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CONJUGACY OF PIECEWISE C^1 -HOMEOMORPHISMS OF CLASS P OF THE CIRCLE

ABDELHAMID ADOUANI AND HABIB MARZOUGUI

ABSTRACT. We give a characterization of piecewise C^1 -homeomorphism of class P of the circle with irrational rotation number and finitely many break points which are piecewise differentiably conjugate to C^1 -diffeomorphisms. The following properties are equivalent:

- i) f is conjugate to a C^1 -diffeomorphism of the circle by a piecewise C^1 -homeomorphism of class P .
- ii) The number of break points of f^n is bounded by some constant that doesn't depend on n .
- iii) The product of jumps of f in the break points contained in a same orbit is equal 1.
- iv) f is conjugate to a C^1 -diffeomorphism of the circle by a piecewise quadratic homeomorphism of class P .

This characterization extend Liousse's Theorem for PL homeomorphisms of the circle ([3]).

1. INTRODUCTION

1.1. Preliminaries. Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $\pi : \mathbb{R} \rightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1 . The homeomorphism f admits a lift $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ that is an increasing homeomorphism of \mathbb{R} such that $\pi \circ \tilde{f} = f \circ \pi$. Conversely, the projection of such a homeomorphism of \mathbb{R} is an orientation preserving homeomorphism of S^1 .

Let $x \in S^1$. We call:

- *orbit* of x by f the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$
- *positive orbit* from x by f the subset $O_f^+(x) = \{f^n(x) : n \in \mathbb{N}\}$
- *negative orbit* from x by f the subset $O_f^-(x) = \{f^n(x) : n \in -\mathbb{N}\}$

A *segment* of the orbit $O_f(d)$ containing d is a subset of the form $\{f^s(d) : -k \leq s \leq n - k\}$, noted $[f^{-k}(d), \dots, f^{n-k}(d)]$, $k, n \in \mathbb{N}$.

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Historically, the dynamic study of circle homeomorphisms was initiated by H. Poincaré ([4], 1886), he introduced the rotation number of a homeomorphism f of S^1 as

$$\rho(f) = \lim_{n \rightarrow +\infty} \frac{\tilde{f}^n(x) - x}{n} (\text{mod } 1)$$

Poincaré shows that this limit exist and does not depends of x and the lift \tilde{f} of f .

Assuming f is a C^r -diffeomorphism ($r \geq 2$) and $\rho(f)$ is irrational, A. Denjoy ([2]) proved:

Theorem 1.1. (*Denjoy*)([2]). *Every C^r -diffeomorphism f ($r \geq 2$) with irrational rotation number $\rho(f)$ is topologically conjugate to rotation $R_{\rho(f)}$.*

This means that there exists an orientation preserving homeomorphism h of S^1 such that $f = h^{-1} \circ R_{\rho(f)} \circ h$.

Denjoy noted that this result can be extended (with the same proof) to a large class of circle homeomorphisms: *the class P*.

1.2. Class P

Definition 1.2. Let f be a piecewise C^1 orientation preserving homeomorphism of S^1 . A *break point* of f is a point of discontinuity of the derivative Df . The homeomorphism f is called of *class P* if it is C^1 except in a finitely or countably many break points admitting left and right derivatives; the derivative $Df : S^1 \rightarrow \mathbb{R}_+^*$ has the following properties :

- there exist two constants $0 < a < b < +\infty$ such that:

$$a < Df(x) < b,$$

for all x where Df exists,

$$a < Df_+(c) < b, \text{ and } a < Df_-(c) < b \text{ at the break point } c,$$

- $\text{Log} Df$ is a bounded variation on S^1

The ratio $\sigma_f(c) := \frac{Df_+(c)}{Df_-(c)}$ is called *jump of f in c* .

Notice that homeomorphisms of class P form a sub-group of the group of orientation preserving homeomorphisms of S^1 .

As examples of homeomorphisms of class P , we mention:

- the C^2 -diffeomorphisms,
- The PL homeomorphisms (i.e. piecewise linear), these are not C^2 -diffeomorphisms. An orientation preserving homeomorphism f of S^1 is called a *PL homeomorphism* if f is differentiable except at finite number of break points $(c_i)_{1 \leq i \leq p}$ of S^1 and such that Df is constant on each $]c_i, c_{i+1}[$.

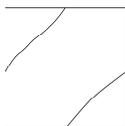


FIGURE 1. General Boshernitzan of last break point 0

The set of PL homeomorphisms is a sub-group of the group of homeomorphisms of class P and contains rotations.

