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| Maurice A. de Gosson. A Weyl Calculus on Symplectic Phase Space. 2005. hal-00007947v1

HAL Id: hal-00007947

<https://hal.science/hal-00007947v1>

Preprint submitted on 11 Aug 2005 (v1), last revised 16 Feb 2006 (v2)

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A Weyl Calculus on Symplectic Phase Space

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August 11, 2005

Abstract

We study the twisted Weyl symbol of metaplectic operators; this requires the definition of an index for symplectic paths related to the Conley–Zehnder index. We thereafter define a metaplectically covariant algebra of pseudo-differential operators acting on functions on symplectic space.

1 Introduction

It is part of the mathematical folklore to describe the metaplectic representation of the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ in terms of unitary operators acting on functions in n variables; these variables are either the “position coordinates” $x = (x_1, \dots, x_n)$ or the dual “momentum coordinates” $p = (p_1, \dots, p_n)$, or a mixture of both containing no “conjugate pairs” x_j, p_j . There is thus a discrepancy between symplectic *geometry*, where $\mathrm{Sp}(2n, \mathbb{R})$ acts on phase-space points depending on $2n$ variables (x, p) , and symplectic *harmonic analysis* where the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ acts on functions of half as many variables. This state of affairs can hardly be questioned by quantum physicists: the metaplectic representation intervening both in an “active” and a “passive” way in quantum mechanics, it is comforting for them that $\mathrm{Mp}(2n, \mathbb{R})$ can only be seen, to paraphrase Dirac, “with the x -eye or the p -eye”: for them the uncertainty principle prohibits the existence of a quantum-mechanical phase space.

It turns out that it is perfectly possible to construct a metaplectic representation of $\mathrm{Sp}(2n, \mathbb{R})$ acting on functions of $z = (x, p)$; there are actually at least two options for doing this. One can use the easy way, which consists in constructing an isometry U of $L^2(\mathbb{R}^n)$ on a subspace of $L^2(\mathbb{R}^{2n})$ (for instance the “coherent state representation”, familiar to physicists), and to make operators $\widehat{S} \in \mathrm{Mp}(2n, \mathbb{R})$ act on $L^2(\mathbb{R}^{2n})$ by intertwining them with U . This straightforward approach (common in quantum mechanics) has the disadvantage that it is tautological: we do not obtain a true action of $\mathrm{Mp}(2n, \mathbb{R})$ on all of $L^2(\mathbb{R}^{2n})$, but only on a subspace; it is certainly not obvious what sense to give to $\widehat{S}f$ for arbitrary $f \in L^2(\mathbb{R}^{2n})$. There is another way, which requires more work, but which is in the end far more rewarding. It consists in two steps: one first writes the elements of a set of generators of $\mathrm{Mp}(2n, \mathbb{R})$ in the form

$$\widehat{S} = \left(\frac{1}{2\pi}\right)^n \int a_\sigma(z_0) \widehat{T}(z_0) dz_0;$$

where

$$\widehat{T}(z_0) = e^{i(\langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle)} T(z_0)$$

is the Heisenberg–Weyl operator with $T(z_0)f(x) = f(x - x_0)$ and a_σ the twisted symbol of \widehat{S} . One then extends the operator \widehat{S} by letting $\widehat{T}(z_0)$ act functions on \mathbb{R}^{2n} via the “obvious” formula

$$\widehat{T}(z_0)F(z) = e^{i(\langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle)} F(z - z_0).$$

This last step actually corresponds to an irreducible unitary representation of the Heisenberg group on a closed subspace of $L^2(\mathbb{R}^{2n})$; in view of Stone–von Neumann’s theorem this representation is thus equivalent to the Heisenberg–Weyl representation. This possibility was actually very explicitly hinted at in the seminal paper by Grossmann *et al.* [20], but not fully exploited; it is a shame that it apparently has fallen in such oblivion.

In this paper we will actually use a slight variant of the procedure outlined above: instead of defining the phase-space operators by bluntly extending the domain of $\widehat{T}(z_0)$, we will use the variant

$$\widehat{S}_{\mathrm{ph}} = \left(\frac{1}{2\pi}\right)^n \int a_\sigma(z_0) \widehat{T}_{\mathrm{ph}}(z_0) dz_0$$

where $\widehat{T}_{\mathrm{ph}}(z_0)$ is the modified Heisenberg–Weyl operators defined by

$$\widehat{T}_{\mathrm{ph}}(z_0)F(z) = e^{-\frac{i}{2}\sigma(z, z_0)} F(z - z_0);$$

these operators satisfy the same commutation and product relations

$$\widehat{T}_{\text{ph}}(z_0)\widehat{T}_{\text{ph}}(z_1) = e^{-i\sigma(z_0, z_1)}\widehat{T}_{\text{ph}}(z_1)\widehat{T}_{\text{ph}}(z_0) \quad (1)$$

$$\widehat{T}_{\text{ph}}(z_0 + z_1) = e^{-\frac{i}{2}\sigma(z_0, z_1)}\widehat{T}_{\text{ph}}(z_0)\widehat{T}_{\text{ph}}(z_1) \quad (2)$$

as the operators $\widehat{T}(z_0)$ and will therefore allow the construction of an irreducible unitary representation of the Heisenberg group \mathbf{H}_n . This being done we can go one step further and associate to an arbitrary Weyl operator

$$\widehat{A} = \left(\frac{1}{2\pi}\right)^n \int a_\sigma(z_0)\widehat{T}(z_0)dz_0$$

the “phase-space operator”

$$\widehat{A}_{\text{ph}} = \left(\frac{1}{2\pi}\right)^n \int a_\sigma(z_0)\widehat{T}_{\text{ph}}dz_0;$$

the usual metaplectic covariance of Weyl calculus is preserved by this procedure: if we replace the symbol a by $a \circ S$ where $S \in \text{Sp}(2n, \mathbb{R})$ then \widehat{A}_{ph} is replaced by $\widehat{S}_{\text{ph}}^{-1}\widehat{A}_{\text{ph}}\widehat{S}_{\text{ph}}$. This choice of definition of phase space operators, using $\widehat{T}_{\text{ph}}(z_0)$ instead of $\widehat{T}(z_0)$, is not arbitrary. It has at least two major advantages:

- The first advantage is that our choice makes the relationship between the operators \widehat{A}_{ph} with the Wigner–Moyal transform $W(f, g)$ of pairs of functions very simple: we will see that the operators \widehat{A}_{ph} and their standard counterparts \widehat{A} are coupled by the formula

$$\widehat{A}_{\text{ph}}W'(f, \overline{g}) = W'(\widehat{A}f, \overline{g}) \quad (3)$$

for all $f, \phi \in \mathcal{S}(\mathbb{R}^n)$; here $W'(f, \overline{g})(z)$ is a re-scaled variant of $W(f, \overline{g})$ guaranteeing that \widehat{A}_{ph} is unitary when \widehat{A} is (Proposition 15);

- The second advantage, of a more technical nature, is that it makes the study of domains somewhat easier. As we will (briefly) discuss in the Conclusion to this article one of the main applications of the theory we sketch might be quantum mechanics (Weyl calculus was after all designed for this purpose). Take for instance $\widehat{A} \in \text{Mp}(2n, \mathbb{R})$. Then \widehat{A} is an automorphism of $\mathcal{S}(\mathbb{R}^n)$. If we fix g in formula (3) above and let f describe all of $\mathcal{S}(\mathbb{R}^n)$ then $\widehat{A}_{\text{ph}}W'(f, \overline{g})$ will describe a certain subspace of $\mathcal{S}(\mathbb{R}^{2n})$. Suppose in particular g is a normalized Gaussian; then that subspace consists of a very simple set of functions, namely those functions F such that $e^{\frac{1}{2}|z|^2}F$ is anti-analytic (Example 16).

This article is structured as follows:

- In §2 we review the main properties of the Arnol’d–Leray–Maslov (ALM) index for pairs of Lagrangian paths, and its by-product, the relative symplectic Maslov index. We take the opportunity to show on a few examples that these indices contain as particular cases other intersection indices used in the literature.
- In §3 we define a new symplectic index, denoted by ν , and related to the familiar Conley–Zehnder index, but relaxed of any non-degeneracy conditions on the endpoint of the path. The properties of a “symplectic Cayley transform” allow us to relate that index ν to the relative Maslov index corresponding to a particular polarization of the symplectic space. This property is interesting *per se* and could perhaps allow applications to the theory of periodic Hamiltonian orbits; this possibility will however not be investigated here in order to keep the article reasonably long.
- In §4 we begin by reviewing the standard theory of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ and of its Maslov index. We then define a family of unitary Weyl operators $\widehat{R}_\nu(S)$ parametrized by $S \in \mathrm{Sp}(2n, \mathbb{R})$ such that $\det(S - I) \neq 0$ and $\nu \in \mathbb{R}$. These operators, which can be written in the very simple form

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z \widehat{T}(Sz) \widehat{T}(-z) dz$$

generate a projective representation of the symplectic group. We then show that if the parameter ν is chosen to be index defined in §3, then these operators generate $\mathrm{Mp}(2n, \mathbb{R})$.

- In §5 we construct a phase-space Weyl calculus along the lines indicated above; that calculus is symplectically covariant with respect to conjugation with the metaplectic operators of §4: an immediate generalization of a deep result of Shale shows that this covariance actually characterizes uniquely the Weyl operators we have constructed.

Let us precise some notations that will be used throughout this paper; we take the opportunity to recall some basic results.

Symplectic notations

Let (E, ω) be a finite-dimensional symplectic space; we denote by $\text{Sp}(E, \omega)$ and $\text{Lag}(E, \omega)$ the symplectic group and Lagrangian Grassmannian and by

$$\begin{aligned}\pi^{\text{Sp}} : \text{Sp}_{\infty}(E, \omega) &\longrightarrow \text{Sp}(E, \omega) \\ \pi^{\text{Lag}} : \text{Lag}_{\infty}(E, \omega) &\longrightarrow \text{Lag}(E, \omega)\end{aligned}$$

the corresponding universal coverings. We will call $\text{Lag}_{\infty}(E, \omega)$ the ‘‘Maslov bundle’’ of the symplectic space (E, ω) .

Let $X = \mathbb{R}^n$; the standard symplectic structure on $Z = X \oplus X^*$ is defined by

$$\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{for } z = (x, p), z' = (x', p').$$

Identifying Z with \mathbb{R}^{2n} we have $\sigma(z, z') = \langle Jz, z' \rangle$ where $\langle z, z' \rangle = \langle x, x' \rangle + \langle p, p' \rangle$ is the usual Euclidean scalar product on \mathbb{R}^{2n} and $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The subgroup $\text{Sp}(Z, \sigma) \cap \text{O}(2n, \mathbb{R})$ is identified with the unitary group $\text{U}(n, \mathbb{C})$ by the mapping

$$\iota : \begin{bmatrix} A & -B \\ B & B \end{bmatrix} \longmapsto A + iB;$$

the action of $\text{U}(n, \mathbb{C})$ on $\text{Lag}(Z, \sigma)$ is denoted by $(u, \ell) \longmapsto u\ell$.

Maslov index on $\text{Sp}(Z, \sigma)$

The Maslov index for loops in $\text{Sp}(Z, \sigma)$ is defined as follows: let $\gamma : [0, 1] \longrightarrow \text{Sp}(Z, \sigma)$ be such that $\gamma(0) = \gamma(1)$, and set $\gamma(t) = S_t$. Then $U_t = (S_t S_t)^{-1/2} S_t$ is the orthogonal part in the polar decomposition of S_t :

$$U_t \in \text{Sp}(Z, \sigma) \cap \text{O}(2n, \mathbb{R});$$

let us denote by u_t its image $\iota(U_t)$ in $\text{U}(n, \mathbb{C})$ and define $\rho(S_t) = \det u_t$. The Maslov index of γ is the degree of the loop $t \longmapsto \rho(S_t)$ in S^1 :

$$m(\gamma) = \deg[t \longmapsto \det(\iota(U_t))], \quad 0 \leq t \leq 1].$$

Generalized Fresnel integral

We will need the following Fresnel-type formula: Let F be the Fourier transform on \mathbb{R}^m

$$Ff(v) = \left(\frac{1}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-i\langle v, u \rangle} f(u) du;$$

if M is a real symmetric $m \times m$ matrix such that $M > 0$ and $f : u \mapsto e^{\frac{i}{2}\langle Mu, u \rangle}$ then we have the Fresnel-type formula

$$Ff(v) = |\det M|^{-1/2} e^{\frac{i\pi}{4} \text{sign } M} e^{-\frac{i}{2}\langle M^{-1}v, v \rangle} \quad (4)$$

where $\text{sign } M$, the “signature” of M , is the number of > 0 eigenvalues of M minus the number of < 0 eigenvalues.

Weyl–Wigner–Moyal formalism

We refer to the standard literature (for instance [8, 37, 41]) for detailed studies of Weyl pseudo-differential calculus. The Wigner–Moyal transform $W(f, g)$ of $f, g \in \mathcal{S}(X)$ is defined by

$$W(f, g)(x, p) = \left(\frac{1}{2\pi}\right)^n \int_X e^{-i\langle p, y \rangle} f\left(x + \frac{1}{2}y\right) \bar{g}\left(x - \frac{1}{2}y\right) dy; \quad (5)$$

it extends to a mapping

$$W : \mathcal{S}(X) \times \mathcal{S}'(X) \longrightarrow \mathcal{S}'(X).$$

The Weyl operator \widehat{A} with “symbol” $a \in \mathcal{S}'(X)$ is defined by

$$\langle \widehat{A}f, \phi \rangle = \langle a, W(f, \bar{\phi}) \rangle$$

for $f, g \in \mathcal{S}(X)$; $\langle \cdot, \cdot \rangle$ denotes the usual distributional bracket. The symplectic Fourier transform of $a \in \mathcal{S}(Z)$ is defined by

$$\mathcal{F}_\sigma a(z) = f_\sigma(z) = \int_Z e^{-i\sigma(z, z')} a(z') dz'$$

and extends to $\mathcal{S}'(Z)$; setting $a_\sigma = \mathcal{F}_\sigma a$ (the twisted symbol) we have

$$\widehat{A}f(x) = \left(\frac{1}{2\pi}\right)^n \int_Z a_\sigma(z) \widehat{T}(z) f(x) dz$$

(interpreted in the distributional sense). Let a and b be the symbols of A and B respectively; then the twisted symbol c_σ of the compose $C = AB$ (when defined) is given by the “twisted convolution”

$$c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n \int_Z e^{\frac{i}{2}\sigma(z, z')} a_\sigma(z - z') b_\sigma(z') dz'.$$

2 The ALM and Maslov Indices

We review, without proofs, the main formulas and results developed in [11, 12]; for an alternative construction due to Dazord see [7]. In [4] Cappell *et al.* compare the ALM index to various other indices used in mathematics. We begin by defining a notion of signature for triples of Lagrangian planes (it is sometimes called “Maslov triple index”).

