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# First order quasilinear equations with boundary conditions in the $L^\infty$ framework

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## Abstract

We study a class of first order quasilinear equations on bounded domains in the  $L^\infty$  framework. Using the "semi Kružkov entropy-flux pairs", we define a weak-entropy solution, state an existence and uniqueness result, and a set preserving result.

*Key words:* Semi Kružkov entropy-flux pair; Set preserving property; BV estimates; Vanishing viscosity method; Doubling variable method.

*1991 MSC:* Mathematics Subject Classifications: 35K65, 35L65, 35L60

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## Introduction

In this paper,  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , is a bounded smooth domain. Let us denote by  $\partial\Omega$  the boundary of  $\Omega$  and by  $n$  the outer normal vector to  $\partial\Omega$ . We denote  $Q_T \equiv (0, T) \times \Omega$  and  $\Sigma_T \equiv (0, T) \times \partial\Omega$ . Let us consider this set of equations:

$$\frac{\partial u}{\partial t} + \nabla \cdot (f(t, x, u)) + g(t, x, u) = 0 \text{ on } Q_T \quad (1)$$

$$u(0, \cdot) = u^0 \quad \text{on } \Omega \quad (2)$$

$$"u = u^D" \quad \text{on } \Sigma_T \quad (3)$$

where the sense of the boundary condition will be precised further. We consider the following assumption:

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**Assumption 1**

(i)  $f$  and  $g$  are two functions defined on  $[0, T] \times \bar{\Omega} \times \mathbb{R}$  such that

$$f \in \left( C^2([0, T] \times \bar{\Omega} \times [a, b]) \right)^d, \quad g \in C^2([0, T] \times \bar{\Omega} \times [a, b])$$

(ii)  $f, \nabla \cdot f$  and  $g$  are Lipschitz continuous w.r.t.  $u$ , uniformly in  $(t, x)$ , the constants of Lipschitz continuity being respectively denoted  $\mathcal{L}_{[f]}$ ,  $\mathcal{L}_{[\nabla \cdot f]}$ ,  $\mathcal{L}_{[g]}$ .

(iii)  $(u^0, u^D) \in L^\infty(\Omega; [a, b]) \times L^\infty(\Sigma_T; [a, b])$ ,

(iv)  $(\nabla \cdot f + g)(\cdot, \cdot, a) \leq 0$  and  $(\nabla \cdot f + g)(\cdot, \cdot, b) \geq 0$  uniformly in  $(t, x)$ .

From a mathematical point of view, numerous works have approached or investigated this field. On unbounded domains, existence and uniqueness of a solution for quasilinear first order equations domains has been solved in the pioneering work of Kruřkov [1] who introduced the concept of weak entropy solutions and related "Kruřkov entropy-flux pairs"

$$\left( |u - k|, \operatorname{sgn}(u - k)(f(t, x, u) - f(t, x, k)) \right).$$

When dealing with bounded domains, under some regularity assumptions on the data, Bardos, Le Roux and Nédélec [2] also proved existence and uniqueness of a weak entropy solution satisfying a "Kruřkov entropy-flux pair" formulation including boundary terms; for this, they introduced an appropriate mathematical boundary condition that must be understood in a particular way. Nevertheless, when considering  $L^\infty$  data, the lack of regularity prevents from using the result of Bardos, Le Roux and Nédélec. This difficulty was overcome, at least in the case of autonomous scalar conservation laws on bounded domains, by Otto [3,4] who introduced "boundary entropy-flux pairs"

$$\left( H(u, k), Q_{[f]}(u, k) \right)$$

satisfying particular properties (to be recalled further), which enable to state existence and uniqueness of a so-called weak entropy solution and a set preserving result for this solution. Finally, using a lemma proposed by Vovelle [5], it appears that a formulation using "semi Kruřkov entropy-flux pairs"

$$\left( (u - k)^\pm, \operatorname{sgn}_\pm(u - k)(f(t, x, u) - f(t, x, k)) \right)$$

is equivalent to a formulation based on "boundary entropy-flux pairs". Here, the functions  $u \mapsto (u - \kappa)^\pm$  are the so-called "semi Kruřkov entropies" (see [6,7,5]), defined by

$$(u - \kappa)^+ = \begin{cases} u - \kappa, & \text{if } u \geq \kappa, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad (u - \kappa)^- = (\kappa - u)^+.$$

The functions  $\operatorname{sgn}_\pm(u - \kappa)(f(\cdot, \cdot, u) - f(\cdot, \cdot, \kappa))$  are the corresponding "semi Kruřkov fluxes", where  $u \mapsto \operatorname{sgn}_\pm(u)$  is the derivative of the function  $u \mapsto u^\pm$

with value 0 at point 0. This "semi Kruřkov entropy-flux pairs" formulation is very similar to the initial one of Kruřkov, and uses simple algebraic expressions. Now, let us consider the following questions:

- What is the appropriate definition of a weak entropy solution for first order quasilinear equations (i.e. including non-autonomous fluxes and source terms) on bounded domains with  $L^\infty$  data ? Answering this question would draw a complete parallel with the results of Bardos, Le Roux and Nédélec [2] and those of Otto [3] and Vovelle [5]: indeed, the analysis of scalar conservation laws with  $L^\infty$  data, initiated by Otto, would be extended to quasilinear first order equations, studied by Bardos, Le Roux and Nédélec.
- What sufficient conditions lead to a set preserving result ? Indeed, such a property is crucial when studying some physical problems

Thus, it is the purpose of this paper to give a general framework which is valid for first order quasilinear equations on bounded domains with  $L^\infty$  data. Among the difficulties, we can observe that, when dealing with non autonomous fluxes and source terms, a formulation with "boundary entropy-flux pairs" is not possible anymore. Fortunately, the concept of "semi Kruřkov entropy-flux pairs" allows to overcome difficulties. This work is organized as follows:

1. Definition, initial and boundary conditions, set preserving property
2. Existence
3. Uniqueness

Existence and uniqueness theorems are based on techniques that have been widely used in [1–4]. But we point out the fact that these arguments have never been gathered with the appropriate definition of a weak entropy solution in this general framework in order to establish an existence and uniqueness theorem along with a set preserving result: in fact, we deeply use the results detailed in [4], up to the following modifications: proofs for existence and uniqueness are adapted to the "semi Kruřkov entropy-flux pairs", dealing with additional terms induced by the source term  $g$  and the fact that the flux  $f$  is non-autonomous.

## 1 Definition, initial / boundary conditions, set preserving property

**Definition 1** *Let us suppose that Assumption 1 holds. A function  $u \in L^\infty(Q_T, [a, b])$  is said to be a weak entropy solution of problem (1)-(2)-(3) if it satisfies*

$$(\mathcal{P}_{SK}) \left\{ \begin{array}{l} \int_{Q_T} \left\{ (u - k)^\pm \frac{\partial \varphi}{\partial t} + (\text{sgn}_\pm(u - k)(f(t, x, u) - f(t, x, k))) \nabla \varphi \right. \\ \left. - \text{sgn}_\pm(u - k)(\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi \right\} dx dt \\ + \int_{\Omega} (u^0 - k)^\pm \varphi(0, x) dx + \mathcal{L}_{[f]} \int_{\Sigma_T} (u^D - k)^\pm \varphi(t, r) d\gamma(r) dt \geq 0 \\ \forall \phi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d), \phi \geq 0, \forall k \in \mathbb{R} \end{array} \right.$$

Let us explain the way the boundary / initial conditions are satisfied for this problem. Interestingly, the concept of "boundary entropy-flux pairs" defined by Otto is still the key point. Thus, let us recall their definition:

**Definition 2** *Let  $(H, Q_{[f]})$  be in  $C^1(\mathbb{R}^2) \times (C^1((0, T) \times \bar{\Omega} \times \mathbb{R}^2))^d$ . The pair  $(H, Q_{[f]})$  is said to be a "boundary entropy-flux pair" (for the flux  $f$ ) if:*

1. for all  $w \in \mathbb{R}$ ,  $s \mapsto H(s, w)$  is a convex function,
2.  $\forall w \in \mathbb{R}$ ,  $\partial_s Q_{[f]}(\cdot, \cdot, s, w) = \partial_s H(s, w) \frac{\partial f}{\partial s}(\cdot, \cdot, s)$ ,
3.  $\forall w \in \mathbb{R}$ ,  $H(w, w) = 0$ ,  $Q_{[f]}(\cdot, \cdot, w, w) = 0$ ,  $\partial_s H(w, w) = 0$ .

Let us recall the lemma provided by Vovelle [5], which gives the link between "semi Kruřkov entropy-flux pairs" and "boundary entropy-flux pairs":

### Lemma 3

- (i) *Let  $\eta \in C^1(\mathbb{R}; \mathbb{R})$  be a convex function such that there exists  $w \in [a, b]$  with  $\eta(w) = 0$  and  $\eta'(w) = 0$ . Then  $\eta$  can be uniformly approximated on  $[a, b]$  by applications of the kind*

$$s \longmapsto \sum_1^p \alpha_i (s - \kappa_i)^- + \sum_1^q \beta_j (s - \tilde{\kappa}_j)^+$$

where  $\alpha_i \geq 0$ ,  $\beta_j \geq 0$ ,  $\kappa_i \in [a, b]$  and  $\tilde{\kappa}_j \in [a, b]$ .

- (ii) *Conversely, there exists a sequence of "boundary entropy-flux pairs" which converges to the "semi Kruřkov entropy-flux pairs" (see Appendix A).*

**Lemma 4 (Boundary condition)** *Let  $u \in L^\infty(Q_T)$  satisfying  $(\mathcal{P}_{SK})$ . Then,*

$$\text{ess } \lim_{s \rightarrow 0^+} \int_{\Sigma_T} Q_{[f]}(t, r, u(t, r - s n(r)), u^D(r)) \cdot n(r) \beta(t, r) \, d\gamma(r) \, dt \geq 0,$$

for all "boundary entropy-flux pair"  $(H, Q_{[f]})$ ,  $\forall \beta \in L^1(\Sigma_T)$ ,  $\beta \geq 0$  a.(4)

**PROOF.** We directly use the proof of Lemma 7.12 in [4], adapted to the particular case of the "semi Kruřkov entropy-flux pairs". Thus, we easily state that if  $u \in L^\infty(Q_T)$  satisfies  $(\mathcal{P}_{SK})$ , then, defining the quantity

$$\text{ess } \lim_{s \rightarrow 0^+} \int_{\Sigma_T} \left\{ \text{sgn}_\pm(u(t, r - sn) - v^D(t, r)) (f(t, r, u(t, r - sn)) - f(t, r, v^D(t, r))) \right\} \cdot n \beta(t, r) \, d\gamma(r) \, dt \quad (5)$$

exists for all  $\beta \in L^1((0, T) \times \mathbb{R}^{d-1})$ ,  $\beta \geq 0$  a. e., and all  $v^D \in L^\infty((0, T) \times \mathbb{R}^{d-1})$ . Moreover, we have:

$$\begin{aligned} & \text{ess } \lim_{s \rightarrow 0^+} \int_{\Sigma_T} \left\{ \text{sgn}_\pm(u(t, r - sn) - v^D(t, r)) (f(t, r, u(t, r - sn)) - f(t, r, v^D(t, r))) \right\} \cdot n \beta(t, r) \, d\gamma(r) \, dt \\ & \geq -\mathcal{L}_{[f]} \int_{\Sigma_T} (u^D(r) - v^D(r))^\pm \beta(t, r) \, d\gamma(r) \, dt \end{aligned}$$

for all  $\beta \in L^1((0, T) \times \mathbb{R}^{d-1})$ ,  $\beta \geq 0$  a. e., and all  $v^D \in L^\infty((0, T) \times \mathbb{R}^{d-1})$ . Then, taking  $v^D = u^D$ , every "boundary flux"  $Q_{[f]}$  is uniformly approximated by a linear combination of "semi Kruřkov fluxes" (see Lemma 3), every coefficient being non-negative, which preserves the inequality and concludes the proof.

Now let us give some comprehensive details on the way the boundary condition is satisfied:

**Remark 5** *The boundary condition 4 is nothing less than the one obtained in [3,4], up to a generalization to non-autonomous fluxes and taking account of a source-term which does not interfere in the boundary condition. It is satisfied, although working only with the "semi Kruřkov entropy-flux pairs" formulation (a "boundary entropy-flux pairs" formulation is not possible anymore). However, the way to understand the boundary condition is given in [4,3,5]: generally speaking, the problem should be overdetermined and the boundary equality cannot be required to be assumed at each point of the boundary, even if the solution is a regular function. But, with additional assumptions, the more comprehensive "BLN" condition is recovered (see Appendix A).*

To complete the scope of boundary / initial conditions, we recall the following result, which is proved with the same arguments than in Lemma 7.41 of [4]:

**Lemma 6 (Initial condition)** *Let  $u \in L^\infty(Q_T)$  satisfying  $(\mathcal{P}_{SK})$ . Then,*

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} |u(t, x) - u^0(x)| \, dx = 0 \quad (6)$$

Now we state the following property:

**Theorem 7 (Set preserving property)** *Under Assumption 1 (we recall that, in particular,  $u^0, u^D$  are functions with values in  $[a, b]$ ), if a function  $u$  satisfies  $(\mathcal{P}_{SK})$ , then  $a \leq u \leq b$  a.e. on  $Q_T$ .*

**PROOF.** Set  $k = a$  in  $(\mathcal{P}_{SK})$ . Since we have by Assumption 1 (iii) and (iv),

$$(u^0 - a)^- = 0, \quad (u^D - a)^- = 0,$$

the boundary / initial terms vanish. Then if we choose a particular test-function which only depends on time  $t$ , we obtain:

$$\int_{Q_T} \left\{ (u - a)^- \phi'(t) - \operatorname{sgn}_-(u - a) (\nabla \cdot f(t, x, a) + g(t, x, u)) \phi(t) \right\} dx dt \geq 0$$

for all  $\phi \in \mathcal{D}([0, T[), \phi \geq 0$ . Now, using

$$(\nabla \cdot f(t, x, a) + g(t, x, u)) = (\nabla \cdot f(t, x, a) + g(t, x, a)) + g(t, x, u) - g(t, x, a)$$

and Assumption 1 (iv), we get

$$\int_{Q_T} (u - a)^- \phi'(t) - \operatorname{sgn}_-(u - a) (g(t, x, u) - g(t, x, a)) \phi(t) \, dx dt \geq 0, \quad (7)$$

for all  $\phi \in \mathcal{D}([0, T[), \phi \geq 0$ . Furthermore, it can be easily proved that the following property holds:

$$-\mathcal{L}_{[g]}(u - a)^- \leq \operatorname{sgn}_-(u - a) (g(t, x, u) - g(t, x, a)) \leq \mathcal{L}_{[g]}(u - a)^-$$

and Inequality (7) implies

$$\int_{Q_T} (u - a)^- (\phi'(t) + \mathcal{L}_{[g]} \phi(t)) \, dx dt \geq 0.$$

Introducing the function

$$q_a(t) = e^{-\mathcal{L}_{[g]}t} \int_{\Omega} (u - a)^-(t, x) dx, \quad (8)$$

the above inequality gives

$$\int_0^T q_a(t) e^{\mathcal{L}_{[g]}t} \left( \phi'(t) + \mathcal{L}_{[g]} \phi(t) \right) dt \geq 0.$$

Denoting  $\psi(t) = e^{\mathcal{L}_{[g]}t} \phi(t)$ , we infer that for all  $\psi \in \mathcal{D}([0, T])$ ,  $\psi \geq 0$ ,

$$\int_0^T q_a(t) \psi'(t) dx dt \geq 0, \quad (9)$$

Let  $\tau < T$ ,  $\delta_\tau = T - \tau$  and  $r \in \mathcal{D}([0, T])$  be such that:  $r$  is non-increasing,  $r \equiv 1$  on  $[0, \tau]$ ,  $r \equiv 0$  on  $[\tau + \delta_\tau/2, T]$ . Choosing

$$\psi(t) = \frac{T-t}{T} r(t)$$

in Inequality (9) gives

$$-\frac{1}{T} \int_0^T q_a(t) r(t) dt + \int_0^T q_a(t) \frac{T-t}{T} r'(t) dt \geq 0$$

Since  $r' \leq 0$ , the second term of the left-hand side of the previous inequality is negative. Since  $r(t) = 1, \forall t \in (0, \tau)$  and  $r \geq 0$ , the first term is upper bounded by

$$-\frac{1}{T} \int_0^\tau q_a(t) dt$$

which is consequently non-negative. But,  $q_a$  is obviously a non-negative function, so that  $q_a \equiv 0$ , on  $(0, \tau)$ . Therefore, we deduce from the definition of  $q_a$  (see Equation (8)) that  $(u - a)^- = 0$  on  $\Omega \times (0, \tau)$ . Letting  $\tau \rightarrow T$ , we have  $u \geq a$  a.e. Similarly, by choosing  $k = b$  in  $(\mathcal{P}_{SK})$  (with the "semi Kružkov entropy"  $u \mapsto (u - b)^+$ ), we prove  $u \leq b$  a.e.