Analytic homeomorphisms are not of class P . By J.C. Yoccoz [5], they satisfy the conclusion of Denjoy theorem but not Denjoy theory.

Definition 1.3. A piecewise C^1 -homeomorphism B of class P of S^1 with two break points x_0 and $B(x_0)$ is called a *general Boshernitzan* of S^1 of *last break point* $B(x_0)$. If $B(x_0) = 0$, B is called a *standard Boshernitzan of last break point 0* (see Figure 1).

If we conjugate a standard Boshernitzan B_0 of last break point 0 by a rotation R_{x_0} , the homeomorphism $R_{x_0} \circ B_0 \circ R_{x_0}^{-1}$ is a general Boshernitzan of last break point $B(x_0)$ and conversely.

The lift $\tilde{B}_0 \pmod{1}$ of B_0 on $[0, 1[$ is a general exchange of 2 intervals on $[0, 1[$. The affine Boshernitzan correspond to affine exchange of 2 intervals on $[0, 1[$.

Notice that Boshernitzan was the first who studied these examples in order to built examples of "rational" PL homeomorphisms with irrational rotation numbers (cf. [1]).

Denote by

- $C(f)$ the set of break points of f .
- $\pi_s(f)$ the product of jumps of f in the break points of f :

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c).$$

Definition 1.4. ([3]) A piecewise C^1 -homeomorphism f of class P of S^1 has the *property D* if the product of jumps of f in the break points of f contained in a same orbit is equal 1.

In particular, if f has the property D then $\pi_s(f) = 1$. Conversely, if $\pi_s(f) = 1$ and if all break points of f are contained in a same orbit then f has the property D . If f is a PL homeomorphism we have always $\pi_s(f) = 1$ and therefore if all the break points of f are contained in a same orbit then f satisfies property D .

Proposition 1.5. (*Invariance of π_s by piecewise C^1 conjugacy*).

Let f, g be two orientation preserving piecewise C^1 -homeomorphisms of class P of S^1 . If f and g are piecewise C^1 conjugate then $\pi_s(f) = \pi_s(g)$.

Proof. Let H be the piecewise C^1 -homeomorphism which conjugate f to g : $Hof = goH$. For $x \in S^1$, we define $\pi_s(f)(x) = \prod_{y \in O_f(x)} \sigma_f(y)$ the product of jumps of f in the break points contained in the orbit of x . Since $O_g(H(x)) = H(O_f(x))$ then

$$\pi_s(g)(H(x)) = \prod_{z \in O_g(H(x))} \sigma_g(z) = \prod_{y \in O_f(x)} \sigma_g(H(y)).$$

Since $\sigma_g(H(y)) = \frac{\sigma_H(f(y))\sigma_f(y)}{\sigma_H(y)}$ then

$$\pi_s(g)(H(x)) = \prod_{y \in O_f(x)} \frac{\sigma_H(f(y))\sigma_f(y)}{\sigma_H(y)} = \pi_s(f)(x) \prod_{y \in O_f(x)} \frac{\sigma_H(f(y))}{\sigma_H(y)}.$$

$$\text{Or } \prod_{y \in O_f(x)} \frac{\sigma_H(f(y))}{\sigma_H(y)} = \prod_{k \in \mathbb{Z}} \frac{\sigma_H(f(f^k(x)))}{\sigma_H(f^k(x))} = \frac{\prod_{k \in \mathbb{Z}} \sigma_H(f^k(x))}{\prod_{k \in \mathbb{Z}} \sigma_H(f^k(x))} = 1.$$

Hence $\pi_s(g)(H(x)) = \pi_s(f)(x)$ and therefore $\pi_s(g) = \pi_s(f)$. \square

In the sequel, we suppose that homeomorphisms f of class P considered have only a finite number of break points i.e. $C(f)$ is finite.

Our main result is the following :

Theorem 1.6. *Let f be a piecewise C^1 -homeomorphism of class P of the circle with irrational rotation number. The following properties are equivalent:*

- i) f is conjugate to a C^1 -diffeomorphism of the circle through a piecewise C^1 -homeomorphism.*
- ii) The number of break points of f^n is bounded by some constant that doesn't depend on n .*
- iii) f satisfies property D .*

iv) f is conjugate to a general Boshernitzan (of last break point $B(x_0)$ such that $\sigma_B(x_0) \times \sigma_B[B(x_0)] = 1$) or a C^1 -diffeomorphism of the circle through a PL homeomorphism.

v) f is conjugate to a C^1 -diffeomorphism of the circle through a piecewise quadratic homeomorphism.

