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ON RESTRICTION PROPERTIES OF EQUIVARIANT K-THEORY RINGS.

ABDELOUAHAB AROUCHE

ABSTRACT. An important ingredient in the completion theorem of equivariant K-theory given by S. Jackowski is that the representation ring $R(\Gamma)$ of a compact Lie group satisfies two restriction properties called (N) and $(R_{\mathcal{F}})$.

We give in this note sufficient conditions on a (compact) Γ -space Z such that these properties hold with $K_{\Gamma}^*(Z)$ instead of $R(\Gamma)$. As an example, we consider the space $Z(\Gamma; G)$ of the so called “elementary cocycles with coefficients in G ” invented by H. Ibisch in his construction of a universal $(\Gamma; G)$ -bundle.

1. INTRODUCTION

In what follows, let Γ be a compact Lie group and Z a compact Γ -space. Consider, for every subgroup $\Lambda \leq \Gamma$, the ring $M_{\Lambda} = K_{\Lambda}^*(Z)$. We have a restriction morphism $res : M_{\Gamma} \rightarrow M_{\Lambda}$ whose kernel is denoted by I_{Λ}^{Γ} or I_{Λ} for short. If \mathcal{F} is a family of (closed) subgroups of Γ (stable under subconjugation), then the \mathcal{F} -adic topology on M_{Γ} is defined by the set of ideals

$$I(\mathcal{F}) = \{I_{\Lambda_1} \cdot I_{\Lambda_2} \cdots I_{\Lambda_n}, \Lambda_i \in \mathcal{F}, i = 1, \dots, n\}.$$

If $\Lambda \leq \Gamma$, then $\mathcal{F} \cap \Lambda = \{\Omega \in \mathcal{F} : \Omega \leq \Lambda\}$ is a family of subgroups of Λ .

Let X be a compact Γ -space with a Γ -map $\sigma : X \rightarrow Z$. Define the two pro-rings

$$K_{\Gamma}^*[\mathcal{F}](X) = \{K_{\Gamma}^*(X \times K) : K \text{ compact} \subseteq E\mathcal{F}\},$$

and

$$(K_{\Gamma}^*/I(\mathcal{F}))(X) = \{K_{\Gamma}^*(X)/I.K_{\Gamma}^*(X) : I \in I(\mathcal{F})\}$$

Here $E\mathcal{F}$ denotes the classifying space of T. tom Dieck [7]. The completion theorem asserts that the pro-homomorphism

$$p_X(\mathcal{F}) : K_{\Gamma}^*[\mathcal{F}](X) \rightarrow (K_{\Gamma}^*/I(\mathcal{F}))(X)$$

induced by the projection $X \times E\mathcal{F} \rightarrow X$, is actually an isomorphism, provided M_{Λ} satisfies the conditions (N) and $(R_{\mathcal{F}})$, and $K_{\Lambda}(X)$ is finitely generated over M_{Λ} (via σ), for all $\Lambda \leq \Gamma$.

In order to prove his completion theorem [4], S. Jackowski required the following two conditions, and showed they are satisfied when $Z = *$ is a point :

(N) : M_{Γ} is noetherian and M_{Λ} is a finitely generated M_{Γ} -module, $\forall \Lambda \leq \Gamma$.

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$(R_{\mathcal{F}})$: For every family \mathcal{F} of subgroups of Γ , and every subgroup $\Lambda \leq \Gamma$, the \mathcal{F} -adic topology on M_{Λ} defined by the restriction $res : M_{\Gamma} \rightarrow M_{\Lambda}$ coincides with the $(\mathcal{F} \cap \Lambda)$ -adic topology.

2. SOME NILPOTENT ELEMENTS IN $K_{\Gamma}^*(Z)$.

Let $z \in Z$, and Γ_z be its isotropy subgroup. Then an element $\zeta \in M_{\Gamma}$ belongs to $\ker \left\{ K_{\Gamma}^*(Z) \xrightarrow{\varphi_z} R(\Gamma_z) \right\}$ if and only if its restriction to the orbit $\Gamma.z$ is zero. It is well known that the elements of $\ker \left\{ K_{\Gamma}^*(Z) \rightarrow \prod_{z \in Z} R(\Gamma_z) \right\}$ are nilpotent [6](5.1). We have the following :

Lemma 1. *Let $z \in Z$ and $\zeta \in \ker \{K_{\Gamma}^*(Z) \rightarrow R(\Gamma_z)\}$. Then there exists a relatively compact open Γ -neighbourhood $V(\zeta)$ of z such that the restriction of ζ to $\overline{V(\zeta)}$ is nilpotent.*