2.1 The Kashiwara signature

For proofs see [4, 26]. Let (E, ω) be a symplectic space, $\dim E = n < \infty$. Let (ℓ, ℓ', ℓ'') be a triple of elements of $\text{Lag}(E, \omega)$. By definition the Kashiwara signature $\tau(\ell, \ell', \ell'')$ of that triple is the signature of the quadratic form

$$(z, z', z'') \longmapsto \omega(z, z') + \omega(z', z'') + \omega(z'', z)$$

on $\ell \oplus \ell' \oplus \ell''$. The kernel of that quadratic form is isomorphic to $(\ell \cap \ell') \oplus (\ell' \cap \ell'') \oplus (\ell'' \cap \ell)$ hence

$$\tau(\ell, \ell', \ell'') \equiv n + \dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \pmod{2}.$$

The Kashiwara signature has the following properties:

[K.1] τ is antisymmetric:

$$\tau(\mathbf{p}(\ell, \ell', \ell'')) = (-1)^{\text{sgn}(\mathbf{p})} \tau(\ell, \ell', \ell'')$$

for any permutation \mathbf{p} of the set $\{\ell, \ell', \ell''\}$; $\text{sgn}(\mathbf{p}) = 0$ if \mathbf{p} is even, 1 if \mathbf{p} is odd. In particular $\tau(\ell, \ell', \ell'') = 0$ if any two of the three Lagrangian planes ℓ, ℓ', ℓ'' are identical;

[K.2] τ is $\text{Sp}(E, \omega)$ -invariant:

$$\tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'')$$

for every $S \in \text{Sp}(E, \omega)$;

[K.3] τ is locally constant on each set set of triples

$$\{(\ell, \ell', \ell'') : \dim \ell \cap \ell' = k; \dim \ell' \cap \ell'' = k'; \dim \ell'' \cap \ell = k''\}$$

where $0 \leq k, k', k'' \leq n$;

K.4 τ is a cocycle:

$$\tau(\ell, \ell', \ell'') - \tau(\ell', \ell'', \ell''') + \tau(\ell', \ell'', \ell''') - \tau(\ell', \ell'', \ell''') = 0 \quad (6)$$

for all $\ell, \ell', \ell'', \ell'''$ in $\text{Lag}(E, \omega)$.

K.5 τ is dimensionally additive: Let $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$. Identifying $\text{Lag}(E', \omega') \oplus \text{Lag}(E'', \omega'')$ with a subset of $\text{Lag}(E, \omega)$ we have

$$\tau(\ell'_1 \oplus \ell''_1, \ell'_2 \oplus \ell''_2, \ell'_3 \oplus \ell''_3) = \tau'(\ell'_1, \ell'_2, \ell'_3) + \tau''(\ell''_1, \ell''_2, \ell''_3) \quad (7)$$

where τ' and τ'' are the Kashiwara signatures on $\text{Lag}(E', \omega')$ and $\text{Lag}(E'', \omega'')$ and $\tau = \tau' \oplus \tau''$ that on $\text{Lag}(E, \omega)$.

In addition to these fundamental properties which characterize τ , the Kashiwara signature enjoys the following subsidiary properties which are very useful for practical calculations:

K.6 If $\ell \cap \ell'' = 0$ then $\tau(\ell, \ell', \ell'')$ is the signature of the quadratic form

$$Q'(z') = \omega(P_{\ell\ell''}z', z') = \omega(z', P_{\ell''\ell}z')$$

on ℓ' , where $P_{\ell\ell''}$ is the projection onto ℓ along ℓ'' and $P_{\ell''\ell} = I - P_{\ell\ell''}$ is the projection on ℓ'' along ℓ .

K.7 Let (ℓ, ℓ', ℓ'') be a triple of Lagrangian planes such that an $\ell = \ell \cap \ell' + \ell \cap \ell''$. Then $\tau(\ell, \ell', \ell'') = 0$.

K.8 Let (E, ω) be the standard symplectic space $(X \oplus X^*, \sigma)$. Let $\ell_A = \{(x, Ax), x \in X\}$ where A is a symmetric linear mapping $X \rightarrow X^*$. Then

$$\tau(X^*, \ell_A, X) = \text{sign}(A). \quad (8)$$

Remark 1 It is proven in [4] that K.1, K.2, K.5, K.8 uniquely characterize the signature τ .

The Kashiwara signature is related to several other indices appearing in the literature. Here are two examples; for more see [4] where, for instance, the relationship between τ and Wall's index [39] is investigated.

Example 2 In [25] Leray defined the index of inertia $\text{Inert}(\ell, \ell', \ell'')$ of a triple of pairwise transverse elements of $\text{Lag}(E, \omega)$ as being the common index of inertia of the three quadratic forms $z \mapsto \omega(z, z')$, $z' \mapsto \omega(z', z'')$, $z'' \mapsto \omega(z'', z')$ where $(z, z', z'') \in \ell \times \ell' \times \ell''$ is such that $z + z' + z'' = 0$. It easily follows from property (K.6) of τ that

$$\tau(\ell, \ell', \ell'') = 2 \text{Inert}(\ell, \ell', \ell'') - n.$$

Example 3 In [34] Robbin and Salamon's define a "composition form" Q for pairs (S, S') of elements of $\text{Sp}(Z, \sigma)$ such that $SX^* \cap X^* = S'X^* \cap X^* = 0$; it is given by

$$Q(S, S') = \text{sign}(B^{-1}B''(B')^{-1})$$

when

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad S' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad S'' = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}.$$

We have shown in [15] that:

$$Q(S, S') = \tau(X^*, SX^*, SS'X^*). \quad (9)$$

2.2 The ALM index

We denote by α and β the generators with index 0 of $\pi_1[\text{Lag}(E, \omega)] \simeq (\mathbb{Z}, +)$ and $\pi_1[\text{Sp}(E, \omega)] \simeq (\mathbb{Z}, +)$, respectively. Assume that $(E, \omega) = (Z, \sigma)$ and identify (x, p) with the vector $(x_1, p_1, \dots, x_n, p_n)$. The direct sum

$$\text{Lag}(1) \otimes \cdots \otimes \text{Lag}(1) \quad (n \text{ terms})$$

is identified with a subset of $\text{Lag}(Z, \sigma)$. Consider the loop $\beta_{(1)} : t \mapsto e^{2\pi it}$, $0 \leq t \leq 1$, in $W(1, \mathbb{C}) \equiv \text{Lag}(1)$. Then $\beta = \beta_{(1)} \oplus I_{2n-2}$ where I_{2n-2} is the identity in $W(n-1, \mathbb{C})$. Similarly, denoting by $\text{Sp}(1)$ the symplectic group acting on pairs (x_j, p_j) the direct sum

$$\text{Sp}(1) \oplus \text{Sp}(1) \oplus \cdots \otimes \text{Sp}(1) \quad (n \text{ terms})$$

is identified with a subgroup of $\text{Sp}(Z, \sigma)$. Let $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then α is identified with

$$\alpha : t \mapsto e^{2\pi t J_1} \oplus I_{n-2}, \quad 0 \leq t \leq 1 \quad (10)$$

where I_{2n-2} is the identity on \mathbb{R}^{2n-2} .

The Arnol'd–Leray–Maslov (for short: "ALM") index on (E, ω) is the unique mapping

$$\text{Lag}_\infty(E, \omega) \times \text{Lag}_\infty(E, \omega) \longrightarrow \mathbb{Z}$$

having the following characteristic property:

ALM.1 *Topological and cocycle condition:* μ is locally constant on the sets

$$\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\} \quad (11)$$

($0 \leq k \leq n$) and satisfies

$$\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell''). \quad (12)$$

The ALM index has the following additional properties:

ALM.2 *Antisymmetry:*

$$\mu(\ell_\infty, \ell'_\infty) = -\mu(\ell'_\infty, \ell_\infty) \quad , \quad \mu(\ell_\infty, \ell_\infty) = 0 \quad (13)$$

ALM.3 *Value modulo 2:* We have

$$\mu(\ell_\infty, \ell'_\infty) \equiv n + \dim \ell \cap \ell' \pmod{2}. \quad (14)$$

ALM.4 *Action of $\pi_1[\text{Lag}(E, \omega)]$:* we have

$$\mu(\beta^r \ell_\infty, \beta^{r'} \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 2(r - r') \quad (15)$$

for all integers r and r' .

(In particular $\mu(\beta^r \ell_\infty, \ell_\infty)$ is twice the Maslov index of any Lagrangian loop homeomorphic to β^r .)

ALM.5 *Dimensional additivity:* Let $E = E' \oplus E''$ and $\omega = \omega' \oplus \omega''$. If μ' and μ'' are the ALM indices on $\text{Lag}_\infty(E', \omega')$, $\text{Lag}_\infty(E'', \omega'')$ then

$$\mu(\ell'_{1,\infty} \oplus \ell''_{1,\infty}, \ell'_{2,\infty} \oplus \ell''_{2,\infty}) = \mu'(\ell'_{1,\infty}, \ell'_{2,\infty}) + \mu''(\ell''_{1,\infty}, \ell''_{2,\infty}). \quad (16)$$

The natural action

$$\text{Sp}(E, \omega) \times \text{Lag}(E, \omega) \longrightarrow \text{Lag}(E, \omega)$$

induces an action

$$\text{Sp}_\infty(E, \omega) \times \text{Lag}_\infty(E, \omega) \longrightarrow \text{Lag}_\infty(E, \omega)$$

such that

$$S_\infty(\beta^2 \ell_\infty) = (\alpha S_\infty) \ell_\infty = \beta^2 (S_\infty \ell_\infty) \quad (17)$$

where α (resp. β) are the generators of $\pi_1[\mathrm{Sp}(E, \omega)]$ and $\pi_1[\mathrm{Lag}(E, \omega)]$ previously defined. The uniqueness of an index satisfying property (ALM.1) together with the symplectic invariance (K.2) of τ imply that:

ALM.6 *Symplectic invariance:* for all $S_\infty \in \mathrm{Sp}_\infty(E, \omega)$ we have

$$\mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty). \quad (18)$$

Let us give a procedure for calculating explicitly the ALM index.

Assume that (E, ω) is the standard symplectic space $(X \oplus X^*, \sigma)$. Identifying $\mathrm{Lag}(Z, \sigma)$ with the set

$$W(n, \mathbb{C}) = \{w \in \mathrm{U}(n, \mathbb{C}) : w = w^T\}$$

using the mapping which to $\ell = uX^*$ ($u \in \mathrm{U}(n, \mathbb{C})$) associates $w = uu^T$, the Maslov bundle $\mathrm{Lag}_\infty(Z, \sigma)$ is identified with

$$W_\infty(n, \mathbb{C}) = \{(w, \theta) : w \in W(n, \mathbb{C}), \det w = e^{i\theta}\};$$

the projection $\pi^{\mathrm{Lag}} : \ell_\infty \mapsto \ell$ becomes $(w, \theta) \mapsto w$. The ALM index is then calculated as follows:

- If $\ell \cap \ell' = 0$ then

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi} [\theta - \theta' + i \mathrm{Tr} \mathrm{Log}(-w(w')^{-1})] \quad (19)$$

(the transversality condition $\ell \cap \ell'$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue);

- If $\ell \cap \ell' \neq 0$ one chooses ℓ'' such that $\ell \cap \ell'' = \ell' \cap \ell'' = 0$ and one then calculates $\mu(\ell_\infty, \ell'_\infty)$ using the formula (12) the values of $\mu(\ell_\infty, \ell''_\infty)$ and $\mu(\ell'_\infty, \ell''_\infty)$ given by (19). ($\mu(\ell_\infty, \ell'_\infty)$ does not depend on the choice of ℓ'' in view of the cocycle property (6) of τ .)

The ALM index is useful for expressing in a simple way various Lagrangian path intersection indices. For instance, in [34] is defined an intersection index for paths in $\mathrm{Lag}(Z, \sigma)$ with arbitrary endpoints, counting algebraically the intersections of a path Λ in $\mathrm{Lag}(Z, \sigma)$ with the caustic $\Sigma_\ell = \{\ell' : \ell \cap \ell' = 0\}$:

Example 4 Let μ_{RS} be the Robbin–Salamon index defined in [34]. That index associates to a continuous path $\Lambda : [a, b] \rightarrow \text{Lag}(Z, \sigma)$ and $\ell \in \text{Lag}(Z, \sigma)$ a number $\mu_{RS}(\Lambda, \ell)$. In [14, 15] we have shown that

$$\mu_{RS}(\Lambda, \ell) = \frac{1}{2}(m(\ell_{b,\infty}, \ell_\infty) - m(\ell_{a,\infty}, \ell_\infty)) \quad (20)$$

where ℓ_∞ is an arbitrary element of $\text{Lag}_\infty(Z, \sigma)$ covering ℓ ; $\ell_{a,\infty}$ is the equivalence class of an arbitrary path Λ_{0a} joining the chosen base point ℓ_0 of $\text{Lag}(Z, \sigma)$ to $\ell_a = \Lambda(a)$, and $\ell_{b,\infty}$ is the equivalence class of the concatenation $\Lambda_{0a} * \Lambda$.