## 2 Existence

Existence is obtained from the parabolic approximation (vanishing viscosity method). We consider the following set of equations:



$$\frac{\partial u_\varepsilon}{\partial t} + \nabla \cdot (f(t, x, u_\varepsilon)) + g(t, x, u_\varepsilon) = \varepsilon \Delta u_\varepsilon \text{ on } Q_T \quad (10)$$

$$u_\varepsilon(0, \cdot) = u_\varepsilon^0 \quad \text{on } \Omega \quad (11)$$

$$u = u_\varepsilon^D \quad \text{on } \Sigma_T \quad (12)$$

where the following assumption holds:

**Assumption 2**

- (i)  $u_\varepsilon^D$  and  $u_\varepsilon^0$  satisfy compatibility conditions on  $\overline{\Sigma}_T \cap \overline{Q}_T$ ,
- (ii)  $u_\varepsilon^D$  and  $u_\varepsilon^0$  are smooth functions: for instance,  $u_\varepsilon^D \in C^2(\Sigma_T; [a, b])$ ,  $u_\varepsilon^0 \in C^2(\overline{\Omega}; [a, b])$ .

Under Assumption 2, the quasilinear parabolic problem (10)-(11)-(12) admits a unique solution  $u_\varepsilon \in C^2(\overline{\Omega} \times ]0, T[)$ . We study the convergence of  $\{u_\varepsilon\}_\varepsilon$  when  $\varepsilon$  tends to 0. As in [4], we introduce the following tools:

**Definition 8** Let  $\mu$  be a sufficient small positive number, and let us define the following function:

$$s(x) = \begin{cases} \min(\text{dist}(x, \partial\Omega), \mu), & \text{if } x \in \Omega \\ -\min(\text{dist}(x, \partial\Omega), \mu), & \text{if } x \in \mathbb{R}^d \setminus \Omega \end{cases}$$

Let  $\xi_\varepsilon$  be defined by

$$\xi_\varepsilon(x) = 1 - \exp\left(-\frac{\mathcal{L}_{[f]} + \varepsilon \mathcal{R}}{\varepsilon} s(x)\right), \quad \text{with } \mathcal{R} = \sup_{0 < s(x) < \mu} |\Delta s(x)|.$$

Notice that  $s$  is Lipschitz continuous in  $\mathbb{R}^d$  and smooth on the closure of the set  $\{x \in \mathbb{R}^d, |s(x)| < \mu\}$ . Moreover, it can be proved (see [4]):

**Proposition 9**  $\xi_\varepsilon$  being defined in Definition 8, for all  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ ,  $\varphi \geq 0$ ,

$$\mathcal{L}_{[f]} \int_{\Omega} |\nabla \xi_\varepsilon| \varphi \leq \varepsilon \int_{\Omega} \nabla \xi_\varepsilon \nabla \varphi + (\mathcal{L}_{[f]} + \varepsilon \mathcal{R}) \int_{\partial\Omega} \varphi \quad (13)$$

**Lemma 10** Let  $(u, u^D, u^0)$  satisfy equations (10)-(11)-(12), the boundary / initial conditions satisfying Assumption 2 (subscripts are dropped for convenience). Then,

- (i) for all  $\varphi \in \mathcal{D}(-\infty, T[\times \mathbb{R}^d)$ , for all  $k \in \mathbb{R}$ ,

$$\int_{Q_T} \left\{ (u - k)^\pm \frac{\partial \varphi}{\partial t} + \text{sgn}_\pm(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi - \text{sgn}_\pm(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi + \varepsilon (u - k)^\pm \Delta \varphi \right\} \xi_\varepsilon \, dx \, dt \quad (14)$$

$$\begin{aligned}
& + \int_{\Omega} (u^0 - k)^{\pm} \varphi(0, x) \xi_{\varepsilon} dx \\
& \geq -2\varepsilon \int_{Q_T} (u - k)^{\pm} \nabla \varphi \nabla \xi_{\varepsilon} dx dt - (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} (u^D - k)^{\pm} \varphi(r) d\gamma(r) dt,
\end{aligned}$$

(ii) the following set preserving property holds:

$$a \leq u \leq b \tag{15}$$

**PROOF.** ■ *Proof of (i):* Let us define the functions:

$$\operatorname{sgn}_{\pm}^{\eta}(z) = \begin{cases} H_{\eta}(z), & \text{if } z \in \mathbb{R}^{\pm} \\ -H_{\eta}(-z), & \text{if } z \in \mathbb{R}^{\mp} \end{cases}, \quad I_{\eta}^{\pm}(z) = \int_0^z \operatorname{sgn}_{\pm}^{\eta}(t) dt$$

where the function  $H_{\eta}$  is a classical approximation of the Heaviside graph:  $H_{\eta}(z) = z/\eta \chi_{[0, \eta]}(z) + \chi_{[\eta, +\infty]}(z)$ . Obviously, the pairs

$$\left( I_{\eta}^{\pm}(z, k), \operatorname{sgn}_{\pm}^{\eta}(z - k) (f(t, x, z) - f(t, x, k)) \right)$$

mimick the behaviour of the "semi Kruřkov entropy-flux pairs". Notice that  $I_{\eta}^{\pm}(\cdot, k) \in C^1(\mathbb{R})$  is piecewise convex. Then, multiplying Equation (10) by  $\operatorname{sgn}_{\pm}^{\eta}(u - k) \varphi \xi_{\varepsilon}$ , with  $\varphi \in \mathcal{D}(-\infty, T] \times \mathbb{R}^d$ , we obtain (after integration by parts):

$$\begin{aligned}
& \int_{Q_T} \left\{ I_{\eta}^{\pm}(u, k) \frac{\partial \varphi}{\partial t} + \operatorname{sgn}_{\pm}^{\eta}(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right. \\
& \quad \left. - \operatorname{sgn}_{\pm}^{\eta}(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi + \varepsilon I_{\eta}^{\pm}(u, k) \Delta \varphi \right\} \xi_{\varepsilon} dx dt \\
& + \int_{Q_T} \operatorname{sgn}_{\pm}^{\eta}(u - k) (f(t, x, u) - f(t, x, k)) \varphi \nabla \xi_{\varepsilon} dx dt \\
& + \int_{\Omega} \left( \int_k^{u^0} \operatorname{sgn}_{\pm}^{\eta}(v - k) dv \right) \varphi(0, x) \xi_{\varepsilon} dx \\
& + \int_{Q_T} (f(t, x, u) - f(t, x, k)) \cdot \nabla u \operatorname{sgn}_{\pm}^{\eta \prime}(u - k) \varphi \xi_{\varepsilon} dx dt \\
& \geq \varepsilon \int_{Q_T} \left\{ \nabla \left( I_{\eta}^{\pm}(u, k) \varphi \right) \nabla \xi_{\varepsilon} - 2I_{\eta}^{\pm}(u, k) \nabla \varphi \nabla \xi_{\varepsilon} \right\} dx dt
\end{aligned}$$

After some computation, we state that:

$$\left| \operatorname{sgn}_{\pm}^{\eta}(u - k) (f(t, x, u) - f(t, x, k)) \right| \leq \mathcal{L}_{[f]} I_{\eta}^{\pm}(u, k) + \mathcal{L}_{[f]} \eta$$

Moreover, using Proposition 9 with  $I_\eta^\pm(u, k)\varphi$  instead of  $\varphi$ , we get:

$$\begin{aligned} \mathcal{L}_{[f]} \int_{Q_T} I_\eta^\pm(u, k)\varphi |\nabla \xi_\varepsilon| dx dt \\ \leq \varepsilon \int_{Q_T} \nabla (I_\eta^\pm(u, k)\varphi) \cdot \nabla \xi_\varepsilon + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} I_\eta^\pm(u^D, k)\varphi(r) d\gamma(r) dt \end{aligned}$$

Using these two results in the previous inequality gives:

$$\begin{aligned} \int_{Q_T} \left\{ I_\eta^\pm(u, k) \frac{\partial \varphi}{\partial t} + \operatorname{sgn}_\pm^\eta(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right. \\ \left. - \operatorname{sgn}_\pm^\eta(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi + \varepsilon I_\eta^\pm(u, k) \Delta \varphi \right\} \xi_\varepsilon dx dt \\ + \int_{\Omega} \left( \int_k^{u^0} \operatorname{sgn}_\pm^\eta(v - k) dv \right) \varphi(0, x) \xi_\varepsilon dx \\ + \int_{Q_T} (f(t, x, u) - f(t, x, k)) \cdot \nabla u \operatorname{sgn}_\pm^{\eta'}(u - k) \varphi \xi_\varepsilon dx dt \\ \geq -2\varepsilon \int_{Q_T} I_\eta^\pm(u, k) \nabla \varphi \nabla \xi_\varepsilon dx dt - (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} I_\eta^\pm(u^D, k)\varphi(r) d\gamma(r) dt \\ - \mathcal{L}_{[f]} \eta \int_{Q_T} \varphi |\nabla \xi_\varepsilon| dx dt \end{aligned}$$

Now, let  $\eta$  tend to 0. The first and second terms of the left-hand side give:

$$\begin{aligned} \int_{Q_T} \left\{ (u - k)^\pm \frac{\partial \varphi}{\partial t} + \operatorname{sgn}_\pm(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right. \\ \left. - \operatorname{sgn}_\pm(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi + \varepsilon (u - k)^\pm \Delta \varphi \right\} \xi_\varepsilon dx dt \\ + \int_{\Omega} (u^0 - k)^\pm \varphi(0, x) \xi_\varepsilon dx \end{aligned}$$

The last term of the left-hand side tends to 0 by Lemma 2 in [2]<sup>2</sup>. Finally, the right-hand side tends to:

$$-2\varepsilon \int_{Q_T} (u - k)^\pm \nabla \varphi \nabla \xi_\varepsilon dx dt - (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} (u^D - k)^\pm \varphi(r) d\gamma(r) dt$$

<sup>2</sup> This lemma, due to Saks [8], says that if  $v \in C^1(\overline{\Omega})$ , then

$$\lim_{\eta \rightarrow 0} \int_{\Omega} |\nabla v| \chi_\eta = 0$$

$\chi_\eta$  being the characteristic function of the set  $\{x \in \Omega; |v(x)| \leq \eta\}$ .

and the proof is concluded.

■ *Proof of (ii):* Taking into account the properties of  $f$  and  $g$  (see Assumption 1) and  $u_\varepsilon^D, u_\varepsilon^0$  (see Assumption 2), we choose in the first inequality stated in this lemma (see (i)) the particular value of  $k$ , namely  $k = a$ , with a test-function which only depends on time  $t$ . Thus, we obtain

$$\int_{Q_T} (u_\varepsilon - a)^- \varphi'(t) \geq 0$$

for all  $\varphi \in \mathcal{D}(-\infty, T]$ ,  $\varphi \geq 0$ . Then, similarly to the proof of Theorem 7, we obtain that  $u_\varepsilon \geq a$  a.e. In the same way, we prove that  $u_\varepsilon \leq b$  a.e.

Now we propose the following  $L^1$ -stability result:

**Lemma 11** *Let  $(u_1, u_1^D, u_1^0), (u_2, u_2^D, u_2^0)$ , satisfy equations (10)-(11)-(12), the corresponding boundary / initial conditions satisfying Assumption 2. Then,  $\mathcal{L}_{[g]}$  denoting the constant Lipschitz continuity w.r.t.  $u$  of  $g$ , for all  $t \in (0, T)$ ,*

$$\int_{\Omega} |u_1(t, \cdot) - u_2(t, \cdot)| \xi_\varepsilon \, dx \leq \left\{ \int_{\Omega} |u_1^0 - u_2^0| \xi_\varepsilon \, dx + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} |u_1^D - u_2^D| \right\} e^{\mathcal{L}_{[g]}T} \quad (16)$$

**PROOF.** Let us denote  $w = u_1 - u_2$ ,  $w^D = u_1^D - u_2^D$ ,  $w^0 = u_1^0 - u_2^0$ . Multiplying the equation

$$\frac{\partial w}{\partial t} + \nabla \cdot (f(t, x, u_1) - f(t, x, u_2)) + (g(t, x, u_1) - g(t, x, u_2)) - \varepsilon \Delta w = 0$$

by  $\varphi_\delta'(w) \xi_\varepsilon$ , where  $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2}$ , and integrating over  $(0, t) \times \Omega$ , we obtain

$$\begin{aligned} & \int_{\Omega} \varphi_\delta(w(t, \cdot)) \xi_\varepsilon \, dx - \int_{\Omega} \varphi_\delta(w^0) \xi_\varepsilon \, dx \\ & - \int_0^t \int_{\Omega} (f(\tau, x, u_1) - f(\tau, x, u_2)) (\varphi_\delta''(w) \nabla w \xi_\varepsilon + \varphi_\delta'(w) \nabla \xi_\varepsilon) \, dx \, d\tau \\ & + \int_0^t \int_{\Omega} (g(\tau, x, u_1) - g(\tau, x, u_2)) \varphi_\delta'(w) \xi_\varepsilon \, dx \, d\tau \\ & + \varepsilon \int_0^t \int_{\Omega} \{ |\nabla w|^2 \varphi_\delta''(w) \xi_\varepsilon + (\nabla \varphi_\delta(w)) \cdot \nabla (\xi_\varepsilon) \} \, dx \, d\tau = 0 \end{aligned}$$

Now we study the behaviour of each term w.r.t  $\delta$ : using the uniform Lipschitz continuity of  $f$ , Young's inequality and the fact that  $z^2 \varphi_\delta''(z) = z^2 \delta^2 (z^2 +$