Proof. If $\zeta = E - \mathbf{T} \in \ker \{K_{\Gamma}^*(Z) \rightarrow R(\Gamma_z)\}$, then $E_z - T = 0 \in R(\Gamma_z)$, where T is seen as a Γ_z -module. Therefore, there exists a Γ_z -module R such that $E_z \oplus R \cong T \oplus R$. Consider the isomorphism between Γ_z - vector bundles :

$$E \oplus \mathbf{R} \big|_{\{z\}} \cong \mathbf{T} \oplus \mathbf{R} \big|_{\{z\}}.$$

According to [6](1.2), there is a Γ_z -neighbourhood U of z and a Γ_z - isomorphism

$$E \oplus \mathbf{R} \big|_U \cong \mathbf{T} \oplus \mathbf{R} \big|_U.$$

Let S be a slice at z ([2] II.4.1). Then $\Gamma.(S \cap U)$ is an open Γ -neighbourhood of z such that $\zeta \in \ker \{K_{\Gamma}^*(Z) \rightarrow R(\Gamma_y)\}$, $\forall y \in \Gamma.(S \cap U)$. Now let $V(\zeta)$ be any open Γ -neighbourhood of z satisfying

$$z \in V(\zeta) \subseteq \overline{V(\zeta)} \subseteq \Gamma.(S \cap U).$$

□

Proposition 1. *Assume $K_{\Gamma}^*(Z)$ is noetherian. Then there exists a finite subset $Y \subseteq Z$ such that the ideal $\ker \left\{ K_{\Gamma}^*(Z) \rightarrow \prod_{y \in Y} R(\Gamma_y) \right\}$ consists of nilpotent elements.*

Proof. For each $z \in Z$, let $\ker \{K_{\Gamma}^*(Z) \rightarrow R(\Gamma_z)\}$ be generated by $\zeta_1, \dots, \zeta_{n_z}$, and for $j = 1, \dots, n_z$, let $V(\zeta_j)$ be as in lemma (2.1). Put

$$V_z = \bigcap_{j=1, \dots, n_z} V(\zeta_j).$$

Since Z is compact, it can be covered by a finite number of such open subsets :

$$Z = \bigcup_{i=1, \dots, n} V_{z_i}.$$

It is easy to show that the ideal $\ker \left\{ K_{\Gamma}^*(Z) \rightarrow \prod_{i=1, \dots, n} R(\Gamma_{z_i}) \right\}$ consists of nilpotent elements. □

3. SUPPORTS OF PRIMES

In [4], G. Segal defined the support *suppy* of a prime $y \in \text{Spec}R(\Gamma)$ to be a minimal subgroup from which y comes. We have shown in [1] that such a notion does exist for primes in M_Γ . Indeed, if $p : Z \rightarrow *$ is the projection of a Γ -space Z on a fixed point and $p^* : R(\Gamma) \rightarrow M_\Gamma$ is the induced homomorphism, we have the following :

Proposition 2. *Let $\Lambda \leq \Gamma$ be a (closed) subgroup. Assume M_Γ is noetherian and let $x \in \text{Spec}M_\Gamma$. Then the following properties are equivalent :*

- (1) *x come from M_Λ , i.e. $\exists y \in \text{Spec}M_\Lambda : x = \text{res}^{-1}(y)$.*
- (2) *x contains I_Λ .*
- (3) *$\text{supp}(p^{*-1}(x))$ is subconjugate to Λ .*

In order to prove proposition (3.1), we need the following lemma :

Lemma 2. *Let $x \in \text{Spec}M_\Gamma$ and $z \in Z$ be such that there exists a prime $t \in \text{Spec}R(\Gamma_z)$, with support Σ , satisfying $x = \varphi_z^{-1}(t)$, $\varphi_z : M_\Gamma \rightarrow R(\Gamma_z)$. Let $\gamma \in \Gamma$ and $z' = \gamma.z$. Then there exists a prime $t' \in \text{Spec}R(\Gamma_{z'})$, with support $\Sigma' = \gamma\Sigma\gamma^{-1}$, satisfying $x = \varphi_{z'}^{-1}(t')$, $\varphi_{z'} : M_\Gamma \rightarrow R(\Gamma_{z'})$.*

