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Abstract. We give an explicit decription for univeral principal U(r)-bundles with periodic map by means of equivariant Stiefel manifolds. We then show that the associated equivariant vector bundle is equivalent to the canonical one given by G. Segal. Finally, we investigate some ideals involved in the equivariant K-theory of this classifying space.

1. Introduction

For a compact Lie group Γ and a topological group G, we call a $(\Gamma; G)$-bundle any principal G-bundle on which Γ acts by bundle maps [D], [L]. It has been shown in [D] and [L] that a universal $(\Gamma; G)$-bundle is given by $E(\Gamma; G) = \underset{H\in F}{*} \frac{\Gamma\times G}{H}$, where $F = F(\Gamma; G)$ is the family of subgroups $H \leq \Gamma \times G$ such that $H \cap G = 1$ (see also [AHJM]). This space is characte rized by the fact that the fixed points set $E(\Gamma; G)^H$ is contractible if $H \in F$ and empty otherwise [J].

2. Two lemmas

In the sequel, we shall need the following

Lemma 1. *if $G = G_1 \times G_2$, then there is a $(\Gamma \times G)$-homotopy equivalence :*

$$E(\Gamma; G) \simeq E(\Gamma; G_1) \times E(\Gamma; G_2).$$

Lemma 2. *If $\Gamma_1 \leq \Gamma_2$ and $G_1 \leq G_2$, then there exists a $(\Gamma_1 \times G_1)$-homotopy equivalence :*

$$E(\Gamma_1; G_1) \simeq res_{(\Gamma_1\times G_1)} E(\Gamma_2; G_2).$$

3. The main theorem

In what follows, we restrict our study to the case $\Gamma = \mathbf{Z}_n$, the cyclic group of order n. Recall that the Stiefel manifolds are defined by

$$V_{r,s} = \{(v_1, ..., v_r), v_i \in \mathbf{C}^s, v_i.v_j = \delta_{ij}, i, j = 1, ..., r\}$$

and

$$V_{r,\infty} = \underset{\overrightarrow{s}}{\lim} V_{r,s}.$$

Let $\mathbf{Z}_n \times U(r)$ act on the Stiefel manifold $V_{r,\infty}$ by :

$$(\gamma, a).(v_1, ..., v_r) = (w_1, ..., w_r)$$

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such that

$$w_j^k = \gamma^k . \sum_{i=1}^{r} \overline{a_{ji}} v_i^k,\ j = 1, ..., r,\ k = 1, ..., \infty$$

where γ is a generator of $\mathbf{Z}_n$, $a \in U(r)$, and $(v_1, ..., v_r) \in V_{r,\infty}$.
We then have :

Theorem 1. *There is a $(\mathbf{Z}_n \times U(r))$-homotopy equivalence :*

$$E(\mathbf{Z}_n; U(r)) \simeq V_{r,\infty}$$

Proof. It is enough to show that $V_{r,\infty}$ endowed with the so defined $(\mathbf{Z}_n \times U(r))$-action satisfies the fixed points set characterization of $E(\mathbf{Z}_n; U(r))$. Now, up to conjugation, we have :

$$F(\mathbf{Z}_n; U(r)) = \left\{ H_{d,s} : d \mid n; s = (s_1, ..., s_d); 0 \leq s_i \leq r, i = 1, ..., d; \sum_{i=1}^{d} s_i = r \right\}$$

with :

$$H_{d,s} = \langle \lambda, \rho(\lambda) = \begin{pmatrix} \lambda & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \lambda & & & & & & & \\ & & & \lambda^2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \lambda^2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda^d = 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda^d = 1 \end{pmatrix} \rangle.$$

The matrix $\rho(\lambda)$ contain s_i diagonal coefficients equal to λ^i (λ is a primitive d-th root of unity), all the other coefficient being zero. It i obvious that if

$$(1, a) . (v_1, ..., v_r) = (v_1, ..., v_r),$$

then $a = \mathbf{I}_r$. Hence, $E(\mathbf{Z}_n; U(r))^H = \emptyset$ when $H \notin F(\mathbf{Z}_n; U(r))$, on one hand. On the other hand, $(v_1, ..., v_r) \in E(\mathbf{Z}_n; U(r))^{H_{d,s}}$ iff

$$\forall j = 1, ..., r : \forall k = 1, ..., \infty : v_j^k = \lambda^{k - m_j} . v_j^k; m_j = \rho(\lambda)_{jj}.$$

Hence, $\forall j = 1, ..., r : v_j^k = 0$ unless $k = m_j mod d$]. It i not difficult to ssee that $E(\mathbf{Z}_n; U(r))^{H_{d,s}}$ is contractible (see *sec.*6. for an explicit example).