$\delta^2)^{-3/2} < \delta$ , we get

$$\begin{aligned} & -\left(f(\tau, x, u_1) - f(\tau, x, u_2)\right) \nabla w \varphi_\delta''(w) \xi_\varepsilon + \varepsilon |\nabla w|^2 \varphi_\delta''(w) \xi_\varepsilon \\ & \geq \left\{ -\mathcal{L}_{[f]} |w| |\nabla w| + \varepsilon |\nabla w|^2 \right\} \varphi_\delta''(w) \xi_\varepsilon \geq -\frac{\mathcal{L}_{[f]}^2}{4\varepsilon} w^2 \varphi_\delta''(w) \xi_\varepsilon \geq -\frac{\mathcal{L}_{[f]}^2}{4\varepsilon} \delta \xi_\varepsilon \end{aligned}$$

Moreover, observing that  $|z| \varphi_\delta'(z) \leq \varphi_\delta(z)$ , we obtain

$$\begin{aligned} & -\left(f(\tau, x, u_1) - f(\tau, x, u_2)\right) \varphi_\delta'(w) \nabla \xi_\varepsilon \geq -\mathcal{L}_{[f]} |w| |\varphi_\delta'(w)| |\nabla \xi_\varepsilon| \\ & \geq -\mathcal{L}_{[f]} \varphi_\delta(w) |\nabla \xi_\varepsilon| \end{aligned}$$

Following the same idea, recalling that  $\mathcal{L}_{[g]}$  denotes the constant of Lipschitz continuity of  $g$ , we get

$$\left(g(\tau, x, u_1) - g(\tau, x, u_2)\right) \varphi_\delta'(w) \xi_\varepsilon \geq -\mathcal{L}_{[g]} |w| |\varphi_\delta'(w)| \xi_\varepsilon \geq -\mathcal{L}_{[g]} \varphi_\delta(w) \xi_\varepsilon$$

Finally, using the previous inequalities, we state that

$$\begin{aligned} & \int_{\Omega} \varphi_\delta(w(t, \cdot)) \xi_\varepsilon dx - \int_{\Omega} \varphi_\delta(w^0) \xi_\varepsilon dx \\ & \quad - \frac{\mathcal{L}_{[f]}^2 \delta T}{4\varepsilon} \int_{\Omega} \xi_\varepsilon dx - \mathcal{L}_{[f]} \int_0^t \int_{\Omega} \varphi_\delta(w) |\nabla \xi_\varepsilon| dx d\tau \\ & \quad - \mathcal{L}_{[g]} \int_0^t \int_{\Omega} \varphi_\delta(w) \xi_\varepsilon dx d\tau + \varepsilon \int_0^t \int_{\Omega} \nabla(\varphi_\delta(w)) \nabla(\xi_\varepsilon) dx d\tau \leq 0 \end{aligned}$$

and therefore, taking  $\varphi = \varphi_\delta(w)$  in Inequality (13) gives

$$\begin{aligned} \int_{\Omega} \varphi_\delta(w(t, \cdot)) \xi_\varepsilon dx & \leq \int_{\Omega} \varphi_\delta(w^0) \xi_\varepsilon dx + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} \varphi_\delta(w) \xi_\varepsilon dx d\tau \\ & \quad + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_0^t \int_{\Omega} \varphi_\delta(w^D) dx d\tau + \frac{\mathcal{L}_{[f]}^2 \delta T}{4\varepsilon} \int_{\Omega} \xi_\varepsilon dx \end{aligned}$$

Now let  $\delta$  tend to 0. We get

$$\begin{aligned} & \int_{\Omega} |w(t, \cdot)| \xi_\varepsilon dx \\ & \leq \int_{\Omega} |w^0| \xi_\varepsilon dx + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_0^t \int_{\Omega} |w^D| dx d\tau + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} |w| \xi_\varepsilon dx d\tau \\ & \leq \int_{\Omega} |w^0| \xi_\varepsilon dx + (\mathcal{L}_{[f]} + \mathcal{R}\varepsilon) \int_{\Sigma_T} |w^D| dx d\tau + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} |w| \xi_\varepsilon dx d\tau \end{aligned}$$

Applying Gronwall's lemma concludes the proof.

**Lemma 12** *Let  $(u, u^D, u^0)$  satisfy equations (10)-(11)-(12), the corresponding boundary / initial conditions satisfying Assumption 2. We suppose furthermore that  $u^D$  has a smooth extension to  $\bar{Q}_T$ , denoted  $\bar{u}^D$ . Then, there exists a constant  $\lambda$  which only depends on  $\|u^0\|_\Omega$ ,  $\|\bar{u}^D\|_{\Sigma_T}$ ,  $T$ ,  $\Omega$ ,  $f$  and  $g$  such that*

$$\sup_{t \in (0, T)} \int_{\Omega} \left\{ \left| \frac{\partial u}{\partial t}(t, \cdot) \right| + \left| \nabla u(t, \cdot) \right| \right\} \leq \lambda \quad (17)$$

Here, we used the notation

$$\begin{aligned} \|u^0\|_\Omega &= \int_{\Omega} |\Delta u^0| + |\nabla u^0| + |u^0| \\ \|\bar{u}^D\|_{\Sigma_T} &= \sup_{Q_T} \left\{ |\Delta \bar{u}^D| + \left| \frac{\partial \bar{u}^D}{\partial t} \right| + |\nabla \bar{u}^D| + |\bar{u}^D| \right\} \\ &\quad + \int_{Q_T} \left| \nabla^2 \frac{\partial \bar{u}^D}{\partial t} \right| + |\nabla^3 \bar{u}^D| + \left| \frac{\partial^2 \bar{u}^D}{\partial t^2} \right| + \left| \nabla \frac{\partial \bar{u}^D}{\partial t} \right| + |\nabla^2 \bar{u}^D| \end{aligned}$$

**PROOF.** In this proof, we will say that a constant "does not depend on  $\varepsilon$ " if it only depends on  $\|u^0\|_\Omega$ ,  $\|\bar{u}^D\|_{\Sigma_T}$ ,  $T$ ,  $\Omega$ ,  $f$  and  $g$ . Moreover, for the sake of simplicity,  $\bar{u}^D$  will be identified to  $u^D$ . The proof is organized in two steps:

■ *Step 1: Boundedness of  $\int_{\Omega} \left| \frac{\partial u}{\partial t}(t, \cdot) \right|$*

Let us still denote  $u^D$  the smooth extension of  $u^D$  onto  $\bar{Q}_T$ . We introduce

$$\begin{aligned} v &= u - u^D, \\ e &= \frac{\partial^2 u^D}{\partial t^2} + \nabla \cdot \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right) + \nabla \cdot \left( \frac{\partial f}{\partial t}(\cdot, \cdot, u) \right) \\ &\quad + \frac{\partial g}{\partial u}(\cdot, \cdot, u) \frac{\partial u^D}{\partial t} + \frac{\partial g}{\partial t}(\cdot, \cdot, u) - \varepsilon \Delta \frac{\partial u^D}{\partial t}, \end{aligned}$$

so that we easily get

$$\frac{\partial^2 v}{\partial t^2} + \nabla \cdot \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) \frac{\partial v}{\partial t} \right) + \frac{\partial g}{\partial u}(\cdot, \cdot, u) \frac{\partial v}{\partial t} - \varepsilon \Delta \left( \frac{\partial v}{\partial t} \right) = -e \quad (18)$$

Multiplying Equation (18) by

$$\varphi_\delta' \left( \frac{\partial v}{\partial t} \right),$$

where  $\varphi_\delta(z) = (z^2 + \delta^2)^{1/2}$ , and integrating over  $(0, t) \times \Omega$ , we obtain, by using the property  $\varphi_\delta'(\partial v / \partial t) = 0$  on  $\Sigma_T$ :

$$\begin{aligned} & \int_{\Omega} \varphi_\delta \left( \frac{\partial v}{\partial t}(t, \cdot) \right) - \int_{\Omega} \varphi_\delta \left( \frac{\partial v}{\partial t}(0, \cdot) \right) - \int_0^t \int_{\Omega} \frac{\partial f}{\partial u}(\tau, x, u) \cdot \nabla \left( \frac{\partial v}{\partial t} \right) \frac{\partial v}{\partial t} \varphi_\delta'' \left( \frac{\partial v}{\partial t} \right) \\ & + \int_0^t \int_{\Omega} \frac{\partial g}{\partial u}(\tau, x, u) \frac{\partial v}{\partial t} \varphi_\delta' \left( \frac{\partial v}{\partial t} \right) + \varepsilon \int_0^t \int_{\Omega} \left| \nabla \left( \frac{\partial v}{\partial t} \right) \right|^2 \varphi_\delta'' \left( \frac{\partial v}{\partial t} \right) \\ & = - \int_0^t \int_{\Omega} e \varphi_\delta' \left( \frac{\partial v}{\partial t} \right). \end{aligned} \quad (19)$$

Further we have

$$\begin{aligned} & - \frac{\partial f}{\partial u}(\tau, x, u) \cdot \nabla \left( \frac{\partial v}{\partial t} \right) \frac{\partial v}{\partial t} \varphi_\delta'' \left( \frac{\partial v}{\partial t} \right) + \varepsilon \left| \nabla \left( \frac{\partial v}{\partial t} \right) \right|^2 \varphi_\delta'' \left( \frac{\partial v}{\partial t} \right) \\ & \geq - \frac{1}{4\varepsilon} \left| \frac{\partial f}{\partial u}(\tau, x, u) \right|^2 \left( \frac{\partial v}{\partial t} \right)^2 \varphi_\delta'' \left( \frac{\partial v}{\partial t} \right) \geq - \frac{\mathcal{L}_{[f]}^2 \delta}{4\varepsilon} \end{aligned}$$

Thus, letting  $\delta \rightarrow 0$  in Equation (19) gives

$$\int_{\Omega} \left| \frac{\partial v}{\partial t}(t, \cdot) \right| \leq \int_{\Omega} \left| \frac{\partial v}{\partial t}(0, \cdot) \right| + \int_0^t \int_{\Omega} |e| + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} \left| \frac{\partial v}{\partial t} \right|$$

which obviously implies:

$$\int_{\Omega} \left| \frac{\partial u}{\partial t}(t, \cdot) \right| \leq \int_{\Omega} \left| \frac{\partial u^D}{\partial t}(t, \cdot) \right| + \int_{\Omega} \left| \frac{\partial v}{\partial t}(0, \cdot) \right| + \int_0^t \int_{\Omega} |e| + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} \left| \frac{\partial v}{\partial t} \right| \quad (20)$$

Now, let us briefly analyze each term of the right-hand side in the previous inequality:

- ▷ *Analysis of*  $\int_{\Omega} \left| \frac{\partial u^D}{\partial t}(t, \cdot) \right|$  - Obviously, it is bounded by  $c_1 = \|u^D\|_{\Sigma_T}$ .
- ▷ *Analysis of*  $\int_{\Omega} \left| \frac{\partial v}{\partial t}(0, \cdot) \right|$  - We obtain from equation (10):

$$\int_{\Omega} \left| \frac{\partial v}{\partial t}(0, \cdot) \right| = \int_{\Omega} \left| - \nabla \cdot (f(0, \cdot, u^0)) - g(0, \cdot, u^0) + \varepsilon \Delta u^0 - \frac{\partial u^D}{\partial t}(0, \cdot) \right|$$

So far, we have

$$\begin{aligned} \int_{\Omega} \left| -\nabla \cdot (f(0, \cdot, u^0)) \right| &= \int_{\Omega} \left\{ \left| (\nabla \cdot f)(0, \cdot, u^0) \right| + \left| \frac{\partial f}{\partial u}(0, \cdot, u^0) \nabla u^0 \right| \right\} \\ &\leq \mathcal{L}_{[\nabla \cdot f]} \int_{\Omega} |u^0| + \mathcal{L}_{[f]} \int_{\Omega} |\nabla u^0| \leq c_2^{(1)} \end{aligned}$$

where  $c_2^{(1)}$  only depends on  $f$  and  $\|u^0\|_{\Omega}$ . Moreover,

$$\int_{\Omega} |g(0, \cdot, u^0)| \leq |\Omega| \sup \left( |g(t, x, s)|, (t, x, s) \in \overline{Q_T} \times [a, b] \right) \leq c_2^{(2)}$$

where  $c_2^{(2)}$  only depends on  $g$  and  $\Omega$ . Further, for  $\varepsilon$  bounded (which can be assumed, for instance  $\varepsilon \leq 1$ ), we get

$$\int_{\Omega} \left| \varepsilon \Delta u^0 - \frac{\partial u^D}{\partial t}(0, \cdot) \right| \leq \int_{\Omega} |\Delta u^0| + |\Omega| \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right| \leq c_2^{(3)}$$

where  $c_2^{(3)}$  only depends on  $\|u^0\|_{\Omega}$ ,  $\|u^D\|_{\Sigma_T}$  and  $\Omega$ . Thus, the analyzed term satisfies:

$$\int_{\Omega} \left| \frac{\partial v}{\partial t}(0, \cdot) \right| dx \leq c_2$$

where  $c_2$  only depends on  $f$ ,  $g$ ,  $\|u^0\|_{\Omega}$ ,  $\|u^D\|_{\Sigma_T}$  and  $\Omega$ .

▷ *Analysis of  $\int_0^t \int_{\Omega} |e|$*  - Let us recall that, from the definition of  $e$ , we get:

$$\begin{aligned} \int_0^t \int_{\Omega} |e| &\leq \int_0^t \int_{\Omega} \left\{ \left| \frac{\partial^2 u^D}{\partial t^2} \right| + \left| \nabla \cdot \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right) \right| + \left| \nabla \cdot \left( \frac{\partial f}{\partial t}(\cdot, \cdot, u) \right) \right| \right. \\ &\quad \left. + \left| \frac{\partial g}{\partial u}(\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right| + \left| \frac{\partial g}{\partial t}(\cdot, \cdot, u) \right| + \left| \varepsilon \Delta \frac{\partial u^D}{\partial t} \right| \right\}. \end{aligned}$$

Now, we have

$$\int_0^t \int_{\Omega} \left| \frac{\partial^2 u^D}{\partial t^2} \right| \leq \int_{Q_T} \left| \frac{\partial^2 u^D}{\partial t^2} \right| \leq c_3^{(1)}$$

with, for instance,  $c_3^{(1)} = \|u^D\|_{\Sigma_T}$ . Moreover, one has

$$\begin{aligned} \nabla \cdot \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right) &= \left( \nabla \cdot \frac{\partial f}{\partial u} \right) (\cdot, \cdot, u) \frac{\partial u^D}{\partial t} + \frac{\partial^2 f}{\partial u^2}(\cdot, \cdot, u) \cdot \nabla u \frac{\partial u^D}{\partial t} \\ &\quad + \frac{\partial f}{\partial u}(\cdot, \cdot, u) \cdot \nabla \frac{\partial u^D}{\partial t} \end{aligned}$$



Thus, each term can be controlled in the following way:

$$\begin{aligned}
\int_0^t \int_{\Omega} \left| \left( \nabla \cdot \frac{\partial f}{\partial u} \right) (\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right| &\leq c_3^{(2)} \\
\int_0^t \int_{\Omega} \left| \frac{\partial^2 f}{\partial u^2} (\cdot, \cdot, u) \cdot \nabla u \frac{\partial u^D}{\partial t} \right| &\leq c_3^{(3)} \int_0^t \int_{\Omega} |\nabla u| \\
\int_0^t \int_{\Omega} \left| \frac{\partial f}{\partial u} (\cdot, \cdot, u) \cdot \nabla \frac{\partial u^D}{\partial t} \right| &\leq c_3^{(4)}
\end{aligned}$$

with, for instance,

$$\begin{aligned}
c_3^{(2)} &= \sup_{Q_T \times [a,b]} \left| \nabla \cdot \frac{\partial f}{\partial u} \right| \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right| T |\Omega| \\
c_3^{(3)} &= \sup_{Q_T \times [a,b]} \left| \frac{\partial^2 f}{\partial u^2} \right| \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right| \\
c_3^{(4)} &= \mathcal{L}_{[f]} \int_{Q_T} \left| \nabla \frac{\partial u^D}{\partial t} \right|
\end{aligned}$$