The theory of that index has been applied and extended with success to problems in functional analysis [2] and in Morse theory where it provides useful “spectral flow” formulas (see Piccione and his collaborators [9, 32]).

2.3 Relative Maslov indices on $\text{Sp}(Z, \sigma)$

The Maslov indices μ_ℓ on $\text{Sp}_\infty(Z, \sigma)$ are defined in terms of the ALM index as follows. Let $\ell_\infty \in \text{Lag}_\infty(Z, \sigma)$ and $S_\infty \in \text{Sp}_\infty(Z, \sigma)$; formulae (17), (15) imply that the integer $\mu(S_\infty \ell_\infty, \ell_\infty)$ only depends on $\ell = \pi^{\text{Lag}}(\ell_\infty)$. The “Maslov index on $\text{Sp}_\infty(Z, \sigma)$ relative to ℓ ” is the mapping $\mu_\ell : \text{Sp}_\infty(Z, \sigma) \rightarrow \mathbb{Z}$ defined by

$$\mu_\ell(S_\infty) = \mu(S_\infty \ell_\infty, \ell_\infty). \quad (21)$$

It follows from the cocycle property (12) in (ALM.1) that:

[M.1] *Uniqueness and product:* μ_ℓ is the only mapping $\text{Sp}_\infty(Z, \sigma) \rightarrow \mathbb{Z}$ which is locally constant on each set

$$\text{Sp}_\ell(n; k) = \{S \in \text{Sp}(Z, \sigma) : \dim(S\ell \cap \ell) = k\} \quad (22)$$

($0 \leq k \leq n$) and such that

$$\mu_\ell(S_\infty S'_\infty) = \mu_\ell(S_\infty) + \mu_\ell(S'_\infty) + \tau(\ell, S\ell, SS'\ell). \quad (23)$$

[M.2] *Antisymmetry:* for all $S_\infty \in \text{Sp}_\infty(Z, \sigma)$

$$\mu_\ell(S_\infty^{-1}) = -\mu_\ell(S_\infty) \quad , \quad \mu_\ell(I_\infty) = 0 \quad (24)$$

(I_∞ the unit of $\text{Sp}_\infty(Z, \sigma)$);

M.3 *Action of $\pi_1[\mathrm{Sp}(Z, \sigma)]$:* let α be the generator of $\pi_1[\mathrm{Sp}(Z, \sigma)]$; the

$$\mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 4r \quad (25)$$

for all $S_\infty \in \mathrm{Sp}_\infty(Z, \sigma)$ and $r \in \mathbb{Z}$.

M.4 *Dimensional additivity:* Let $Z' = \mathbb{R}^{2n'}$, $Z'' = \mathbb{R}^{2n''}$, $n' + n'' = n$. Identifying $\mathrm{Sp}(Z', \sigma') \oplus \mathrm{Sp}(Z'', \sigma'')$ with a subgroup of $\mathrm{Sp}(Z, \sigma)$ we have

$$\mu_{\ell' \oplus \ell''}(S'_\infty \oplus S''_\infty) = \mu_{\ell'}(S'_\infty) + \mu_{\ell''}(S''_\infty).$$

Notice that it follows from formula (14) that

$$\mu_\ell(S_\infty) \equiv n + \dim(S\ell \cap \ell) \pmod{2} \quad (26)$$

Following formula, which immediately follows from the cocycle property (K.4) of τ , describes the behavior of the Maslov index under changes of ℓ :

$$\begin{aligned} \mu_\ell(S_\infty) - \mu_{\ell'}(S_\infty) &= \tau(S\ell, \ell, \ell') - \tau(S\ell, S\ell', \ell') \\ &= \tau(S\ell, \ell, S\ell') - \tau(\ell, S\ell', \ell'). \end{aligned} \quad (27)$$

It is sometimes advantageous to work with the “reduced Maslov index” relative to $\ell \in \mathrm{Lag}(Z, \sigma)$; it is the function $m_\ell : \mathrm{Sp}_\infty(Z, \sigma) \rightarrow \mathbb{Z}$ defined by

$$\begin{aligned} m_\ell(S_\infty) &= m(S_\infty \ell_\infty, \ell_\infty) \\ &= \frac{1}{2}(\mu_\ell(S_\infty) + n + \dim(S\ell \cap \ell)). \end{aligned}$$

Notice that in view of (26) we have

$$m_\ell(S_\infty) \equiv n + \dim(S\ell \cap \ell) \pmod{2}.$$

The properties of the reduced index m_ℓ are immediately deduced from those of μ_ℓ ; for instance

$$m_\ell(S_\infty S'_\infty) = m_\ell(S_\infty) + m_\ell(S'_\infty) + \mathrm{Inert}(\ell, S\ell, S S'\ell) \quad (28)$$

and

$$m_\ell(\alpha^r S_\infty) = m_\ell(S_\infty) + 2r \quad (29)$$

for $r \in \mathbb{Z}$.

Exactly as the ALM index allows an easy construction of Lagrangian path intersection indices (Example 4) the relative Maslov index allows to construct symplectic path intersection indices:

Example 5 Let Σ be a continuous path $[a, b] \longrightarrow \mathrm{Sp}(Z, \sigma)$; set $S_t = \Sigma(t)$. Let $\ell \in \mathrm{Lag}(Z, \sigma)$. The intersection index of Σ with the subvariety $\{S : S\ell \cap \ell \neq 0\}$ of $\mathrm{Sp}(Z, \sigma)$ is by definition

$$\mu(\Sigma, \ell) = \frac{1}{2}(m_\ell(S_{b,\infty}) - m(S_{a,\infty}))$$

where $S_{a,\infty}$ is the homotopy class in $\mathrm{Sp}(Z, \sigma)$ of an arbitrary path Σ_{0a} joining the identity to S_a and $S_{b,\infty}$ that of the concatenation $\Sigma_{0a} * \Sigma$. Choosing $\ell = X^*$ one obtains the symplectic path intersection studied in [34] (see [15]).

3 The index ν on $\mathrm{Sp}_\infty(Z, \sigma)$

We are going to study in some detail an index $\nu : \mathrm{Sp}_\infty(Z, \sigma) \longrightarrow \mathbb{Z}$ which will be fundamental in defining the correct phase of the Weyl symbol of a metaplectic operator. That index may be seen as an extension of the Conley–Zehnder index [6, 22, 33] which plays an important role in the theory of periodic Hamiltonian orbits and their bifurcations [5], and in Floer homology.

3.1 Symplectic Cayley transform

We will need a notion of Cayley transform for symplectic matrices (a similar transform has been considered by Howe in [24]). Let $S \in \mathrm{Sp}(Z, \sigma)$. If $\det(S - I) \neq 0$ the matrix

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1} \tag{30}$$

that is, equivalently,

$$M_S = \frac{1}{2}J + J(S - I)^{-1} \tag{31}$$

is called the “symplectic Cayley transform of S ”.

The following Lemma summarizes the properties of the symplectic Cayley transform:

Lemma 6 Let $\mathrm{Sp}_0(Z, \sigma)$ be the set of all $S \in \mathrm{Sp}(Z, \sigma)$ with $\det(S - I) \neq 0$ and $\mathrm{Sym}_0(2n, \mathbb{R})$ the set of all real $2n \times 2n$ symmetric matrices M such that $\det(M - \frac{1}{2}J) \neq 0$.

(i) The symplectic Cayley transform is a bijection

$$\mathrm{Sp}_0(Z, \sigma) \longrightarrow \mathrm{Sym}_0(2n, \mathbb{R})$$

whose inverse is given by the formula

$$S = (M - \frac{1}{2}J)^{-1}(M + \frac{1}{2}J) \quad (32)$$

if $M = M_S$.

(ii) The symplectic Cayley transform of the product SS' is (when defined) given by the formula

$$M_{SS'} = M_S + (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1}. \quad (33)$$

(iii) The symplectic Cayley transform of S and S^{-1} are related by

$$M_{S^{-1}} = -M_S. \quad (34)$$

We omit the proof since the formulae above are obtained by elementary algebraic manipulations involving the use of the relations $SJS^T = S^TJS = J$ characterizing symplectic matrices; alternatively it is *mutatis mutandis* the same as proof of Howe's [24] Cayley transform for symplectic matrices (also see [8], pp. 242–243).

3.2 Definition of $\nu(S_\infty)$ and first properties

We define on $Z \oplus Z$ the symplectic form σ^\ominus by

$$\sigma^\ominus(z_1, z_2; z'_1, z'_2) = \sigma(z_1, z'_1) - \sigma(z_2, z'_2)$$

and denote by $\text{Sp}^\ominus(2n)$ and $\text{Lag}^\ominus(2n)$ the corresponding symplectic group and Lagrangian Grassmannian. Let μ^\ominus the ALM index on $\text{Lag}^\ominus_\infty(2n)$ and μ_L^\ominus the Maslov index on $\text{Sp}^\ominus_\infty(2n)$ relative to $L \in \text{Lag}^\ominus(2n)$.

For $S_\infty \in \text{Sp}^\ominus_\infty(Z, \sigma)$ we define

$$\nu(S_\infty) = \frac{1}{2}\mu^\ominus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty) \quad (35)$$

where $(I \oplus S)_\infty$ is the homotopy class in $\text{Sp}^\ominus(2n)$ of the path

$$t \longmapsto \{(z, S_t z) : z \in Z\} \quad , \quad 0 \leq t \leq 1$$

and $\Delta = \{(z, z) : z \in Z\}$ the diagonal of $Z \oplus Z$. Setting $S_t^\ominus = I \oplus S_t$ we have $S_t^\ominus \in \text{Sp}^\ominus(2n)$ hence formulae (35) is equivalent to

$$\nu(S_\infty) = \frac{1}{2}\mu^\ominus_\Delta(S_\infty^\ominus) \quad (36)$$

where μ_{Δ}^{\ominus} is the Maslov index on $\mathrm{Sp}_{\infty}^{\ominus}(2n)$ corresponding to $\Delta \in \mathrm{Lag}^{\ominus}(2n)$.

Note that replacing n by $2n$ in the congruence (26) we have

$$\begin{aligned}\mu^{\ominus}((I \oplus S)_{\infty} \Delta_{\infty}, \Delta_{\infty}) &\equiv \dim((I \oplus S)\Delta, \Delta) \pmod{2} \\ &\equiv \dim \mathrm{Ker}(S - I) \pmod{2}\end{aligned}$$

and hence

$$\nu(S_{\infty}) \equiv \frac{1}{2} \dim \mathrm{Ker}(S - I) \pmod{1}$$

so that $\nu(S_{\infty})$ is always an integer since the eigenvalue 1 of S has even multiplicity.

The index ν has the following rather straightforward properties:

$\nu.1$ *Antisymmetry:* For all $S_{\infty} \in \mathrm{Sp}_{\infty}(Z, \sigma)$ we have

$$\nu(S_{\infty}^{-1}) = -\nu(S_{\infty}).$$

This property immediately follows from the equality $(S_{\infty}^{\ominus})^{-1} = (I \oplus S^{-1})_{\infty}$ and the antisymmetry of μ_{Δ}^{\ominus} .

$\nu.2$ *Action of $\pi_1[\mathrm{Sp}(Z, \sigma)]$:* For all $r \in \mathbb{Z}$ we have

$$\nu(\alpha^r S_{\infty}) = \nu(S_{\infty}) + 2r$$

To see this it suffices to observe that to the generator α of $\pi_1[\mathrm{Sp}(Z, \sigma)]$ corresponds the generator $I_{\infty} \oplus \alpha$ of $\pi_1[\mathrm{Sp}^{\ominus}(2n)]$; in view property (25) of the Maslov index it follows that

$$\begin{aligned}\nu(\alpha^r S_{\infty}) &= \frac{1}{2} \mu_{\Delta}^{\ominus}((I_{\infty} \oplus \alpha)^r S_{\infty}^{\ominus}) \\ &= \frac{1}{2} (\mu_{\Delta}^{\ominus}(S_{\infty}^{\ominus}) + 4r) \\ &= \nu(S_{\infty}) + 2r.\end{aligned}$$

Let us now prove a formula for the index of a product. This formula will be instrumental in identifying the twisted Weyl symbol of a metaplectic operator.

$\nu.3$ *Product.* If S_{∞} , S'_{∞} , and $S_{\infty} S'_{\infty}$ are such that $\det(S - I) \neq 0$, $\det(S' - I) \neq 0$, and $\det(SS' - I) \neq 0$ then

$$\nu(S_{\infty} S'_{\infty}) = \nu(S_{\infty}) + \nu(S'_{\infty}) + \frac{1}{2} \mathrm{sign} M_S \quad (37)$$

where M_S is the symplectic Cayley transform of S .

