Permutation representations of the braid group commutator subgroup

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PERMUTATION REPRESENTATIONS OF THE BRAID GROUP COMMUTATOR SUBGROUP.

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ABSTRACT. We study the representations of the commutator subgroup K_n of the braid group B_n into the symmetric group S_r. Motivated by some experimental results, we conjecture that every such a representation with $n > r$ must be trivial.

1. Introduction

In [SiWi1], D. Silver and S. Williams exploited the structure of the kernel subgroup K of an epimorphism $\chi : G \to \mathbb{Z}$, where G is a finitely presented group, to show that the set $Hom\,(K, \Sigma)$ of representations of K into a finite group Σ has a structure of a subshift of finite type (SFT), a symbolic dynamical system described by a graph Γ; namely, there is a one to one correspondence between representations $\rho : K \to \Sigma$ and bi-infinite paths in Γ.

We apply this method to the group B_n of braids with n-strands, with χ being the abelianization homomorphism and Σ the symmetric group of degree r. The subgroup $K_n = \ker \chi$ is then the commutator subgroup of B_n.

It is a well known fact that for a given group K, there is a finite to one correspondence between its subgroups of index no greater than r and representations $\rho : K \to S_r$. This correspondence can be described by

$$\rho \longmapsto \{g \in K : \rho\,(g)\,(1) = 1\}\,.$$

The pre-image of a subgroup of index exactly r consists of $(r-1)!$ transitive representations ρ. (ρ is said to be transitive if $\rho\,(K)$ operates transitively on $\{1, 2, ..., r\}$). This will allow us to draw some conclusions about the subgroups of finite index of K_n.

We give an algorithm to compute $Hom\,(K_n, S_r)$, for $n \geq 5$ and $r \geq n$. Motivated by some experimental results, we conjecture that $Hom\,(K_n, S_r)$ is trivial for $n \geq 5$ and $r < n$. Since every representation in $Hom\,(B_n, S_r)$ restricts to an element of $Hom\,(K_n, S_r)$, we enhance the given algorithm in order to compute $Hom\,(B_n, S_r)$.

2. Generalities

Let B_n be the braid group given by the presentation:

$$\left\langle \sigma_1, ..., \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i; & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j; & |i-j| = 1 \end{array} \right\rangle,$$

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(see [BuZi] for additional background). Let $\beta \in B_n$ be a braid. Then β can be written as :

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k},$$

with $i_1, \cdots, i_k \in \{1, \cdots, n-1\}$ and $\varepsilon_i = \pm 1$. Define the exponent sum of β (in terms of the σ_i's) denoted by $\exp(\beta)$, as:

$$\exp(\beta) = \varepsilon_1 + \cdots + \varepsilon_k.$$

Then $\exp(\beta)$ is an invariant of the braid group, that is, it doesn't depend on the writing of β. Moreover, $\exp(\beta) : B_n \to \mathbb{Z}$ is a homomorphism. Let H_n denote its kernel. Then the Reidemeister-Schreier theorem [LySc] enables us to find a presentation for H_n. We choose the set

$$\left\{ \cdots, \sigma_1^{-m}, \sigma_1^{-m+1}, \cdots, \sigma_1^{-1}, 1, \sigma_1, \sigma_1^2, \cdots, \sigma_1^m, \cdots \right\}$$

as a Schreier system of right coset representatives of H_n in B_n. Putting $z_m = \sigma_1^m \left(\sigma_2 \sigma_1^{-1}\right) \sigma_1^{-m}$ for $m \in \mathbb{Z}$, and $x_i = \sigma_i \sigma_1^{-1}$ for $i = 3, \cdots, n-1$, we get the following presentation of H_n :

$$H_n = \left\langle \begin{array}{ll} z_m, & m \in \mathbb{Z} \\ x_i, & i = 3, \cdots, n-1 \end{array} \;\middle|\; \begin{array}{cc} x_i x_j = x_j x_i, & |i-j| \geq 2; \\ x_i x_j x_i = x_j x_i x_j, & |i-j| = 1; \\ z_m z_{m+2} = z_{m+1}, & \\ z_m x_3 z_{m+2} = x_3 z_{m+1} x_3, & m \in \mathbb{Z}; \\ z_m x_i = x_i z_{m+1}, & i = 4, \cdots, n-1; \\ & m \in \mathbb{Z}. \end{array} \right\rangle$$