Proof. Let $\Gamma.z \xrightarrow{i} Z$ be the inclusion of the orbit of z . We have the following commutative diagram :

$$\begin{array}{ccc}
 & R(\Gamma_z) & \\
 \varphi_z \nearrow & & \nwarrow \chi_z \\
 K_\Gamma^*(Z) & \xrightarrow{i^*} & K_\Gamma^*(\Gamma.z) \\
 \searrow \varphi_{\gamma.z} & & \nwarrow \chi_{\gamma.z} \\
 & R(\Gamma_{\gamma.z}) &
 \end{array}$$

It is clear that both χ_z and $\chi_{\gamma.z}$ are isomorphisms. Put $h = \chi_z^{-1} \circ \chi_{\gamma.z}$. Then h admits a restriction $\tilde{h} : R(\Sigma) \rightarrow R(\gamma\Sigma\gamma^{-1})$, which makes the following diagram commute and ends the proof :

$$\begin{array}{ccccc}
 & R(\Gamma_z) & \xrightarrow{\text{res}} & R(\Sigma) & \\
 & \nearrow \chi_z & & & \\
 K_\Gamma^*(Z) & \xrightarrow{i^*} & K_\Gamma^*(\Gamma.z) & \cong \downarrow h & \cong \downarrow \tilde{h} \\
 & \searrow \chi_{\gamma.z} & & & \\
 & R(\Gamma_{\gamma.z}) & \xrightarrow{\text{res}} & R(\gamma\Sigma\gamma^{-1}) &
 \end{array}$$

□

Proof. Now we proceed to prove proposition (3.1) (see [1]). $1) \implies 2)$ and $2) \implies 3)$ are obvious. Let us prove $3) \implies 1)$. To this end, let z_1, \dots, z_n be elements of Z such that $\ker \left\{ \varphi : K_\Gamma^*(Z) \rightarrow \prod_{i=1, \dots, n} R(\Gamma_{z_i}) \right\}$ consists of nilpotent elements. Since every $R(\Gamma_{z_i})$ is finitely generated over M_Γ (it is even finitely generated over $R(\Gamma)$ [5]), the induced map on spectra $\varphi_* : \prod_{i=1, \dots, n} \text{Spec}R(\Gamma_{z_i}) \rightarrow \text{Spec}M_\Gamma$ is onto. So there

is an element $z \in Z$ and a prime $t \in \text{Spec}R(\Gamma_z)$ such that $x = \varphi_z^{-1}(t)$. Following [5] (3.7), we have $\text{supp}(p^{*-1}(x)) = \text{supp}(t)$. Let $\Sigma = \text{supp}(p^{*-1}(x))$. By assumption, there exists $\gamma \in \Gamma$ such that $\gamma\Sigma\gamma^{-1} \leq \Lambda$. Put $z' = \gamma.z$. According to lemma (3.2), there exists a prime $t' \in \text{Spec}R(\Gamma_{z'})$, with support $\Sigma' = \gamma\Sigma\gamma^{-1}$, and satisfying

$$x = \varphi_{z'}^{-1}(t'), \quad \varphi_{z'} : M_\Gamma \rightarrow R(\Gamma_{\gamma z}).$$

Therefore, $x \in \text{Spec}M_\Gamma$ comes from Λ , as shown by the commutative diagram :

$$\begin{array}{ccc} M_\Gamma & \rightarrow & M_\Lambda \\ \downarrow & & \downarrow \\ R(\Gamma_{\gamma.z}) & \rightarrow & R(\Lambda_{\gamma.z}) \\ & \searrow & \swarrow \\ & R(\Sigma') & \end{array}$$

□

Hence, $\text{supp}(p^{*-1}(x))$ is, up to conjugation, the minimal subgroup of Γ from which x comes. We denote it by $\text{supp}(x)$. Consequently, if $(x, y) \in \text{Spec}M_\Gamma \times \text{Spec}M_\Lambda$ is such that $x = \text{res}^{-1}(y)$, then $\text{supp}(x) = \text{supp}(y)$.

4. THE RESTRICTION PROPERTIES

Let us first consider the condition $(R_{\mathcal{F}})$. We have the following :

Theorem 1. *Assume M_Λ is noetherian $\forall \Lambda \leq \Gamma$. Then, for every family \mathcal{F} of subgroups of Γ , the condition $(R_{\mathcal{F}})$ is satisfied.*