Corollary 1.7. *Under the hypothesis of Theorem 1.6, if the number of break points of f^n is bounded by some constant that doesn't depend on n then f is conjugate to a C^1 -diffeomorphism of the circle through a piecewise quadratic homeomorphism.*

It follows from Theorem 1.6:

Corollary 1.8. ([3]) *Let f be a PL homeomorphism of the circle with irrational rotation number α . The following properties are equivalent:*

- i) f is conjugate to the rotation R_α through a piecewise C^1 -homeomorphism,*
- ii) The number of break points of f^n is bounded by some constant that doesn't depend on n .*
- iii) f is conjugate to an affine Boshernitzan of last break point 0 or a rotation with rotation number α through a PL homeomorphism,*
- iv) f is conjugate to R_α through a piecewise C^∞ (analytic) homeomorphism.*

2. HOMEOMORPHISMS OF CLASS P WITH PROPERTY D

Let $n \in \mathbb{N}$. Denote by $C(f^n)$ the set of break points of f^n . We have $C(f^n) \subset \{f^{-k}(c), k = 0, 1, \dots, n-1; c \in C(f)\}$.

Let $x \in S^1$, we have :

$$Df^n(x) = Df(x) \times Df(f(x)) \times \dots \times Df(f^{n-1}(x)).$$

The jump of f^n in x is then :

$$\sigma_{f^n}(x) = \sigma_f(x) \times \sigma_f(f(x)) \times \dots \times \sigma_f(f^{n-1}(x)).$$

Definition 2.1. Let $c \in C(f)$. A *maximal connection* of c is a segment $M = [f^{-p}(c), \dots, f^q(c)]$ of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (resp. $f^q(c)$) is the first (resp. last) break point of f on $O_f(c)$.

Hence, the negative orbit $O_f^-(f^{-p}(c))$ (resp. positive orbit $O_f^+(f^q(c))$) from $f^{-p}(c)$ (resp. from $f^q(c)$) doesn't contain any break point of f .

The following properties are trivial:

- Two break points of f are on the same maximal connection, if and only if, they are on the same orbit.

- Two distinct maximal connections are disjoint.

There is a finite number of maximal connections ; denoted by

$$M_1 = [f^{-N_1}(c_1), \dots, c_1], \dots, M_p = [f^{-N_p}(c_p), \dots, c_p] \text{ where } c_1, \dots, c_p \in C(f).$$

So, we have the decomposition :

$$C(f) = \coprod_{i=1}^p C_i \text{ where } C_i = C(f) \cap M_i, \quad i = 1, \dots, p.$$

This is equivalent to say that: for every $i = 1, \dots, p$, the product of jumps of f in the break points of C_i is equal 1 (i.e. $\prod_{d \in C_i} \sigma_f(d) = 1$). We have also

$$\prod_{d \in C_i} \sigma_f(d) = \prod_{d \in M_i} \sigma_f(d). \text{ So, } f \text{ satisfies the property } D \text{ means that}$$

$$\prod_{d \in C_i} \sigma_f(d) = 1.$$

Proposition 2.2. *Let f be a homeomorphism of class P of the circle with irrational rotation number α . Then f has property D if and only if the number of break points of f^n is bounded by some constant N_0 that doesn't depend on n .*

Proof. Suppose that the number of break points of f^n is bounded by a constant N_0 . Let $d \in C(f)$ and $M = [f^{-p}(d), \dots, f^q(d)]$ be the maximal connection of d . Lets show that $\prod_{\delta \in M} \sigma_f(\delta) = 1$:

Let $n \in \mathbb{N}$ fixed. We have :

$$C(f^{n+1}) \subset \{f^{-k}(c) : c \in C(f), 0 \leq k \leq n\},$$

and

$$\sigma_{f^{n+1}}[f^{-k}(d)] = \sigma_f[f^{-k}(d)] \times \dots \times \sigma_f[f^{n-k}(d)]$$

We let :

$$J_n := \{0 \leq k \leq n : \sigma_{f^{n+1}}[f^{-k}(d)] = 1\}.$$

By hypothesis :

$$\text{card}\{0 \leq k \leq n ; \sigma_{f^{n+1}}[f^{-k}(d)] \neq 1\} \leq N_0.$$

Then

$$\text{card}(J_n) \geq n + 1 - N_0.$$

For n such that $n > N_0 + p$, we have $\text{card}(J_n) > p + 1 \geq \text{card}(J)$ where $J = \{0 \leq k \leq p : \sigma_{f^{n+1}}[f^{-k}(d)] = 1\}$. Therefore, J est strictly contained in J_n . Let $K_n := J_n - J$ and let k_n be the minimum of K_n . Since $n - k_n + 1 \geq \text{card}(K_n)$ and $\text{card}(K_n) \geq n - (N_0 + p)$ then $p + 1 \leq k_n \leq p + N_0 + 1$ for $n > p + N_0$.