Example 1. *For $n = 3, 4$, we have:*

$$H_3 = \langle z_m \,|\, z_m z_{m+2} = z_{m+1}; \forall m \in \mathbb{Z} \rangle$$

is a free group on two generators $z_0 = \sigma_2 \sigma_1^{-1}$ and $z_{-1} = \sigma_1^{-1} \sigma_2$, and

$$H_4 = \left\langle z_m, t \;\middle|\; \begin{array}{c} z_m z_{m+2} = z_{m+1}, \\ z_m t z_{m+2} = t z_{m+1} t, \end{array} \quad m \in \mathbb{Z} \right\rangle.$$

Note that $H_2 = \{1\}$, since $B_2 = \langle \sigma_1 \,|\, \rangle \cong \mathbb{Z}$.

Now, every commutator in B_n has exponent sum zero. Conversely, every generator of H_n is a product of commutators. Hence, we have $H_n = K_n$, and exp is the abelianization homomorphism.

Each K_n fits into a split exact sequence:

$$1 \to K_n \to B_n \to \mathbb{Z} \to 0,$$

and there are "natural" inclusions $B_n \subset B_{n+1}$ and $K_n \subset K_{n+1}$.

3. The representation shift

Our goal is to sudy representations of K_n into the symmetric group S_r. Note that there is a natural homomorphism:

$$\pi : B_n \to S_n,$$

for all $n \geq 2$, given by $\sigma_i \longmapsto (i\,i+1)$. This restricts to K_n to give a non trivial homomorphism:

$$\begin{array}{lll} x_i & \longmapsto & (12)\,(i\,i+1) \quad i = 3, \cdots, n-1 \\ z_m & \longmapsto & \begin{cases} (132)\,, & m \text{ is even} \\ (123)\,, & m \text{ is odd} \end{cases} \end{array}$$

We start with $n = 3$ and describe $Hom\,(K_3, S_r)$ by means of a graph Γ that we will construct in a step by step fashion [SiWi2]. Note that $Hom\,(K_3, S_r)$ contains $\pi\,|_{K_3}$, for $r \geq 3$. Later we will see that $Hom\,(K_3, S_2)$ is not trivial.

A representation $\rho : K_n \to S_r$ is a function ρ from the set of generators z_m to S_r such that for each $m \in \mathbb{Z}$, the relation:

$$\rho\,(z_m)\,\rho\,(z_{m+2}) = \rho\,(z_{m+1})$$

holds in S_r. Any such function can be constructed as follows, beginning with step 0 and proceeding to steps $\pm 1, \pm 2, \cdots$

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(step -1) Choose $\rho\,(z_{-1})$ if possible such that $\rho\,(z_{-1})\,\rho\,(z_1) = \rho\,(z_0)$.
(step 0) Choose values $\rho\,(z_0)$ and $\rho\,(z_1)$.
(step 1) Choose $\rho\,(z_2)$ if possible such that $\rho\,(z_0)\,\rho\,(z_2) = \rho\,(z_1)$.
(step 2) Choose $\rho\,(z_3)$ if possible such that $\rho\,(z_1)\,\rho\,(z_3) = \rho\,(z_2)$.
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This process leads to a bi-infinite graph whose vertices are the maps $\rho : \{z_0, z_1\} \to S_r$, each of which can be regarded as an ordered pair $(\rho\,(z_0)\,, \rho\,(z_1))$. There is a directed edge from ρ to ρ' if and only if $\rho\,(z_1) = \rho'\,(z_0)$ and $\rho\,(z_0)\,\rho'\,(z_1) = \rho\,(z_1)$. In such a case, we can extend $\rho : \{z_0, z_1\} \to S_r$ by defining $\rho\,(z_2)$ to be equal to $\rho'\,(z_1)$. Now if there is an edge from ρ' to ρ", we can likewise extend ρ by defining $\rho\,(z_3)$ to be ρ" (z_1). We implement this process by starting with an ordered pair (a_0, a_1) of elements of S_r, and computing at each step a new ordered pair from the old one, so that every edge in the graph looks like:

$$(a_m, a_{m+1}) \to (a_{m+1}, a_{m+2})\,,$$

with

$$a_{m+2} = a_m^{-1} a_{m+1}.$$

In our case, the graph Γ we obtain consists necessarily of disjoint cycles. This gives an algorithm for finding $Hom(K_3, S_r)$. Observe that $Hom(K_3, S_r)$ is endowed with a shift map

$$\sigma : \rho \longmapsto \sigma\,(\rho)$$

defined by

$$\sigma\,(\rho) : x \longmapsto \rho\left(\sigma_1 x \sigma_1^{-1}\right).$$

If we regard ρ as a bi-infinite path in the graph Γ, then σ correspond to the shift map $(a_m, a_{m+1}) \longmapsto (a_{m+1}, a_{m+2})$, since $a_{m+1} = \sigma_1 a_m \sigma_1^{-1}$. Any cycle in the graph Γ with length p corresponds to p representations having least period p. These are the iterates of some representation $\rho \in Hom(K_3, S_r)$ satisfying $\rho\,(z_m) = a_m$ and $\sigma^p\,(\rho) = \rho$, since $a_{m+p} = a_m$.

In order to minimize calculations, we extract some foreseeable behaviour for various choices of the ordered pair (a_0, a_1) in the previous algorithm.

First, the dynamics of ordered pairs (a_0, a_1) such that $a_0 = 1$ or $a_1 = 1$ or $a_0 = a_1$ is entirely known. To be precise, we get a cycle of length 6 unless $a = a^{-1}$, in which case it is of length 3 (or 1 if and only if $a=1$).

$$\begin{aligned}(1,a) &\rightarrow (a,a)\rightarrow(a,1)\rightarrow\left(1,a^{-1}\right)\\ &\rightarrow \left(a^{-1},a^{-1}\right)\rightarrow\left(a^{-1},1\right)\rightarrow(1,a)\,.\end{aligned}$$

Second, when we proceed to a new step, we do not need to take a pair we have already got in a previous cycle, since we would get indeed the same cycle. The following dichotomy will prove useful in the sequel:

Definition 1. *If a vertex of a cycle in* Γ *has equal components, then the cycle is said of type I. Otherwise, it is of type II.*

Note that a cycle is determined by any of its vertices. Furthermore, the type I cycles are determined by elements of S_r. Recall that a non trivial element $a\in S_r$ has order two if and only if it is a product of disjoint transpositions. Let n_r be the number of such elements. This gives us a means to compute the number of type I cycles to be $\frac{1}{2}\left(1+n_r+r!\right)$ and of representations coming from them to be $3r!-2$.

Example 2. *Since* $S_2=\{1,(12)\}$, $Hom\left(K_3,S_2\right)$ *consists only of the following type I cycle:*

$$(1,(12))\rightarrow((12)\,,(12))\rightarrow((12)\,,1)\rightarrow(1,(12))\,,$$

along with the trivial representation. So $\left|Hom\left(K_3,S_2\right)\right|=4$. *As for* $Hom\left(K_3,S_3\right)$, *there are three type I cycles of length* 3 *corresponding to transpositions and one type I cycle of length* 6 *corresponding to the* 3*-cycle* (123) *(and its inverse). Looking at type II cycles, we find two cycles of length* 9 *corresponding to the pairs* $((23),(12))$ *and* $((23)\,,(123))$ *and one cycle of length* 2 *corresponding to the pair* $((123)\,,(132))$. *This last one is exactly the orbit (under the shift map* σ*) of* $\pi\left|_{K_3}\right.$. *All by all, we have* $\left|Hom\left(K_3,S_3\right)\right|=36$.

