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# The universal cover of an algebra without double bypass

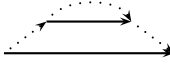
Patrick Le Meur

## Abstract

Let  $A$  be a basic and connected finite dimensional algebra over a field  $k$  of characteristic zero. We show that if the quiver of  $A$  has no double bypass then the fundamental group (as defined in [17]) of any presentation of  $A$  by quiver and relations is the quotient of the fundamental group of a privileged presentation of  $A$ . Then we show that the Galois covering of  $A$  associated with this privileged presentation satisfies a universal property with respect to the connected Galois coverings of  $A$  in a similar fashion to the universal cover of a topological space.

## Introduction

In this text,  $k$  will be an algebraically closed field. Let  $A$  be a finite dimensional algebra over  $k$ . In order to study left  $A$ -modules we may assume that  $A$  is basic and connected, where basic means that  $A$  is the direct sum of pairwise non isomorphic indecomposable projective left  $A$ -modules. For such an algebra, the study of the Galois coverings of  $A$  gives some information on the representation theory of  $A$  (see [7], [11] and [17]) and is a particular case of the covering techniques introduced in [6], [10] and [18]. Recall that in order to manipulate coverings of  $A$  we will always consider, unless otherwise stated,  $A$  as a  $k$ -category with set of objects a complete set  $\{e_i\}_i$  of primitive pairwise orthogonal idempotents and with morphisms space  $e_i \rightarrow e_j$  the vector space  $e_j A e_i$ . The covering techniques have led to the definition (see [11] and [17]) of a fundamental group associated with any presentation of  $A$  by quiver and admissible relations, and which satisfies many topological flavoured properties (see [3], [11] and [17]). This construction and its associated properties depend on the choice of a presentation of  $A$ . In particular, one can find algebras for which there exist different presentations giving rise to non isomorphic fundamental groups. In this text we compare the fundamental groups of the presentations of  $A$  as defined in [17], and we study the coverings of  $A$  with the following question in mind: does  $A$  have a universal Galois covering? i.e. does  $A$  admit a Galois covering which is factorised by any other Galois covering? This question has been successfully treated in the case  $A$  is representation finite (see [6] and [10]). The present study will involve quivers “without double bypass”. In simple terms, a quiver without double bypass is a quiver which has no distinct parallel arrows, no oriented

cycle and has no subquiver of the following form  where continued (resp. dotted) arrows represent arrows (resp. oriented paths) in the quiver. Assuming that  $k$  is a characteristic zero field and that the ordinary quiver  $Q$  of  $A$  has no double bypass, we prove the following result announced in [16]:

**Theorem 1.** *Assuming the above conditions, there exists a presentation  $kQ/I_0 \simeq A$  by quiver and relations such that for any other presentation  $kQ/I \simeq A$  the identity map on the walks in  $Q$  induces a surjective group morphism  $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, I)$ .*

The proof of the above theorem allows us to recover the following fact proven in [5]: if  $A$  is a basic triangular connected and constricted finite dimensional  $k$ -algebra, then different presentations of  $A$  give rise to isomorphic fundamental groups. Under the hypotheses made before stating Theorem 1 and with the same notations, if  $k\tilde{Q}/\tilde{I}_0 \xrightarrow{F_0} kQ/I_0$  is the Galois covering with group  $\pi_1(Q, I_0)$  induced by the universal Galois covering of  $(Q, I_0)$  (see [17]), we show the following result.

**Theorem 2.** *For any connected Galois covering  $F: \mathcal{C}' \rightarrow A$  with group  $G$  there exist an isomorphism  $kQ/I_0 \xrightarrow{\sim} A$ , a Galois covering  $p: k\tilde{Q}/\tilde{I}_0 \rightarrow \mathcal{C}'$  with group a normal subgroup  $N$  of  $\pi_1(Q, I_0)$  and a commutative diagram:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I}_0 & \xrightarrow{p} & \mathcal{C}' \\ F_0 \downarrow & & \downarrow F \\ kQ/I_0 & \xrightarrow{\sim} & A \end{array}$$

together with an exact sequence of groups:  $1 \rightarrow N \rightarrow \pi_1(Q, I_0) \rightarrow G \rightarrow 1$

Hence Theorem 2 partially answers the question concerning the existence of a universal Galois covering. The text is organised as follows: in Section 1 we define the notions we will use, in Section 2 we prove Theorem 1, in Section 3 we give useful results on covering functors, these results will be used in the proof of Theorem 2 to which Section 4 is devoted. Section 2 gives the proofs of all the results that were announced by the author in [16]. This text is part of the author's thesis made at Université Montpellier 2 under the supervision of Claude Cibils.

## 1 Basic definitions

### $k$ -categories, covering functors, Galois coverings

A  $k$ -category is a category  $\mathcal{C}$  such that the objects class  $\mathcal{C}_0$  of  $\mathcal{C}$  is a non empty set and such that each set  ${}_y\mathcal{C}_x$  of morphisms  $x \rightarrow y$  of  $\mathcal{C}$  is a  $k$ -vector space with  $k$ -bilinear composition. Let  $\mathcal{C}$  be a  $k$ -category. We will say that  $\mathcal{C}$  is **locally bounded** if the following properties are satisfied:

- a) distinct objects are not isomorphic,
- b) for each  $x \in \mathcal{C}_0$ , the  $k$ -algebra  ${}_x\mathcal{C}_x$  is local,
- c)  $\bigoplus_{y \in \mathcal{C}_0} {}_y\mathcal{C}_x$  is finite dimensional for any  $x \in \mathcal{C}_0$ ,
- d)  $\bigoplus_{x \in \mathcal{C}_0} {}_y\mathcal{C}_x$  is finite dimensional for any  $y \in \mathcal{C}_0$ .