Further again,

$$\begin{aligned}
\int_0^t \int_{\Omega} \left| \nabla \cdot \left( \frac{\partial f}{\partial t} (\cdot, \cdot, u) \right) \right| &\leq \int_0^t \int_{\Omega} \left| \nabla \cdot \frac{\partial f}{\partial t} (\cdot, \cdot, u) \right| + \left| \frac{\partial^2 f}{\partial t \partial u} (\cdot, \cdot, u) \cdot \nabla u \right| \\
&\leq c_3^{(5)} + c_3^{(6)} \int_0^t \int_{\Omega} |\nabla u|
\end{aligned}$$

with, for instance,

$$c_3^{(5)} = \sup_{Q_T \times [a,b]} \left| \nabla \cdot \frac{\partial f}{\partial t} \right| T |\Omega|, \quad c_3^{(6)} = \sup_{Q_T \times [a,b]} \left| \frac{\partial^2 f}{\partial u^2} \right| \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right|.$$

Finally, we have also

$$\int_0^t \int_{\Omega} \left| \frac{\partial g}{\partial u} (\cdot, \cdot, u) \frac{\partial u^D}{\partial t} \right| + \left| \frac{\partial g}{\partial t} (\cdot, \cdot, u) \right| + \left| \varepsilon \Delta \frac{\partial u^D}{\partial t} \right| \leq c_3^{(7)}$$

with

$$c_3^{(7)} = \left( \mathcal{L}_{[g]} \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right| + \sup_{Q_T \times [a,b]} \left| \frac{\partial g}{\partial t} \right| + \sup_{Q_T} \left| \Delta \frac{\partial u^D}{\partial t} \right| \right) T |\Omega|$$

Taking  $c_3 = \max \left( c_3^{(1)} + c_3^{(2)} + c_3^{(4)} + c_3^{(5)} + c_3^{(7)}, c_3^{(3)} + c_3^{(6)} \right)$  and using the

previous inequalities together gives, we get

$$\int_0^t \int_{\Omega} |e| \leq c_3 \left( 1 + \int_0^t \int_{\Omega} |\nabla u| \right)$$

$c_3$  being a constant only depending on  $f, g, \Omega, T$  and  $\|u^D\|_{\Sigma_T}$ .

▷ *Analysis of  $\mathcal{L}_{[g]} \int_0^t \int_{\Omega} \left| \frac{\partial v}{\partial t} \right|$*  - We have:

$$\mathcal{L}_{[g]} \int_0^t \int_{\Omega} \left| \frac{\partial v}{\partial t} \right| \leq \mathcal{L}_{[g]} \left( \int_0^t \int_{\Omega} \left| \frac{\partial u}{\partial t} \right| + \int_0^t \int_{\Omega} \left| \frac{\partial u^D}{\partial t} \right| \right) \leq c_4 \left( 1 + \int_0^t \int_{\Omega} \left| \frac{\partial u}{\partial t} \right| \right)$$

with

$$c_4 = \mathcal{L}_{[g]} \max \left( 1, \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} \right| |\Omega| T \right).$$

Thus, recalling Inequality (20) along with the previous results, we obtain

$$\int_{\Omega} \left| \frac{\partial u}{\partial t}(t, \cdot) \right| dx \leq c_5 \left( 1 + \int_0^t \int_{\Omega} |\nabla u(\tau, x)| dx d\tau + \int_0^t \int_{\Omega} \left| \frac{\partial u}{\partial t}(\tau, x) \right| dx d\tau \right) \quad (21)$$

by taking, for instance,  $c_5 = \sum_{i=1}^4 c_i$  which does not depend on  $\varepsilon$ .

■ *Step 2: Boundedness of  $\int_{\Omega} |\nabla u(t, \cdot)|$*

Let us state a similar inequality for  $\nabla u(t, \cdot)$ . This property is obtained using a three-step method:

• **Step 2-1:** Recalling that  $v = u - u^D$  and denoting

$$h_1 = \frac{\partial u^D}{\partial t} + \nabla \cdot (f(t, x, u^D)) + g(t, x, u^D) - \varepsilon \Delta u^D \quad (22)$$

we have

$$\frac{\partial v}{\partial t} + \nabla \cdot (f(t, x, u) - f(t, x, u^D)) + g(t, x, u) - g(t, x, u^D) - \varepsilon \Delta v = -h_1 \quad (23)$$

We multiply Equation (23) by  $\varphi_{\delta}'(v) \beta$ , where  $\beta \in \mathcal{D}(\mathbb{R})$ ,  $\beta \geq 0$ , depends only on the space variable and  $\varphi_{\delta}(z) = (z^2 + \delta^2)^{1/2} - \delta$ . After integration over  $(0, t) \times \Omega$ , partial integration and using the fact that

$$\varphi_{\delta}'(v) = 0, \quad \varphi_{\delta}(v) = 0, \quad \nabla \varphi_{\delta}(v) \cdot n = 0$$

on  $\Sigma_T$ , we obtain

$$\begin{aligned}
& \int_{\Omega} \varphi_{\delta}(v(t, \cdot)) \beta - \int_{\Omega} \varphi_{\delta}(v(0, \cdot)) \beta - \varepsilon \int_0^t \int_{\Omega} \varphi_{\delta}(v) \Delta \beta \\
& \quad - \int_0^t \int_{\Omega} \varphi_{\delta}'(v) (f(\tau, x, u) - f(\tau, x, u^D)) \nabla \beta \\
& \quad - \int_0^t \int_{\Omega} (f(\tau, x, u) - f(\tau, x, u^D)) \nabla v \varphi_{\delta}''(v) \beta \\
& \quad + \int_0^t \int_{\Omega} \varphi_{\delta}'(v) (g(\tau, x, u) - g(\tau, x, u^D)) \beta \\
& \quad + \varepsilon \int_0^t \int_{\Omega} |\nabla v|^2 \varphi_{\delta}''(v) \beta = - \int_0^t \int_{\Omega} \varphi_{\delta}'(v) h_1 \beta
\end{aligned}$$

We let  $\delta \rightarrow 0$  and thus

$$\begin{aligned}
& \int_{\Omega} |v(t, \cdot)| \beta - \int_{\Omega} |v(0, \cdot)| \beta - \varepsilon \int_0^t \int_{\Omega} |v| \Delta \beta \\
& \quad - \int_0^t \int_{\Omega} \operatorname{sgn}(u - u^D) (f(\tau, x, u) - f(\tau, x, u^D)) \nabla \beta \\
& \quad + \int_0^t \int_{\Omega} \operatorname{sgn}(u - u^D) (g(\tau, x, u) - g(\tau, x, u^D)) \beta \\
& \leq - \int_0^t \int_{\Omega} \operatorname{sgn}(u - u^D) h_1 \beta
\end{aligned} \tag{24}$$

Now we choose

$$\beta(x) = \gamma \left( \frac{s(x)}{\rho} \right)$$

where  $s(x)$  is defined as before,  $\rho$  is a strictly positive number and  $\gamma \in \mathcal{D}(\mathbb{R})$  is a fixed non-negative function such that

$$\gamma(0) = 0, \quad \gamma(\sigma) = 1, \quad \text{for } \sigma \geq 1.$$

Let us study the behaviour w.r.t.  $\rho$  of each term:

$$\triangleright \text{Behaviour w.r.t. } \rho \text{ of } \int_0^t \int_{\Omega} \operatorname{sgn}(u - u^D) (f(t, x, u) - f(t, x, u^D)) \nabla \beta:$$

Obviously, one has

$$\nabla \beta = \gamma' \left( \frac{s(x)}{\rho} \right) \frac{1}{\rho} \nabla s(x)$$

and

$$\nabla s(x) = 0, \quad \text{on } \Omega \setminus K_\mu$$

with  $K_\mu = \{x \in \Omega, \text{dist}(x, \partial\Omega) < \mu\}$ . Thus, each point  $x \in K_\mu$  (for  $\mu$  small enough) can be described as  $x = r(x) - s(x) n(r)$ , where  $r(x)$  is the nearest boundary point to  $x$ , and  $n(r)$  is the outer vector to  $\partial\Omega$  at point  $r(x)$ . Let us notice that  $\nabla s(x) = -n(r)$ , if  $x \in K_\mu$ . From the previous observations, we deduce the following equality (for the sake of simplicity,  $F(u, u^D)(\tau, x)$  denotes the value of the function

$$\text{sgn}(u - u^D) (f(\cdot, \cdot, u) - f(\cdot, \cdot, u^D))$$

at point  $(\tau, x) \in Q_T$ :

$$\begin{aligned} & \int_0^t \int_\Omega F(u, u^D)(\tau, x) \nabla \beta(x) \\ &= \int_0^t \int_{K_\mu} F(u, u^D)(\tau, x) \gamma' \left( \frac{s(x)}{\rho} \right) \frac{1}{\rho} \nabla s(x) dx d\tau \\ &= \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} F(u, u^D)(\tau, r - sn(r)) \gamma' \left( \frac{s}{\rho} \right) \frac{1}{\rho} (-n(r)) d\gamma(r) ds d\tau \\ &= - \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} F(u, u^D)(\tau, r - \sigma\rho n(r)) \gamma'(\sigma) n(r) d\gamma(r) d\sigma d\tau \\ &= - \int_0^{\mu/\rho} \gamma'(\sigma) \left( \int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r - \sigma\rho n(r)) n(r) d\gamma(r) d\tau \right) d\sigma \end{aligned}$$

Thus, letting  $\rho \rightarrow 0$ , we obtain

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \int_0^t \int_\Omega F(u, u^D)(\tau, x) \nabla \beta(x) \\ &= - \int_0^{+\infty} \gamma'(\sigma) \left( \int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r) n(r) d\gamma(r) d\tau \right) d\sigma \\ &= - \int_0^{+\infty} \gamma'(\sigma) d\sigma \left( \int_0^t \int_{\partial\Omega} F(u, u^D)(\tau, r) n(r) d\gamma(r) d\tau \right) \\ &= 0 \end{aligned}$$

since  $F(u, u^D) = 0$  on  $\Sigma_T$ .

▷ Behaviour w.r.t.  $\rho$  of  $\int_0^t \int_\Omega |v| \Delta \beta$ :

For the particular choice of  $\beta$ , we have:

$$\begin{aligned}\Delta\beta &= \nabla \cdot \left( \frac{1}{\rho} \gamma' \left( \frac{s(x)}{\rho} \right) \nabla s(x) \right) \\ &= \frac{1}{\rho} \sum_{i=1}^d \left\{ \frac{1}{\rho} \gamma'' \left( \frac{s(x)}{\rho} \right) \left( \frac{\partial s(x)}{\partial x_i} \right)^2 + \gamma' \left( \frac{s(x)}{\rho} \right) \frac{\partial^2 s(x)}{\partial x_i^2} \right\}\end{aligned}$$

Moreover, if  $x \in K_\mu$ , then  $\frac{\partial s(x)}{\partial x_i} = -n_i(r)$ ,  $n_i$  being the  $i$ th component of  $n(r)$ , so that

$$\frac{1}{\rho^2} \gamma'' \left( \frac{s(x)}{\rho} \right) \sum_{i=1}^d \left( \frac{\partial s(x)}{\partial x_i} \right)^2 = \frac{1}{\rho^2} \gamma'' \left( \frac{s(x)}{\rho} \right) \|n(r)\|^2 = \frac{1}{\rho^2} \gamma'' \left( \frac{s(x)}{\rho} \right)$$

and, as a consequence,

$$\Delta\beta = \begin{cases} \frac{1}{\rho^2} \gamma'' \left( \frac{s(x)}{\rho} \right) + \frac{1}{\rho} \gamma' \left( \frac{s(x)}{\rho} \right) \Delta s(x), & \text{on } K_\mu \\ 0, & \text{elsewhere} \end{cases}$$

Thus, since  $v(\tau, r(x)) = 0$  ( $r(x)$  being a boundary point) and using the above expression of  $\Delta\beta$ , we have

$$\begin{aligned}\int_0^t \int_\Omega |v| \Delta\beta &= \int_0^t \int_\Omega |v(\tau, x) - v(\tau, r(x))| \Delta\beta \\ &= \int_0^t \int_{K_\mu} |v(\tau, x) - v(\tau, r(x))| \left( \gamma'' \left( \frac{s(x)}{\rho} \right) \frac{1}{\rho^2} + \frac{1}{\rho} \gamma' \left( \frac{s(x)}{\rho} \right) \Delta s(x) \right)\end{aligned}$$

Let us focus on the first right-hand side of the previous equality:

$$\begin{aligned}&\int_0^t \int_0^\mu \int_{\partial\Omega} |v(\tau, r - s n(r)) - v(\tau, r)| \gamma'' \left( \frac{s}{\rho} \right) \frac{1}{\rho^2} d\gamma(r) ds d\tau \\ &= \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} \frac{|v(\tau, r - \sigma\rho n(r)) - v(\tau, r)|}{\rho} \gamma''(\sigma) d\gamma(r) d\sigma d\tau \\ &= \int_0^{\mu/\rho} \sigma \gamma''(\sigma) \left( \int_0^t \int_{\partial\Omega} \frac{|v(\tau, r - \sigma\rho n(r)) - v(\tau, r)|}{\sigma\rho} d\gamma(r) d\tau \right) d\sigma\end{aligned}$$

Now let us focus on the second right-hand side: since  $|\Delta s| \leq \mathcal{R}$  on  $K_\mu$ ,

$$\begin{aligned}
& \left| \int_0^t \int_{K_\mu} |v(\tau, x) - v(\tau, r(x))| \frac{1}{\rho} \gamma' \left( \frac{s(x)}{\rho} \right) \Delta s(x) dx d\tau \right| \\
& \leq \mathcal{R} \int_0^t \int_0^\mu \int_{\partial\Omega} |v(\tau, r - s n(r)) - v(\tau, r)| \frac{1}{\rho} \left| \gamma' \left( \frac{s}{\rho} \right) \right| d\gamma(r) ds d\tau \\
& \leq \mathcal{R} \int_0^t \int_0^{\mu/\rho} \int_{\partial\Omega} |v(\tau, r - \sigma \rho n(r)) - v(\tau, r)| \gamma'(\sigma) d\gamma(r) d\sigma d\tau \\
& = \mathcal{R} \rho \int_0^{\mu/\rho} \sigma \gamma'(\sigma) \left( \int_0^t \int_{\partial\Omega} \frac{|v(\tau, r - \sigma \rho n(r)) - v(\tau, r)|}{\sigma \rho} d\gamma(r) d\tau \right) d\sigma.
\end{aligned}$$

Letting  $\rho \rightarrow 0$  (notice that the second right-hand side tends to 0) gives

$$\begin{aligned}
\lim_{\rho \rightarrow 0} \int_0^t \int_{\Omega} |v| \Delta \beta &= \int_0^{+\infty} \sigma \gamma''(\sigma) \left( \int_0^t \int_{\partial\Omega} |\nabla v(\tau, r) \cdot n(r)| d\gamma(r) d\tau \right) d\sigma \\
&= - \int_0^t \int_{\partial\Omega} |\nabla v(\tau, r) \cdot n(r)| d\gamma(r) d\tau
\end{aligned}$$