In the last section we present, among other things, the results of computer calculations of type II cycles in the graphs of $Hom(K_3,S_4)$ *using Maple.*

Now let us proceed to compute $Hom(K_4,S_r)$. Since $K_3\subset K_4$, every representation $\rho\in Hom(K_4,S_r)$ restricts to a representation $\rho\left|_{K_3}\right.\in Hom(K_3,S_r)$, the latter being described by a cycle. All we have to do is then to check which representation in $Hom(K_3,S_r)$ does extend to K_4. To this end, observe that K_4 is gotten from K_3 by adjunction of a generator x_3 subject to the relations

$$z_m x_3 z_{m+2}=x_3 z_{m+1} x_3;m\in\mathbb{Z}.$$

Hence we may proceed as follows. Take a cycle in $Hom(K_3,S_r)$ (by abuse of language, i.e. identify each representation with its orbit, since a representation in $Hom(K_3,S_r)$ extends to K_4 if and only if every element in its orbit does), and choose if possible a value $b_3\in S_r$ for $\rho\left(x_3\right)$. This value must satisfy the relations

$$a_m b_3 a_{m+2}=b_3 a_{m+1} b_3;$$

for $m=0,\cdots,p-1$, where p is the cycle's length and the indexation is $modp$. Observe that the choice $b_3=1$ is convenient, so all cycles extend to K_4. However, this is the only possibility for type I cycles to extend, for if b_3 commute with some a_m, then $b_3=1$. For type II cycles, we find for example that no cycle in $Hom(K_3,S_3)$ extends to K_4 with non trivial b_3 and that out of 71 cycles in $Hom(K_3,S_4)$ only ten do extend to K_4, each with three possibilities for b_3 (the same for all; see the last section).

Before giving the general procedure, let us proceed one further step to show that all type I cycles will vanish for $n \geq 5$. Take a cycle in $Hom(K_3, S_r)$, $r \geq 2$ along with a convenient value b_3 of $\rho(x_3)$. We look for an element $b_4 \in S_r$ satisfying :

$$\begin{aligned} a_m b_4 &= b_4 a_{m+1}, \quad m = 0, \cdots, p; \\ b_3 b_4 b_3 &= b_4 b_3 b_4 \end{aligned}$$

Hence, if

$$b_3 b_4 = b_4 b_3$$

then using

$$a_m a_{m+2} = a_{m+1},$$

and

$$a_m b_3 a_{m+2} = b_3 a_{m+1} b_3,$$

we get

$$b_3 = b_4 = 1,$$

and

$$a_m = a_{m+1}, \forall m = 0, \cdots, p-1,$$

so that the representation is trivial. So only type II cycles, with non trivial b_3 possibly extend to K_5 (beside the trivial one). So no (type II) cycle in $Hom(K_3, S_3)$ extends to K_5. It turns out that no type II cycle in $Hom(K_3, S_4)$ extends to K_5 .

Algorithm 1. *The general procedure for $Hom(K_n, S_r)$, $n \geq 5$ is to consider only type II cycles along with convenient non trivial values $b_3, \cdots, b_{n-2}$, which correspond to representations in $Hom(K_{n-1}, S_r)$ and find a non trivial element $b_{n-1} \in S_r$ such that the following relations are satisfied:*

$$\begin{aligned} a_m b_{n-1} &= b_{n-1} a_{m+1}, & m &= 0, \cdots, p; \\ b_{n-1} b_i &= b_i b_{n-1} & i &= 3, \cdots, n-2; \\ b_{n-1} b_{n-2} b_{n-1} &= b_{n-2} b_{n-1} b_{n-2} \end{aligned}$$

The element b_{n-1} has to be non trivial, otherwise the representation is trivial. Experimental results lead us to conjecture that the process will stop at step $n = r$. That is:

Conjecture 1. *$Hom(K_n, S_r)$ is trivial for $n \geq r+1$.*

It is obvious that a cycle (of any type) can not extend to K_n if it doesn't extend to K_{n-1} it is enough for the conjecture to be true that $Hom(K_{r+1}, S_r)$ be trivial. Recall that for $n \leq r$, $Hom(K_n, S_r)$ is not trivial since it contains the homomorphism $\pi|_{K_n} : K_n \to S_n$.