As

$$\sigma_{f^{n+1}}[f^{-k_n}(d)] = \sigma_f[f^{-p}(d)] \times \dots \times \sigma_f[f^{n-k_n}(d)]$$

then

$$\sigma_f[f^{-p}(d)] \times \sigma_f[f^{-p+1}(d)] \times \dots \times \sigma_f[f(d)] = \sigma_{f^{n+1}}[f^{-kn}(d)]$$

for every $n > p + q + N_0$. It follows,

$$\sigma_f[f^{-p}(d)] \times \sigma_f[f^{-p+1}(d)] \times \dots \times \sigma_f[f^q(d)] = \prod_{\delta \in M} \sigma_f(\delta) = 1.$$

Conversely, suppose that for every $d \in C(f)$, the product of jumps of f in the break points of the maximal connection of d is equal 1. Lets show that the number of break points of f^n is bounded.

Let $n \in \mathbb{N}$. The point of discontinuity of Df^n are among $f^{-k}(c_i)$, $k = 0, 1, \dots, n + N_i - 1$, $i = 1, \dots, p$.

The jump of f^n in these points are :

$$\sigma_{f^n}[f^{-k}(c_i)] = \sigma_f[f^{-k}(c_i)] \times \dots \times \sigma_f(c_i) \times \dots \times \sigma_f[f^{n-k-1}(c_i)].$$

If $k \geq N_i$ and $n-k-1 \geq 0$ then the segment of orbit $[f^{-k}(c_i), \dots, f^{n-k-1}(c_i)]$ contains the maximal connection of c_i . Hence, $\sigma_{f^n}[f^{-k}(c_i)] = 1$. So, a necessary condition for $f^{-k}(c_i)$ to be a discontinuity point of Df^n is that $0 \leq k < N_i$ or $n-1 < k \leq n+N_i+1$. Therefore, the number of discontinuity points of Df^n which are contained in the orbit of c_i is less then $2N_i$.

We conclude that the number of discontinuity points of Df^n is bounded by $2(N_1 + \dots + N_p)$. \square

To a piecewise C^1 -homeomorphism f of class P of S^1 that satisfies property D , we associate the real :

$$\pi(f) = \prod_{i=1}^p \prod_{k=0}^{N_i-1} \sigma_{f^{N+1}}(f^{-k}(c_i))$$

where $N = \max_{1 \leq i \leq p} (N_i)$.

Remark. If f is a homeomorphism that satisfies the property D , then $\pi_s(f) = 1$ but one can have $\pi(f) \neq 1$ as can be shown by the following example:

If B is a standard Boshernitzan of last break point 0 and which satisfies the property D , we have : $\pi_s(B) = 1$ and $\pi(B) = \sigma_{B^2}(0) = \sigma_B(0) \times \sigma_B(B(0))$.

If $B(0) \neq B^{-1}(0)$, then $\sigma_B(B(0)) = 1$ and $\pi(B) = \sigma_B(0) \neq 1$.

If $B(0) = B^{-1}(0)$, then $\pi(B) = \sigma_B(0)\sigma_B(B^{-1}(0)) = \sigma_B(0)\sigma_B(B(0)) = 1$.

Proposition 2.3. *Let f be a piecewise C^1 -homeomorphism of class P of the circle. If f satisfies property D then :*

i) if $\pi(f) = 1$ then f is conjugate to a C^1 -diffeomorphism of the circle through a PL homeomorphism.

ii) if $\pi(f) \neq 1$ then f is conjugate to a general Boshernitzan B (of last break point $B(x_0)$ such that $\pi_s(B) = \sigma_B(x_0) \times \sigma_B[B(x_0)] = 1$ and $\pi(B) = \pi(f)$) through a PL homeomorphism.

Proof. Case i) $\pi(f) = 1$.

Let H be the PL homeomorphism with the following properties :

- i) $H(0) = 0$,
- ii) $f^{-k}(c_i)$, $i = 1, \dots, p$; $k = 0, \dots, N_i - 1$ are the break points of H ,
- iii) the jumps of H are given by : $\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i))$ where $N = \max_{1 \leq i \leq p} N_i$.