As a consequence, Inequality (24) becomes

$$\begin{aligned}
& \int_{\Omega} |v(t, x)| dx + \varepsilon \int_{\partial\Omega} |\nabla v(\tau, r) \cdot n(r)| d\gamma(r) d\tau \\
& \leq \int_{\Omega} |v(0, x)| dx + \mathcal{L}_{[g]} \int_0^t \int_{\Omega} |v(\tau, x)| d\tau dx + \int_0^t \int_{\Omega} |h_1(\tau, x)| d\tau dx \quad (25)
\end{aligned}$$

- **Step 2-2:** Now, let us denote

$$z_i = \frac{\partial v}{\partial x_i}, \quad z = \nabla v$$

Then we have

$$\frac{\partial z_i}{\partial t} + \nabla \cdot \left( \frac{\partial f}{\partial u}(t, x, u) z_i \right) - \varepsilon \Delta z_i = -h_2^{(i)}$$

with

$$\begin{aligned}
h_2^{(i)} &= \frac{\partial^2 u^D}{\partial x_i \partial t} + \nabla \cdot \left( \frac{\partial f}{\partial u}(t, x, u) \frac{\partial u^D}{\partial x_i} \right) + \nabla \cdot \left( \frac{\partial f}{\partial x_i}(t, x, u) \right) \\
&\quad + \frac{\partial}{\partial x_i} (g(t, x, u)) - \varepsilon \Delta \frac{\partial u^D}{\partial x_i}
\end{aligned}$$

Multiplying the above equation by  $\partial\phi_\delta/\partial\xi_i(z)$ , with  $\phi_\delta(\xi) = (|\xi|^2 + \delta^2)^{1/2}$ , adding the terms ( $i = 1, d$ ), we have, using the usual Einstein summation convention (i.e. whenever an index appears twice in one expression, the summation over this index is performed):

$$\begin{aligned} \int_0^t \int_\Omega \frac{\partial z_i}{\partial t} \frac{\partial \phi_\delta}{\partial \xi_i}(z) dx d\tau &= \int_\Omega \phi_\delta(v(t, \cdot)) dx - \int_\Omega \phi_\delta(v(0, \cdot)) dx, \\ -\varepsilon \int_0^t \int_\Omega \Delta z_i \frac{\partial \phi_\delta}{\partial \xi_i}(z) dx d\tau &= \varepsilon \int_0^t \int_\Omega \frac{\partial z_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(z) \frac{\partial z_k}{\partial x_j} - \varepsilon \int_0^t \int_{\partial\Omega} \frac{\partial z_i}{\partial x_j} n_j \frac{\partial \phi_\delta}{\partial \xi_i}(z), \\ \int_0^t \int_\Omega \nabla \cdot \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) z_i \right) \frac{\partial \phi_\delta}{\partial \xi_i}(z) dx d\tau \\ &= - \int_0^t \int_\Omega \frac{\partial f_j}{\partial u}(\cdot, \cdot, u) z_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(z) \frac{\partial z_k}{\partial x_j} + \int_0^t \int_{\partial\Omega} \frac{\partial f_j}{\partial u}(\cdot, \cdot, u) n_j z_i \frac{\partial \phi_\delta}{\partial \xi_i}(z) \end{aligned}$$

Due to the estimate

$$\begin{aligned} \varepsilon \frac{\partial z_i}{\partial x_j} \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(z) \frac{\partial z_k}{\partial x_j} - \frac{\partial f_j}{\partial u}(\cdot, \cdot, u) z_i \frac{\partial^2 \phi_\delta}{\partial \xi_i \partial \xi_k}(z) \frac{\partial z_k}{\partial x_j} \\ = \frac{\delta^2}{(|z|^2 + \delta^2)^{3/2}} \left[ \varepsilon |\nabla z|^2 - \frac{\partial f_j}{\partial u}(\cdot, \cdot, u) z_i \frac{\partial z_i}{\partial x_j} \right] \\ \geq -\frac{1}{4\varepsilon} \left| \frac{\partial f}{\partial u}(\cdot, \cdot, u) \right|^2 \frac{\delta^2 |z|^2}{(|z|^2 + \delta^2)^{3/2}} \geq -\frac{\mathcal{L}_{[f]} \delta}{4\varepsilon}, \end{aligned}$$

we obtain for  $\delta \rightarrow 0$

$$\begin{aligned} \int_\Omega |z(t, \cdot)| &\leq \int_\Omega |z(0, \cdot)| + \int_0^t \int_\Omega |h_2| \\ &\quad + \limsup_{\delta \rightarrow 0} \int_0^t \int_{\partial\Omega} \left| \frac{\partial f}{\partial u}(\cdot, \cdot, u) \cdot n z_j \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n \frac{\partial \phi_\delta}{\partial \xi_i}(z) \right| \end{aligned}$$

Due to  $z = 0$  on  $\Sigma_T$ , we have on  $\Sigma_T$

$$z = \nabla v = (\nabla v \cdot n) n, \quad \Delta v = D^2 v(n, n) + \Delta s \nabla v \cdot n$$

where  $D^2 v$  is the bilinear form of the second differential of  $v$ . Therefore, the integrand can be rewritten as

$$\begin{aligned} \frac{\partial f}{\partial u}(\cdot, \cdot, u) \cdot n z_j \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n \frac{\partial \phi_\delta}{\partial \xi_i}(z) \\ = \frac{\partial f}{\partial u}(\cdot, \cdot, u) \cdot n \frac{|\nabla v|^2}{(|\nabla v|^2 + \delta^2)^{1/2}} - \varepsilon D^2 v \left( n, \frac{\nabla v}{(|\nabla v|^2 + \delta^2)^{1/2}} \right) \\ = \left( \frac{\partial f}{\partial u}(\cdot, \cdot, u) \cdot \nabla v - \varepsilon D^2 v(n, n) \right) \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}} \end{aligned}$$

Moreover,

$$\begin{aligned}
& \nabla \cdot (f(t, x, u) - f(t, x, u^D)) \\
&= (\nabla \cdot f)(t, x, u) - (\nabla \cdot f)(t, x, u^D) + \frac{\partial f}{\partial y} \cdot \nabla u - \frac{\partial f}{\partial u} \cdot \nabla u^D \\
&= (\nabla \cdot f)(t, x, u) - (\nabla \cdot f)(t, x, u^D) + \frac{\partial f}{\partial u}(t, x, u) \cdot \nabla v \\
&\quad - \left( \frac{\partial f}{\partial u}(t, x, u^D) - \frac{\partial f}{\partial u}(t, x, u) \right) \cdot \nabla u^D.
\end{aligned}$$

Thus, for  $(t, x) \in \Sigma_T$ , one has (we recall that  $u = u^D$ ):

$$\left( \frac{\partial f}{\partial u}(t, x, u^D) - \frac{\partial f}{\partial u}(t, x, u) \right) = 0,$$

and we obtain

$$\frac{\partial f}{\partial u}(t, x, u) \cdot \nabla v = \nabla \cdot (f(t, x, u) - f(t, x, u^D)) - \left\{ (\nabla \cdot f)(t, x, u) - (\nabla \cdot f)(t, x, u^D) \right\}.$$

Since  $\frac{\partial v}{\partial t} = 0$  on  $\Sigma_T$ , we have for  $(t, x) \in \Sigma_T$

$$\begin{aligned}
& \frac{\partial f}{\partial u}(t, x, u) \cdot n \ z_j \ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n \ \frac{\partial \phi_\delta}{\partial \xi_i}(z) \\
&= \left( \frac{\partial v}{\partial t} + \nabla \cdot (f(t, x, u) - f(t, x, u^D)) - \varepsilon \Delta v + \varepsilon \Delta s \nabla v \cdot n \right) \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}} \\
&\quad - \left\{ \nabla \cdot f(t, x, u) - \nabla \cdot f(t, x, u^D) \right\} \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}} \\
&= (-h_1 + \varepsilon \Delta s \nabla v \cdot n) \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}} \\
&\quad - \left\{ \nabla \cdot f(t, x, u) - \nabla \cdot f(t, x, u^D) + g(t, x, u) - g(t, x, u^D) \right\} \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}} \\
&\quad - \left\{ g(t, x, u) - g(t, x, u^D) \right\} \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}}
\end{aligned}$$

Since  $(t, x) \in \Sigma_T$ , we obtain ( $u = u^D$ ):

$$\frac{\partial f}{\partial u}(t, x, u) \cdot n \ z_j \ \frac{\partial \phi_\delta}{\partial \xi_j}(z) - \varepsilon \nabla z_i \cdot n \ \frac{\partial \phi_\delta}{\partial \xi_i}(z) = (-h_1 + \varepsilon \Delta s \nabla v \cdot n) \frac{\nabla v \cdot n}{(|\nabla v|^2 + \delta^2)^{1/2}}$$

Putting this in the last inequality gives:

$$\int_{\Omega} |z(t, \cdot)| \leq \int_{\Omega} |z(0, \cdot)| + \int_0^t \int_{\Omega} |h_2| + \int_0^t \int_{\partial \Omega} \left\{ |h_1| + \varepsilon \mathcal{R} |\nabla v \cdot n| \right\}$$



which, together with Inequality (25) (stated at step 2-1) implies

$$\int_{\Omega} \{|\nabla v(t, \cdot)| + \mathcal{R}|v(t, \cdot)|\} \leq \int_{\Omega} \{|\nabla v(0, \cdot)| + \mathcal{R}|v(0, \cdot)|\} + \int_0^t \int_{\Omega} |h_2| + \int_0^t \int_{\partial\Omega} |h_1|$$

and, as a consequence,

$$\begin{aligned} \int_{\Omega} \{|\nabla u(t, \cdot)| + \mathcal{R}|u(t, \cdot)|\} &\leq \int_{\Omega} \{|\nabla u^D(t, \cdot)| + \mathcal{R}|u^D(t, \cdot)|\} \\ &\quad + \int_{\Omega} \{|\nabla v(0, \cdot)| + \mathcal{R}|v(0, \cdot)|\} \\ &\quad + \int_0^t \int_{\Omega} |h_2| + \int_0^t \int_{\partial\Omega} |h_1| \end{aligned} \quad (26)$$

• **Step 2-3:** Let us analyse each term of the Inequality (26).

▷ *Analysis of*  $\int_{\Omega} \{|\nabla u^D(t, \cdot)| + \mathcal{R}|u^D(t, \cdot)|\}$  - We easily state that

$$\int_{\Omega} \{|\nabla u^D(t, \cdot)| + \mathcal{R}|u^D(t, \cdot)|\} \leq c_6 = \left( \sup_{Q_T} |\nabla u^D| + \mathcal{R} \sup_{Q_T} |u^D| \right) |\Omega| T$$

▷ *Analysis of*  $\int_{\Omega} \{|\nabla v(0, \cdot)| + \mathcal{R}|v(0, \cdot)|\}$  - Clearly, one has:

$$\begin{aligned} \int_{\Omega} \{|\nabla v(0, \cdot)| + \mathcal{R}|v(0, \cdot)|\} &\leq \int_{\Omega} \{|\nabla u^0| + |\nabla u^D(0, \cdot)| + \mathcal{R}(|u^0| + |u^D(0, \cdot)|)\} \\ &\leq \int_{\Omega} (|\nabla u^0| + \mathcal{R}|u^0|) + \left( \sup_{Q_T} |\nabla u^D| + \mathcal{R} \sup_{Q_T} |u^D| \right) = c_7 \end{aligned}$$

▷ *Analysis of*  $\int_0^t \int_{\Omega} |h_1|$  - Recalling the expression of  $h_1$ ,

$$\begin{aligned} \int_0^t \int_{\Omega} |h_1| &= \int_0^t \int_{\Omega} \left| \frac{\partial u^D}{\partial t} + \nabla \cdot (f(t, x, u^D)) + g(t, x, u^D) - \varepsilon \Delta u^D \right| \\ &\leq \left( \sup_{Q_T} \left| \frac{\partial u^D}{\partial t} + \Delta u^D \right| + \sup_{Q_T \times [a, b]} |g| \right) |\Omega| T + \int_0^t \int_{\Omega} \nabla \cdot (f(t, x, u^D)) \end{aligned}$$

Since  $\nabla \cdot (f(t, x, u^D)) = \nabla \cdot f(t, x, u^D) + \frac{\partial f}{\partial u}(t, x, u^D) \nabla u^D$ , we have

$$\int_0^t \int_{\Omega} |h_1| \leq c_8$$

with  $c_8 = \left( \sup_{Q_T} \left\{ \left| \frac{\partial u^D}{\partial t} \right| + |\Delta u^D| + \mathcal{L}_{[f]} |\nabla u^D| \right\} + \sup_{Q_T \times [a,b]} |g + \nabla \cdot f| \right) |\Omega| T$ .

▷ *Analysis of  $\int_0^t \int_{\Omega} |h_2|$*  - First, let us develop the expression of  $h_2$ :

$$\begin{aligned} h_2^{(i)} = & \frac{\partial^2 u^D}{\partial x_i \partial t} + \nabla \cdot \frac{\partial f}{\partial u}(t, x, u) \frac{\partial u^D}{\partial x_i} + \frac{\partial^2 f}{\partial u^2}(t, x, u) \nabla u \frac{\partial u^D}{\partial x_i} \\ & + \frac{\partial f}{\partial u}(t, x, u) \nabla \frac{\partial u^D}{\partial x_i} + \nabla \cdot \frac{\partial f}{\partial x_i}(t, x, u) + \frac{\partial^2 f}{\partial x_i \partial u}(t, x, u) \nabla u \\ & + \frac{\partial g}{\partial x_i}(t, x, u) + \frac{\partial g}{\partial u}(t, x, u) \frac{\partial u}{\partial x_i} - \varepsilon \Delta \frac{\partial u^D}{\partial x_i} \end{aligned}$$

From this, we get:

$$\int_0^t \int_{\Omega} |h_2| \leq c_9^{(1)} + c_9^{(2)} \int_0^t \int_{\Omega} |\nabla u| \leq c_9 \left( 1 + \int_0^t \int_{\Omega} |\nabla u| \right)$$

with the following constants

$$\begin{aligned} c_9^{(1)} = & \left( \sup_{Q_T \times [a,b]} \left| \nabla \cdot \frac{\partial f}{\partial u} \right| \sup_{Q_T} |\nabla u^D| + \sup_{Q_T \times [a,b]} |\nabla^2 f| + \sup_{Q_T \times [a,b]} |\nabla g| \right) |\Omega| T \\ & + \mathcal{L}_{[f]} \int_{Q_T} |\nabla^2 u^D| + \int_{Q_T} |\nabla^3 u^D|, \end{aligned}$$

$$c_9^{(2)} = \sup_{Q_T \times [a,b]} \left| \frac{\partial^2 f}{\partial u^2} \right| \sup_{Q_T} |\nabla u^D| + \mathcal{L}_{[g]} + \sup_{Q_T \times [a,b]} \left| \nabla \cdot \frac{\partial f}{\partial u} \right|,$$

$$c_9 = \max(c_9^{(1)}, c_9^{(2)}).$$

To conclude Step 2, we gather Inequality (26) with all the previous bounds:

$$\int_{\Omega} \left\{ |\nabla u(t, \cdot)| + \mathcal{R}|u(t, \cdot)| \right\} \leq c_{10} \left( 1 + \int_0^t \int_{\Omega} |\nabla u| \right)$$

where  $c_{10} = \sum_{i=6}^9 c_i$  does not depend on  $\varepsilon$ .