4. Extension to the braid group

In this section, we address the question of extending representations

$$\rho \in Hom(K_n, S_r)$$

to representations

$$\tilde{\rho} \in Hom(B_n, S_r).$$

Applying [SiWi1 (3.5)], the extension is possible if and only if there is an element $c \in S_r$ such that

$$\begin{aligned} a_m c &= c a_{m+1}, & m &= 0, \cdots, p-1; \\ c b_i &= b_i c & i &= 3, \cdots, n-1; \end{aligned}$$

Observe that a necessary condition for a representation $\rho \in Hom(K_n, S_r)$ to extend to $\tilde{\rho} \in Hom(B_n, S_r)$ is that $\rho \in Hom(K_n, A_r)$, since the alternating group A_r is the commutator subgroup of S_r. A sufficient condition is that ρ be the restriction of some representation $\hat{\rho} \in Hom(K_{n+2}, S_r)$ for if this is the case, the choice $c = b_{n+1}$ will do. In this case, since $\hat{\rho}$ maps K_n into A_r, it also maps K_{n+2} into A_r, for the values b_i are conjugate and A_r is normal in S_r. As a result, we get the following

Proposition 1. *for $n \geq 5$, $Hom(K_n, S_r) = Hom(K_n, A_r)$.*

Actually we can enhance our algorithm to one which gives for fixed $n \geq 5$ and $r \geq n$ the sets $Hom(K_n, S_r)$ and $Hom(B_n, S_r)$.

Step one: find all cycles of both types. This gives $Hom(K_3, S_r)$.

Step two: For the trivial cycle, take any c. For a type II cycle C, find $c \neq 1$ such that $a_m c = c a_{m+1}$, for $m = 0, \cdots, p-1$. This gives $Hom(B_3, S_r)$.

Step three: For a cycle of any type , take $b_3 = 1$. For a type II cycle C, find $b_3 \neq 1$ such that $a_m b_3 a_{m+2} = b_3 a_{m+1} b_3$, for $m = 0, \cdots, p-1$. This gives $Hom(K_4, S_r)$.

Step four: take a type II cycle C, along with a convenient b_3. If this cycle occurs in $Hom(B_3, S_r)$ with some convenient c then :

if $cb_3 = b_3 c$, then the representation $[C, b_3]$ moves up to a representation $[C, b_3, c]$ in $Hom(B_4, S_r)$;

if $cb_3c = b_3cb_3$, then the representation $[C, b_3]$ moves up to a representation $[C, b_3, b_4]$ in $Hom(K_5, S_r)$ by taking $b_4 = c$.

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Step i: take a representation ρ in $Hom(K_i, S_r)$, encoded by a type II cycle C along with convenient values $b_3, \cdots, b_{i-1}$. if $[C, b_3, b_4, \cdots, b_{i-2}, c]$ occurs in $Hom(B_{i-1}, S_r)$ with some convenient c then:

if $cb_{i-1} = b_{i-1}c$, then the representation $[C, b_3, b_4, \cdots, b_{i-1}]$ moves up to a representation $[C, b_3, b_4, \cdots, b_{i-1}, c]$ in $Hom(B_i, S_r)$;

if $cb_{i-1}c = b_{i-1}cb_{i-1}$, then the representation $[C, b_3, b_4, \cdots, b_{i-1}]$ moves up to a representation $[C, b_3, b_4, \cdots, b_{i-1}, b_i]$ in $Hom(K_{i+1}, S_r)$ by taking $b_i = c$.

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Note that if the conjecture is true, then for $n \geq r+1$, every representation $\tilde{\rho} : B_n \to S_r$ factorizes through the abelianized group $(B_n)_{ab}$, and has a cyclic image. Hence, there are $r!$ possible choices for $\tilde{\rho}$.