Unless otherwise stated, all the  $k$ -categories we will introduce will be locally bounded. As an example, let  $A$  be a basic finite dimensional  $k$ -algebra. If  $1 = \sum_{i=1}^n e_i$  is a decomposition of the unit into a sum of pairwise orthogonal primitive idempotents, then  $A = \bigoplus_{i,j} e_j A e_i$  and  $A$  is a locally bounded  $k$ -category with set of objects  $\{e_1, \dots, e_n\}$  and with morphisms space  $e_i \rightarrow e_j$  equal to  $e_j A e_i$ . We will say that the  $k$ -category  $\mathcal{C}$  is **connected** if for any  $x, y \in \mathcal{C}_0$  there exists a sequence  $x_0 = x, \dots, x_n = y$  in  $\mathcal{C}_0$  such that  ${}_{x_i}\mathcal{C}_{x_{i+1}} \neq 0$  or  ${}_{x_{i+1}}\mathcal{C}_{x_i} \neq 0$  for any  $i$ . Recall that an **ideal**  $I$  of  $\mathcal{C}$  is the data of vector subspaces  ${}_y I_x \subseteq {}_y\mathcal{C}_x$  for each  $x, y \in \mathcal{C}_0$ , such that the composition of a morphism in  $I$  with any morphism in  $\mathcal{C}$  lies in  $I$ . The **radical** (see [6]) of  $\mathcal{C}$  is the ideal  $\mathcal{R}\mathcal{C}$  of

$\mathcal{C}$  such that  ${}_y\mathcal{RC}_x$  is the set of non invertible morphisms  $x \rightarrow y$  for any  $x, y \in \mathcal{C}_0$ . If  $n \geq 2$  we set  $\mathcal{R}^n\mathcal{C} = (\mathcal{RC})^n$ . The **ordinary quiver** of  $\mathcal{C}$  has set of vertices  $\mathcal{C}_0$ , and for  $x, y \in \mathcal{C}_0$  the number of arrows  $x \rightarrow y$  is exactly  $\dim_k {}_y\mathcal{RC}_x / {}_y\mathcal{R}^2\mathcal{C}_x$ . Finally, we say that  $\mathcal{C}$  is **triangular** if  $Q$  has no oriented cycles. All functors are assumed to be  $k$ -linear functors between  $k$ -categories.

A functor  $F: \mathcal{E} \rightarrow \mathcal{B}$  is called a **covering functor** (see [6]) if the following properties are satisfied:

- a)  $F^{-1}(x) \neq \emptyset$  for any  $x \in \mathcal{B}_0$ ,
- b) for any  $x_0, y_0 \in \mathcal{C}$  and any  $\hat{x}_0, \hat{y}_0 \in \mathcal{E}_0$  such that  $F(\hat{x}_0) = x_0$  and  $F(\hat{y}_0) = y_0$ , the following maps induced by  $F$  are isomorphisms:

$$\bigoplus_{F(\hat{y})=y} {}_{\hat{y}}\mathcal{E}_{\hat{x}_0} \rightarrow {}_{y_0}\mathcal{B}_{x_0} \quad \text{and} \quad \bigoplus_{F(\hat{x})=x} {}_{\hat{y}_0}\mathcal{E}_{\hat{x}} \rightarrow {}_{y_0}\mathcal{B}_{x_0} .$$

In particular, if  $u \in {}_{y_0}\mathcal{B}_{x_0}$ , the inverse images of  $u$  by these isomorphisms will be called **the lifting of  $u$**  (w.r.t.  $F$ ) with source (resp. target)  $\hat{x}_0$  (resp.  $\hat{y}_0$ ). Recall that if  $\mathcal{E}$  is locally bounded (resp. connected) then so is  $\mathcal{B}$ .

A  **$G$ -category** is a  $k$ -category  $\mathcal{C}$  endowed with a group morphism  $G \rightarrow \text{Aut}(\mathcal{C})$ . Moreover, if the induced action of  $G$  on  $\mathcal{C}_0$  is free, then  $\mathcal{C}$  is called a **free  $G$ -category**. The **quotient category**  $\mathcal{C}/G$  of a free  $G$ -category  $\mathcal{C}$  (see [7] for instance) has set of objects  $\mathcal{C}_0/G$ . For any  $\alpha, \beta \in \mathcal{C}_0/G$  we set:

$${}_{\beta}(\mathcal{C}/G)_{\alpha} = \left( \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \right) / G$$

and the composition is induced by the composition in  $\mathcal{C}$ . The natural projection  $\mathcal{C} \rightarrow \mathcal{C}/G$  is a covering functor. A **Galois covering with group  $G$**  is a functor  $F: \mathcal{E} \rightarrow \mathcal{B}$  with  $\mathcal{E}$  a free  $G$ -category and such that there exists a commutative diagram:

$$\begin{array}{ccc} & \mathcal{E} & \\ & \swarrow & \searrow F \\ \mathcal{E}/G & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where  $\mathcal{E} \rightarrow \mathcal{E}/G$  is the natural projection and the horizontal arrow is an isomorphism. In particular a Galois covering is a covering functor. A **connected Galois covering** is a Galois covering  $\mathcal{E} \rightarrow \mathcal{B}$  where  $\mathcal{E}$  is connected.