Such homeomorphism exist since

$$\prod_{i=1}^p \prod_{k=0}^{N_i-1} \sigma_H(f^{-k}(c_i)) = \pi(f) = 1$$

Take $G = H \circ f \circ H^{-1}$. A priori, the break points of G are :

- the break points of H^{-1} :

$$H(f^{-k}(c_i)), ; k = 0, \dots, N_i - 1, i = 1, \dots, p,$$

- the image by H of break points of f :

$$H(f^{-k}(c_i)), k = 0, \dots, N_i, i = 1, \dots, p,$$

- the image by $H \circ f^{-1}$ of break points of H :

$$H \circ f^{-1}(f^{-k}(c_i)) = H(f^{-k}(c_i)) \text{ with } k = 1, \dots, N_i.$$

Therefore the possible break points of G are among :

$$H(f^{-k}(c_i)), k = 0, \dots, N_i, i = 1, \dots, p.$$

Compute the jumps of G in these points:

$$\sigma_G[H(f^{-k}(c_i))] = \frac{\sigma_H[f(f^{-k}(c_i))] \times \sigma_f(f^{-k}(c_i))}{\sigma_H(f^{-k}(c_i))}$$

- If $k = 1, \dots, N_i - 1$, we have by definition :

$$\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i))$$

and since $N \geq k$ then :

$$\sigma_{f^{N+1}}(f^{-k}(c_i)) = \prod_{s=0}^N \sigma_f(f^{s-k}(c_i)) = \prod_{s=-k}^{N-k} \sigma_f(f^s(c_i)) = \prod_{s=-k}^0 \sigma_f(f^s(c_i)).$$

Hence :

$$\sigma_H[f(f^{-k}(c_i))] = \sigma_H[f^{-(k-1)}(c_i)] = \prod_{s=-k+1}^0 \sigma_f(f^s(c_i)).$$

Then : $\sigma_G[H(f^{-k}(c_i))] = 1$.

• If $k = 0$, we obtain :

$$\sigma_G[H(f^{-k}(c_i))] = \sigma_G[H(c_i)] = \frac{\sigma_H[f(c_i)] \times \sigma_f(c_i)}{\sigma_H(c_i)}$$

We have $\sigma_G[H(c_i)] = 1$ since $\sigma_H(c_i) = \sigma_f(c_i)$ and $\sigma_H[f(c_i)] = 1$ (since $f(c_i)$ is not a break point of H .)

• If $k = N_i$, we have :

$$\sigma_G[H(f^{-k}(c_i))] = \sigma_G[H(f^{-N_i}(c_i))] = \frac{\sigma_H[f(f^{-N_i}(c_i))] \times \sigma_f(f^{-N_i}(c_i))}{\sigma_H(f^{-N_i}(c_i))}$$

We have $\sigma_H(f^{-N_i}(c_i)) = 1$ (since $f^{-N_i}(c_i)$ is not a break point of H).

Also,

$$\sigma_H[f(f^{-N_i}(c_i))] \times \sigma_f(f^{-N_i}(c_i)) = \prod_{s=-N_i+1}^0 \sigma_f(f^s(c_i)) \times \sigma_f(f^{-N_i}(c_i)) = \prod_{s=0}^{N_i} \sigma_f(f^{-s}(c_i)) = 1$$

since

$$\sigma_H[f(f^{-N_i}(c_i))] = \sigma_H[f^{-(N_i-1)}(c_i)] = \prod_{s=-N_i+1}^0 \sigma_f(f^s(c_i))$$

It follows that : $\sigma_G[H(f^{-N_i}(c_i))] = 1$.

We conclude that G has no break point, G is then a C^1 -diffeomorphism.

Case ii) $\pi(f) \neq 1$.

We choose a point c such that :

$$c \neq f^{-k}(c_i), c \neq f(c), c \neq f^{N+1}(f^{-k}(c_i)), i = 1, \dots, p, k = 0, \dots, N_i.$$

To obtain the PL homeomorphism H that satisfies properties i), ii), and iii), we need to add a break point c to H such that: $\sigma_H(c) = \frac{1}{\pi(f)}$.

The break points of H are :

$$f^{-k}(c_i), (i = 1, \dots, p, k = 0, \dots, N_i - 1), \text{ and } c$$

and the jumps in these points are :

$$\sigma_H(f^{-k}(c_i)) = \sigma_{f^{N+1}}(f^{-k}(c_i)), \sigma_H(c) = \frac{1}{\pi(f)}.$$

We let $G = H \circ f \circ H^{-1}$. The possible break points of G are among:
 $H(c)$, $H \circ f^{-1}(c)$, $H(f^{-k}(c_i))$, $k = 0, \dots, N_i$, $i = 1, \dots, p$.