5. Consequences

Regarding the correspondence between subgroups of finite index of a group K and its representations into symmetric groups, we investigate the subgroups of index r of K_n for low degrees r. The general principle is to compute the number of transitive representatations of K into S_r to deduce the number of subgroups of K with index exactly r. We start with K_3 as usual. Note that since K_3 is freely generated by z_0 and z_{-1}, it maps onto any symmetric group, and hence, has subgroups of every index. Now, if a representation in $Hom(K_3, S_r)$ is transitive, then so are the representations in its orbit. Consider a type I cycle in $Hom(K_3, S_r)$. Then the representations it defines are transitive if and only if the defining element

a is (with respect to the action of S_r on $\{1, \cdots, r\}$). This exactly means that a is an r-cycle. If $r > 2$ then $a^2 \neq 1$ and the cycle has length 6.

Claim 1. *The number of transitive representations $\rho \in Hom(K_3, S_r)$, $r \geq 2$ coming from type I cycles is $3\,(r-1)!$.*

For $r = 2$, there are only type I cycles and there is only one 2-cycle, which has length 3; Hence, The number of transitive representations $\rho \in Hom(K_3, S_2)$ is 3. The kernels of these representations give rise to subgroups of K_3 with index 2.

Claim 2. *There are three subgroups of K_3 with index 2.*

Now we compute the number of subgroups of K_3 with index 3. Among all representations we have seen in example 2, there are six transitive representations coming from the only type I cycle and all representations coming from type II cycles are transitive. Hence:

Claim 3. *The number of transitive representations in $Hom\,(K_3, S_3)$ is 26, consequently there are thirteen subgroups of K_3 with index 3.*

We can proceed in this way for every degree r. To compute the number of transitive representations of K_3 into S_r , we need only consider those coming from type II cycles, since we already know the number of those coming from type I cycles. This can be done using a computer algebra system, by taking any cycle $C = (a_0, \cdots, a_{p-1})$ and checking if the subgroup $\langle a_0, \cdots, a_{p-1}\rangle$ of S_r acts transitively on $\{1, \cdots, r\}$. If so, this gives rise to p transitive representations in $Hom\,(K_3, S_3)$. Then we divide the total number by $(r-1)!$ to find the number of subgroups of K_3 of index r.

Now let us consider $Hom\,(K_4, S_r)$. For $r = 2$ we have, as previously:

Claim 4. *There are three subgroups of K_4 with index 2.*

As for transitive representations in $Hom\,(K_4, S_3)$, since all cycles in $Hom\,(K_3, S_3)$ extend to K_4 with only $b_3 = 1$, we have:

Claim 5. *There are twenty six transitive representations in $Hom\,(K_4, S_3)$, hence thirteen subgroups of K_3 with index 3.*

For $r \geq 4$, we have $3\,(r-1)!$ transitive representations coming from type I cycles, and we must check which representation coming from a type II cycle is transitive. For a cycle $C = (a_0, \cdots, a_{p-1})$ such that $\langle a_0, \cdots, a_{p-1}\rangle$ failed to be transitive, we check if $\langle a_0, \cdots, a_{p-1}, b_3\rangle$ (with b_3 non trivial) is transitive. Indeed, if $\langle a_0, \cdots, a_{p-1}\rangle$ is transitive, then so is $\langle a_0, \cdots, a_{p-1}, b_3\rangle$. Finally, we divide the total number by $(r-1)!$ to find the number of subgroups of K_3 of index r.