In view of (31) we have

$$\langle M_S z, z \rangle = \sigma((S - I)^{-1} z, z);$$

since the quadratic forms $z \mapsto \sigma((S - I)^{-1} z, z)$ and $z \mapsto \sigma(z, (S - I)z) = \sigma(z, Sz)$ are equivalent they have same signature, and formula (37) is therefore equivalent to

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \text{sign } \sigma(Sz, z) \quad (38)$$

where $\text{sign } \sigma(Sz, z)$ is the signature of the quadratic form $z \mapsto \sigma(Sz, z)$. Let us prove (38). In view of (36) and the product property (23) of the Maslov index on $\text{Sp}_\infty^\ominus(2n)$ we have

$$\begin{aligned} \nu(S_\infty S'_\infty) &= \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \tau^\ominus(\Delta, S^\ominus \Delta, S^\ominus S'^\ominus \Delta) \\ &= \nu(S_\infty) + \nu(S'_\infty) - \frac{1}{2} \tau^\ominus(S^\ominus S'^\ominus \Delta, S^\ominus \Delta, \Delta) \end{aligned}$$

where $S^\ominus = I \oplus S$, $S'^\ominus = I \oplus S'$ and τ^\ominus is the Kashiwara signature on the symplectic space $(Z \oplus Z, \sigma^\ominus)$. The condition $\det(SS' - I) \neq 0$ is equivalent to $S^\ominus \Delta \cap \Delta = 0$ hence we can apply property (K.6) of τ with $\ell = S^\ominus S'^\ominus \Delta$ and $\ell'' = \Delta$. The projection operator onto $S^\ominus S'^\ominus \Delta$ along Δ is

$$\text{Pr}(S^\ominus S'^\ominus \Delta, \Delta) = \begin{bmatrix} (I - SS')^{-1} & -(I - SS')^{-1} \\ SS'(I - SS')^{-1} & -SS'(I - SS')^{-1} \end{bmatrix}$$

hence $\tau^\ominus(S^\ominus S'^\ominus \Delta, S^\ominus \Delta, \Delta)$ is the signature of the quadratic form

$$Q(z) = \sigma^\ominus(\text{Pr}(S^\ominus S'^\ominus \Delta, \Delta)(z, Sz); (z, Sz))$$

that is, since $\sigma^\ominus = \sigma \ominus \sigma$:

$$\begin{aligned} Q(z) &= \sigma((I - SS')^{-1}(I - S)z, Sz) - \sigma(SS'(I - SS')^{-1}(I - S)z, Sz) \\ &= \sigma((SS' - I)(I - SS')^{-1}(I - S)z, Sz) \\ &= \sigma(Sz, z). \end{aligned}$$

The index ν has in addition the following topological property. Let

$$\begin{aligned} \text{Sp}^+(Z, \sigma) &= \{S \in \text{Sp}(Z, \sigma) : \det(S - I) > 0\} \\ \text{Sp}^-(Z, \sigma) &= \{S \in \text{Sp}(Z, \sigma) : \det(S - I) < 0\} \\ \text{Sp}_0(Z, \sigma) &= \text{Sp}(Z, \sigma) \setminus (\text{Sp}^+(Z, \sigma) \cup \text{Sp}^-(Z, \sigma)); \end{aligned}$$

the sets $\mathrm{Sp}^\pm(Z, \sigma)$ are connected and disjoint. We have:

[$\nu.4$] Let S_∞ be the homotopy class of a path Σ in $\mathrm{Sp}(Z, \sigma)$ joining the identity to $S \in \mathrm{Sp}_0(Z, \sigma)$, and let $S' \in \mathrm{Sp}(Z, \sigma)$ be in the same connected component $\mathrm{Sp}^\pm(Z, \sigma)$ as S . Then $\nu(S'_\infty) = \nu(S_\infty)$ where S'_∞ is the homotopy class in $\mathrm{Sp}(Z, \sigma)$ of the concatenation of Σ and a path joining S to S' in $\mathrm{Sp}_0(Z, \sigma)$.

Assume in fact that S and S' belong to, say, $\mathrm{Sp}^+(Z, \sigma)$ and let Σ be a symplectic path representing S_∞ and $t \mapsto \Sigma'(t)$ $0 \leq t \leq 1$, a path joining S to S' . Let $S_\infty(t)$ be the homotopy class of $\Sigma * \Sigma'(t)$. We have $\det(S(t) - I) > 0$ for all $t \in [0, 1]$ hence $S_\infty^\ominus(t)\Delta \cap \Delta \neq 0$ as t varies from 0 to 1. It follows from the continuity property (M.1) of the Maslov index that the function $t \mapsto \mu_\Delta^\ominus(S_\infty^\ominus(t))$ is constant, hence

$$\mu_\Delta^\ominus(S_\infty^\ominus) = \mu_\Delta^\ominus(S_\infty^\ominus(0)) = \mu_\Delta^\ominus(S_\infty^\ominus(1)) = \mu_\Delta^\ominus(S'^\ominus)$$

which was to be proven.

3.3 Relation between ν and μ_{X^*}

The index ν can be expressed in simple way in terms of the Maslov index μ_{X^*} on $\mathrm{Sp}_\infty(Z, \sigma)$. The following technical result will be helpful in establishing this important relation. Recall that $S \in \mathrm{Sp}(Z, \sigma)$ is said to be “free” if $SX^* \cap X^* = 0$; this condition is equivalent to $\det B \neq 0$ when S is identified with the matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (39)$$

in the canonical basis. The set of all free automorphisms is dense in $\mathrm{Sp}(Z, \sigma)$. The quadratic form W on $X \oplus X$ defined by

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle$$

where $P = DB^{-1}$, $L = B^{-1}$, $Q = B^{-1}A$ then generates S in the sense that $(x, p) = S(x', p')$ is equivalent to $p = \partial_x W(x, x')$, $p' = \partial_{x'} W(x, x')$.

Lemma 7 *Let $S_W \in \mathrm{Sp}(Z, \sigma)$ be given by (39). We have*

$$\det(S_W - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}) \quad (40)$$

that is:

$$\det(S_W - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T).$$

In particular the symmetric matrix

$$P + Q - L - L^T = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1}$$

is invertible.

Proof. Since B is invertible we can factorize $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

and hence

$$\begin{aligned} \det(S_W - I) &= \det(-B) \det(C - (D - I)B^{-1}(A - I)) \\ &= (-1)^n \det B \det(C - (D - I)B^{-1}(A - I)). \end{aligned}$$

Since S is symplectic we have $C - DB^{-1}A = -(B^T)^{-1}$ (cf. Step 3 in the proof of Proposition 8) and hence

$$C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows ■

Proposition 8 *Let $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ have projection $S = \pi^{\text{Sp}}(S_\infty)$ such that $\det(S - I) \neq 0$ and $SX^* \cap X^* = 0$. Then*

$$\nu(S_\infty) = \frac{1}{2}(\mu_{X^*}(S_\infty) + \text{sign } W_S) \quad (41)$$

where W_S is the symmetric matrix defined by

$$W_S = DB^{-1} + AB^{-1} - B^{-1} - (B^T)^{-1} \quad \text{if } S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Proof. We will divide the proof in three steps.

Step 1. Let $L \in \text{Lag}^\ominus(4n, \mathbb{R})$. Using successively formulae (36) and (27) we have

$$\nu(S_\infty) = \frac{1}{2}(\mu_L^\ominus(S_\infty^\ominus) + \tau^\ominus(S^\ominus \Delta, \Delta, L) - \tau^\ominus(S^\ominus \Delta, S^\ominus L, L)).$$

Choosing in particular $L = L_0 = X^* \oplus X^*$ we get

$$\begin{aligned} \mu_{L_0}^\ominus(S_\infty^\ominus) &= \mu^\ominus((I \oplus S)_\infty(X_\infty^* \oplus X_\infty^*), (X_\infty^* \oplus X_\infty^*)) \\ &= \mu(X_\infty^*, X_\infty^*) - \mu(X_\infty^*, S_\infty X_\infty^*) \\ &= -\mu(X_\infty^*, S_\infty X_\infty^*) \\ &= \mu_{X^*}(S_\infty) \end{aligned}$$

so that there remains to prove that

$$\tau^\ominus(S^\ominus \Delta, \Delta, L_0) - \tau^\ominus(S^\ominus \Delta, S^\ominus L_0, L_0) = -\text{sign } W_S.$$

Step 2. We are going to show that

$$\tau^\ominus(S^\ominus \Delta, S^\ominus L_0, L_0) = 0;$$

in view of the symplectic invariance (K.2) and the antisymmetry (K.1) of τ^\ominus this is equivalent to

$$\tau^\ominus(L_0, \Delta, L_0, (S^\ominus)^{-1}L_0) = 0. \quad (42)$$

We have

$$\Delta \cap L_0 = \{(0, p; 0, p) : p \in \mathbb{R}^n\}$$

and $(S^\ominus)^{-1}L_0 \cap L_0$ consists of all $(0, p', S^{-1}(0, p''))$ with $S^{-1}(0, p'') = (0, p')$; since S (and hence S^{-1}) is free we must have $p' = p'' = 0$ so that

$$(S^\ominus)^{-1}L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.$$

It follows that we have

$$L_0 = \Delta \cap L_0 + (S^\ominus)^{-1}L_0 \cap L_0$$

hence (42) in view of property (K.7) of τ .

Step 3. Let us finally prove that.

$$\tau^\ominus(S^\ominus \Delta, \Delta, L_0) = -\text{sign } W_S;$$

this will complete the proof of the proposition. The condition $\det(S - I) \neq 0$ is equivalent to $S^\ominus \Delta \cap \Delta = 0$ hence, using property (K.6) of τ ,

$$\tau^\ominus(S^\ominus \Delta, \Delta, L_0) = -\tau^\ominus(S^\ominus \Delta, L_0, \Delta)$$

is the signature of the quadratic form Q on L_0 defined by

$$Q(0, p, 0, p') = -\sigma^\ominus(P_\Delta(0, p, 0, p'); 0, p, 0, p')$$

where

$$P_\Delta = \begin{bmatrix} (S - I)^{-1} & -(S - I)^{-1} \\ S(S - I)^{-1} & -S(S - I)^{-1} \end{bmatrix}$$

is the projection on $S^\ominus \Delta$ along Δ in $Z \oplus Z$. It follows that the quadratic form Q is given by

$$Q(0, p, 0, p') = -\sigma^\ominus((I - S)^{-1}(0, p''), S(I - S)^{-1}(0, p''); 0, p, 0, p')$$

where we have set $p'' = p - p'$; by definition of σ^\ominus this is

$$\begin{aligned} Q(0, p, 0, p') &= \\ &= -\sigma((I - S)^{-1}(0, p''), (0, p)) + \sigma(S(I - S)^{-1}(0, p''), (0, p')). \end{aligned}$$

Let now M_S be the symplectic Cayley transform (30) of S ; we have

$$(I - S)^{-1} = JM_S + \frac{1}{2}I \quad , \quad S(I - S)^{-1} = JM_S - \frac{1}{2}I$$

and hence

$$\begin{aligned} Q(0, p, 0, p') &= -\sigma((JM_S + \frac{1}{2}I)(0, p''), (0, p)) + \sigma((JM_S - \frac{1}{2}I)(0, p''), (0, p')) \\ &= -\sigma(JM_S(0, p''), (0, p)) + \sigma(JM_S(0, p''), (0, p')) \\ &= \sigma(JM_S(0, p''), (0, p'')) \\ &= -\langle M_S(0, p''), (0, p'') \rangle. \end{aligned}$$

Let us calculate explicitly M_S . Writing S in usual block-form we have

$$S - I = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

that is

$$S - I = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} W_S & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

where we have used the identity

$$C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}$$

which follows from the relation $C - DB^{-1}A = -(B^T)^{-1}$ (the latter is a rephrasing of the equalities $D^T A - B^T C = I$ and $D^T B = B^T D$, consequences of the fact that $S^T JS = S^T JS$ since $S \in \text{Sp}(Z, \sigma)$). It follows that

$$\begin{aligned} (S - I)^{-1} &= \begin{bmatrix} W_S^{-1} & 0 \\ B^{-1}(I - A)W_S^{-1} & I \end{bmatrix} \begin{bmatrix} (I - D)B^{-1} & I \\ B^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} W_S^{-1}(I - D)B^{-1} & W_S^{-1} \\ B^{-1}(I - A)W_S^{-1}(I - D)B^{-1} + B^{-1} & B^{-1}(I - A)W_S^{-1} \end{bmatrix} \end{aligned}$$

and hence

$$M_S = \begin{bmatrix} B^{-1}(I - A)W_S^{-1}(I - D)B^{-1} + B^{-1} & \frac{1}{2}I + B^{-1}(I - A)W_S^{-1} \\ -\frac{1}{2}I - W_S^{-1}(I - D)B^{-1} & -W_S^{-1} \end{bmatrix}$$

from which follows that

$$Q(0, p, 0, p') = \langle W_S^{-1}p'', p'' \rangle = \langle W_S^{-1}(p - p'), (p - p') \rangle.$$

The matrix of the quadratic form Q is thus

$$2 \begin{bmatrix} W_S^{-1} & -W_S^{-1} \\ -W_S^{-1} & W_S^{-1} \end{bmatrix}$$

and this matrix has signature $\text{sign}(W_S^{-1}) = \text{sign } W_S$, concluding the proof.

■

4 The Metaplectic Group

We denote by $\text{Mp}(Z, \sigma)$ the unitary representation in $L^2(X)$ of the two-sheeted covering group $\text{Sp}_2(Z, \sigma)$ of $\text{Sp}(Z, \sigma)$. That group, called the *metaplectic group* in the literature [8, 25, 40], is generated by the operators $\widehat{S}_{W,m}$ defined by

$$\widehat{S}_{W,m}f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \Delta(W) \int_X e^{-iW(x,x')} f(x') dx'$$

where

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle \quad (43)$$

with $P = P^T$, $Q = Q^T$, $\det L \neq 0$, and

$$\Delta(W) = i^m \sqrt{|\det L|} \quad , \quad m\pi = \arg \det L$$

(note that the integer m is only defined modulo 4). The projection $\pi^{\text{Mp}} : \text{Mp}(Z, \sigma) \rightarrow \text{Sp}(Z, \sigma)$ is determined by the action on the generators $\widehat{S}_{W,m}$ which is given by $S_W = \pi^{\text{Mp}}(\widehat{S}_{W,m})$ where S_W is the free symplectic matrix generated by W .