Finally, since  $u$  is a function with values in  $[a, b]$ , from the above inequality, we infer that there exists  $c_{11}$  which does not depend on  $\varepsilon$  such that

$$\int_{\Omega} |\nabla u(t, \cdot)| \leq c_{11} \left( 1 + \int_0^t \int_{\Omega} |\nabla u| \, d\tau \, dx \right) \quad (27)$$

Now, we gather results obtained in Steps 1 and 2: using Inequalities (21) and (27) gives

$$\int_{\Omega} \left\{ \left| \frac{\partial u}{\partial t}(t, \cdot) \right| + |\nabla u(t, \cdot)| \right\} \leq c_{12} \left( 1 + \int_0^t \int_{\Omega} \left\{ \left| \frac{\partial u}{\partial t} \right| + |\nabla u| \right\} \right) \quad (28)$$

where  $c_{12}$  ( $= c_5 + c_{11}$ , for instance) does not depend on  $\varepsilon$ . Applying Gronwall's lemma concludes the proof of Lemma 12.

**Theorem 13 (Existence)** *Let us suppose that Assumption 1 holds. Let  $u_\varepsilon$  be the unique solution of Equations (10)-(11)-(12) corresponding to initial / boundary conditions  $(u_\varepsilon^0, u_\varepsilon^D)$  satisfying Assumption 2 and let*

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon^D = u^D \quad \text{in } L^1(\Sigma_T), \quad \lim_{\varepsilon \rightarrow 0} u_\varepsilon^0 = u^0 \quad \text{in } L^1(\Omega)$$

where  $u^D \in L^\infty(\Sigma_T; [a, b])$  and  $u^0 \in L^\infty(\Omega; [a, b])$ . Then, the sequence  $\{u_\varepsilon\}_\varepsilon$  converges to some function  $u \in L^\infty(Q_T; [a, b])$  in  $C^0([0, T], L^1(\Omega))$ . Moreover  $u$  is a weak entropy solution of problem (1)-(2)-(3).

**PROOF.** Before entering into technical details, let us give the sketch of this proof. Our goal is to let  $\varepsilon$  tend to 0. Nevertheless, we cannot apply estimates stated in Lemma 12 on  $u_\varepsilon$  because  $u_\varepsilon^D, u_\varepsilon^0$  satisfy compatibility conditions but do not necessarily have an extension over  $\overline{Q_T}$  with sufficient regularity. Thus, we introduce, by means of construction,  $(u_{\varepsilon,h}^D, u_{\varepsilon,h}^0)$  which both satisfy compatibility conditions and have an extension over  $\overline{Q_T}$  with sufficient regularity. Moreover,  $(u_{\varepsilon,h}^D, u_{\varepsilon,h}^0)$  are uniformly "close" to  $(u_\varepsilon^D, u_\varepsilon^0)$  (as  $h \rightarrow 0$ , uniformly w.r.t.  $\varepsilon$ ), which implies that  $u_{\varepsilon,h}$  is "close" to  $u_\varepsilon$  (in a sense which will be precised further). Then, we apply Arzelà-Ascoli theorem on the sequence  $\{u_\varepsilon\}$  in order to prove that it is relatively compact in  $C^0([0, T]; L^1(\Omega))$ . Of course, we have to verify that the sequence satisfies the hypotheses of the theorem (equicontinuity and pointwise relative compactness): for this, we use the properties of  $u_{\varepsilon,h}$  and the fact that  $u_\varepsilon$  is "close" to  $u_{\varepsilon,h}$ .

In order to use Lemma 12, we need some extension of  $u_\varepsilon^D$  and  $u_\varepsilon^0$  to  $\overline{Q}_T$ , with sufficient regularity. Let us define the function  $u_\varepsilon^{D,0}$  by

$$\begin{aligned} u_\varepsilon^{D,0}(t, r + s n(r)) &= u_\varepsilon^D(t, r), & t \in (0, T), \quad r \in \partial\Omega, \quad |s| \leq \min(t, \delta) \\ u_\varepsilon^{D,0}(t, x) &= u_\varepsilon^0(x), & -\delta < t < \min(\text{dist}(x, \partial\Omega), \delta), \quad x \in \Omega \\ u_\varepsilon^{D,0}(t, x) &= 0, & \text{elsewhere.} \end{aligned}$$

Moreover, we mollify the above function (with a usual mollifier) which provides regularity on  $\overline{Q}_T$ :

$$u_{\varepsilon,h}^{D,0}(t, x) = \int_{\mathbb{R}^{d+1}} u_\varepsilon^{D,0}(t', x') \phi_h(t - t', x - x') dt' dx'$$

Now we denote by  $u_{\varepsilon,h}^D$  (resp.  $u_{\varepsilon,h}^0$ ) the restriction of  $u_{\varepsilon,h}^{D,0}$  to  $\Sigma_T$  (resp.  $\{0\} \times \Omega$ ). Let  $u_{\varepsilon,h}$  be the solution of Equations (10)-(11)-(12) corresponding to the boundary and initial conditions  $u_{\varepsilon,h}^D$  and  $u_{\varepsilon,h}^0$ . On one hand, the uniform boundedness of  $u_\varepsilon^D, u_\varepsilon^0$  implies the uniform boundedness of  $u_{\varepsilon,h}^D, u_{\varepsilon,h}^0$  which provides (see Inequality (15) of Lemma 10) the uniform boundedness of  $u_\varepsilon, u_{\varepsilon,h}$ . Obviously, the following (strong) convergences hold:

$$\lim_{h \rightarrow 0} u_{\varepsilon,h}^D = u_\varepsilon^D \quad \text{in } L^1(\Sigma_T), \quad \lim_{h \rightarrow 0} u_{\varepsilon,h}^0 = u_\varepsilon^0 \quad \text{in } L^1(\Omega)$$

uniformly w.r.t.  $\varepsilon$ . This and Inequality (16) (see Lemma 11) imply

$$\lim_{h \rightarrow 0} u_{\varepsilon,h} = u_\varepsilon \quad \text{in } C^0([0, T], L^1(\Omega)),$$

uniformly w.r.t.  $\varepsilon$ . On the other hand, it follows from the boundedness of  $u_\varepsilon^D \in L^1(\Sigma_T)$  and  $u_\varepsilon^0 \in L^1(\Omega)$  that

$$\|u_{\varepsilon,h}^D\|_{\Sigma_T} \leq \frac{c}{h^3}, \quad \|u_{\varepsilon,h}^0\|_{\Omega} \leq \frac{c}{h^2}.$$

For fixed  $h > 0$ , it follows from Inequality (17) that the sequences

$$\left\{ \frac{\partial u_{\varepsilon,h}}{\partial t} \right\}, \quad \{ \nabla u_{\varepsilon,h} \}$$

are bounded in  $C^0([0, T], L^1(\Omega))$ . Now we propose to state that  $\{u_\varepsilon\}_\varepsilon$  is precompact in  $C^0([0, T], L^1(\Omega))$  with the Arzelà-Ascoli theorem:

(i) **Equicontinuity of  $\{u_\varepsilon\}_\varepsilon$ :** Let  $\alpha > 0$ . There exists some  $h > 0$  such that

$$2 \int_{\Omega} |u_{\varepsilon,h}(t, \cdot) - u_\varepsilon(t, \cdot)| dx < \alpha/2, \quad \forall t \in [0, T], \quad \forall \varepsilon > 0$$

and, from the uniform boundedness of  $\partial u_{\varepsilon,h}/\partial t$ , there exists  $\delta > 0$  such that

$$\delta \int_{\Omega} \left| \frac{\partial u_{\varepsilon,h}}{\partial t}(t, \cdot) \right| dx < \alpha/2, \quad \forall t \in [0, T], \quad \forall \varepsilon > 0 \quad (29)$$

Thus, for all  $\varepsilon > 0$  and all  $t_1, t_2 \in [0, T]$  such that  $|t_1 - t_2| \leq \delta$ , we have

$$\begin{aligned} \int_{\Omega} |u_{\varepsilon}(t_1, \cdot) - u_{\varepsilon}(t_2, \cdot)| &\leq \sum_{i=1}^2 \int_{\Omega} |u_{\varepsilon,h}(t_i, \cdot) - u_{\varepsilon}(t_i, \cdot)| + \int_{\Omega} |u_{\varepsilon,h}(t_1, \cdot) - u_{\varepsilon,h}(t_2, \cdot)| \\ &\leq \sum_{i=1}^2 \int_{\Omega} |u_{\varepsilon,h}(t_i, \cdot) - u_{\varepsilon}(t_i, \cdot)| + |t_1 - t_2| \sup_{t \in [t_1, t_2]} \int_{\Omega} \left| \frac{\partial u_{\varepsilon,h}}{\partial t}(t, \cdot) \right| \leq \alpha \end{aligned}$$

Thus, the sequence  $u_{\varepsilon}$  is equicontinuous in  $C^0([0, T], L^1(\Omega))$ .

(ii) **Pointwise relative compactness of  $\{u_{\varepsilon}\}_{\varepsilon}$ :**

For this, we use the Kolmogorov-Fréchet-Weil theorem:

- ▷ Since  $\{u_{\varepsilon}\}$  is uniformly bounded in  $L^{\infty}(Q_T)$ ,  $\{u_{\varepsilon}(t, \cdot)\}$  is also bounded in  $L^1(\Omega)$  (uniformly w.r.t.  $t \in [0, T]$  and  $\varepsilon$ ).
- ▷ Let  $\eta > 0$ . Let us consider  $K_{\eta} \subset \Omega$ , defined by  $K_{\eta} = \{x \in \Omega, \text{dist}(x, \partial\Omega) \geq \eta\}$ . Obviously,  $K_{\eta}$  is compact and

$$\sup_{u_{\varepsilon}(t, \cdot)} \int_{\Omega \setminus K_{\eta}} |u_{\varepsilon}(t, \cdot)| \leq \max(|a|, |b|) \text{meas}(\Omega \setminus K_{\eta}) = C(a, b, \partial\Omega) \eta.$$

- ▷ Recalling the existence of  $\delta > 0$  such that Inequality (29) holds, we get uniformly in  $t \in [0, T]$  and  $\varepsilon > 0$ ,

$$\int_{\Omega^{\Delta x}} |u_{\varepsilon}(t, \cdot + \Delta x) - u_{\varepsilon}(t, \cdot)| \leq 2 \int_{\Omega} |u_{\varepsilon,h}(t, \cdot) - u_{\varepsilon}(t, \cdot)| + |\Delta x| \int_{\Omega} |\nabla u_{\varepsilon,h}(t, \cdot)|$$

which is smaller than  $\alpha$  for  $|\Delta x| \leq \delta$  and  $\Omega^{\Delta x} = \{x \in \Omega, x + \Delta x \in \Omega\}$ .

Thus, the sequence  $\{u_{\varepsilon}(t, \cdot)\}_{t \in [0, T], \varepsilon > 0}$  is relatively compact in  $L^1(\Omega)$ .

Thus, by the Arzelà-Ascoli theorem,  $\{u_{\varepsilon}\}_{\varepsilon}$  is precompact in  $C^0([0, T], L^1(\Omega))$ , and since  $C^0([0, T], L^1(\Omega))$  is complete, we infer that, up to a subsequence,

$$\lim_{\varepsilon \rightarrow 0} u_{\varepsilon} = u \quad \text{in } C^0([0, T], L^1(\Omega))$$

Moreover,  $u \in L^{\infty}(Q_T; [a, b])$  (see Inequality (15)). Finally,  $u$  is a weak entropy solution of (1)-(2)-(3): recalling that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |1 - \xi_{\varepsilon}| = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} |\nabla \xi_{\varepsilon}| = 0,$$

passing to the limit w.r.t.  $\varepsilon$  in Inequality (14) concludes the proof.

### 3 Uniqueness

**Definition 14** For any  $k \in \mathbb{R}$ , let us denote:

$$\begin{aligned} (\widetilde{H}^k, \widetilde{Q}_{[f]}^k) : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (z, w) &\longrightarrow \left( \text{dist}(z, \mathcal{I}(w, k)), \mathcal{F}_{[f]}(\cdot, \cdot, z, w, k) \right) \end{aligned}$$

with  $\mathcal{I}(w, k) = [\min(w, k), \max(w, k)]$  and  $\mathcal{F}_{[f]} \in \mathcal{C}(\mathbb{R} \times \Omega \times \mathbb{R}^3)$  defined as:

$$\mathcal{F}_{[f]}(\cdot, \cdot, z, w, k) = \begin{cases} f(\cdot, \cdot, w) - f(\cdot, \cdot, z) & \text{for } z \leq w \leq k \\ 0 & \text{for } k \leq z \leq w \\ f(\cdot, \cdot, z) - f(\cdot, \cdot, k) & \text{for } w \leq k \leq z \\ f(\cdot, \cdot, k) - f(\cdot, \cdot, z) & \text{for } z \leq k \leq w \\ 0 & \text{for } w \leq z \leq k \\ f(\cdot, \cdot, z) - f(\cdot, \cdot, w) & \text{for } k \leq w \leq z \end{cases}$$

**Lemma 15** Let  $u \in L^\infty(Q_T)$  satisfy  $(\mathcal{P}_{SK})$ ; then one has:

■ for all  $\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d)$ , for all  $k \in \mathbb{R}$ ,

$$\begin{aligned} \int_{Q_T} \left\{ |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right. \\ \left. - \text{sgn}(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi \right\} dx dt \\ \geq \text{ess} \lim_{s \rightarrow 0^-} \int_{\Sigma_T} \left\{ \text{sgn}(u(t, r + sn) - k) (f(t, x, u(t, r + sn)) \right. \\ \left. - f(t, x, k)) \right\} \cdot n \varphi(t, r) d\gamma(r) dt \end{aligned}$$

■ for all  $\beta \in L^1(\Sigma_T)$ ,  $\beta \geq 0$  a.e., and for all  $k \in \mathbb{R}$ ,

$$\text{ess} \lim_{s \rightarrow 0^-} \int_{\Sigma_T} \mathcal{F}_{[f]}(t, r, u(t, r + sn), u^D, k) \cdot n \beta(t, r) d\gamma(r) dt \geq 0$$

■ for all  $\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d)$ , for all  $k \in \mathbb{R}$ ,

$$\int_{Q_T} \left\{ |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right.$$

$$\begin{aligned}
& -\operatorname{sgn}(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi \Big\} dx dt \\
& \geq \int_{\Sigma_T} \operatorname{sgn}(k - u^D) (f(t, r, k) - f(t, r, u^D)) \cdot n \varphi(t, r) d\gamma(r) dt \\
& -\operatorname{ess} \lim_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r + sn) - k) (f(t, r, u(t, r + sn)) - f(t, r, k)) \cdot n \varphi(t, r) d\gamma(r) dt
\end{aligned} \tag{30}$$

**PROOF.**

▷ 1<sup>st</sup> *inequality*: - Adding the two inequalities defined by  $(\mathcal{P}_{SK})$  with each "semi Kruřkov entropy-flux pair" gives the following inequality

$$\begin{aligned}
& \int_{Q_T} \left\{ |u - k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u - k) (f(t, x, u) - f(t, x, k)) \nabla \varphi \right. \\
& \left. -\operatorname{sgn}(u - k) (\nabla \cdot f(t, x, k) + g(t, x, u)) \varphi \right\} dx dt \geq \textcircled{31}
\end{aligned}$$

for any  $\varphi \in \mathcal{D}(Q_T)$ . Thus, since  $u$  satisfies Inequality (31) along with the initial condition (6) (see Lemma 6), the result is obtained by following the same lines of the proof of Lemma 7.12. in [4].