A  **$G$ -graded category** is a  $k$ -category  $\mathcal{C}$  such that each morphism space has a decomposition  ${}_y\mathcal{C}_x = \bigoplus_{g \in G} {}_y\mathcal{C}_x^g$  satisfying  ${}_z\mathcal{C}_y \cdot {}_y\mathcal{C}_x^h \subseteq {}_z\mathcal{C}_x^{gh}$ . The **smash-product category** (see [7])  $\mathcal{C}\sharp G$  has set of objects  $(\mathcal{C}\sharp G)_0 = \mathcal{C}_0 \times G$ , and  ${}_{(y,t)}(\mathcal{C}\sharp G)_{(x,s)} = {}_y\mathcal{C}_x^{t^{-1}s}$  for  $(x, s)$  and  $(y, t)$  in  $(\mathcal{C}\sharp G)_0$ . The composition in  $\mathcal{C}\sharp G$  is induced by the composition in  $\mathcal{C}$ . The natural projection  $F: \mathcal{C}\sharp G \rightarrow \mathcal{C}$ , defined by  $F(x, s) = x$  and  $F(u) = u$  for  $u \in {}_{(y,t)}(\mathcal{C}\sharp G)_{(x,s)} \subseteq {}_y\mathcal{C}_x$ , is a Galois covering with group  $G$ . It has been shown in [7] that if  $p: \mathcal{E} \rightarrow \mathcal{B}$  is a Galois covering with group  $G$ , then  $\mathcal{B}$  is a  $G$ -graded category and there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow[\varphi]{\sim} & \mathcal{B}\sharp G \\ & \searrow p & \swarrow \\ & \mathcal{B} & \end{array}$$

where  $\mathcal{B}\sharp G \rightarrow \mathcal{B}$  is the natural projection and  $\varphi$  is an isomorphism.

### Quivers with admissible relations

Let  $Q$  be a **locally finite** quiver with set of vertices  $Q_0$ , set of arrows  $Q_1$  and source and target maps  $s, t: Q_1 \rightarrow Q_0$  respectively. Recall that locally finite means that  $s^{-1}(x)$  and  $t^{-1}(x)$  are finite sets for any  $x \in Q_0$ . For simplicity we will write  $x^+$  (resp.  $x^-$ ) for the set  $s^{-1}(x)$  (resp.  $t^{-1}(x)$ ). A (non trivial) **oriented path** in  $Q$  is a non empty sequence  $\alpha_1, \dots, \alpha_n$  of arrows of  $Q$  such that  $s(\alpha_{i+1}) = t(\alpha_i)$  for any  $1 \leq i \leq n-1$ . Such a path is written  $\alpha_n \dots \alpha_1$ , its source (resp. target) is  $s(\alpha_1)$  (resp.  $t(\alpha_n)$ ). For each  $x \in Q_0$  we will write  $e_x$  for the (trivial) path of length 0 and with source and target equal to  $x$ . The **path category**  $kQ$  has set of objects  $Q_0$ , the morphism space  ${}_y kQ_x$  is the vector space with basis the set of oriented paths in  $Q$  with source  $x$  and target  $y$  (including  $e_x$  in case  $x = y$ ). The composition of morphisms in  $kQ$  is induced by the concatenation of paths. Notice that  $kQ$  is a free  $k$ -category in the following sense: for any  $k$ -category  $\mathcal{C}$ , a functor  $kQ \xrightarrow{F} \mathcal{C}$  is uniquely determined by the family of morphisms  $\{F(\alpha) \in {}_{F(y)}\mathcal{C}_{F(x)} \mid x \xrightarrow{\alpha} y \in Q_1\}$ . We will denote by  $kQ^+$  the ideal of  $kQ$  generated by  $Q_1$ . Notice also that if  $Q_0$  is finite then  $kQ$  is also a  $k$ -algebra,  $kQ = \bigoplus_{x,y} {}_y kQ_x$ , with unit  $1 = \sum_{x \in Q_0} e_x$ , and  $kQ^+$  becomes an ideal of this  $k$ -algebra. If  $r \in {}_y kQ_x$  we define the **support of  $r$**  (denoted by  $\text{supp}(r)$ ) to be the set of paths in  $Q$  which appear in  $r$  with a non zero coefficient. Moreover, we define **normal form of  $r$**  as an equality of the type  $r = \sum_i \lambda_i u_i$  such that  $\lambda_i \in k^*$  for any  $i$  and where the paths  $u_i$  are pairwise distinct. An **admissible ideal** of  $kQ$  is an ideal  $I \subseteq kQ$  such that  $I \subseteq (kQ^+)^2$  and such that for any  $x \in Q_0$  there exists  $n \geq 2$  such that  $I$  contains all the paths with length at least  $n$  and with source or target  $x$ . The couple  $(Q, I)$  is then called a **quiver with admissible relations** and the quotient category  $kQ/I$  is locally bounded. When  $Q_0$  is finite, an admissible ideal  $I$  of  $kQ$  is exactly an ideal  $I$  of the  $k$ -algebra  $kQ$  such that  $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$  for some integer  $n \geq 2$ . Recall from [6] that if  $\mathcal{C}$  is a locally bounded  $k$ -category then there exists an admissible ideal  $I$  for the ordinary quiver  $Q$  of  $\mathcal{C}$  and there exists an isomorphism  $kQ/I \xrightarrow{\sim} \mathcal{C}$ . Such an isomorphism is called a **presentation of  $\mathcal{C}$  with quiver and (admissible) relations** (or an **admissible presentation** for short). Similarly, if  $A$  is a finite dimensional and basic  $k$ -algebra, an admissible presentation of  $A$  is an isomorphism of  $k$ -algebras  $kQ/I \xrightarrow{\sim} A$  where  $(Q, I)$  is a bound quiver.