Proof. For each $\Omega \in \mathcal{F}$ we have $I_\Omega^\Lambda \supseteq I_\Omega^\Gamma.M_\Lambda$. So it remains only to show that for any ideal $I.M_\Lambda$, where $I \in I(\mathcal{F})$, there is an ideal $J \in I(\mathcal{F} \cap \Lambda)$ such that $J \subseteq r(I.M_\Lambda)$, since M_Λ is noetherian. It is enough to do that for $I = I_\Omega^\Gamma$. So let p_1, \dots, p_n be minimal prime ideals in M_Λ containing $I.M_\Lambda$, and let $\Sigma_1, \dots, \Sigma_n$ be their supports. Put $J = I_{\Sigma_1} \dots I_{\Sigma_n}$. Since p_i comes from Σ_i , we have $J \subseteq I_{\Sigma_1} \cap \dots \cap I_{\Sigma_n} \subseteq p_1 \cap \dots \cap p_n = r(I)$. Now, if we put $q_i = \text{res}^{-1}(p_i)$, then $\Sigma_i = \text{supp}(q_i)$. But q_i comes from Ω because it contains I_Ω^Γ . Hence, Σ_i is subconjugate to Ω . Since $\Omega \in \mathcal{F}$, so does Σ_i , for $i = 1, \dots, n$, that is, $J \in I(\mathcal{F} \cap \Lambda)$.

Corollary 1. *If the condition (N) is satisfied, then so is the condition $(R_{\mathcal{F}})$.*

□

It is easy to see that the condition (N) is fulfilled if Z is a (compact) differentiable Γ -manifold, and more generally, for any finite Γ -CW-complex, for M_Λ is a finitely generated module over $R(\Lambda)$, $\forall \Lambda \leq \Gamma$. This can be shown by induction and use of Mayer-Vietoris sequence.

Corollary 2. *Assume M_Λ is finitely generated over $R(\Gamma)$, $\forall \Lambda \leq \Gamma$. Then the conditions (N) and $(R_{\mathcal{F}})$ are satisfied for every family \mathcal{F} of subgroups of Γ .*

5. AN EXAMPLE

For a topological group G and a family \mathcal{F} of subgroups of a compact Lie group Γ , H.Ibisch has constructed in [2] a universal $(\Gamma; G)$ -bundle $E(\mathcal{F}; G)$ whose “components” are the spaces

$$Z(\Lambda; G) = \{f : \Lambda \times \Lambda \rightarrow G : f(\lambda_1, \lambda_2, \lambda_3) = f(\lambda_1, \lambda_2, \lambda_3), \forall \lambda_1, \lambda_2, \lambda_3 \in \Lambda\}, \Lambda \leq \Gamma.$$

The crucial fact is that for a Γ -space X , trivial $(\Gamma; G)$ -bundles over X correspond to Γ -maps $\sigma : X \rightarrow Z(\Gamma; G)$. We then take $Z = Z(\Gamma; G)$. Actually, the Γ -space $Z(\Gamma; G)$ turns to be Γ -homeomorphic to the space of maps $\Gamma \rightarrow G$ sending 1_Γ to 1_G . Moreover, $Z(\Gamma; G)^\Gamma$ is the space of (continuous) homomorphisms $\Gamma \rightarrow G$. When Γ is finite, the space $Z(\Gamma; G)$ has a further description. For instance, if $\Gamma = \mathbf{Z}_n$ is the cyclic group of order n , then $Z(\Gamma; G)$ is the product $G^{n-1} \times \{1_G\}$ with the action $\gamma^i \cdot (g_1, \dots, g_n = 1_G) = (g_{1+i}g_i^{-1}, \dots, g_{n+i}g_i^{-1})$, where γ is a generator of \mathbf{Z}_n and the indexation is mod $[n]$, hence $Z(\mathbf{Z}_2; G)$ is no but G with involution $g \mapsto g^{-1}$. If G is moreover a compact Lie group, then $Z(\Gamma; G)$ is a compact Γ -manifold. Accordingly, the conditions (N) and $R(\mathcal{F})$ are satisfied.

REFERENCES

- [1] A. Arouche, Supports of primes in equivariant K-theory rings, Arab. J. Sc. Eng., **vol 25**, n 2C (2000), 129-132.
- [2] G.E. Bredon, Introduction to compact transformation groups, Academic Press (1972).
- [3] H. Ibisch, Classifying spaces for principal bundles with compact Lie group actions, Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht **39** (1981), 4-5.
- [4] S. Jackowski, Families of subgroups and completion, J. Pure Appl. Algebra **37** (1985), 167-179.
- [5] G. Segal, The representation ring for a compact Lie group, Publ. Math. IHES **34** (1968).
- [6] G. Segal, Equivariant K-theory, Publ. Math. IHES **34** (1968).
- [7] T. tom Dieck, Transformation groups, Walter de Gruyter, Berlin, New York (1987).

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