/\omega_x$ and the corresponding monodromy matrix is

$$S_1=\begin{bmatrix}1&0&0&0\\0&\cos\chi&0&\sin\chi\\0&0&1&0\\0&-\sin\chi&0&\cos\chi\end{bmatrix}\quad,\ \chi=2\pi\frac{\omega_y}{\omega_x};$$

it is the endpoint of the symplectic path $t\longmapsto S_t$, $0\leq t\leq 1$, consisting of the matrices

$$S_t=\begin{bmatrix}\cos 2\pi t&0&\sin 2\pi t&0\\0&\cos\chi t&0&\sin\chi t\\-\sin 2\pi t&0&\cos 2\pi t&0\\0&-\sin\chi t&0&\cos\chi t\end{bmatrix}.$$

In Gutzwiller's formula (1) the sum is taken over periodic orbits, including their repetitions; we are thus led to calculate the Conley–Zehnder index of

the path $t \longmapsto S_t$ with $0 \leq t \leq r$ where the integer r indicates the number of repetitions of the orbit. Let us calculate the Conley–Zehnder index $\nu(\tilde{S}_{r,\infty})$ of this path. We have $S_t = \Sigma_t \oplus \tilde{S}_t$ where

$$\Sigma_t = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix} \ , \ \tilde{S}_t = \begin{bmatrix} \cos \chi t & \sin \chi t \\ -\sin \chi t & \cos \chi t \end{bmatrix};$$

in view of the additivity property of the relative Maslov index we thus have

$$\nu(S_{r,\infty}) = \nu(\Sigma_{r,\infty}) + \nu(\tilde{S}_{r,\infty}).$$

The homotopy class $\Sigma_{r,\infty}$ of the path $t \longmapsto \Sigma_t$, $0 \leq t \leq r$ is just α^{-r}, α the generator of $\pi_1[\mathrm{Sp}(1)]$, hence

$$\nu(\Sigma_{T,\infty}) = -2r$$

in view of formula (26). Let us next calculate $\nu(\tilde{S}_{r,\infty})$. We will use formula (32) so we begin by calculating the relative Maslov index

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = m(\tilde{S}_{r,\infty}\ell_{P,\infty}, \ell_{P,\infty}).$$

When t goes from 0 to r the line $\tilde{S}_t\ell_P$ describes a loop in $\mathrm{Lag}(1)$ going from ℓ_P to $\tilde{S}_r\ell_P$. We have $\tilde{S}_t \in U(1)$; its image in $U(1,\mathbb{C})$ is $e^{-i\chi t}$ hence the Souriau mapping identifies $\tilde{S}_t\ell_P$ with $e^{-2i\chi t}$. It follows, using formula (9), that

$$\begin{aligned} m_{\ell_P}(\tilde{S}_{r,\infty}) &= \frac{1}{2\pi}\left(-2r\chi + i\,\mathrm{Log}(-e^{-2ir\chi})\right) + \frac{1}{2} \\ &= \frac{1}{2\pi}\left(-2r\chi + i\,\mathrm{Log}(e^{i(-2r\chi+\pi)})\right) + \frac{1}{2} \end{aligned}$$

The logarithm is calculated as follows: for $\theta \neq (2k+1)\pi$ $(k \in \mathbb{Z})$

$$\mathrm{Log}\, e^{i\theta} = i\theta - 2\pi i \left[\frac{\theta+\pi}{2\pi}\right]$$

and hence

$$\mathrm{Log}(e^{i(-2r\chi+\pi)}) = -i(2r\chi + \pi + 2\pi\left[\frac{r\chi}{\pi}\right]);$$

it follows that the Maslov index is

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = -\left[\frac{r\chi}{\pi}\right]. \tag{35}$$