Every $\widehat{S} \in \text{Mp}(Z, \sigma)$ can be written (in infinitely many ways) as a product $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$ (see [25, 12] for a proof); if

$$\widehat{S}_{W,m} \widehat{S}_{W',m'} = \widehat{S}_{W'',m''} \widehat{S}_{W''',m'''}$$

then we have

$$\begin{aligned} m + m' - \text{Inert}(P' + Q) &\equiv m'' + m''' - \text{Inert}(P''' + Q'') \pmod{4} \\ \text{rank}(P' + Q) &\equiv \text{rank}(P''' + Q'') \pmod{4}. \end{aligned}$$

In [10] we have shown that if $\widehat{S} = \widehat{S}_{W,m}\widehat{S}_{W',m'}$ is the projection on $\text{Mp}(Z, \sigma)$ of $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ then

$$\begin{aligned} m_{X^*}(S_\infty) &\equiv m + m' - \text{Inert}(P' + Q) \pmod{4} \\ &\equiv m + m' - \text{Inert}(X^*, S_W X^*, S_W S_{W'} X^*) \pmod{4}; \end{aligned}$$

it follows that the class of $m_{X^*}(S_\infty)$ modulo 4 only depends on the projection \widehat{S} ; denoting that class by $\widehat{m}(\widehat{S})$ the function $\widehat{m} : \text{Mp}(Z, \sigma) \rightarrow \mathbb{Z}/4\mathbb{Z}$ is called ‘‘Maslov index on $\text{Mp}(Z, \sigma)$ ’’. One proves [10, 12] that $\widehat{m}(\widehat{S}_{W,m}) = \widehat{m}$ and that

$$\widehat{m}(\widehat{S}\widehat{S}') = \widehat{m}(\widehat{S}) + \widehat{m}(\widehat{S}') + \widehat{\text{Inert}}(X^*, SX^*, SS'X^*)$$

for all $\widehat{S}, \widehat{S}' \in \text{Mp}(Z, \sigma)$.

The operators $\widehat{S}_{W,m}$ generate $\text{Mp}(Z, \sigma)$; so do the operators $\widehat{V}_P, \widehat{M}_{L,m}$, and \widehat{J} defined by

$$\widehat{V}_P f(x) = e^{-\frac{i}{2}\langle Px, x \rangle} f(x) \quad , \quad \widehat{M}_{L,m} f(x) = i^m \sqrt{|\det L|} f(Lx)$$

when $P = P^T$ and $\det L \neq 0$, and

$$\widehat{J} f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \int_X e^{-i\langle x, x' \rangle} f(x') dx'.$$

Notice that if W is given by (43) then

$$\widehat{S}_{W,m} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{J} \widehat{V}_{-Q}. \quad (44)$$

4.1 A class of unitary operators on $L^2(X)$

We are going to construct a class of Weyl operators $\widehat{R}_\nu(S)$ parametrized by $(S, \nu) \in \text{Sp}(Z, \sigma) \times \mathbb{R}$; we will see that these operators generate a projective representation of $\text{Sp}(Z, \sigma)$, containing the metaplectic group $\text{Mp}(Z, \sigma)$ (this last step will be achieved by identifying the parameter ν with the index introduced in last section).

Let $S \in \text{Sp}(Z, \sigma)$ be such that $\det(S - I) \neq 0$ and define

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z \widehat{T}(Sz) \widehat{T}(-z) dz \quad (45)$$

where the integral is interpreted in the sense of Bochner. Taking into account the relation (2) we have

$$\widehat{T}((S - I)z) = e^{-\frac{i}{2}\sigma(Sz, z)}\widehat{T}(Sz)\widehat{T}(-z)$$

so that we can rewrite definition (45) as

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n i^\nu \sqrt{|\det(S - I)|} \int_Z e^{-\frac{i}{2}\sigma(Sz, z)}\widehat{T}((S - I)z)dz. \quad (46)$$

Let us write this formula in Weyl form:

Proposition 9 *The operator $\widehat{R}_\nu(S)$ is given by*

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int_Z e^{\frac{i}{2}\langle M_S z, z \rangle} \widehat{T}(z) dz \quad (47)$$

where M_S is the symplectic Cayley transform of S .

Proof. In view of (31) and the antisymmetry of J we have

$$\langle M_S z, z \rangle = \langle J(S - I)^{-1}z, z \rangle = \sigma((S - I)^{-1}z, z).$$

Performing the change of variables $z \mapsto (S - I)^{-1}z$ we can rewrite the integral in the right-hand side of (46) as

$$\begin{aligned} \int_Z e^{-\frac{i}{2}\sigma(Sz, z)}\widehat{T}((S - I)z)dz &= \int_Z e^{\frac{i}{2}\sigma(z, (S - I)z)}\widehat{T}((S - I)z)dz \\ &= \int_Z e^{\frac{i}{2}\langle M_S z, z \rangle}\widehat{T}(z)dz \end{aligned}$$

hence the result. ■

Formula (47) defines a Weyl operator with twisted symbol

$$a_\sigma(z) = \frac{i^\nu}{\sqrt{|\det(S - I)|}} e^{\frac{i}{2}\langle M_S z, z \rangle}. \quad (48)$$

If in addition that $\det(S + I) \neq 0$ we easily deduce from this formula the usual Weyl symbol a . In fact, $a = \mathcal{F}_\sigma a_\sigma$ that is

$$a(z) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int_Z e^{-i\sigma(z, z')} e^{\frac{i}{2}\langle M_S z', z' \rangle} dz';$$

applying the Fresnel formula (4) with $m = 2n$ we then get

$$a(z) = \frac{i^{\nu+\frac{1}{2}\text{sign } M_S}}{\sqrt{|\det(S-I)|}} |\det M_S|^{-1/2} e^{\frac{i}{2}\langle JM_S^{-1}Jz, z \rangle}.$$

Since by definition of M_S

$$\det M_S = 2^{-n} \det(S+I) \det(S-I)$$

we can rewrite the formula above as

$$a(z) = 2^{n/2} \frac{i^{\nu+\frac{1}{2}\text{sign } M_S}}{\sqrt{|\det(S+I)|}} e^{\frac{i}{2}\langle JM_S^{-1}Jz, z \rangle}. \quad (49)$$

(Behold: this formula is only valid when S has no eigenvalue ± 1 .)

Let us begin by studying composition and inversion for the operators $\widehat{R}_\nu(S)$. This will allow us to establish that the operators $\widehat{R}_\nu(S)$ are unitary.

Proposition 10 *Let S and S' in $\text{Sp}(Z, \sigma)$ be such that $\det(S-I) \neq 0$, $\det(S'-I) \neq 0$. (i) If $\det(SS'-I) \neq 0$ then*

$$\widehat{R}_\nu(S)\widehat{R}_\nu(S') = \widehat{R}_{\nu+\nu'+\frac{1}{2}\text{sign } M}(SS'). \quad (50)$$

(ii) *The operator $\widehat{R}_\nu(S)$ is invertible and its inverse is*

$$\widehat{R}_\nu(S)^{-1} = \widehat{R}_{-\nu}(S^{-1}). \quad (51)$$

Proof. (i) The twisted symbols of $\widehat{R}_\nu(S)$ and $\widehat{R}_\nu(S')$ are, respectively,

$$\begin{aligned} a_\sigma(z) &= \frac{i^\nu}{\sqrt{|\det(S-I)|}} e^{\frac{i}{2}\langle M_S z, z \rangle} \\ b_\sigma(z) &= \frac{i^\nu}{\sqrt{|\det(S'-I)|}} e^{\frac{i}{2}\langle M_{S'} z, z \rangle}. \end{aligned}$$

The twisted symbol c_σ of the compose $\widehat{R}_\nu(S)\widehat{R}_\nu(S')$ is given by

$$c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n \int_Z e^{\frac{i}{2}\sigma(z, z')} a_\sigma(z-z') b_\sigma(z') dz'$$

that is

$$c_\sigma(z) = K \int_Z e^{\frac{i}{2}(\sigma(z, z') + \Phi(z, z'))} dz'$$

where the constant in front of the integral is

$$K = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu+\nu'}}{\sqrt{|\det(S-I)(S'-I)|}}$$

and the phase $\Phi(z, z')$ is given by

$$\Phi(z, z') = \langle M_S(z - z'), z - z' \rangle + \langle M_{S'}z', z' \rangle$$

that is

$$\Phi(z, z') = \langle M_S z, z \rangle - 2 \langle M_S z, z' \rangle + \langle (M_S + M_{S'})z', z' \rangle.$$

Observing that

$$\begin{aligned} \sigma(z, z') - 2 \langle M_S z, z' \rangle &= \langle (J - 2M_S)z, z' \rangle \\ &= -2 \langle J(S - I)^{-1}z, z' \rangle \end{aligned}$$

we have

$$\begin{aligned} \sigma(z, z') + \Phi(z, z') &= -2 \langle J(S - I)^{-1}z, z' \rangle \\ &\quad + \langle M_S z, z \rangle + \langle (M_S + M_{S'})z', z' \rangle \end{aligned}$$

and hence

$$c_\sigma(z) = K e^{\frac{i}{2}\langle M_S z, z \rangle} \int_Z e^{-i\langle J(S-I)^{-1}z, z' \rangle} e^{\frac{i}{2}\langle (M_S + M_{S'})z', z' \rangle} dz'. \quad (52)$$

Applying the Fresnel formula (4) with $m = 2n$ to the formula above and replacing K with its value we get

$$c_\sigma(z) = \left(\frac{1}{2\pi}\right)^n |\det[(M_S + M_{S'})(S - I)(S' - I)]|^{-1/2} e^{\frac{i\pi}{4} \text{sign } M} e^{i\Theta(z)} \quad (53)$$

where the phase Θ is given by

$$\begin{aligned} \Theta(z) &= \langle M_S z, z \rangle - \langle (M_S + M_{S'})^{-1}J(S - I)^{-1}z, J(S - I)^{-1}z \rangle \\ &= \langle M_S + (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1}z, z \rangle \end{aligned}$$

that is $\Theta(z) = M_{SS'}$ in view of part (ii) of Lemma 6. Noting that by definition (31) of the symplectic Cayley transform we have

$$M_S + M_{S'} = J(I + (S - I)^{-1} + (S' - I)^{-1})$$

it follows, using property (33) of the symplectic Cayley transform, that

$$\begin{aligned}\det[(M_S + M_{S'})(S - I)(S' - I)] &= \det[(S - I)(M_S + M_{S'})(S' - I)] \\ &= \det[(S - I)(M_S + M_{S'})(S' - I)] \\ &= |\det(SS' - I)|\end{aligned}$$

which concludes the proof of the first part of proposition. Proof of (ii). Since $\det(S - I) \neq 0$ we also have $\det(S^{-1} - I) \neq 0$. Formula (52) in the proof of part (i) shows that the symbol of $\widehat{C} = \widehat{R}_\nu(S)\widehat{R}_{-\nu}(S^{-1})$ is

$$c_\sigma(z) = K e^{\frac{i}{2}\langle M_S z, z \rangle} \int_Z e^{-i\langle J(S-I)^{-1}z, z' \rangle} e^{\frac{i}{2}\langle (M_S + M_{S^{-1}})z', z' \rangle} dz'$$

where the constant K is this time

$$\begin{aligned}K &= \left(\frac{1}{2\pi}\right)^n \frac{1}{\sqrt{|\det(S - I)(S^{-1} - I)|}} \\ &= \left(\frac{1}{2\pi}\right)^n \frac{1}{|\det(S - I)|}\end{aligned}$$

since $\det(S^{-1} - I) = \det(I - S)$. Using again Lemma 6 we have $M_S + M_{S^{-1}} = 0$ hence, setting $z'' = (S^T - I)^{-1}Jz'$,

$$\begin{aligned}c_\sigma(z) &= \left(\frac{1}{2\pi}\right)^n \frac{e^{\frac{i}{2}\langle M_S z, z \rangle}}{|\det(S - I)|} \int_Z e^{-i\langle J(S-I)^{-1}z, z' \rangle} dz' \\ &= \left(\frac{1}{2\pi}\right)^n e^{\frac{i}{2}\langle M_S z, z \rangle} \int_Z e^{i\langle z, z'' \rangle} dz'' \\ &= (2\pi)^n \delta(z)\end{aligned}$$

and \widehat{C} is thus the identity operator. ■

The composition formula above allows us to prove that the operators $\widehat{R}_\nu(S)$ are unitary:

Corollary 11 *Let $S \in \text{Sp}(Z, \sigma)$ be such that $\det(S - I) \neq 0$. The operators $\widehat{R}_\nu(S)$ are unitary: $\widehat{R}_\nu(S)^* = \widehat{R}_\nu(S)^{-1}$.*

Proof. The symbol of the adjoint of a Weyl operator is the complex conjugate of the symbol of that operator. Since the twisted and Weyl symbol are symplectic Fourier transforms of each other the symbol a of $\widehat{R}_\nu(S)$ is thus given by

$$(2\pi)^n a(z) = \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int_Z e^{-i\sigma(z, z')} e^{\frac{i}{2}\langle M_S z', z' \rangle} dz'.$$

We have

$$(2\pi)^n \overline{a(z)} = \frac{i^{-\nu}}{\sqrt{|\det(S - I)|}} \int_Z e^{i\sigma(z, z')} e^{-\frac{i}{2}\langle M_S z', z' \rangle} dz'.$$

Since $M_{S^{-1}} = -M_S$ and $|\det(S - I)| = |\det(S^{-1} - I)|$ we have

$$\begin{aligned} (2\pi)^n \overline{a(z)} &= \frac{i^{-\nu}}{\sqrt{|\det(S^{-1} - I)|}} \int_Z e^{-i\sigma(z, z')} e^{\frac{i}{2}\langle M_{S^{-1}} z', z' \rangle} dz' \\ &= \frac{i^{-\nu}}{\sqrt{|\det(S^{-1} - I)|}} \int_Z e^{i\sigma(z, z')} e^{\frac{i}{2}\langle M_{S^{-1}} z', z' \rangle} dz' \end{aligned}$$

hence $\overline{a(z)}$ is the symbol of $\widehat{R}_\nu(S)^{-1}$ and this concludes the proof. \blacksquare

4.2 Relation with $\text{Mp}(Z, \sigma)$

Let $S_\infty \in \text{Sp}(Z, \sigma)$ have projection $\pi^{\text{Sp}}(S_\infty) = S$. Proposition 10 and its Corollary will allow us to prove that if we choose $\nu = \nu(S_\infty)$ in $\widehat{R}_\nu(S)$ then this operator is in the metaplectic group $\text{Mp}(Z, \sigma)$. The proof of this property will however require some work. Let us begin by giving a definition: Let $\widehat{S} \in \text{Mp}(Z, \sigma)$ have projection $S \in \text{Sp}(Z, \sigma)$ such that $\det(S - I) \neq 0$ and choose $S_\infty \in \text{Sp}_\infty(Z, \sigma)$ covering \widehat{S} . We define

$$\widehat{\nu}(\widehat{S}) \equiv \nu(S_\infty) \pmod{4}. \quad (54)$$

The index $\widehat{\nu}$ is well-defined: assume in fact that S'_∞ is a second element of $\text{Sp}_\infty(Z, \sigma)$ covering \widehat{S} ; we have $S'_\infty = \alpha^r S_\infty$ for some $r \in \mathbb{Z}$ (α the generator of $\pi_1[\text{Sp}(Z, \sigma)]$); since $\text{Mp}(Z, \sigma)$ is a double covering of $\text{Sp}(Z, \sigma)$ the integer r must be even. Recalling that

$$\nu(\alpha^r S_\infty) = \nu(S_\infty) + 2r$$

the left-hand side of (54) only depends on \widehat{S} and not on the element of $\text{Sp}_\infty(Z, \sigma)$ covering it.