- the jump of G in $H(c)$:

$$\sigma_G[H(c)] = \frac{\sigma_H[f(c)] \times \sigma_f(c)}{\sigma_H(c)}.$$

We have : $\sigma_f(c) = 1$ since $c \neq f^{-k}(c_i)$, $i = 1, \dots, p$, $k = 0, \dots, N_i$.

We have also $\sigma_H[f(c)] = 1$ since $f(c)$ is not a break point of H ; indeed :

- $f(c) \neq c$
- $f(c) \neq f^{-k}(c_i)$, $k = 0, \dots, N_i - 1$ (otherwise, $c = f^{-s}(c_i)$, $s = 1, \dots, N_i$, which is impossible by hypothesis.).

It follows that: $\sigma_G[H(c)] = \frac{1}{\sigma_H(c)} \neq 1$. $H(c)$ is then a break point of G .

- The jump of G in $H \circ f^{-1}(c)$:

$$\sigma_G[H \circ f^{-1}(c)] = \frac{\sigma_H(c) \times \sigma_f[f^{-1}(c)]}{\sigma_H[f^{-1}(c)]}$$

We have : $\sigma_H[f^{-1}(c)] = 1$ since :

- $f^{-1}(c) \neq c$
- $f^{-1}(c) \neq f^{-k}(c_i)$ since :

Otherwise, $c = f^{1-k}(c_i)$, $k = 0, \dots, N_i - 1$. For $k = 1, \dots, N_i - 1$, this is impossible by hypothesis. For $k = 0$, $c = f(c_i)$ is also impossible since by hypothesis, $c \neq f^{N+1}(f^{-k}(c_i))$, $i = 1, \dots, p$, $k = 0, \dots, N_i$, in particular for $k = N_i$, $c \neq f(c_i)$. It follows that $\sigma_G[H \circ f^{-1}(c)] = \sigma_H(c) \neq 1$. So,

$H \circ f^{-1}(c)$ is a break point of G .

- the jump of G in $H \circ f^{-k}(c_i)$: by the compute above, we have $\sigma_G[H \circ f^{-k}(c_i)] = 1$, $i = 1, \dots, p$.

Therefore, G has exactly two break points :

$x_0 := H \circ f^{-1}(c)$ and $G(x_0) = H(c)$ with $\sigma_G(x_0) \times \sigma_G[G(x_0)] = 1$. The homeomorphism G is then a Boshernitzan of S^1 of last break point $H(c)$. Let's show that $\pi(B) = \pi(f)$:

We have

$$\pi(B) = \sigma_B(H(c)) = \sigma_{H \circ f \circ H^{-1}}(H(c)) = \frac{\sigma_H(f(c)) \times \sigma_f(c)}{\sigma_H(c)} = \frac{1}{\frac{1}{\pi(f)}} = \pi(f). \quad \square$$

Corollary 2.4. *Let f be a piecewise C^1 -homeomorphism of class P of the circle. If the break points of f are on the same orbit with*

$\pi_s(f) = \pi(f) = 1$ then f is conjugate to a C^1 -diffeomorphism of the circle through a PL homeomorphism.

Proof. This follows from the fact that if all the break points of f are on the same orbit and $\pi_s(f) = 1$ then f has the property D and so, we conclude by proposition 2.3, i). \square

3. THE CONJUGACY LEMMAS

Lemma 3.1. *For every $b \in S^1$, for every $\delta \in]0, 1[$, there exist a piecewise quadratic orientation preserving homeomorphism $H_{\delta,b}$ of S^1 with only one break point b such that: $\sigma_{H_{\delta,b}}(b) = \delta$.*

Proof. If $\delta \in]0, 1[$ we put

$$\tilde{H}_\delta(x) = \left(\frac{1-\delta}{1+\delta}\right)x(x + \frac{2\delta}{1-\delta}), \quad x \in [0, 1[.$$

One can check that $\tilde{H}_\delta(0) = 0$, $\tilde{H}_\delta(1) = 1$ and that \tilde{H}_δ is an increasing homeomorphism of $[0, 1[$ with $(D\tilde{H}_\delta)_+(0) = \frac{2\delta}{1+\delta}$ and $(D\tilde{H}_\delta)_-(1) = \frac{2}{1+\delta}$.