### Transvections, dilatations

A **bypass** (see [2]) in  $Q$  is a couple  $(\alpha, u)$  where  $\alpha \in Q_1$  and  $u \neq \alpha$  is a path in  $Q$  parallel to  $\alpha$  (this means that  $\alpha$  and  $u$  share the same source and the same target). A **double bypass** is a 4-tuple  $(\alpha, u, \beta, v)$  such that  $(\alpha, u)$  and  $(\beta, v)$  are bypasses and such that the arrow  $\beta$  appears in the path  $u$ . Notice that if  $\alpha, \beta$  are distinct parallel arrows of  $Q$ , then  $(\alpha, \beta, \beta, \alpha)$  is a double bypass. Notice also that if  $u = va$  is an oriented cycle in  $Q$  with first arrow  $a$ , then  $(a, au, a, au)$  is a double bypass. Hence, if  $Q$  has no double bypass, then  $Q$  has no distinct parallel arrows and no oriented cycle. If  $A$  is a basic  $k$ -algebra with quiver  $Q$ , we will say that  $A$  **has no double bypass** if  $Q$  has no double bypass. A **transvection** is an automorphism of the  $k$ -category  $kQ$  of the form  $\varphi_{\alpha, u, \tau}$  where  $(\alpha, u)$  is a bypass,  $\tau \in k$  and  $\varphi_{\alpha, u, \tau}$  is given by  $\varphi_{\alpha, u, \tau}(\alpha) = \alpha + \tau u$  and  $\varphi_{\alpha, u, \tau}(\beta) = \beta$  for any arrow  $\beta \neq \alpha$  (this uniquely defines  $\varphi_{\alpha, u, \tau}$  since  $kQ$  is a free  $k$ -category). Notice that  $Q$  has no double bypass if and only if any two transvections commute. A **dilatation** is an automorphism  $D: kQ \xrightarrow{\sim} kQ$  such that  $D(\alpha) \in k^* \alpha$  for any arrow  $\alpha$ . Notice that the definitions of transvections and dilatations are analogous to the ones of transvection and dilatation matrices (see [15,

Chap. XIII, § 9] for instance). Recall that a dilatation matrix of  $GL_n(k)$  is a diagonal invertible matrix with at most one diagonal entry different from 1 and a transvection matrix is a matrix with diagonal entries equal to 1 and which has at most one non diagonal entry different from 0.

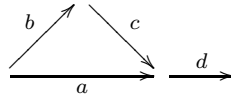
### Fundamental group, coverings of quivers with relations

Let  $(Q, I)$  be a quiver with admissible relations. For each arrow  $\alpha \in Q_1$  we will write  $\alpha^{-1}$  for its formal inverse with source (resp. target)  $s(\alpha^{-1}) = t(\alpha)$  (resp.  $t(\alpha^{-1}) = s(\alpha)$ ). A walk is an unoriented path in  $Q$ . More precisely it is a formal product  $u_n \dots u_1$  of arrows and of formal inverses of arrows such that  $s(u_{i+1}) = t(u_i)$  for any  $1 \leq i \leq n-1$ . Let  $r = t_1 u_1 + \dots + t_n u_n \in {}_y I_x$  where  $t_i \in k^*$  and where the  $u_i$ 's are distinct paths. Then  $r$  is called a **minimal relation** if  $n \geq 1$  and if for any non empty proper subset  $E$  of  $\{1, \dots, n\}$  we have  $\sum_{i \in E} t_i u_i \notin {}_y I_x$ . With this definition, any  $r \in I$  can be written as the sum of minimal relations with pairwise disjoint supports. Notice that in this definition we do not assume  $n$  to be greater than or equal to 2 as done usually (see [17]). This change is done for simplicity and does not affect the constructions which follow. The **homotopy relation** of  $(Q, I)$  is the smallest equivalence relation  $\sim_I$  on the set of walks (in  $Q$ ) which is compatible with the concatenation of walks and such that:

- .  $\alpha \alpha^{-1} \sim_I e_y$  and  $\alpha^{-1} \alpha \sim_I e_x$  for any arrow  $x \xrightarrow{\alpha} y$ ,
- .  $u_1 \sim_I u_2$  for any minimal relation  $t_1 u_1 + \dots + t_n u_n$ .

Notice that in order to compute  $\sim_I$  we may restrict ourselves to any set of minimal relations generating the ideal  $I$  (see [9]). Assume that  $Q$  is connected (i.e.  $Q$  is connected as an unoriented graph) and let  $x_0 \in Q_0$ . The **fundamental group** (see [17])  $\pi_1(Q, I, x_0)$  of  $(Q, I)$  at  $x_0$  is the set of  $\sim_I$ -classes of walks starting and ending at  $x_0$ . The composition is induced by the concatenation of walks and the unit is the  $\sim_I$ -class of  $e_{x_0}$ . Since different choices for  $x_0$  give rise to isomorphic fundamental groups (since  $Q$  is connected) we shall write  $\pi_1(Q, I)$  for short.

**Example 1.** (see [3]) Assume that  $Q$  is the following quiver:



and set  $I = \langle da \rangle$  and  $J = \langle da - dc \rangle$ . Then  $kQ/I \simeq kQ/J$  whereas  $\pi_1(Q, I) \simeq \mathbb{Z}$  and  $\pi_1(Q, J) = 0$ .

A **covering**  $(Q', I') \xrightarrow{p} (Q, I)$  of quivers with admissible relations (see [17]) is a quiver morphism  $Q' \xrightarrow{p} Q$  such that  $p(I') \subseteq I$  and such that:

- a)  $p^{-1}(x) \neq \emptyset$  for any  $x \in Q_0$ ,
- b)  $x^+ \xrightarrow{p} p(x)^+$  and  $x^- \xrightarrow{p} p(x)^-$  are bijective for any  $x \in Q'_0$ ,
- c) for any minimal relation  $r \in {}_y I_x$  and for any  $x' \in p^{-1}(x)$  there exist  $y' \in p^{-1}(y)$  and  $r' \in {}_{y'} I'_{x'}$  such that  $p(r') = r$ ,
- d) same statement as c) after interchanging  $x$  and  $y$ .