Let S and S' in $\text{Sp}(Z, \sigma)$ be such that $\det(S - I) \neq 0$. Let \widehat{S} and \widehat{S}' in $\text{Mp}(Z, \sigma)$ have projections S and S' : $\pi^{\text{Mp}}(\widehat{S}) = S$ and $\pi^{\text{Mp}}(\widehat{S}') = S'$ (there are two possible choices in each case). We have

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \text{sign } M_S$$

hence, taking classes modulo 4,

$$\widehat{\nu}(\widehat{S}\widehat{S}') = \widehat{\nu}(\widehat{S}) + \widehat{\nu}(\widehat{S}') + \frac{1}{2} \widehat{\text{sign}} M_S.$$

Choosing $\nu = \nu(\widehat{S})$, $\nu' = \nu(\widehat{S}')$ formula (50) becomes

$$\widehat{R}_{\nu(\widehat{S})}(S)\widehat{R}_{\nu(\widehat{S}')} (S') = \widehat{R}_{\nu(\widehat{S}\widehat{S}')} (SS') \quad (55)$$

which suggests that the operators $\widehat{R}_{\nu(\widehat{S})}(S)$ generate a true (two-sheeted) unitary representation of the symplectic group, that is the metaplectic group. Formula (55) is however not sufficient to prove this, because the $\widehat{R}_{\nu(\widehat{S})}(S)$ have only been defined for $\det(S - I) \neq 0$. We are going to show that these operator generate a group, and that this group is indeed the metaplectic group $\text{Mp}(Z, \sigma)$.

Recall that if W is a quadratic form (43) we denoted by W_S the Hessian matrix of the function $x \mapsto W(x, x)$:

$$W_S = P + Q - L - L^T \quad (56)$$

that is

$$W_S = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1} \quad (57)$$

where $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is the free symplectic matrix generated by W . Also recall (Lemma 7) that

$$\begin{aligned} \det(S - I) &= (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}) \\ &= (-1)^n \det L^{-1} \det(P + Q - L - L^T). \end{aligned} \quad (58)$$

We begin by proving that $\widehat{R}_{\nu}(S_W)$ can be identified with $\widehat{S}_{W,m}$ if ν is chosen in a suitable way:

Proposition 12 *Let $\widehat{S}_{W,m} \in \text{Mp}(Z, \sigma)$ be one of the two operators with projection $S = S_W$. (i) We have $\widehat{R}_{\nu}(S_W) = \widehat{S}_{W,m}$ provided that*

$$\nu \equiv \nu(\widehat{S}) \pmod{4}; \quad (59)$$

(ii) *When this is the case we have*

$$\arg \det(S - I) \equiv (\nu(\widehat{S}) - n)\pi \pmod{2\pi}. \quad (60)$$

Proof. Proof of (i). Let $\delta \in \mathcal{S}'(\mathbb{R}^n)$ be the Dirac distribution centered at $x = 0$; setting

$$C_{W,\nu} = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}}$$

we have, by definition of $\widehat{R}_\nu(S)$,

$$\begin{aligned}\widehat{R}_\nu(S)\delta(x) &= C_{W,\nu} \int_Z e^{\frac{i}{2}\langle M_S z_0, z_0 \rangle} e^{i(\langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle)} \delta(x - x_0) dz_0 \\ &= C_{W,\nu} \int_Z e^{\frac{i}{2}\langle M_S(x, p_0), (x, p_0) \rangle} e^{\frac{i}{2}\langle p, x \rangle} \delta(x - x_0) dz_0\end{aligned}$$

hence, setting $x = 0$,

$$\widehat{R}_\nu(S)\delta(0) = C_{W,\nu} \int_Z e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} \delta(-x_0) dz_0$$

that is, since $\int_X \delta(-x_0) dx_0 = 1$,

$$\widehat{R}_\nu(S)\delta(0) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int_Z e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} dp_0. \quad (61)$$

Let us next calculate the scalar product

$$\langle M_S(0, p_0), (0, p_0) \rangle = \sigma((S - I)^{-1}0, p_0), (0, p_0)).$$

The relation $(x, p) = (S - I)^{-1}(0, p_0)$ is equivalent to $S(x, p) = (x, p + p_0)$ that is to

$$p + p_0 = \partial_x W(x, x) \quad \text{and} \quad p = -\partial_{x'} W(x, x).$$

these relations yield after a few calculations

$$x = (P + Q - L - L^T)^{-1} p_0 \quad ; \quad p = (L - Q)(P + Q - L - L^T)^{-1} p_0$$

and hence

$$\langle M_S(0, p_0), (0, p_0) \rangle = -\langle W_S^{-1} p_0, p_0 \rangle \quad (62)$$

where W_S is the symmetric matrix (56). Applying the Fresnel formula (4) to the integral in (61) we get

$$\left(\frac{1}{2\pi}\right)^n \int_{X^*} e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} dp_0 = e^{-\frac{i\pi}{4} \text{sign } W_S} |\det W_S|^{1/2};$$

observing that in view of formula (58) we have

$$\frac{1}{\sqrt{|\det(S_W - I)|}} = |\det L|^{1/2} |\det W_S|^{-1/2}$$

we obtain

$$\widehat{R}_\nu(S_W)\delta(0) = \left(\frac{1}{2\pi}\right)^n i^\nu e^{-\frac{i\pi}{4} \text{sign } W_S} |\det L|^{1/2}.$$

Now, by definition of $\widehat{S}_{W,m}$,

$$\begin{aligned}\widehat{S}_{W,m}\delta(0) &= \left(\frac{1}{2\pi i}\right)^n i^m \sqrt{|\det L|} \int_X e^{iW(0,x')} \delta(x') dx' \\ &= \left(\frac{1}{2\pi}\right)^n i^{m-n/2} \sqrt{|\det L|}\end{aligned}$$

and hence

$$i^\nu e^{-\frac{i\pi}{4} \text{sign } W_S} = i^{m-n/2}.$$

It follows that we have

$$\nu - \frac{1}{2} \text{sign } W_S \equiv m - \frac{n}{2} \pmod{4}$$

which is equivalent to formula (59) since W_S has rank n . Proof of (ii). In view of formula (58) we have

$$\arg \det(S - I) = n\pi + \arg \det B + \arg \det W_S \pmod{2\pi}.$$

Taking into account the obvious relations

$$\begin{aligned}\arg \det B &\equiv \pi \widehat{m}(\widehat{S}) \pmod{2\pi} \\ \arg \det W_S &\equiv \pi \text{Inert } W_S \pmod{2\pi}\end{aligned}$$

formula (60) follows. ■

Recall that $\widehat{S} \in \text{Mp}(Z, \sigma)$ can be written (in infinitely many ways) as a product $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$. We are going to show that $\widehat{S}_{W,m}$ and $\widehat{S}_{W',m'}$ always can be chosen such that $\det(\widehat{S}_{W,m} - I) \neq 0$ and $\det(\widehat{S}_{W',m'} - I) \neq 0$.

Corollary 13 *The operators $\widehat{R}_\nu(S_W)$ generate $\text{Mp}(Z, \sigma)$. In fact, every $\widehat{S} \in \text{Mp}(Z, \sigma)$ can be written as a product*

$$\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'} = \widehat{R}_\nu(S_W) \widehat{R}_{\nu'}(S_{W'}) \quad (63)$$

where $\det(S_W - I) \neq 0$, $\det(S_{W'} - I) \neq 0$, and ν, ν' are given by (59).

Proof. Let $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$. In view of Proposition 12 it suffices to show that W and W' can be chosen so that $S_W = \pi^{\text{Mp}}(\widehat{S}_{W,m})$ and $S_{W'} = \pi^{\text{Mp}}(\widehat{S}_{W',m'})$ satisfy $\det(S_W - I) \neq 0$, $\det(S_{W'} - I) \neq 0$. That the $\widehat{R}_\nu(S_W)$ indeed generate $\text{Mp}(Z, \sigma)$ follows from formula (63). Let us write $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$ and apply the factorization (44) to each of the factors:

$$\widehat{S} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{J} \widehat{V}_{-(P'+Q)} \widehat{M}_{L',m'} \widehat{J} \widehat{V}_{-Q'}. \quad (64)$$

We claim that $\widehat{S}_{W,m}$ and $\widehat{S}_{W',m'}$ can be chosen in such a way that $\det(S_W - I) \neq 0$ and $\det(S_{W'} - I) \neq 0$ that is,

$$\det(P + Q - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + Q' - L' - L'^T) \neq 0;$$

this will prove the assertion. We first remark that the right hand-side of (64) obviously does not change if we replace P' by $P' + \lambda I$ and Q by $Q - \lambda I$ where $\lambda \in \mathbb{R}$. Choose now λ such that it is not an eigenvalue of $P + Q - L - L^T$ and $-\lambda$ is not an eigenvalue of $P' + Q' - L' - L'^T$; then

$$\begin{aligned} \det(P + Q - \lambda I - L - L^T) &\neq 0 \\ \det(P' + \lambda I + Q' - L' - L'^T) &\neq 0 \end{aligned}$$

and we have $\widehat{S} = \widehat{S}_{W_1, m_1} \widehat{S}_{W'_1, m'_1}$ with

$$\begin{aligned} W_1(x, x') &= \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle (Q - \lambda I)x', x' \rangle \\ W'_1(x, x') &= \frac{1}{2} \langle (P' + \lambda I)x, x \rangle - \langle L'x, x' \rangle + \frac{1}{2} \langle Q'x', x' \rangle; \end{aligned}$$

this concludes the proof. ■

There remains to prove that every $\widehat{S} \in \text{Mp}(Z, \sigma)$ such that $\det(S - I) \neq 0$ can be written in the form $\widehat{R}_\nu(S)$:

Proposition 14 *For every $\widehat{S} \in \text{Mp}(Z, \sigma)$ such that $\det(S - I) \neq 0$ we have $\widehat{S} = \widehat{R}_{\nu(\widehat{S})}(S)$ with*

$$\nu(\widehat{S}) = \nu + \nu' + \frac{1}{2} \text{sign}(M + M') \quad (65)$$

if $\widehat{S} = \widehat{R}_\nu(S_W) \widehat{R}_{\nu'}(S_{W'})$ and $M = M_{S_W}$, $M' = M_{S_{W'}}$.

Proof. Let us write $\widehat{S} = \widehat{R}_\nu(S_W) \widehat{R}_{\nu'}(S_{W'})$. A straightforward calculation using the composition formula (50) and the Fresnel integral (4) shows that

$$\widehat{S} = \left(\frac{1}{2\pi} \right)^n \frac{i^{\nu + \nu' + \frac{1}{2} \text{sgn}(M + M')}}{\sqrt{|\det(S_W - I)(S_{W'} - I)(M + M')|}} \int_Z e^{\frac{i}{2} \langle Nz, z \rangle} \widehat{T}(z) dz \quad (66)$$

where N is given by

$$N = M - (M + \frac{1}{2}J)(M + M')^{-1}(M - \frac{1}{2}J).$$

We claim that

$$\det(S_W - I)(S_{W'} - I)(M + M') = \det(S - I) \quad (67)$$

(hence $M + M'$ is indeed invertible), and that

$$N = \frac{1}{2}J(S + I)(S - I)^{-1} = M_S. \quad (68)$$

The first of these identities is easy to check by a direct calculation: by definition of M and M' we have, since $\det J = 1$,

$$\begin{aligned} \det(S_W - I)(S_{W'} - I)(M + M') &= \\ \det(S_W - I)(I + (S_W - I)^{-1} + (S_W - I)^{-1})(S_{W'} - I) \end{aligned}$$

that is

$$\det(S_W - I)(S_{W'} - I)(M + M') = \det(S_W S_{W'} - I)$$

which is precisely (67). Formula (68) is at first sight more cumbersome; there is however an easy way out: assume that $\widehat{S} = \widehat{S}_{W'', m''}$; in view of Lemma 6 we have in this case

$$N = \frac{1}{2}J(S_W S_{W'} + I)(S_W S_{W'} - I)^{-1}$$

and this algebraic identity then holds for all $S = S_W S_{W'}$ since the free symplectic matrices are dense in $\mathrm{Sp}(Z, \sigma)$. Thus,

$$\widehat{S} = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu+\nu'+\frac{1}{2}\mathrm{sgn}(M+M')}}{\sqrt{|\det(S-I)|}} \int_Z e^{\frac{i}{2}\langle M_S z, z \rangle} \widehat{T}(z) dz$$

and to conclude the proof there remains to prove that

$$\nu(S)\pi = (\nu + \nu' + \frac{1}{2}\mathrm{sign}(M + M'))\pi$$

is effectively one of the two possible choices for $\arg \det(S - I)$. We have

$$\begin{aligned} (\nu + \nu' + \frac{1}{2}\mathrm{sign}(M + M'))\pi &= \\ -\arg \det(S_W - I) - \arg \det(S_{W'} - I) + \frac{1}{2}\pi \mathrm{sign}(M + M'); \end{aligned}$$

we next note that if R is any real invertible $2n \times 2n$ symmetric matrix with q negative eigenvalues we have $\arg \det R = q\pi \bmod 2\pi$ and $\frac{1}{2}\mathrm{sign} R = 2n - q$ and hence

$$\arg \det R = \frac{1}{2}\pi \mathrm{sign} R \bmod 2\pi.$$

It follows, taking (67) into account, that

$$(\nu + \nu' + \frac{1}{2}\mathrm{sign}(M + M'))\pi = \arg \det(S - I) \bmod 2\pi$$

which concludes the proof. ■

5 Weyl Calculus on Symplectic Space

Let us now define a class of pseudo-differential operators acting on functions defined on (Z, σ) . The passage from the usual Weyl calculus is made explicit using a family of isometries of $L^2(X)$ onto closed subspaces of $L^2(Z)$. Using the results of previous section we will establish that the calculus thus constructed enjoys a property of metaplectic covariance which makes it into a true generalization of the usual Weyl calculus.