▷ 2<sup>nd</sup> *inequality*: - The result is easily obtained by Lemma 4 applied to the particular "boundary fluxes"  $\mathcal{F}_{[f]}$  (see Definition 14).

▷ 3<sup>rd</sup> *inequality*: - On the one hand, the function  $2\mathcal{F}_{[f]}(\cdot, \cdot, z, w, k)$  is equal to

$$\begin{aligned}
& \operatorname{sgn}(z - w) (f(\cdot, \cdot, z) - f(\cdot, \cdot, w)) - \operatorname{sgn}(k - w) (f(\cdot, \cdot, k) - f(\cdot, \cdot, w)) \\
& + \operatorname{sgn}(z - k) (f(\cdot, \cdot, z) - f(\cdot, \cdot, k)).
\end{aligned}$$

On the second hand, terms of the form

$$\operatorname{ess} \lim_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - v^D(r)) (f(t, r, u(t, r - sn)) - f(t, r, v^D(r))) \cdot n \beta(r) d\gamma(r) dt$$

exists for all  $\beta \in L^1(\Sigma_T)$ , all  $v^D \in L^\infty(\Sigma_T)$ . Indeed, this term is obtained by using the proof of Lemma 4: it is sufficient to add the terms of (5) corresponding to each "semi Kruřkov entropy-flux pair". Therefore, from the second inequality, each term of the following inequality exists:

$$\begin{aligned}
& -\operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - k) \left( f(t, r, u(t, r - sn)) - f(t, r, k) \right) \cdot n \beta(t, r) \, d\gamma(r) \, dt \\
& \leq \operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - u^D(r)) \left( f(t, r, u(t, r - sn)) - f(t, r, u^D(r)) \right) \cdot n \beta(t, r) \, d\gamma(r) \, dt \\
& - \int_{\Sigma_T} \operatorname{sgn}(k - u^D(r)) \left( f(t, r, k) - f(t, r, u^D(r)) \right) \cdot n \beta(t, r) \, d\gamma(r) \, dt
\end{aligned}$$

and the result is straightforward.

**Lemma 16** *Let  $u \in L^\infty(Q_T)$  (resp.  $v \in L^\infty(Q_T)$ ) be a solution of  $(\mathcal{P}_{SK})$  with initial and boundary conditions  $(u^0, u^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$  (resp.  $(v^0, v^D) \in L^\infty(\Omega) \times L^\infty(\Sigma_T)$ ); then*

$$\begin{aligned}
& - \int_{Q_T} \left\{ |u - v| \frac{\partial \beta}{\partial t} + \operatorname{sgn}(u - v) \left( f(t, x, u) - f(t, x, v) \right) \nabla \beta \right. \\
& \quad - \operatorname{sgn}(u - v) \left( \nabla \cdot f(t, x, v) - \nabla \cdot f(t, x, u) \right) \beta \\
& \quad \left. - \operatorname{sgn}(u - v) \left( g(t, x, u) - g(t, x, v) \right) \beta \right\} dx \, dt \\
& \leq \int_{\Omega} |u^0(x) - v^0(x)| \beta(0, x) \, dx + \mathcal{L}_{[f]} \int_{\Sigma_T} |u^D(r) - v^D(r)| \beta(r) \, d\gamma(r) \, dt
\end{aligned}$$

for all  $\beta \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$ .

**PROOF.** As it was already pointed out, each term that can be written under the form

$$\operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - v^D(r)) \left( f(t, r, u(t, r - sn)) - f(t, r, v^D(r)) \right) \cdot n \beta(r) \, d\gamma(r) \, dt$$

exists for all  $\beta \in L^1(\Sigma_T)$ , all  $v^D \in L^\infty(\Sigma_T)$ . Thus, we infer that there exists  $\theta_{i,j} \in L^\infty(\Sigma_T)$  such that:

$$\begin{aligned}
& \int_{\Sigma_T} \theta_{1,1}(t, r) \beta(t, r) \, d\gamma(r) \, dt \\
& = \operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - u^D(r)) \left( f(t, r, u(t, r - sn)) - f(t, r, u^D(r)) \right) \cdot n \beta \, d\gamma(r) \, dt
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma_T} \theta_{1,2}(t, r) \beta(t, r) \, d\gamma(r) \, dt \\
& = \operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(u(t, r - sn) - v^D(r)) \left( f(t, r, u(t, r - sn)) - f(t, r, v^D(r)) \right) \cdot n \beta \, d\gamma(r) \, dt
\end{aligned}$$



$$\begin{aligned}
& \int_{\Sigma_T} \theta_{2,2}(t, r) \beta(t, r) d\gamma(r) dt \\
&= \operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(v(t, r - sn) - v^D) \left( f(t, r, v(t, r - sn)) - f(t, r, v^D) \right) \cdot n \beta d\gamma(r) dt
\end{aligned}$$

$$\begin{aligned}
& \int_{\Sigma_T} \theta_{2,1}(t, r) \beta(t, r) d\gamma(r) dt \\
&= \operatorname{ess\,lim}_{s \rightarrow 0^-} \int_{\Sigma_T} \operatorname{sgn}(v(t, r - sn) - u^D) \left( f(t, r, v(t, r - sn)) - f(t, r, u^D) \right) \cdot n \beta d\gamma(r) dt
\end{aligned}$$

After this introduction of notations, we now apply the double variable method, initiated by Kruřkov [1], to Inequality (30). Let  $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^{d+1})$  be a symmetric regularizing sequence. For the sake of simplicity, we denote

$$\begin{aligned}
p &= (t, x) \in Q_T, & p' &= (t', x') \in Q_T \\
\gamma(p) &= (t, r) \in \Sigma_T, & \gamma(p') &= (t', r') \in \Sigma_T
\end{aligned}$$

and let

$$\beta_\varepsilon(p, p') = \beta \left( \frac{p + p'}{2} \right) \rho_\varepsilon(p - p'),$$

for all  $p, p' \in (Q_T)^2$ , for a given  $\beta \in \mathcal{D}((0, T) \times \mathbb{R}^d)$ ,  $\beta \geq 0$ . Hold  $p' \in Q_T$  fixed and replace in Inequality (30)  $k$  by  $v(p')$  and  $\beta(p)$  by  $\beta_\varepsilon(p, p')$ . After integration over  $Q_T$  (w.r.t. the variable  $p'$ ), and using the notation

$$F(p, u(p), v(p')) = \operatorname{sgn}(u(p) - v(p')) (f(p, u(p)) - f(p, v(p'))),$$

we easily get:

$$I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon + I_5^\varepsilon \leq I_6^\varepsilon + I_7^\varepsilon$$

with

$$\begin{aligned}
I_1^\varepsilon &= -\frac{1}{2} \int_{Q_T} \int_{Q_T} |u(p) - v(p')| \frac{\partial \beta}{\partial t} \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
I_2^\varepsilon &= -\frac{1}{2} \int_{Q_T} \int_{Q_T} F(p, u(p), v(p')) \nabla \beta \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
I_3^\varepsilon &= -\int_{Q_T} \int_{Q_T} |u(p) - v(p')| \frac{\partial \rho_\varepsilon}{\partial t} (p-p') \beta \left( \frac{p+p'}{2} \right) dp dp' \\
I_4^\varepsilon &= -\int_{Q_T} \int_{Q_T} F(p, u(p), v(p')) \nabla \rho_\varepsilon(p-p') \beta \left( \frac{p+p'}{2} \right) dp dp' \\
I_5^\varepsilon &= \int_{Q_T} \int_{Q_T} \left\{ \nabla \cdot f(p, v(p')) + g(p, u(p)) \right\} \beta \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
I_6^\varepsilon &= \int_{Q_T} \int_{\Sigma_T} \theta_{1,1}(\gamma(p)) \beta \left( \frac{\gamma(p)+p'}{2} \right) \rho_\varepsilon(\gamma(p)-p') d\gamma(p) dp' \\
I_7^\varepsilon &= -\int_{Q_T} \int_{\Sigma_T} F(t, r, v(p'), u^D(\gamma(p))) \cdot n \beta \left( \frac{\gamma(p)+p'}{2} \right) \rho_\varepsilon(\gamma(p)-p') d\gamma(p) dp'
\end{aligned}$$

Now changing the role of  $(u(p), p)$  and  $(v(p'), p')$ , we get similarly

$$J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon + J_5^\varepsilon \leq J_6^\varepsilon + J_7^\varepsilon$$

with

$$\begin{aligned}
J_1^\varepsilon &= -\frac{1}{2} \int_{Q_T} \int_{Q_T} |v(p') - u(p)| \frac{\partial \beta}{\partial t} \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
J_2^\varepsilon &= -\frac{1}{2} \int_{Q_T} \int_{Q_T} F(p', v(p'), u(p)) \nabla \beta \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
J_3^\varepsilon &= \int_{Q_T} \int_{Q_T} |v(p') - u(p)| \frac{\partial \rho_\varepsilon}{\partial t} (p-p') \beta \left( \frac{p+p'}{2} \right) dp dp' \\
J_4^\varepsilon &= \int_{Q_T} \int_{Q_T} F(p', v(p'), u(p)) \nabla \rho_\varepsilon(p-p') \beta \left( \frac{p+p'}{2} \right) dp dp' \\
J_5^\varepsilon &= \int_{Q_T} \int_{Q_T} \left\{ \nabla \cdot f(p', u(p)) + g(p', v(p')) \right\} \beta \left( \frac{p+p'}{2} \right) \rho_\varepsilon(p-p') dp dp' \\
J_6^\varepsilon &= \int_{Q_T} \int_{\Sigma_T} \theta_{2,2}(\gamma(p')) \beta \left( \frac{p+\gamma(p')}{2} \right) \rho_\varepsilon(p-\gamma(p')) d\gamma(p') dp \\
J_7^\varepsilon &= -\int_{Q_T} \int_{\Sigma_T} F(t', r', u(p), v^D(\gamma(p'))) \cdot n \beta \left( \frac{p+\gamma(p')}{2} \right) \rho_\varepsilon(p-\gamma(p')) d\gamma(p') dp
\end{aligned}$$

Adding the two inequalities, let us remark that  $I_1^\varepsilon = J_1^\varepsilon$  and  $I_3^\varepsilon = -J_3^\varepsilon$  so that we have:

$$2I_1^\varepsilon + (I_2^\varepsilon + J_2^\varepsilon) + (I_4^\varepsilon + J_4^\varepsilon) + (I_5^\varepsilon + J_5^\varepsilon) \leq (I_6^\varepsilon + J_6^\varepsilon) + (I_7^\varepsilon + J_7^\varepsilon).$$

We are now ready to pass to the limit on  $\varepsilon$ : for convenience, proofs are omitted. Let us mention that this method has been widely used in the works related to hyperbolic problems [1–4] but also parabolic problems [6] or elliptic-hyperbolic problems (in free boundary problems applied to lubrication theory, [9]). Thus,

$$\begin{aligned} 2 \lim_{\varepsilon \rightarrow 0} I_1^\varepsilon &= - \int_{Q_T} |u - v| \frac{\partial \beta}{\partial t}, \quad \lim_{\varepsilon \rightarrow 0} (I_4^\varepsilon + J_4^\varepsilon) = 0, \\ \lim_{\varepsilon \rightarrow 0} I_2^\varepsilon &= \lim_{\varepsilon \rightarrow 0} J_2^\varepsilon = - \frac{1}{2} \int_{Q_T} \operatorname{sgn}(u - v) (f(t, x, u) - f(t, x, v)) \nabla \beta, \\ \lim_{\varepsilon \rightarrow 0} I_5^\varepsilon &= \int_{Q_T} \left\{ \nabla \cdot f(t, x, v) + g(t, x, u) \right\} \beta, \quad \lim_{\varepsilon \rightarrow 0} J_5^\varepsilon = - \int_{Q_T} \left\{ \nabla \cdot f(t, x, u) + g(t, x, v) \right\} \beta, \\ \lim_{\varepsilon \rightarrow 0} I_6^\varepsilon &= \frac{1}{2} \int_{\Sigma_T} \theta_{1,1}(t, r) \beta(t, r) d\gamma(r) dt, \quad \lim_{\varepsilon \rightarrow 0} J_6^\varepsilon = \frac{1}{2} \int_{\Sigma_T} \theta_{2,2}(t, r) \beta(t, r) d\gamma(r) dt, \\ \lim_{\varepsilon \rightarrow 0} I_7^\varepsilon &= - \frac{1}{2} \int_{\Sigma_T} \theta_{1,2}(t, r) \beta(t, r) d\gamma(r) dt, \quad \lim_{\varepsilon \rightarrow 0} J_7^\varepsilon = - \frac{1}{2} \int_{\Sigma_T} \theta_{2,1}(t, r) \beta(t, r) d\gamma(r) dt. \end{aligned}$$

Finally we obtain:

$$\begin{aligned} & - \int_{Q_T} \left\{ |u - v| \frac{\partial \beta}{\partial t} + \operatorname{sgn}(u - v) (f(t, x, u) - f(t, x, v)) \nabla \beta \right. \\ & \quad \left. - \operatorname{sgn}(u - v) (\nabla \cdot f(t, x, v) - \nabla \cdot f(t, x, u)) \beta \right. \\ & \quad \left. - \operatorname{sgn}(u - v) (g(t, x, u) - g(t, x, v)) \beta \right\} dx dt \\ & \leq \frac{1}{2} \int_{\Sigma_T} (-\theta_{1,1}(r) + \theta_{2,1}(r) - \theta_{2,2}(r) + \theta_{1,2}(r)) \beta(r) d\gamma(r) dt \end{aligned}$$

for all  $\beta \in \mathcal{D}((0, T) \times \mathbb{R}^d)$ , the terms containing  $\theta_{i,j}(t, r)$  being defined by:

As in [4], let us introduce the following definition:

$$\forall (t, r) \in \Sigma_T, \operatorname{diam}(f(t, r, \cdot) \cdot n, \mathcal{I}(a, b)) = \sup_{z_1, z_2 \in \mathcal{I}(a, b)} \left( |f(t, r, z_1) \cdot n - f(t, r, z_2) \cdot n| \right)$$

Then, if one discusses the cases, one sees that for all  $z_1, z_2, w_1, w_2 \in \mathbb{R}$ , the inequalities

$$\left| \sum_{i,j=1}^2 (-1)^{i+j} \operatorname{sgn}(z_i - w_j) (f(t, r, z_i) - f(t, r, w_j)) \cdot n \right| \leq 2 \operatorname{diam}(f(t, r, \cdot) \cdot n, \mathcal{I}(w_1, w_2))$$

hold and using the property

$$\text{diam}(f(t, r, \cdot) \cdot n, \mathcal{I}(u^D(t, r), v^D(t, r))) \leq \mathcal{L}_{[f]} |u^D(r) - v^D(r)|, \quad \forall (t, r) \in \Sigma_T$$

one easily concludes that

$$\begin{aligned} \frac{1}{2} \left| \int_{\Sigma_T} (-\theta_{1,1}(t, r) + \theta_{2,1}(t, r) - \theta_{2,2}(t, r) + \theta_{1,2}(t, r)) \beta(t, r) d\gamma(r) dt \right| \\ \leq \mathcal{L}_{[f]} \int_{\Sigma_T} |u^D(r) - v^D(r)| \beta(t, r) d\gamma(r) dt \end{aligned}$$

The initial term is obtained by slightly modifying the proof, with test-functions in the appropriate space, namely  $\mathcal{D}((-\infty, T) \times \mathbb{R}^d)$ .