Recall that the automorphism group  $Aut(Q, I)$  of a bound quiver  $(Q, I)$  is the group of automorphisms  $g: Q \xrightarrow{\sim} Q$  of the quiver  $Q$  such that  $g(I) \subseteq I$ . Assume that  $(Q', I') \xrightarrow{p} (Q, I)$  is a covering, then the **group of automorphisms** of  $p$  is defined by  $Aut(p) = \{g \in Aut(Q', I') \mid p \circ g = p\}$ . If  $(Q', I') \xrightarrow{p} (Q, I)$  is a covering and if  $G$  is a subgroup of  $Aut(p)$ , then  $p$  is called a **Galois covering with group**  $G$  if  $Q$  and  $Q'$  are connected

and if  $G$  acts transitively on  $p^{-1}(x)$  for any  $x \in Q_0$ . Notice that if  $(Q', I') \xrightarrow{p} (Q, I)$  is a covering (resp. a Galois covering with group  $G$ ) then the induced functor  $kQ'/I' \xrightarrow{\bar{p}} kQ/I$  is a covering functor (resp. a Galois covering with group  $G$ ). Let  $(Q, I)$  be a connected quiver with admissible relations and let  $x_0 \in Q_0$ . The **universal cover** of  $(Q, I)$  is a Galois covering  $(\tilde{Q}, \tilde{I}) \xrightarrow{\pi} (Q, I)$  with group  $\pi_1(Q, I, x_0)$  as defined in [17]. One can describe it as follows:  $\tilde{Q}_0$  is the set of  $\sim_I$ -classes  $[w]$  of walks  $w$  starting at  $x_0$ . The arrows of  $\tilde{Q}$  are the couples  $(\alpha, [w])$  where  $\alpha \in Q_1$  and  $[w] \in \tilde{Q}_0$  are such that  $s(\alpha) = t(w)$ . The source (resp. target) of the arrow  $(\alpha, [w])$  is  $[w]$  (resp.  $[\alpha w]$ ). The map  $\tilde{Q} \xrightarrow{p} Q$  is defined by  $p([w]) = t(w)$  and  $p(\alpha, [w]) = \alpha$ . The ideal  $\tilde{I}$  is equal to  $p^{-1}(I)$ . Finally, the action of  $\pi_1(Q, I)$  on  $(\tilde{Q}, \tilde{I})$  is the following: if  $g \in \pi_1(Q, I)$  we may write  $g = [\gamma]$  with  $\gamma$  some walk with source and target equal to  $x_0$ . Then for any  $[w] \in \tilde{Q}_0$  (resp.  $(\alpha, [w]) \in \tilde{Q}_1$ ) we have  $g.[w] = [w\gamma^{-1}]$  (resp.  $g.(\alpha, [w]) = (\alpha, [w\gamma^{-1}])$ ).

### Some linear algebra

We introduce here some notions that will be useful in the sequel and that will be used without reference. Let  $E$  be a finite dimensional  $k$ -vector space with basis  $(e_1, \dots, e_n)$  and let  $(e_1^*, \dots, e_n^*)$  be the corresponding dual basis (i.e.  $e_i^*(e_i) = 1$  and  $e_i^*(e_j) = 0$  if  $j \neq i$ ). If  $\{r_t\}_{t \in T}$  is a family in  $E$ , then  $Span(r_t ; t \in T)$  will denote the subspace of  $E$  generated by this family. If  $r \in E$  we will write  $supp(r)$  (the support of  $r$ ) for the set of those  $e_i$ 's appearing in  $r$  with a non zero coefficient. Therefore  $e_i \in supp(r)$  is equivalent to  $e_i^*(r) \neq 0$ . Let  $F \subseteq E$  be a subspace. A non zero element  $r \in F$  is called minimal if it cannot be written as the sum of two non zero elements of  $F$  with disjoint supports. We will denote by  $\equiv_F$  the smallest equivalence relation on  $\{e_1, \dots, e_n\}$  such that  $e_i \equiv_F e_j$  for any  $r \in F$  minimal and any  $e_i, e_j \in supp(r)$ . Like in the situation of the homotopy relation of a bound quiver, the equivalence relation  $\equiv_F$  is determined by any generating family of  $F$  made of minimal elements. Notice that if  $E$  is the vector space with basis the set of oriented paths in a finite quiver  $Q$  (without oriented cycle) and if  $I$  is an admissible ideal of  $kQ$ , then for any paths  $u$  and  $v$  we have:  $u \equiv_I v \Rightarrow u \sim_I v$ . The converse is usually false as one can see in Example 1 where  $a \sim_J cb$  and  $a \not\equiv_J cb$ . Assume now that the basis of  $E$  is totally ordered:  $e_1 < \dots < e_n$ . A **Gröbner** basis of  $F$  is a basis  $(r_1, \dots, r_t)$  of  $F$  such that:

- . for any  $j$  there is some  $i_j$  such that  $r_j \in e_{i_j} + Span(e_i ; i < i_j)$ .
- .  $e_{i_j} \notin supp(r_{j'})$  unless  $j = j'$ .
- . if  $r = e_l + \sum_{i < l} \tau_i e_i \in F$  then  $e_l = e_{i_j}$  for some  $j$ .

With this definition,  $F$  has a unique Gröbner basis which has a natural total order:  $r_1 < \dots < r_t$  if we assume that  $i_1 < \dots < i_t$ . Moreover,  $r_1, \dots, r_t$  are minimal elements of  $F$ . This last property implies in particular that  $e_i \equiv_F e_j$  if and only if there exists a sequence of integers  $m_1, \dots, m_p$  such that  $e_i \in supp(r_{m_1})$ ,  $e_j \in supp(r_{m_p})$  and  $supp(r_{m_j}) \cap supp(r_{m_{j+1}}) \neq \emptyset$  for each  $j$ . Notice that our definition of Gröbner basis is slightly different from the classical one (see e.g. [1]) since we do not use any multiplicative structure. Moreover, our definition is linked with the notion of reduced echelon form matrix (see [14, p. 65]). We may also point out that a study of Gröbner bases in path algebras of quivers has been made in [8].