Let $t \in [0, 1[$ such that $p(t) = b$ and consider the map :

$$\tilde{H}_{\delta,t}(x) = \tilde{H}_\delta(x - t) + t, \quad \text{if } x \in [t, t + 1[.$$

$\tilde{H}_{\delta,t}$ is extended to an increasing homeomorphism, denoted also by $\tilde{H}_{\delta,t}$ on \mathbb{R} and defined by :

$$\tilde{H}_{\delta,t}(x) = \tilde{H}_{\delta,t}(x - k), \quad \text{if } x \in [t + k, t + k + 1[, \quad k \in \mathbb{Z}.$$

Then $\tilde{H}_{\delta,t}$ is an increasing homeomorphism of \mathbb{R} which satisfies :

- $\tilde{H}_{\delta,t}(x + 1) = \tilde{H}_{\delta,t}(x) + 1$,
- $t + k$, $k \in \mathbb{Z}$ are the break points of $\tilde{H}_{\delta,t}$,
- $\sigma_{\tilde{H}_{\delta,t}}(t + k) = \delta$.

As a consequence, $\tilde{H}_{\delta,t}$ induces by projection the piecewise quadratic homeomorphism $H_{\delta,t}$ of S^1 with only one break point b such that

$$\sigma_H(b) = \delta. \quad \square$$

Lemma 3.2. *Let f be a piecewise C^1 -homeomorphism of class P of S^1 with irrational rotation number and having $p + 1$ break points x_0, x_1, \dots, x_p ($p \geq 1$) contained in a same orbit. Then there exists a piecewise quadratic homeomorphism H with one break point x_1 such that for $F = H \circ f \circ H^{-1}$ we have :*

a) *if there exist $i_0 \in \{0, 1, \dots, p\}$ such that $f(x_{i_0}) = x_1$ then :*

a-1) $C(F) = \{y_i = H(x_i), \quad i = 0, 2, \dots, p\}$ if $\sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$.

a-2) $C(F) = \{y_i = H(x_i), \quad i = 0, 2, \dots, p, i \neq i_0\}$ if $\sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

b) *if for every $i = 0, 1, \dots, p$, $f(x_i) \neq x_1$ then*

$C(F) = \{y_i = H(x_i), \quad i = 0, 2, \dots, p\} \cup \{H \circ f^{-1}(x_1)\}$. Moreover, if

$x_1 = f^k(x_0)$, $k \geq 2$ then $y_1 = F^{k-1}(y_0)$.

Proof. By Lemma 3.1, let H be the piecewise quadratic homeomorphism with one break point x_1 such that $\sigma_H(x_1) = \sigma_f(x_1)$. Take $F = H \circ f \circ H^{-1}$. The eventually break points of F are:

$$H(x_i), \quad i = 0, 1, \dots, p, \quad H \circ f^{-1}(x_1).$$

- The jumps of F in these points are:

$$\sigma_F[H(x_i)] = \frac{\sigma_H[f(x_i)] \times \sigma_f(x_i)}{\sigma_H(x_i)}$$

Case a-1): there exist $i_0 \in \{0, 1, \dots, p\}$ such that $f(x_{i_0}) = x_1$ with $\sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$.

In this case, $f(x_i) \neq x_1$ for $i \neq i_0$ and $H \circ f^{-1}(x_1) = H(x_{i_0})$. Hence

- $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$, for $i \notin \{1, i_0\}$:

indeed, $\sigma_H(x_i) = 1$ since x_i is not a break point of H , and $\sigma_H[f(x_i)] = 1$ since $f(x_i) \neq x_1$.

- $\sigma_F[H(x_1)] = 1$:

indeed, $\sigma_F[H(x_1)] = \frac{\sigma_H[f(x_1)] \times \sigma_f(x_1)}{\sigma_H(x_1)}$, $\sigma_H[f(x_1)] = 1$ since $f(x_1) \neq x_1$ (x_1 is not a periodic point) and $\frac{\sigma_f(x_1)}{\sigma_H(x_1)} = 1$.

- $\sigma_F[H(x_{i_0})] = \sigma_f(x_1) \times \sigma_f(x_{i_0}) \neq 1$:

indeed, $\sigma_F[H(x_{i_0})] = \frac{\sigma_H(x_1) \times \sigma_f(x_{i_0})}{\sigma_H(x_{i_0})}$, $\sigma_H(x_{i_0}) = 1$ since $x_{i_0} \neq x_1$ and $\sigma_H(x_1) = \sigma_f(x_1)$.

One conclude that F has p break points:

$$y_i = H(x_i), \quad i = 0, 2, \dots, p.$$

contained in the orbit of y_0 .