Now, we consider $n \geq 5$, where we get rid of type I cycles. Suppose we have found the transitive representations in $Hom\,(K_{n-1}, S_r)$. We then take, for fixed r, a type II cycle $C = (a_0, \cdots, a_{p-1})$ along with values $b_3, \cdots, b_{n-1}$, such that $\langle a_0, \cdots, a_{p-1}, b_3, \cdots, b_{n-2}\rangle$ failed to be transitive and check if $\langle a_0, \cdots, a_{p-1}, b_3, \cdots, b_{n-1}\rangle$ is transitive. We may enhance algorithm 1 by checking, each time we get a new type II cycle, if it is transitive, and if not, we re-check at each time the cycle extends from K_i to K_{i+1}, $i = 3, \cdots, n-1$, after having augmented it with b_i. Dividing by $(r-1)!$ the number of transitive representations in $Hom\,(K_n, S_r)$ we find the number of subgroups of K_n with index r. As a consequence of conjecture 1, we get the following:

Conjecture 2. *For $n \geq 5$ and $2 \leq r \leq n-1$, there are no subgroups of K_n with index r. Moreover, every nontrivial representation ρ of K_n into S_n is transitive.*

Remark 1. *We can likewise investigate the number of subgroups of B_n with a given index r by looking at transitive representations of B_n into S_r. Namely, if conjecture 1 is true, then there is exactly one subgroup of index r in B_n, for $1 \leq r \leq n-1$. Moreover, if $\rho : B_n \to S_n$ is a representation, then $\rho|_{K_n}$ is either trivial or transitive, according to conjecture 2. In the first case, ρ has a cyclic image and we know when it is transitive. In the second case, ρ is transitive.*

6. Experimental facts

In what follows, we list the type II cycles of $Hom(K_n, S_r)$ for various (small) n and r. A word about the notation: each cycle $B[a_0, a_1] = [a_2, a_3, \cdots, a_{p-1}, a_0, a_1]$ is indexed by its first vertex (a_0, a_1) and is followed by its length p. Elements $\tau \in S_r$ are ordered from 1 to $r!$ with repect to the lexicographic order on the vectors $(\tau(1), \cdots, \tau(r))$. It would have taken too much space to list the cycles for $r = 5$. We found that there were no (type II) cycles in $Hom(K_5, S_4)$ nor in $Hom(K_6, S_5)$. Furthermore, $Hom(K_4, S_3)$ contains no type II cycles with non trivial b_3. This motivated our conjecture 1.

$n = 3; r = 3:$
B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]
9
B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]
9
B[4, 5] = [4, 5]
2

$n = 3; r = 4:$
B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]
9
B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]
9
B[2, 7] = [8, 2, 7]
3
B[2, 8] = [7, 2, 8]
3
B[2, 9] = [11, 6, 16, 18, 3, 20, 19, 2, 9]
9
B[2, 10] = [12, 3, 23, 21, 6, 14, 13, 2, 10]
9
B[2, 11] = [9, 6, 18, 16, 3, 19, 20, 2, 11]
9
B[2, 12] = [10, 3, 21, 23, 6, 13, 14, 2, 12]
9
B[2, 13] = [19, 22, 4, 23, 15, 12, 11, 2, 13]
9
B[2, 14] = [20, 15, 18, 5, 22, 10, 9, 2, 14]
9

B[2, 15] = [21, 22, 2, 21, 15, 22, 21, 2, 15]
9
B[2, 16] = [22, 15, 16, 2, 22, 16, 15, 2, 16]
9
B[2, 17] = [23, 7, 24, 23, 2, 17]
6
B[2, 18] = [24, 7, 18, 17, 2, 18]
6
B[2, 19] = [13, 22, 23, 4, 15, 11, 12, 2, 19]
9
B[2, 20] = [14, 15, 5, 18, 22, 9, 10, 2, 20]
9
B[2, 23] = [17, 7, 23, 24, 2, 23]
6
B[2, 24] = [18, 7, 17, 18, 2, 24]
6
B[3, 7] = [13, 15, 3, 13, 7, 15, 13, 3, 7]
9
B[3, 8] = [14, 22, 17, 14, 3, 8]
6
B[3, 9] = [15, 7, 9, 3, 15, 9, 7, 3, 9]
9
B[3, 10] = [16, 7, 11, 5, 15, 23, 20, 3, 10]
9
B[3, 11] = [17, 22, 11, 8, 3, 11]
6
B[3, 12] = [18, 15, 4, 14, 7, 21, 19, 3, 12]
9
B[3, 14] = [8, 22, 14, 17, 3, 14]
6
B[3, 16] = [10, 7, 5, 11, 15, 20, 23, 3, 16]
9
B[3, 17] = [11, 22, 8, 11, 3, 17]
6
B[3, 18] = [12, 15, 14, 4, 7, 19, 21, 3, 18]
9
B[3, 22] = [24, 3, 22]
3
B[3, 24] = [22, 3, 24]
3
B[4, 5] = [4, 5]
2
B[4, 8] = [20, 16, 17, 4, 13, 8, 16, 20, 17, 13, 4, 8]
12
B[4, 9] = [21, 20, 4, 9]
4
B[4, 10] = [22, 13, 18, 6, 21, 11, 7, 4, 10]
9