We end this paragraph with a reminder on the exponential and on the logarithm of an endomorphism. If  $u: E \rightarrow E$  is a nilpotent endomorphism, we define the **exponential** of  $u$  to be  $exp(u) = \sum_{l \geq 0} \frac{1}{l!} u^l$ . Thus,  $exp(u): E \rightarrow E$  is a well defined linear isomorphism such that  $exp(u) - Id$  is nilpotent. If  $v: E \rightarrow E$  is an isomorphism such that  $v - Id$  is nilpotent, we define the **logarithm** of  $v$  to be  $log(v) = \sum_{l \geq 0} (-1)^{l+1} \frac{1}{l} (v - Id)^l$ .

Recall that if  $u: E \rightarrow E$  is a nilpotent endomorphism, then  $\log(\exp(u)) = u$ .

## 2 Proof of Theorem 1

In this section we provide the proof of Theorem 1 (see also [16, Thm 1.1]). We fix  $A$  a finite dimensional basic and connected  $k$ -algebra with quiver  $Q$ . **Throughout this section we will assume that  $Q$  has no oriented cycle.** The proof of Theorem 1 decomposes into 4 steps as follows, and we will devote a subsection to each step:

a) If  $kQ/I$  and  $kQ/J$  are isomorphic to  $A$  as  $k$ -algebras, then there exists  $\varphi: kQ \xrightarrow{\sim} kQ$  a product of transvections and of a dilatation such that  $\varphi(I) = J$ .

b) If  $\varphi(I) = J$  and if  $\varphi$  is a dilatation then  $\pi_1(Q, I) \simeq \pi_1(Q, J)$ . If  $\varphi$  is a transvection, then there exists a surjective group morphism  $\pi_1(Q, I) \rightarrow \pi_1(Q, J)$  or  $\pi_1(Q, J) \rightarrow \pi_1(Q, I)$ , induced by the identity map on the walks in  $Q$ .

c) The homotopy relations  $\sim_I$  of the admissible presentations  $kQ/I$  of  $A$  can be displayed as the vertices of a quiver  $\Gamma$  such that for any arrow  $\sim_I \rightarrow \sim_J$  the identity map on walks induces a surjective group morphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ .

d) If  $k$  has characteristic zero and if  $Q$  has no double bypass, then the quiver  $\Gamma$  has a unique source. Moreover, if  $\sim_{I_0}$  is the source of  $\Gamma$  then  $I_0$  fits Theorem 1.

### 2.1 Different presentations of an algebra are linked by products of transvections and dilatations

In order to consider  $A$  as a  $k$ -category we need to choose a decomposition of the unit into a sum of pairwise orthogonal primitive idempotents. The following proposition shows that this choice is irrelevant and that we may fix these idempotents once and for all. We will omit the proof which is basic linear algebra.

**Proposition 2.1.** [16, 3.1] *Let  $I$  and  $J$  be admissible ideals of  $kQ$ . If  $kQ/I \simeq kQ/J$  as  $k$ -algebras then there exists  $\varphi: kQ \xrightarrow{\sim} kQ$  an automorphism extending the identity map on  $Q_0$  and such that  $\varphi(I) = J$ .*

Recall that  $GL_n(k)$  is generated by transvection and dilatation matrices. The following proposition states an analogous result for the group of automorphisms of  $kQ$  extending the identity map on  $Q_0$ .

**Proposition 2.2.** *Let  $\mathcal{G}$  be the group of automorphisms of  $kQ$  extending the identity map on  $Q_0$ . Let  $\mathcal{D} \subseteq \mathcal{G}$  be the subgroup of the dilatations of  $kQ$  and let  $\mathcal{T} \subseteq \mathcal{G}$  be the subgroup generated by the transvections. Then  $\mathcal{T}$  is a normal subgroup and  $\mathcal{G} = \mathcal{DT} = \mathcal{T}\mathcal{D}$ .*

**Remark 1.** *The group of automorphism of an algebra has already been studied. More precisely the reader can find in [13], [20] and [21] a study of the group of outer automorphisms of an algebra.*

**Proof of Proposition 2.2:** For any transvection  $\varphi = \varphi_{\alpha, u, \tau}$  and any dilatation  $D$  we have  $D\varphi D^{-1} = \varphi_{\alpha, u, \frac{\tau\lambda}{\mu}}$  where  $\lambda \in k^*$  and  $\mu \in k^*$  are such that  $D(u) = \lambda u$  and  $D(\alpha) = \mu \alpha$ . Hence, in order to prove the proposition, it is enough to prove that  $\mathcal{G} = \mathcal{T}\mathcal{D}$ . If  $\psi \in \mathcal{G}$  we shall write  $n(\psi)$  for the number of arrows  $\alpha \in Q_1$  such that  $\psi(\alpha) \notin k^*\alpha$ . Notice that  $n(\psi) = 0$  if and only if  $\psi \in \mathcal{D}$ . Let us prove by induction