Example 1. *Consider the case $r = 1$. We have $E(\mathbf{Z}_n; \mathbf{S}^1) \simeq \mathbf{S}^\infty$. Using lemma 2.1, we get :*

$$E(\mathbf{Z}_n; \mathbf{T}^r) \simeq \prod_{i=1}^{r} \mathbf{S}^\infty.$$

If we suppose that G is a compact Lie group, we can embed it in a $U(r)$, for some suitable r. Then, by use of lemma 2.2, there is a $(\mathbf{Z}_n \times G)$-action on $V_{r,\infty}$ and a $(\mathbf{Z}_n \times G)$-homotopy equivalence :

$$E(\mathbf{Z}_n; G) \simeq V_{r,\infty}.$$

□

4. The associated $\mathbf{Z}_n$-vector bundle.

Let $\mathbf{M}$ be the $\mathbf{Z}_n$-module defined by : $\mathbf{M} = \mathbf{C}^\infty$ and

$$\gamma.(z_1, z_2, ...) = \left(\gamma.z_1, \gamma^2.z_2, ...\right)$$

. Let $G_{r,\infty}$ be the Grassmannian manifold of r-dimensional subspaces of $\mathbf{M}$. Then the right $U(r)$-action on $V_{r,\infty}$ defined by

$$(v_1, ..., v_r).a = \left(1, a^{-1}\right).(v_1, ..., v_r)$$

has as a quotient space $V_{r,\infty}/U(r) = G_{r,\infty}$. There is a canonical $\mathbf{Z}_n$-vector bundle on $G_{r,\infty}$ whose totalspace is

$$E_{\mathbf{M}} = \{(N, z) : z \in N\} \subseteq G_{r,\infty} \times \mathbf{M}$$

and which is universal [S].

Theorem 2. *The $\mathbf{Z}_n$-vector bundle associated with the principal $(\mathbf{Z}_n; U(r))$-bundle $V_{r,\infty} \to G_{r,\infty}$ is equivalent to $E_{\mathbf{M}} \to G_{r,\infty}$*

Proof. The total space of the asociated $\mathbf{Z}_n$-vector bundle is $V_{r,\infty} \times_{U(r)} \mathbf{C}^r$, with the $\mathbf{Z}_n$-action :

$$\gamma.[(v_1, ..., v_r), y] = [(\gamma, \mathbf{I}_r).(v_1, ..., v_r), y].$$

Following [H, ch. 7, (7.1)], we define a map

$$f : V_{r,\infty} \times_{U(r)} \mathbf{C}^r \to E_{\mathbf{M}},$$

by

$$f[(v_1, ..., v_r), y] = \left(\langle v_1, ..., v_r\rangle, \sum_{i=1}^{r} y_i v_i\right)$$

. According to [H], f is a vector bundle isomorphism. It is easy to check its $\mathbf{Z}_n$-equivariance. □

5. Characteristic classes

Let Γ be a compact Lie group and G a topological group. Then $E(\Gamma; G)$ is a right G-space by $e.g = (1, g).e$; we denote by $B(\Gamma; G)$ the quotient space $E(\Gamma; G)/G$. According to [AHJM], the equivariant K-theory ring $K^*_\Gamma(B(\Gamma; G))$ is isomorphic to the completion $\hat{R(\Gamma \times G)}_F$ of the complex representation ring with respect to the F-adic topology. Recall that the F-adic topology is defined on $R(\Gamma \times G)$ by the set of ideals $I_H = \ker\{R(\Gamma \times G) \to R(H)\}$, for $H \in F$. In the case of $\Gamma = \mathbf{Z}_2$ and $G = \mathbf{T}^r$, we have :

$$F = H_k : k = -1, 0, ..., r,$$

where

$$H_{-1} = 1,$$

$$H_0 = \{1, (-1, \mathbf{I}_r)\}, H_r = \{1, (-1, -\mathbf{I}_r)\}$$

and for $k = 1, ..., r-1$:

$$H_k = \left\{1, \left(-1, \begin{pmatrix} -\mathbf{I}_k & 0 \\ 0 & \mathbf{I}_{r-k} \end{pmatrix}\right)\right\}.$$

Putting

$$I_k = \ker\{R(\mathbf{Z}_2 \times \mathbf{T}^r) \to R(H_k)\},$$

the F-adic topology is defined on $R(\mathbf{Z}_2 \times \mathbf{T}^r)$ by the ideal :

$$I = \bigcap_{k=-1}^{r} I_k.$$

Proposition 1. *The the F-adic topology is defined on $R(\mathbf{Z}_2 \times \mathbf{T}^r)$ by the ideal :*

$$I = I_0 \cong R(\mathbf{Z}_2) \otimes I_{\mathbf{T}^r},$$

where $I_{\mathbf{T}^r}$ denotes the augmentation ideal, and that on $R(\mathbf{Z}_2 \times \mathbf{U}(r))$ is defined by :

$$J = I \cap R(\mathbf{Z}_2 \times \mathbf{U}(r)).$$

In general, the F-adic topology on $R(\mathbf{Z}_n \times \mathbf{U}(r))$ is defined by the ideals $I_{d,j}$ coming from the subgroups :

$$K_{d,j} = \langle \lambda, \begin{pmatrix} \lambda^j & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rangle;\ d \mid n;\ j = 1, ..., d;\ \left(\lambda^d = 1\right).$$

Proof. Easy. The general case follows from the description of $F(\mathbf{Z}_2; U(r))$ given in thm (3.1).