5.1 The isometries U_ϕ

In what follows $\phi \in \mathcal{S}(X)$ is normalized to the unity: $\|\phi\|_{L^2(X)}^2 = 1$. We associate to ϕ the integral operator $U_\phi : L^2(X) \longrightarrow L^2(Z)$ defined by

$$U_\phi f(z) = \left(\frac{\pi}{2}\right)^{n/2} W(f, \bar{\phi})\left(\frac{1}{2}z\right). \quad (69)$$

where $W(f, \bar{\phi})$ is the Wigner–Moyal transform (5) of the pair $(f, \bar{\phi})$. A standard –but by no means mandatory– choice is to take for ϕ the real Gaussian

$$\phi_0(x) = \left(\frac{1}{\pi}\right)^{n/4} e^{-\frac{1}{2}|x|^2}; \quad (70)$$

the corresponding operator U_ϕ is then (up to an exponential factor) the “coherent state representation” familiar to quantum physicists.

Proposition 15 *The transform U_ϕ has the following properties: (i) U_ϕ is an isometry: the Parseval formula*

$$(U_\phi f, U_\phi f')_{L^2(Z)} = (f, f')_{L^2(X)} \quad (71)$$

holds for all $f, f' \in \mathcal{S}(X)$. In particular $U_\phi^ U_\phi = I$ on $L^2(X)$. (ii) The range \mathcal{H}_ϕ of U_ϕ is closed in $L^2(Z)$ (and is hence a Hilbert space), and the operator $P_\phi = U_\phi U_\phi^*$ is the orthogonal projection in $L^2(Z)$ onto \mathcal{H}_ϕ . (iii) Let $\widehat{S} \in \text{Mp}(Z, \sigma)$, $\pi^{\text{Mp}}(\widehat{S}) = S$. We have*

$$U_\phi(\widehat{S}f) = (U_{\phi_{\widehat{S}}} f) \circ S^{-1}, \quad \phi_{\widehat{S}} = \overline{\widehat{S}^{-1}\phi}. \quad (72)$$

Proof. (i) Formula (71) is an immediate consequence of the property

$$(W(f, \phi), W(f', \phi'))_{L^2(Z)} = \left(\frac{1}{2\pi}\right)^n (f, f')_{L^2(X)} \overline{(\phi, \phi')_{L^2(X)}} \quad (73)$$

of the Wigner–Moyal transform (see *e.g.* Folland [8] p. 56). (ii) It is clear that $P_\phi^2 = P_\phi$. Let us show that the range of P_ϕ is \mathcal{H}_ϕ ; the closedness of \mathcal{H}_ϕ

will follow. Since $U_\phi^*U_\phi = I$ on $L^2(X)$ we have $U_\phi^*U_\phi f = f$ for every f in $L^2(X)$ and hence the range of U_ϕ^* is $L^2(X)$. It follows that the range of U_ϕ is that of $U_\phi U_\phi^* = P_\phi$ and is hence closed. Recalling that the Wigner–Moyal transform is such that

$$W(\widehat{S}f, \widehat{S}\phi) = W(f, \phi) \circ S^{-1} \quad (74)$$

for every $\widehat{S} \in \text{Mp}(Z, \sigma)$ with $\pi^{\text{Mp}}(\widehat{S}) = S$ we have, using definition (69) of U_ϕ ,

$$\begin{aligned} U_\phi(\widehat{S}f) &= \left(\frac{\pi}{2}\right)^{n/2} W(\widehat{S}f, \overline{\phi})(\tfrac{1}{2}z) \\ &= \left(\frac{\pi}{2}\right)^{n/2} W(\widehat{S}f, \widehat{S}(\widehat{S}^{-1}\overline{\phi}))(\tfrac{1}{2}z) \\ &= \left(\frac{\pi}{2}\right)^{n/2} W(f, \widehat{S}^{-1}\overline{\phi})(\tfrac{1}{2}S^{-1}(z)) \end{aligned}$$

hence (\cdot) . ■

One should be aware of the fact that the Hilbert space \mathcal{H}_ϕ is smaller than $L^2(Z)$:

Example 16 *Assume that $\phi = \phi_0$, the Gaussian (70). It then follows adapting the argument in [31] that $\mathcal{H}_{\phi_0} \cap \mathcal{S}(Z)$ consists of all function F such that*

$$\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial p_j}\right)(e^{\frac{1}{2}|z|^2}F(z)) = 0 \quad (75)$$

for $1 \leq j \leq n$. For arbitrary ϕ the space $\mathcal{H}_\phi \cap \mathcal{S}(Z)$ is isometric to $\mathcal{H}_{\phi_0} \cap \mathcal{S}(Z)$.

Let \mathbf{H}_n be the $(2n+1)$ -dimensional Heisenberg group; it is the set $Z \times \mathbb{R}$ equipped with the multiplicative law

$$(z, t)(z', t') = (z + z', t + t' + \tfrac{1}{2}\sigma(z, z')).$$

The Schrödinger representation of \mathbf{H}_n is, by definition, the mapping \widehat{T} which to every $(z_0, t_0) \in \mathbf{H}_n$ associates the unitary operator $\widehat{T}(z_0, t_0)$ on $L^2(X)$ defined by

$$\widehat{T}(z_0, t_0)f(x) = \exp\left[i(-t_0 + \langle p_0, x \rangle - \tfrac{1}{2}\langle p_0, x_0 \rangle)\right] f(x - x_0). \quad (76)$$

Recall that a classical theorem of Stone and von Neumann (see for instance Wallach [40] for a modern detailed proof) says that the Schrödinger representation is irreducible and that every irreducible unitary representation of \mathbf{H}_n is unitarily equivalent to \widehat{T} .

5.2 The operators \widehat{A}_{ph}

Let us define operators $\widehat{T}_{\text{ph}}(z_0)$ and \widehat{A}_{ph} on $S'(Z)$ by

$$\widehat{T}_{\text{ph}}(z_0) = e^{-\frac{i}{2}\sigma(\cdot, z_0)}T(z_0) \quad (77)$$

($T(z_0)$ the translation operator in Z) and

$$\widehat{A}_{\text{ph}} = \left(\frac{1}{2\pi}\right)^n \int_Z a_\sigma(z_0)\widehat{T}_{\text{ph}}(z_0)dz_0 \quad (78)$$

with $a_\sigma = \mathcal{F}_\sigma a$.

Example 17 Let $a = H$ be given by

$$H = \frac{1}{2}(p^2 + x^2). \quad (79)$$

The corresponding operator is

$$\widehat{H}_{\text{ph}} = -\frac{1}{2}\partial_z^2 - i\frac{1}{2}\sigma(z, \partial_z) + \frac{1}{8}|z|^2. \quad (80)$$

Observe that the operators \widehat{T}_{ph} satisfy the same commutation relation as the usual Weyl–Heisenberg operators:

$$\widehat{T}_{\text{ph}}(z_1)\widehat{T}_{\text{ph}}(z_0) = e^{-i\sigma(z_0, z_1)}\widehat{T}_{\text{ph}}(z_0)\widehat{T}_{\text{ph}}(z_1) \quad (81)$$

and we have

$$\widehat{T}_{\text{ph}}(z_0)\widehat{T}_{\text{ph}}(z_1) = e^{\frac{i}{2}\sigma(z_0, z_1)}\widehat{T}_{\text{ph}}(z_0 + z_1). \quad (82)$$

These properties suggest that we define the phase-space representation \widehat{T}_{ph} of \mathbf{H}_n in analogy with (76) by setting for $F \in L^2(Z)$

$$\widehat{T}_{\text{ph}}(z_0, t_0)F(z) = e^{it_0}\widehat{T}_{\text{ph}}(z_0)F(z). \quad (83)$$

Clearly $\widehat{T}_{\text{ph}}(z_0, t_0)$ is a unitary operator in $L^2(Z)$; moreover a straightforward calculation shows that

$$\widehat{T}_{\text{ph}}(z_0, t_0)\widehat{T}_{\text{ph}}(z_1, t_1) = \widehat{T}_{\text{ph}}(z_0 + z_1, t_0 + t_1 + \frac{1}{2}\sigma(z_0, z_1)) \quad (84)$$

hence \widehat{T}_{ph} is indeed a representation of the Heisenberg group in $L^2(Z)$. We claim that:

Proposition 18 (i) *We have*

$$\widehat{T}_{ph}(z_0, t_0)U_\phi = U_\phi\widehat{T}(z_0, t_0) \quad (85)$$

hence the representation \widehat{T}_{ph} is unitarily equivalent to the Schrödinger representation, and hence irreducible. (ii) *The following intertwining formula holds for every operator \widehat{A}_{ph} :*

$$\widehat{A}_{ph}U_\phi = U_\phi\widehat{A}. \quad (86)$$

Proof. Proof of (i). It suffices to prove that

$$\widehat{T}_{ph}(z_0)U_\phi = U_\phi\widehat{T}(z_0). \quad (87)$$

Let us write the operator U_ϕ in the form $U_\phi = e^{\frac{i}{2}\langle p, x \rangle}W_\phi$ that is

$$W_\phi f(z) = \left(\frac{1}{2\pi}\right)^{n/2} \int_X e^{-i\langle p, x' \rangle} \phi(x - x') f(x') dx'. \quad (88)$$

We have, by definition of $\widehat{T}_{ph}(z_0)$

$$\begin{aligned} \widehat{T}_{ph}(z_0)U_\phi f(z) &= \exp\left[-\frac{i}{2}\sigma(z, z_0) + \langle p - p_0, x - x_0 \rangle\right] W_\phi f(z - z_0) \\ &= \exp\left[\frac{i}{2}(-2\langle p, x_0 \rangle + \langle p_0, x_0 \rangle + \langle p, x \rangle)\right] W_\phi f(z - z_0) \end{aligned}$$

and, by definition of $W_\phi f$,

$$\begin{aligned} W_\phi f(z - z_0) &= \left(\frac{1}{2\pi}\right)^{n/2} \int_X e^{-i\langle p - p_0, x' \rangle} \overline{\phi}(x - x' - x_0) f(x') dx' \\ &= \left(\frac{1}{2\pi}\right)^{n/2} e^{i\langle p - p_0, x_0 \rangle} \int_X e^{-i\langle p - p_0, x'' \rangle} \overline{\phi}(x - x'') f(x'') dx'' \end{aligned}$$

where we have set $x'' = x' + x_0$. The overall exponential in $\widehat{T}_{ph}(z_0)U_\phi f(z)$ is thus

$$u_1 = \exp\left[\frac{i}{2}(-\langle p_0, x_0 \rangle + \langle p, x \rangle - 2\langle p, x'' \rangle + 2\langle p_0, x'' \rangle)\right].$$

Similarly,

$$\begin{aligned} U_\phi(\widehat{T}(z_0)f)(z) &= \left(\frac{1}{2\pi}\right)^{n/2} e^{\frac{i}{2}\langle p, x \rangle} \times \\ &\quad \int_X e^{-i\langle p, x'' \rangle} \overline{\phi}(x - x'') e^{i(\langle p_0, x'' \rangle - \frac{1}{2}\langle p_0, x_0 \rangle)} f(x'' - x_0) dx'' \end{aligned}$$

yielding the overall exponential

$$u_2 = \exp \left[i \left(\frac{1}{2} \langle p, x \rangle - \langle p, x'' \rangle + \langle p_0, x'' \rangle - \frac{1}{2} \langle p_0, x_0 \rangle \right) \right] = u_1$$

which proves (87). It follows from Stone–von Neumann’s theorem that \widehat{T}_{ph} is an irreducible representation of \mathbf{H}_n on each of the Hilbert spaces \mathcal{H}_ϕ . Proof of (ii). In view of formula (87) we have

$$\begin{aligned} \widehat{A}_{\text{ph}} U_\phi f &= \left(\frac{1}{2\pi} \right)^n \int_Z a_\sigma(z_0) \widehat{T}_{\text{ph}}(z_0) U_\phi f(z) dz_0 \\ &= \left(\frac{1}{2\pi} \right)^n \int_Z a_\sigma(z_0) U_\phi(\widehat{T}(z_0)f)(z) dz_0 \\ &= \left(\frac{1}{2\pi} \right)^n U_\phi \left(\int_Z a_\sigma(z_0) \widehat{T}(z_0)f(z) dz_0 \right) \\ &= U_\phi(\widehat{A}f)(z) \end{aligned}$$

hence (86). ■

Phase-space Weyl operators are composed in the usual way:

Proposition 19 *Let a_σ and b_σ be the twisted symbols of the Weyl operators \widehat{A}_{ph} and \widehat{B}_{ph} . The twisted symbol c_σ of the compose $\widehat{A}_{\text{ph}}\widehat{B}_{\text{ph}}$ is the same as that of $\widehat{A}\widehat{B}$, that is*

$$c_\sigma(z) = \left(\frac{1}{2\pi} \right)^n \int e^{\frac{i}{2}\sigma(z, z')} a_\sigma(z - z') b_\sigma(z') d^{2n}z.$$

Proof. By repeated use of (86) we have

$$\begin{aligned} (\widehat{A}_{\text{ph}}\widehat{B}_{\text{ph}})U_\phi &= \widehat{A}_{\text{ph}}(\widehat{B}_{\text{ph}}U_\phi) \\ &= \widehat{A}_{\text{ph}}U_\phi\widehat{B} \\ &= U_\phi(\widehat{A}\widehat{B}) \end{aligned}$$

hence $\widehat{A}_{\text{ph}}\widehat{B}_{\text{ph}} = (\widehat{A}\widehat{B})_{\text{ph}}$; the twisted symbol of $\widehat{A}\widehat{B}$ is precisely c_σ . ■

5.3 Metaplectic covariance

Let us now prove that the phase-space calculus enjoys a metaplectic covariance property which is similar, *mutandis mutatis*, to the familiar corresponding property for usual Weyl operators (and which we will discuss below); the latter is actually a straightforward consequence of the intertwining relation

$$\widehat{S}\widehat{T}(z_0)\widehat{S}^{-1} = \widehat{T}(Sz_0) \quad (89)$$

valid for all $\widehat{S} \in \text{Mp}(Z, \sigma)$ and $z_0 \in Z$.