on  $n \geq 0$  that  $R_n : "n(\psi) \leq n \Rightarrow \psi \in \mathcal{TD}"$  is true. Obviously  $R_0$  is true. Let  $n \geq 1$ , assume that  $R_{n-1}$  is true, and let  $\psi \in \mathcal{G}$  such that  $n(\psi) = n$ . Hence, there exists  $x \xrightarrow{\alpha_1} y \in Q_1$  such that  $\psi(\alpha_1) \notin k^*\alpha_1$ . Let  $\alpha_1, \dots, \alpha_d$  be the arrows  $x \rightarrow y$  of  $Q$  and let  $E = \text{Span}(\alpha_1, \dots, \alpha_d) \simeq {}_y(kQ^+/(kQ^+)^2)_x$ . Since  $kQ \xrightarrow{\psi} kQ$  is an automorphism, the composition  $f: E \hookrightarrow {}_y kQ_x \xrightarrow{\psi} {}_y kQ_x \rightarrow E$  of  $\psi$  with the natural inclusion and the natural projection is a  $k$ -linear isomorphism hence an element of  $GL_d(k)$ . Thus (see [15, Chap. XIII Prop. 9.1]) there exist transvections matrices  $f_1, \dots, f_l \in GL_d(k)$  such that  $f_1 \dots f_l f(\alpha_i) \in k^*\alpha_i$  for each  $i \in \{1, \dots, d\}$ . For each  $f_j$ , let  $\bar{f}_j: kQ \rightarrow kQ$  be the automorphism such that  $\bar{f}_j(\alpha_i) = f_j(\alpha_i)$  for each  $i \in \{1, \dots, d\}$  and such that  $\bar{f}_j(\beta) = \beta$  for any arrow  $\beta$  not parallel to  $\alpha_1$ . In particular,  $\bar{f}_j$  is a transvection with respect to some  $\alpha_{i_j}$ . Let  $g_1 = \bar{f}_1 \dots \bar{f}_l \in \mathcal{T}$ . Then,  $g_1\psi(\alpha_i) \in k^*\alpha_i + (kQ^+)^2$  and if  $\beta \in Q_1$  is not parallel to  $\alpha_1$  and satisfies  $\psi(\beta) \in k^*\beta$  then  $g_1\psi(\beta) \in k^*\beta$ . Let  $\psi_1 = g_1\psi$ . By construction, for each  $i \in \{1, \dots, d\}$ , we have  $\psi_1(\alpha_i) = \lambda_i \alpha_i + \sum_{j=1}^{n_i} \tau_{i,j} u_{i,j}$  with  $u_{i,j}$  paths of length at least 2. Let  $\varphi_{i,j}$  be the transvection  $\varphi_{\alpha_i, u_{i,j}, -\tau_{i,j}/\lambda_i}$  for each  $i \in \{1, \dots, d\}$  and each  $j \in \{1, \dots, n_i\}$ , and let  $g_2 \in \mathcal{T}$  be the product of the  $\varphi_{i,j}$ 's (for any  $i \in \{1, \dots, d\}$  and any  $j \in \{1, \dots, n_i\}$ ). It is easy to check that the  $\varphi_{i,j}$ 's are pairwise commuting so that the definition of  $g_2$  is unambiguous. Since  $Q$  has no oriented cycle, we have  $g_2\psi_1(\alpha_i) = \lambda_i \alpha_i$  for each  $i$ , and  $g_2\psi_1(\beta) \in k^*\beta$  if  $\beta \in Q_1$  is not parallel to  $\alpha_1$  and satisfies  $\psi_1(\beta) \in k^*\beta$ . In particular:  $n(g_2g_1\psi) < n(\psi) = n$ . Since  $R_{n-1}$  is true,  $g_2g_1\psi$  lies in  $\mathcal{TD}$  and so does  $\psi$  (recall  $g_1, g_2 \in \mathcal{T}$ ). Hence,  $R_n$  is true. This achieves the proof of Proposition 2.2. □

**Remark 2.** *Proposition 2.1 and Proposition 2.2 imply that if  $I$  and  $J$  are admissible ideals of  $kQ$  such that  $kQ/I \simeq kQ/J$  as  $k$ -algebras, then there exist  $\varphi_1, \dots, \varphi_n$  (resp.  $\varphi'_1, \dots, \varphi'_m$ ) a sequence of transvections of  $kQ$ , together with  $D$  a dilatation such that  $J = D\varphi_n \dots \varphi_1(I)$  (resp.  $J = \varphi'_m \dots \varphi'_1 D(I)$ ).*

## 2.2 Comparison of the fundamental group of two presentations of an algebra linked by a transvection or a dilatation

If  $I$  is an ideal and  $\varphi$  is a dilatation or a transvection, then  $I$  and  $\varphi(I)$  are similar enough in order to compare the associated homotopy relations. Before stating this comparison we prove two useful lemmas. In these lemmas we fix  $I$  an admissible ideal of  $kQ$ , we fix  $\varphi = \varphi_{\alpha, u, \tau}$  a transvection ( $\tau \neq 0$ ) and we set  $J = \varphi(I)$ .

**Lemma 2.3.** *Assume that  $\alpha \not\sim_I u$  and let  $r \in {}_y I_x$  be a minimal relation with normal form  $r = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b$  such that  $\alpha$  does not appear in the path  $\theta_c$  for any  $c \in C$ . Then there exists a minimal relation  $r' \in {}_y J_x$  with normal form  $r' = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + \sum_{B'} \lambda_{b'} v_{b'} u_{b'}$  where  $B' \subseteq B$ .*