To obtain $\nu(\tilde{S}_{r,\infty})$ we note that by (32)

$$\nu(\tilde{S}_{r,\infty}) = m_{\ell_P}(\tilde{S}_{1,\infty}) - \mathrm{Inert}\, W_S$$

where Inert W'' is the concavity index corresponding to the generating function of $\tilde{S}_t$; the latter is

$$W(x,x',t) = \frac{1}{2\sin\chi t}((x^2+x'^2)\cos\chi t - 2xx')$$

hence $W'' = -\tan(\chi t/2)$. We thus have, taking (35) into account,

$$\nu(\tilde{S}_{r,\infty}) = -\left[\frac{r\chi}{\pi}\right] - \text{Inert}\left(-\tan\frac{r\chi}{2}\right);$$

a straightforward induction on r shows that this can be rewritten more conveniently as

$$\nu(\tilde{S}_{r,\infty}) = -1 - 2\left[\frac{r\chi}{2\pi}\right].$$

Summarizing, we have

$$\begin{aligned}\nu(S_{r,\infty}) &= \nu(\Sigma_{r,\infty}) + \nu(\tilde{S}_{r,\infty}) \\ &= -2r - 1 - 2\left[\frac{r\chi}{2\pi}\right])\end{aligned}$$

hence the index in Gutzwiller's formula corresponding to the r-th repetition is

$$\mu_{x,r} = -\nu(S_{r,\infty}) = 1 + 2r + 2\left[\frac{r\chi}{2\pi}\right]$$

that is, by definition of χ,

$$\mu_{x,r} = 1 + 2r + 2\left[r\frac{\omega_y}{\omega_x}\right]$$

confirming the calculations in [1, 4, 18, 21].

These calculations are valid when the frequencies are incommensurate. If, say, $\omega_x = \omega_y$, the calculations are much simpler: in this case the homotopy class of the loop $t \longmapsto S_t$, $0 \leq t \leq 1$, is $\alpha^{-1} \oplus \alpha^{-1}$ and by the second formula (26),

$$\mu_{x,r} = -\nu(S_{r,\infty}) = 4r$$

which is zero modulo 4 (cf. [18]).

Acknowledgement 9 *We wish to express our gratitude to M. Brack for having patiently explained to us Gutzwiller's trace formula. One of the authors (M. de Gosson) has been supported by the grant 2005/51766-7 of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).*

References

[1] M Brack and R K Bhaduri 1997 *Semiclassical Physics.* Addison–Wesley

[2] S E Cappell, R Lee and E Y Miller 1994 *Comm. Pure and Appl. Math.* **17** 121

[3] C Conley and E Zehnder 1984 *Comm. Pure and Appl. Math.* **37** 207

[4] S C Creagh, J M Robbins, and R G Littlejohn 1990 *Phys. Rev. A*, **4**2(4) 1907

[5] M de Gosson 1990 *Ann. Inst. Fourier* **40**(3) 537

[6] M de Gosson 1992 *J. Geom. Phys.* **9** 255

[7] M de Gosson 1992 *J. Math. Pures et Appl.* **71** 429

[8] M de Gosson 1997 *Maslov Classes, Metaplectic Representation and Lagrangian Quantization* (Research Notes in Mathematics **95**) Wiley–VCH, Berlin

[9] M de Gosson and S de Gosson 20 *J. Phys A: Math. and Gen.* **36** 615

[10] M de Gosson 2005 *Letters in Mathematical Physics* **72** 129

[11] M C Gutzwiller 1990 *Chaos in Classical and Quantum Mechanics.* Interdisciplinary Applied Mathematics, Springer-Verlag

[12] H. Hofer, K. Wysocki, and E. Zehnder 1995 *Geometric and Functional Analysis* **2**(5) 270

[13] J Leray 1981 *Lagrangian Analysis and Quantum Mechanics, a mathematical structure related to asymptotic expansions and the Maslov index* (the MIT Press, Cambridge, Mass.)

[14] G Lion, and M Vergne 1980 *The Weil representation, Maslov index and Theta series*. Progress in mathematics **6**, Birkhäuser

[15] B Mehlig and M Wilkinson 2001 *Ann. Phys.* **18**(10), 541

[16] M Morse 1935 The Calculus of Variations in the Large, AMS, Providence, R. I.

[17] P Muratore-Ginnaneschi 2003 *Phys. Rep.* **383** 299

[18] M Pletyukhov and M Brack 2003 *J. Phys A: Math. and Gen.* **36** 9449

[19] J M Robbin 1991 *Nonlinearity* **4** 343

[20] J M Souriau 1975 In: Group Theoretical Methods in Physics, Lecture Notes in Physics, Springer-Verlag, **50** 17

[21] A Sugita 2001 *Ann. Phys.* **288** 277

[22] C T C Wall 1969 *Invent. Math.* **7** 269.