We begin by noting that the restriction of the mapping $\widehat{A} \longrightarrow \widehat{A}_{\text{ph}}$ to $\text{Mp}(Z, \sigma)$ is an isomorphism of $\text{Mp}(Z, \sigma)$ onto a subgroup $\text{Mp}_{\text{ph}}(Z, \sigma)$ of the group of unitary operators on $L^2(Z)$. This subgroup can thus be identified with the metaplectic group; the projection $\pi^{\text{Mp}_{\text{ph}}} : \text{Mp}_{\text{ph}}(Z, \sigma) \longrightarrow \text{Mp}(Z, \sigma)$ is defined by

$$\pi^{\text{Mp}_{\text{ph}}}(\widehat{S}_{\text{ph}}) = \pi^{\text{Mp}}(\widehat{S}) = S.$$

Proposition 20 *Let $\widehat{S}_{\text{ph}} \in \text{Mp}_{\text{ph}}(Z, \sigma)$ have projection $S \in \text{Sp}(Z, \sigma)$. Let \widehat{A} have symbol a and \widehat{A}_S symbol $a \circ S$, $S \in \text{Sp}(Z, \sigma)$. We have:*

$$\widehat{S}_{\text{ph}} \widehat{T}_{\text{ph}}(z_0) \widehat{S}_{\text{ph}}^{-1} = \widehat{T}_{\text{ph}}(Sz) \quad , \quad \widehat{A}_{S_{\text{ph}}} = \widehat{S}_{\text{ph}}^{-1} \widehat{A}_{\text{ph}} \widehat{S}_{\text{ph}}. \quad (90)$$

Proof. Recall (formula (86) that $\widehat{A}_{\text{ph}} U_\phi = U_\phi \widehat{A}$; in particular we thus have $\widehat{S}_{\text{ph}} = U_\phi \widehat{S} U_\phi^*$ for every $\widehat{S} \in \text{Mp}(Z, \sigma)$; it follows that

$$\widehat{S}_{\text{ph}} \widehat{T}_{\text{ph}}(z_0) \widehat{S}_{\text{ph}}^{-1} = U_\phi \widehat{S} (U_\phi^* \widehat{T}_{\text{ph}}(z_0) U_\phi) \widehat{S}^{-1} U_\phi^*.$$

In view of formula () we have

$$U_\phi^* \widehat{T}_{\text{ph}}(z_0) U_\phi = \widehat{T}(z_0)$$

and hence, by (89),

$$\begin{aligned} \widehat{S}_{\text{ph}} \widehat{T}_{\text{ph}}(z_0) \widehat{S}_{\text{ph}}^{-1} &= U_\phi \widehat{S} \widehat{T}(z_0) \widehat{S}^{-1} U_\phi^* \\ &= U_\phi \widehat{T}(Sz_0) U_\phi^* \\ &= \widehat{T}_{\text{ph}}(Sz) \end{aligned}$$

which proves the first formula (90). The second formula is proven in the same way using the equalities $\widehat{A}_{\text{ph}} = U_\phi \widehat{A} U_\phi^*$: we have

$$\begin{aligned} \widehat{S}_{\text{ph}}^{-1} \widehat{A}_{\text{ph}} \widehat{S}_{\text{ph}} &= (\widehat{S}_{\text{ph}}^{-1} U_\phi) \widehat{A} (U_\phi^* \widehat{S}_{\text{ph}}) \\ &= U_\phi (\widehat{S}^{-1} \widehat{A} \widehat{S}) U_\phi^* \\ &= U_\phi^* \widehat{A}_S U_\phi^* \end{aligned}$$

hence the result since $U_\phi^* \widehat{A}_S U_\phi^* = \widehat{A}_{S_{\text{ph}}}$. (Alternatively we could have proven the second formula (90) using the first together with definition (78) of \widehat{A}_{ph} .

■

Let us shortly discuss the meaning of this result for the uniqueness of the phase-space Weyl calculus we have constructed in this paper.

In [35] Shale proves the following result (see [41], Chapter 30) for a detailed proof): let

$$\mathcal{L}_X = \mathcal{L}(\mathcal{S}(X), \mathcal{S}'(X))$$

be the set of all continuous linear mappings $\mathcal{S}(X) \longrightarrow \mathcal{S}'(X)$. Let $\text{Op} : \mathcal{S}'(Z) \longrightarrow \mathcal{L}_X$ be a sequentially continuous mapping such that:

$$\text{Op}(a)f(x) = a(x)f(x)$$

if $f \in \mathcal{S}(X)$ and $a \in L^\infty(X) \subset \mathcal{S}'(Z)$; we have

$$\widehat{S} \text{Op}(a) \widehat{S}^{-1} = \text{Op}(a \circ S^{-1})$$

for every $\widehat{S} \in \text{Mp}(Z, \sigma)$ with $S = \pi^{\text{Mp}}(\widehat{S})$. Then we have $\text{Op}(a) = \widehat{A}$, the Weyl operator associated with a .

A straightforward duplication of Shale's proof shows that the

6 Conclusion: Perspectives and Comparisons

Let us begin with the perspectives. The Weyl–Wigner–Moyal formalism (and in particular Weyl calculus in its modern form) originates in the efforts of generations of mathematicians (and physicists) to provide quantum mechanics with an efficient and rigorous framework to “quantize” functions into operators (or, on a subtler level, to “dequantize” operators, see [29]). What could be the advantages (or disadvantages) of using the phase-space calculus introduced in this article? Let us return to Example 17 where the harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + x^2)$ becomes the phase-space operator

$$\widehat{H}_{\text{ph}} = -\frac{1}{2}\partial_z^2 - \frac{i}{2}\sigma(z, \partial_z) + \frac{1}{8}|z|^2. \quad (91)$$

The observant reader will perhaps have noticed that this operator is obtained by replacing formally x and p by the operators

$$\widehat{X}_{\text{ph}} = \frac{1}{2}x + i\partial_p \quad \text{and} \quad \widehat{P}_{\text{ph}} = \frac{1}{2}p - i\partial_x \quad (92)$$

whereas traditional Weyl quantization consists in using the prescription $x \longmapsto x$, $p \longmapsto -i\partial_x$, leading to

$$\widehat{H} = \frac{1}{2}(-\partial_x^2 + x^2).$$

The rule (92) is in fact justified by the fact that for every U_ϕ we have

$$U_\phi(xf) = \widehat{X}_{\text{ph}}U_\phi(f) , U_\phi(-i\partial_x f) = \widehat{P}_{\text{ph}}U_\phi(f)$$

for all $f \in \mathcal{S}(X)$ (the proof is purely computational and left to the reader). We may thus write formally

$$\widehat{H}_{\text{ph}} = H(\frac{1}{2}x + i\partial_p, \frac{1}{2}p - i\partial_x)$$

which is of course no worse than the usual notation

$$\widehat{H} = H(x, -i\partial_x)$$

for the usual Weyl operators. It is of course difficult to argue that there are practical advantages in solving the phase-space Schrödinger equation

$$i\partial_t \Psi(z) = H(\frac{1}{2}x + i\partial_p, \frac{1}{2}p - i\partial_x)\Psi(z) \quad (93)$$

instead of the usual

$$i\partial_t \psi(x) = H(x, -i\partial_x)\psi(x). \quad (94)$$

For arbitrary H defined on $Z = \mathbb{R}^{2n}$ (93) is an equation depending on $2n + 1$ variables whereas (94) only involves $n + 1$ variables. There are, however, important conceptual advantages in using (93). We first remark that every solution of (94) is taken into a solution of equation (93) using an arbitrary transform U_ϕ (this immediately follows from the intertwining property $\widehat{H}_{\text{ph}}U_\phi = U_\phi\widehat{H}$ (formula (86)). On the other hand (93) has solutions which are not in the range of U_ϕ , for any U_ϕ . it suffices to choose an initial function $\Psi_0 \notin \mathcal{H}_\phi$; then $\Psi(\cdot, t) \notin \mathcal{H}_\phi$ for every t ; such solutions do thus not correspond to any solution of the usual Schrödinger equation (93). One should however not dismiss them as uninteresting. Suppose for instance that Ψ_0 is the Gaussian

$$\Psi_G(z) = e^{-\frac{1}{2}\langle Gz, z \rangle} , G = G^T > 0. \quad (95)$$

One verifies after a few calculations that there exist $\psi, \phi \in \mathcal{S}(X)$ such that $U_\phi\psi = \Psi_G$ if and only if $G \in Sp(Z, \sigma)$, and that we have in this case

$$\phi = \alpha\phi_0 , \psi = 2^{N/2}\overline{\alpha}\pi^{n/4}\phi_0$$

where ϕ_0 is the Gaussian (70) and α an arbitrary complex constant with modulus one; the square Ψ_G^2 is then Wigner transform $W\psi = W(\psi, \psi)$ of the Gaussian

$$\psi(x) = (\det X)^{1/4} \pi^{3N/4} e^{-\frac{1}{2}\langle (X+iY)x, x \rangle}. \quad (96)$$

where X and Y are real and symmetric, $X > 0$, determined by

$$G = \begin{bmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{bmatrix}$$

(see [19]). What about Gaussians (95) with $G \notin Sp(Z, \sigma)$? A remarkable fact (or a trite one, depending on one's quantum mechanical sensitivity and philosophy) is that it can be proven that $\Psi_G(z)$ is the Wigner transform of a density operator if and only if the Gromov area of the ellipsoid $\langle Gz, z \rangle \leq 1$ is at least $\frac{1}{2}$ (in physicist's units this would be $\frac{1}{2}h$); this is a geometric version of the uncertainty principle of quantum mechanics which we have discussed in some detail in [17]. It thus appears that the phase-space Schrödinger equations propagate Gaussian "mixed states" when the Hamiltonian is a quadratic polynomial, for instance (91). It would certainly be interesting to push this analysis further and generalize these observations to the non-linear case and to functions which are not Gaussians.

We also notice that the choice we did not make for reasons explained in the beginning of the Introduction –namely the use of the standard Heisenberg–Weyl operators $\hat{T}(z_0)$ extended to phase space leads on the quantum-mechanical level to the Schrödinger equation written formally as

$$i\partial_t\Psi(z) = H(x + i\partial_p, -i\partial_x)\Psi(z);$$

the latter has been obtained using "coherent state representation" arguments by Torres–Vega and Frederick [38] and is currently being an object of lively discussions in physics circles; see our comments in [18] and the references therein.

Let us finally indicate a few connections between our approach and results from other authors.

In [24] Howe defines and studies the "oscillator semi-group" Ω . It is the semi-group of Weyl operators whose symbols are general centered Gaussians; we we have only considered symbols which are Gaussians having purely imaginary exponents. One of the main differences between our approach and Howe's lies in the treatment of the metaplectic representation: in [24] its study is performed by moving to Fock space, which allows Howe to bypass the difficulties occurring when $\hat{S} \in Mp(Z, \sigma)$ is no longer of the type $\hat{S}_{W,m}$ (see the comments in [8], p. 246). In the present work these difficulties are solved in a more explicit way by writing \hat{S} as a product $\hat{S}_{W,m}\hat{S}_{W',m'}$ with $\det(S_W - I) \neq 0$, $\det(S_{W'} - I) \neq 0$ (Corollary 13); this allows us to determine explicitly the correct phase factor i^ν in the Weyl representation of \hat{S} (which is closely related to the Conley–Zehnder index) by using the

powerful machinery of the ALM index. (Let us mention, in passing, that the factorization $\widehat{S} = \widehat{S}_{W,m}\widehat{S}_{W',m'}$, which goes back to Leray [25], does not seem to be widely known by mathematicians working on the metaplectic representation; it can however easily be proven using the fact that the symplectic group acts transitively on pairs of transverse Lagrangian planes; see [12]).

An early version of the operators $\widehat{R}_\nu(S)$ has appeared in the work of Mehlig and Wilkinson [28]; it was this paper which triggered the present author's interest in the study of the Weyl symbol of metaplectic operators; for an early version see [16]; Mehlig and Wilkinson's primary goal is to establish trace formulae related to the Gutzwiller approach to semi-classical quantum systems: the precise determination of the Conley–Zehnder-type index ν could certainly be of some use in such a project (but the roadblocks on the way to a rigorous and complete theory are still immense, in spite of many attempts and some advances).

It would perhaps be interesting to recast some of our results in the more general setting of abstract harmonic analysis and representation theory considered in [3, 21], where formal similarities with the present work are to be found (I take the opportunity to thank K. Hannabuss for having drawn my attention to his work on the topic). The “quantization rules” (92) also have a definite resemblance with formulae appearing in deformation quantization *à la* Bayen *et al.* [1]; since the latter is (in its simplest case) based on the notion of Moyal star-product, itself related to the Wigner–Moyal–Weyl formalism, this is not *a priori* surprising: it is very possible that both approaches are cousins, even if obtained by very different methods.

Acknowledgements. This work has been supported by the grant 2005/51766-7 of the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). I thank P. Piccione (São Paulo) for his kind and congenial hospitality.

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