**Theorem 17 (Uniqueness)** *Under Assumption 1, problem  $(\mathcal{P}_{SK})$  admits a unique weak entropy solution.*

**PROOF.** Considering the integral inequality of Lemma 16 with  $v^D = u^D$  and  $v^0 = u^0$  and a test-function which only depends on time  $t$ , we get:

$$\begin{aligned} \int_{Q_T} \left\{ |u - v| \alpha'(t) - \text{sgn}(u - v) (\nabla \cdot f(t, x, v) - \nabla \cdot f(t, x, u)) \alpha(t) \right. \\ \left. + (g(t, x, u) - g(t, x, v)) \alpha(t) \right\} dx dt \geq 0 \quad (32) \end{aligned}$$

for all  $\alpha \in \mathcal{D}(-\infty, T)$ . Then, for an interval  $[t_0, t_1] \subset ]0, T[$ , we can use in Inequality (32) the characteristic function of  $[t_0, t_1]$ , properly mollified, and pass to the limit on the mollifier parameter:

$$\begin{aligned} \int_{\Omega} |u(t_1, \cdot) - v(t_1, \cdot)| &\leq \int_{\Omega} |u(t_0, \cdot) - v(t_0, \cdot)| \\ &+ \int_{t_0}^{t_1} \int_{\Omega} \text{sgn}(u - v) (\nabla \cdot f(t, x, u) - \nabla \cdot f(t, x, v)) dx dt \\ &+ \int_{t_0}^{t_1} \int_{\Omega} \text{sgn}(u - v) (g(t, x, v) - g(t, x, u)) dx dt \end{aligned}$$

Now, since we have, for all  $(t, x) \in (0, T) \times \Omega$ :

$$\text{sgn}(u - v) (\nabla \cdot f(t, x, u) - \nabla \cdot f(t, x, v) + g(t, x, v) - g(t, x, u)) \leq \mathcal{L}_{[g + \nabla \cdot f]} |u - v|$$

where  $\mathcal{L}_{[g+\nabla\cdot f]}$  is the Lipschitz constant of continuity w.r.t.  $u$  of  $\nabla\cdot f + g$ , we obtain:

$$\int_{\Omega} |u(t_1, \cdot) - v(t_1, \cdot)| \leq \int_{\Omega} |u(t_0, \cdot) - v(t_0, \cdot)| + \mathcal{L}_{[g+\nabla\cdot f]} \int_{t_0}^{t_1} \int_{\Omega} |u(t, \cdot) - v(t, \cdot)|_{L^1(\Omega)} dt$$

From Gronwall's lemma, we conclude that:

$$|u(t_1, \cdot) - v(t_1, \cdot)|_{L^1(\Omega)} \leq |u(t_0, \cdot) - v(t_0, \cdot)|_{L^1(\Omega)} e^{\mathcal{L}_{[g+\nabla\cdot f]}(t_1-t_0)}$$

As  $t_0$  tends to 0, and using the fact that  $v^0 = u^0$  along with the initial condition (6), the uniqueness is straightforward.

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## Appendix A: Boundary conditions

The boundary condition 4 is nothing less than the one obtained in [3,4], up to a generalization to non-autonomous fluxes and taking account of a source-term which does not interfere in the boundary condition. But, with additional assumptions, the more comprehensive "BLN" condition is recovered:

(i) If  $u$  admits a trace, meaning that there exists  $u|_{\Sigma_T} \in L^\infty(\Sigma_T)$  such that

$$\operatorname{ess\,lim}_{s \rightarrow 0^+} \int_{\Sigma_T} |u(\tau, r - s n(r)) - u|_{\Sigma_T}(\tau, r)| \, d\gamma(r) \, d\tau = 0$$

then Equation (4) is equivalent to the following equation (see [10,3])

$$Q_{[f]}(\cdot, \cdot, u|_{\Sigma_T}, u^D) \cdot n \geq 0, \quad \text{a.e. on } \Sigma_T.$$

Considering the particular boundary fluxes

$$H_\delta^+(z, \kappa) = \left( (\max(z - \kappa, 0))^2 + \delta^2 \right)^{1/2} - \delta, \quad (33)$$

$$Q_{[f],\delta}^+(\cdot, \cdot, z, \kappa) = \int_\kappa^z \partial_1 H_\delta^+(\lambda, k) \frac{\partial f}{\partial u}(\cdot, \cdot, \lambda) \, d\lambda \quad (34)$$

and

$$H_\delta^-(z, \kappa) = \left( (\min(\kappa - z, 0))^2 + \delta^2 \right)^{1/2} - \delta, \quad (35)$$

$$Q_{[f],\delta}^-(\cdot, \cdot, z, \kappa) = \int_\kappa^z \partial_1 H_\delta^-(\lambda, k) \frac{\partial f}{\partial u}(\cdot, \cdot, \lambda) \, d\lambda, \quad (36)$$

letting  $\delta \rightarrow 0$ , we obtain the following uniform convergences:

$$Q_{[f],\delta}^\pm(\cdot, \cdot, z, \kappa) \rightarrow \operatorname{sgn}_\pm(z - \kappa)(f(\cdot, \cdot, z) - f(\cdot, \cdot, \kappa))$$

Finally taking the boundary flux

$$\begin{aligned} Q_{[f]}(\cdot, \cdot, s, w) = & \operatorname{sgn}_+(s - \max(w, k))(f(\cdot, \cdot, s) - f(\cdot, \cdot, \max(w, k))) \\ & + \operatorname{sgn}_-(s - \min(w, k))(f(\cdot, \cdot, s) - f(\cdot, \cdot, \min(w, k))) \end{aligned}$$

yields the classical condition given by Bardos, Le Roux and Nédélec [2], that is:

$$\begin{aligned} \text{for a.e. } (t, r) \in \Sigma_T, \forall k \in [\min(u|_{\Sigma_T}, u^D), \max(u|_{\Sigma_T}, u^D)], \\ \operatorname{sgn}(u|_{\Sigma_T}(t, r) - u^D(t, r))(f(t, r, u|_{\Sigma_T}(t, r)) - f(t, r, k)) \cdot n(r) \geq \mathbf{07} \end{aligned}$$

(ii) Assume furthermore that for almost every  $(t, r) \in \Sigma_T$ ,  $s \mapsto f(t, r, s) \cdot n(r)$  is a monotone function. Then, Inequality (37) can be simplified in specific cases: indeed, one has

$$u = u^D, \quad \text{on } \Sigma_T^D = \left\{ (t, r) \in \Sigma_T, \frac{\partial f}{\partial u}(t, r, u) \cdot n(r) < 0 \right\}$$

and nothing is imposed at  $\Sigma_T^N = \left\{ (t, r) \in \Sigma_T, \frac{\partial f}{\partial u}(t, r, u) \cdot n(r) \geq 0 \right\}$ .

Thus the boundary condition is "active" only on a part of the boundary.



## Appendix B: Applications to some physical problems

Due to the presence of a non-autonomous flux  $f$  or source term  $g$ , interest in first order quasilinear equations on bounded domains may be related to many problems applied to different mathematical and physical areas: oil engineering (two-phase flows in porous media [11–14]), lubrication theory (two-phase flow in a thin domain [15], modelling of cavitation phenomena [16]), environmental sciences (pollutant transport in manure-like fields [17]), chemical engineering and wastewater treatment (sedimentation of flocculated suspensions in clarifier-thickener units [18]). It is of course an important feature to analyse the related phenomena in order to guarantee the well-posedness of the problem and also qualitative properties of the (possible) solution. Let us introduce in details some physical problems which fall into the scope of our study:

- **Lubrication theory** - The flow of two miscible fluids in a thin film has been derived in [15] and further detailed in [16]. Thus, it allows the modelling of a multifluid flow in a one dimensional lubricated device as a slider or journal bearing, for instance. The behaviour of the saturation  $s$  of the reference fluid obeys the following law:

$$h(x) \frac{\partial s}{\partial t} + \frac{\partial}{\partial x} (Q_{in}(t) f_1(s) + v_0 h(x) f_2(s)) = 0. \quad (38)$$

Here,  $h$  is the normalized gap between the two surfaces in relative motion,  $v_0$  is the shear velocity of the lower surface (the upper one being fixed, for instance) and  $Q_{in}(t)$  is the flow input. Moreover, the auxiliary flux functions  $f_1$  and  $f_2$  have a particular shape:  $f_1(0) = 0$ ,  $f_1(1) = 1$  (typically,  $f_1$  is  $S$ -shaped on  $[0, 1]$ ) and  $f_2(0) = f_2(1) = 0$ . Equation (38) is completed with initial and boundary conditions  $s^0$  and  $s^D$  satisfying, for obvious physical relevance,

$$\left[ \min \left( \inf_{\Omega} s^0, \inf_{\Sigma_T} s^D \right), \max \left( \sup_{\Omega} s^0, \sup_{\Sigma_T} s^D \right) \right] \subset [0, 1].$$

Using an appropriate change of variables  $(t, x) \mapsto (\mathcal{T}, Y)$ , it has been proved in [16] that the problem can be reduced to a scalar conservation law w.r.t. the function  $u$  defined by  $u(\mathcal{T}(t), Y(x)) = s(t, x)$ , namely

$$\frac{\partial u}{\partial \tau} + \frac{\partial}{\partial y} \left( f_1(s) + v_0 \frac{h \circ Y^{-1}(y)}{Q_{in}(\mathcal{T}^{-1}(\tau))} f_2(s) \right) = 0, \quad (39)$$

with corresponding initial and boundary conditions  $u^0$  and  $u^D$  defined by  $u^0(y) = s^0(Y^{-1}(y))$  and  $u^D(\tau, y) = s^D(\mathcal{T}^{-1}(\tau), Y^{-1}(y))$ . Equation (39) is obviously a scalar conservation law w.r.t. a saturation function  $u$ , unlike Equation (38). Notice that the non-autonomous property of the flux function is induced by the shear effects ( $v_0 \neq 0$ ) and spatial variations of  $h$

(typically,  $h$  has a convergent-divergent profile). Moreover, the boundary conditions may lack regularity (typically,  $s^D \in L^\infty(\Sigma_T)$ ), due to fast changes of the supply regime. For physical relevance, it is an important feature to state that Equation (38) (or, equivalently, Equation (39)), with the corresponding initial and boundary conditions, admits a (unique) solution, in a sense that has to be precised, and that it takes its values in the set  $[0, 1]$ .

- **Environmental sciences** - The modelling of pollutant transport taking into account a surface source during rainfall-runoff is described in [17]. It allows to model some step of the pollution process due to the runoff from manure spread fields, an important mode of non-point source pollution. Actually, pollutant release involves two processes, horizontal convection (which occurs in the bottom region of the source) and vertical convective diffusion and/or dispersion from the upper region. The pollutant transport mechanisms for the bottom region and upper regions of the source are modelled as independant processes<sup>3</sup>. The convective process is described by the following transport equation including a source term

$$\frac{\partial c}{\partial t} + \nabla \cdot (U c) + \frac{1}{h}(ic - J_b) = 0, \quad (40)$$

where  $c$  denotes the concentration of pollutant,  $U$  the convective velocity of the flow,  $h$  the depth of flow,  $i$  the effective rainfall intensity and  $J_b$  the rate of solute uptake from the source into the flow. Here, for the sake of simplicity, we suppose that  $h$  is constant and  $U$  satisfies  $\nabla \cdot U = 0$ . However, this assumption may be relaxed<sup>4</sup>. Moreover, Equation (40) is completed with some initial condition  $c^0 \equiv 0$  and an homogeneous boundary condition  $c^D \equiv 0$  on the boundary part for which  $U \cdot n < 0$ . The analysis of this set of equations allows to ensure the well-posedness of the physical problem and guarantees the boundedness of the concentration (see Remark ??).

**Remark 18** *Let us conclude this Appendix with the results obtained in the case of the physical problems that we have introduced.*

- **Lubrication theory** - *The generalized lubrication problem (Equation (38) with its corresponding initial and boundary conditions  $s^0, s^D$ ) admits a unique weak entropy solution  $s \in L^\infty(Q_T)$  in the following sense:*

<sup>3</sup> The justification for the uncoupling of these processes lies in the assumption that the time required to flush out contaminates from the lower region is much shorter than the time required to flush out contaminates from the upper region

<sup>4</sup> The more general case with spatially varying functions  $h$  and  $U$  (with  $q = hU$  and  $\nabla \cdot q = 0$ ) can be considered, leading to a conservation law w.r.t.  $hc$ . Then, as in the lubrication problem, the equation can be reduced to a conservation law w.r.t. a concentration function.

$$\begin{aligned}
& \int_{Q_T} \left\{ h(x) (s - k)^\pm \frac{\partial \varphi}{\partial t} \right. \\
& \quad + \operatorname{sgn}_\pm(s - k) \left( Q_{in}(t) (f_1(s) - f_1(k)) + v_0 h(x) (f_2(s) - f_2(k)) \right) \frac{\partial \varphi}{\partial x} \\
& \quad \left. - \operatorname{sgn}_\pm(s - k) v_0 h'(x) f_2(k) \varphi \right\} dx dt \\
& \quad + \int_{\Omega} (s^0 - k)^\pm \varphi(0, x) dx + \mathcal{L}_{[f]} \int_{\Sigma_T} (s^D - k)^\pm \varphi(t, r) d\gamma(r) dt \geq 0
\end{aligned}$$

$\forall \phi \in \mathcal{D}((-\infty, T) \times \mathbb{R})$ ,  $\phi \geq 0$ ,  $\forall k \in \mathbb{R}$

Here,  $\mathcal{L}_{[f]} = \max(Q_{in})\mathcal{L}_{[f_1]} + v_0 \max(h) \mathcal{L}_{[f_2]}$ . Moreover,  $s$  is a function with values in the set  $[0, 1]$ .

- **Environmental sciences** - The transport problem (Equation (40) with the corresponding homogeneous initial and boundary conditions  $c^0 \equiv 0$ ,  $c^D \equiv 0$ ) admits a unique weak entropy solution  $c \in L^\infty(Q_T)$  in the following sense:

$$\begin{aligned}
& \int_{Q_T} \left\{ (c - k)^\pm \frac{\partial \varphi}{\partial t} + (c - k)^\pm U(t, x) \nabla \varphi \right. \\
& \quad \left. - \operatorname{sgn}_\pm(c - k) \frac{1}{h} (ic - J_b) \varphi \right\} dx dt \\
& \quad + \int_{\Omega} (c^0 - k)^\pm \varphi(0, x) dx \\
& \quad + \mathcal{L}_{[f]} \int_{\Sigma_T} (c^D - k)^\pm \varphi(t, r) d\gamma(r) dt \geq 0
\end{aligned}$$

$\forall \phi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^2)$ ,  $\phi \geq 0$ ,  $\forall k \in \mathbb{R}$

with, for instance,  $\mathcal{L}_{[f]} = \|U\|_{L^\infty}$ . The analysis also provides a critical (worst) value for the concentration, which satisfies  $0 \leq c \leq J_b/i$ .