Example 2. *$K_{d,d} \cong \mathbf{Z}_d$, $I_{d,d} \cong R(\mathbf{Z}_d) \otimes I_{U(r)}$ and $I_{d,d-1} \cong I_{(\mathbf{Z}_d \times \mathbf{T}^r)} \cap R(\mathbf{Z}_d \times U(r))$.*

□

6. Homotopies in Stiefel manifolds

Let $\mathbf{C}^\infty$ be the telescope $\bigcup_{n \geq 1} \mathbf{C}^n$, and let

$$\begin{aligned} (\mathbf{C}^\infty)^{even} &= \{v \in \mathbf{C}^\infty : v_{2n+1} = 0; \quad \forall n = 1, ..., \infty.\}, \\ (\mathbf{C}^\infty)^{odd} &= \{v \in \mathbf{C}^\infty : v_{2n} = 0; \quad \forall n = 1, ..., \infty.\}. \end{aligned}$$

Two maps :

$$g^{even}, g^{odd} : \mathbf{C}^n \times \mathbf{I} \to \mathbf{C}^{2n}$$

are defined in the following way :

$$g_t^{even}(v_1, ..., v_n) = (1-t)(v_1, ..., v_n) + t(0, v_1, ..., 0, v_n)$$

and

$$g_t^{odd}(v_1, ..., v_n) = (1-t)(v_1, ..., v_n) + t(v_1, 0, ..., v_n, 0).$$

These maps extend to :

$$g^{even}, g^{odd} : \mathbf{C}^\infty \times \mathbf{I} \to \mathbf{C}^\infty,$$

and satisfy :

$$g_0^{even} = g_0^{odd} = Id_{\mathbf{C}^\infty};$$
$$g_1^{even}(\mathbf{C}^\infty) \subseteq (\mathbf{C}^\infty)^{even}; g_1^{odd}(\mathbf{C}^\infty) \subseteq (\mathbf{C}^\infty)^{odd}.$$

[H, ch. 3, (6.1)].

Similarly, we define

$$(\mathbf{V}_{r,\infty})^{even} = \left\{(v_1, ..., v_r) \in \mathbf{V}_{r,\infty} : v_j^{2n+1} = 0, \quad \forall j = 1, ..., r; \quad \forall n = 1, ..., \infty\right\},$$

and

$$(\mathbf{V}_{r,\infty})^{odd} = \left\{(v_1, ..., v_r) \in \mathbf{V}_{r,\infty} : v_j^{2n} = 0, \quad \forall j = 1, ..., r; \quad \forall n = 1, ..., \infty\right\}.$$

Proposition 2. *Ther are maps :*

$$G^{even}, G^{odd} : V_{r,\infty} \times \mathbf{I} \to \mathbf{V}_{r,\infty},$$

such that

$$G_0^{even} = G_0^{odd} = Id_{V_{r,\infty}};$$
$$G_1^{even}\left(\mathbf{V}_{r,\infty}\right) \subseteq \left(\mathbf{V}_{r,\infty}\right)^{even}; G_1^{odd}\left(\mathbf{V}_{r,\infty}\right) \subseteq \left(\mathbf{V}_{r,\infty}\right)^{odd}.$$

Proof. we start by the case $r = 1$. It is easy to see that $g_t^{even}(v)$ (resp. $g_t^{odd}(v)$) is not zero if v is not. We define the desired maps :

$$G^{even}, G^{odd} : \mathbf{S}^{\infty} \times \mathbf{I} \to \mathbf{S}^{\infty}.$$

by :

$$G_t^*(v) = \frac{g_t^*(v)}{\|g_t^*(v)\|},$$

for $* = even, odd$. In general, let $v \in \mathbf{C}^n$, and denote by w the firt projection of $g_t^*(v)$ on $\mathbf{C}^n$. The corresponding matrix is :

$$\begin{pmatrix}
1 & & & & & & & & & & \\
& 1-t & & & & & & & & & \\
& t & 1-t & & & & & & & & \\
& & 0 & 1-t & & & & & & & \\
& & t & 0 & 1-t & & & & & & \\
& & & 0 & 0 & 1-t & & & & & \\
& & & t & 0 & 0 & 1-t & & & & \\
& & & & & & & \ddots & & & \\
& & & & & & & & 1-t & & \\
& & & & & t & 0 & \cdots & 0 & 1-t & \\
& & & & & & & & & & 1-t
\end{pmatrix}.$$

We may delete the first or the last row and column according to the parity of n and $*$. We can prove by induction that this matrix determinant is $(1-t)^n$. We define the desired map by use of the Gram-Schmidt map GS :

$$G_t^*(v_1, ..., v_r) = GS\left(g_t^*(v_1), ..., g_t^*(v_r)\right).$$

□

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