B[4, 11] = [23, 12, 19, 11, 13, 23, 19, 4, 11]
9
B[4, 12] = [24, 9, 16, 8, 12, 4, 24, 16, 9, 8, 4, 12]
12
B[4, 16] = [12, 9, 4, 16]
4
B[4, 17] = [9, 20, 24, 4, 21, 17, 20, 9, 24, 21, 4, 17]
12
B[4, 18] = [10, 13, 11, 18, 21, 10, 11, 4, 18]
9
B[4, 19] = [14, 21, 18, 19, 12, 14, 18, 4, 19]
9
B[4, 20] = [13, 16, 4, 20]
4
B[4, 22] = [18, 13, 6, 11, 21, 7, 10, 4, 22]
9
B[5, 7] = [14, 16, 6, 23, 9, 22, 19, 5, 7]
9
B[5, 8] = [13, 21, 24, 5, 20, 8, 21, 13, 24, 20, 5, 8]
12
B[5, 9] = [17, 12, 21, 8, 9, 5, 17, 21, 12, 8, 5, 9]
12
B[5, 10] = [18, 9, 14, 10, 20, 18, 14, 5, 10]
9
B[5, 12] = [16, 13, 5, 12]
4
B[5, 13] = [20, 21, 5, 13]
4
B[5, 14] = [19, 16, 23, 14, 9, 19, 23, 5, 14]
9
B[5, 16] = [24, 13, 12, 17, 16, 5, 24, 12, 13, 17, 5, 16]
12
B[5, 19] = [7, 16, 14, 6, 9, 23, 22, 5, 19]
9
B[5, 21] = [9, 12, 5, 21]
4
B[5, 23] = [11, 20, 10, 23, 16, 11, 10, 5, 23]
9
B[6, 7] = [20, 22, 6, 20, 7, 22, 20, 6, 7]
9
B[6, 8] = [19, 15, 24, 19, 6, 8]
6
B[6, 10] = [24, 15, 10, 8, 6, 10]
6
B[6, 12] = [22, 7, 12, 6, 22, 12, 7, 6, 12]
9
B[6, 15] = [17, 6, 15]
3

B[6, 17] = [15, 6, 17]
3
B[6, 19] = [8, 15, 19, 24, 6, 19]
6
B[6, 24] = [10, 15, 8, 10, 6, 24]
6
B[8, 17] = [24, 8, 17]
3
B[8, 18] = [23, 8, 23, 18, 8, 18]
6
B[8, 24] = [17, 8, 24]
3
B[9, 13] = [9, 13]
2
B[9, 18] = [11, 16, 19, 18, 20, 11, 19, 9, 18]
9
B[10, 14] = [12, 23, 10, 21, 14, 23, 13, 10, 14]
9
B[10, 17] = [10, 19, 17, 19, 10, 17]
6
B[11, 14] = [24, 14, 11, 24, 11, 14]
6
B[12, 20] = [12, 20]
2
B[16, 21] = [16, 21]
2

$n = 4; r = 4$:
ten cycles of length 2 along with three values $b_3 = 8, 17, 24$.
[8, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]
[17, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]
[24, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]

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