Case a-2): there exist $i_0 \in \{0, 1, \dots, p\}$ such that $f(x_{i_0}) = x_1$ with $\sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

In this case, we have as in the case a-1), $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$, for $i \neq 1, i_0$, $\sigma_F[H(x_1)] = 1$ and $\sigma_F[H(x_{i_0})] = \sigma_f(x_1) \times \sigma_f(x_{i_0}) = 1$.

One conclude that F has $p - 1$ break points:

$$y_i = H(x_i), \quad 0 \leq i \leq p, i \neq 1, i_0$$

contained in the orbit of y_0 .

Case b): for every $i = 0, 1, \dots, p$, $f(x_i) \neq x_1$.

- we have $\sigma_F[H(x_i)] = \sigma_f(x_i) \neq 1$:

indeed, $\sigma_H(x_i) = 1$ since x_i is not a break point of H , and $\sigma_H[f(x_i)] = 1$ since $f(x_i) \neq x_1$.

• $\sigma_F[H(x_1)] = 1$:
 indeed, $\sigma_F[H(x_1)] = \frac{\sigma_H[f(x_1)] \times \sigma_f(x_1)}{\sigma_H(x_1)}$, $\sigma_H[f(x_1)] = 1$ since $f(x_1) \neq x_1$ and $\frac{\sigma_f(x_1)}{\sigma_H(x_1)} = 1$.

• $\sigma_F[H \circ f^{-1}(x_1)] = \sigma_H(x_1) \neq 1$:
 indeed, $\sigma_F[H \circ f^{-1}(x_1)] = \frac{\sigma_H(x_1) \times \sigma_f(f^{-1}(x_1))}{\sigma_H(f^{-1}(x_1))}$, $\sigma_H(f^{-1}(x_1)) = 1$ since $f(x_1) \neq x_1$ and $\sigma_f(f^{-1}(x_1)) = 1$ since $f^{-1}(x_1) \neq x_i$, $i = 0, 1, \dots, p$.

Therefore F has $p + 1$ break points:

$$y_1 = H \circ f^{-1}(x_1), \quad y_i = H(x_i), \quad i = 0, 2, \dots, p,$$

Moreover, if $x_1 = f^k(x_0)$ ($k \geq 2$) then $y_1 = F^{k-1}(y_0)$ since

$$y_1 = H \circ f^{-1}(x_1) = H \circ f^{k-1}(x_0) = F^{k-1}(H(x_0)) = F^{k-1}(y_0).$$

□

Corollary 3.3. *Let B be a general Boshernitzan of last break point $x_1 = B(x_0)$ and with irrational rotation number. Then:*

i) if $\sigma_B(x_0) \times \sigma_B(x_1) = 1$ then B is conjugate to a C^1 -diffeomorphism through a piecewise quadratic homeomorphism with one break point x_1 .

ii) if $\sigma_B(x_0) \times \sigma_B(x_1) \neq 1$ then B is conjugate to a homeomorphism of class P with a unique break point through a piecewise quadratic homeomorphism with one break point x_1 .

Proof. This follows from Lemma 3.2, case a) for $p = 1$. □

4. PROOF OF THEOREM 1.6 AND COROLLARY 1.8

Proof of Theorem 1.6.

i) \implies ii). If H is the piecewise C^1 -homeomorphism which conjugate f to a diffeomorphism F : $f = H^{-1} \circ F \circ H$ then $f^n = H^{-1} \circ F^n \circ H$. As H and H^{-1} have the same number l of break points then f^n has at most $2l$ break points for every $n \in \mathbb{Z}$.

ii) \implies iii) is the proposition 2.2.

iii) \implies iv) is the proposition 2.3.

iv) \implies v) follows from proposition 2.3 and corollary 3.3, i).

v) \implies i) is obvious.

Proof of Corollary 1.8. The implications *i) \implies ii)* and *ii) \implies iii)* are consequence of Theorem 1.6. The implication *iv) \implies i)* is clear. The implication *iii) \implies iv)* follows from the fact that if B is the affine Boshernitzan of last break point 0 with slopes (λ, λ') then the piecewise analytic homeomorphism h with one break point defined by its restriction to $[0, 1[$ of its lift

(denoted also by h) $h(x) := \frac{(\frac{\lambda}{\lambda'})^x - 1}{\frac{\lambda}{\lambda'} - 1}$ conjugate B to the rotation R_α where $\alpha = \frac{\text{Log}\lambda}{\text{Log}\lambda - \text{Log}\lambda'}$.

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