**Proof:** Let us assume that  $B \neq \emptyset$  (if  $B = \emptyset$ , the conclusion is immediate). Since  $Q$  has no oriented cycle, the paths  $v_b$  and  $u_b$  do not contain  $\alpha$ . Since  $r$  is a minimal relation of  $I$  and since  $\alpha \not\sim_I u$ , we have  $\theta_c \neq v_b u_{b'}$  for any  $c \in C, b \in B$ . Therefore,  $\varphi(r)$  has a normal form  $\varphi(r) = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + \sum_B \lambda_{b'} v_{b'} u_{b'} \in {}_y J_x \setminus \{0\}$ . Thus there exists a minimal relation  $r' \in {}_y J_x$  with normal form  $r' = \sum_{C'} \lambda_c \theta_c + \sum_{B'} \lambda_b v_b \alpha u_b + \sum_{B'} \lambda_{b'} v_{b'} u_{b'}$  such that  $\emptyset \neq B'_1 \subseteq B, C' \subseteq C$  and  $B' \subseteq B$ . Hence  $\varphi^{-1}(r')$  has a normal

form  $\varphi^{-1}(r') = \sum_{C'} \lambda_c \theta_c + \sum_{B'_1} \lambda_b v_b \alpha u_b + \sum_{B' \setminus B'_1} \lambda_b \tau v_b u u_b - \sum_{B'_1 \setminus B'} \lambda_b \tau v_b u u_b \in {}_y I_x \setminus \{0\}$ . Since  $r \in {}_y I_x$  is a minimal relation and since  $\alpha \not\sim_I u$  we infer that there exists a minimal relation  $r'' \in {}_y I_x$  with normal form  $r'' = \sum_{C''} \lambda_c \theta_c + \sum_{B''} \lambda_b v_b \alpha u_b$  such that  $C'' \subseteq C' \subseteq C$  and  $\emptyset \neq B'' \subseteq B'_1$ . This forces  $C'' = C$  and  $B'' = B$  because  $r \in {}_y I_x$  is a minimal relation. Thus  $C' = C$  and  $B'_1 = B$ . Hence we have a minimal relation  $r' \in {}_y J_x$  with normal form  $r' = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + \sum_{B'} \lambda_b \tau v_b u u_b$  as announced.  $\square$

**Lemma 2.4.** *Assume that  $\alpha \sim_J u$  and let  $r \in {}_y J_x$  be a minimal relation. Then  $v \sim_J w$  for any  $v, w \in \text{supp}(r)$ .*

**Proof:** We may write  $r = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + \mu_b v_b u u_b$  where:

- .  $\lambda_c, \lambda_b \in k^*$  and  $\mu_b \in k$  for any  $c \in C$  and  $b \in B$ ,
- . the paths  $\theta_c, v_b \alpha u_b, v_{b'} u u_{b'}$  ( $c \in C, b, b' \in B$ ) are pairwise distinct,
- . for any  $c \in C$ , the path  $\theta_c$  does not contain  $\alpha$ .

Hence  $\varphi(r) = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + (\mu_b + \tau \lambda_b) v_b u u_b \in {}_y J_x$  and there exists a decomposition  $\varphi(r) = r_1 + \dots + r_n$  where  $r_i \in {}_y J_x$  is a minimal relation and  $\text{supp}(r_i) \cap \text{supp}(r_j) = \emptyset$  if  $i \neq j$ . If  $B = \emptyset$  then  $\varphi(r) = r \in {}_y J_x$  is a minimal relation and the lemma is proved. Hence we may assume that  $B \neq \emptyset$ . This implies that for any  $i \in \{1, \dots, n\}$  there exists  $b \in B$  such that  $v_b \alpha u_b \in \text{supp}(r_i)$  or  $v_b u u_b \in \text{supp}(r_i)$  (if this is not the case then  $r_i = \sum_{C'} \lambda_c \theta_c$  for some non empty subset  $C'$  of  $C$ , thus  $\varphi^{-1}(r_i) = \sum_{C'} \lambda_c \theta_c \in {}_y I_x$  which contradicts the minimality of  $r$ ). Let  $\equiv$  be the smallest equivalence relation on the set  $\{1, \dots, n\}$  such that:  $i \equiv j$  if there exists  $b \in B$  such that  $v_b \alpha u_b \in \text{supp}(r_i)$  and  $v_b u u_b \in \text{supp}(r_j)$ . Since the  $r_i$ 's are minimal relations of  $J$  and since  $\alpha \sim_J u$ , we get: if  $i \equiv j$  then  $v \sim_J w$  for any  $v, w \in \text{supp}(r_i) \sqcup \text{supp}(r_j)$ . Let  $\mathcal{O} \subseteq \{1, \dots, n\}$  be a  $\equiv$ -orbit and let  $r' = \sum_{i \in \mathcal{O}} r_i \in {}_y J_x$ . Hence  $r' = \sum_{C'} \lambda_c \theta_c + \sum_{B'} \lambda_b v_b \alpha u_b + (\mu_b + \tau \lambda_b) v_b u u_b$  where  $C' \subseteq C$  and  $\emptyset \neq B' \subseteq B$ . This implies that  $\varphi^{-1}(r') = \sum_{C'} \lambda_c \theta_c + \sum_{B'} \lambda_b v_b \alpha u_b + \mu_b v_b u u_b \in {}_y I_x$  and the minimality of  $r$  yields  $C' = C$ ,  $B' = B$ ,  $r' = \varphi(r)$  and  $\mathcal{O} = \{1, \dots, n\}$ . Hence  $\{1, \dots, n\}$  is an  $\equiv$ -orbit. Therefore  $v \sim_J w$  for any  $v, w \in \text{supp}(\varphi(r))$ . And since  $\alpha \sim_J u$  we infer that  $v \sim_J w$  for any  $v, w \in \text{supp}(r)$ .  $\square$

We can state the announced comparison now. For short, the word *generated* stands for: *generated as an equivalence relation compatible with the concatenation of walks and such that  $\alpha^{-1} \alpha \sim_I e_x$ ,  $\alpha \alpha^{-1} \sim_I e_y$  for any arrow  $x \xrightarrow{\alpha} y$ .*

**Proposition 2.5.** [16, 3.2] *Let  $I$  be an admissible ideal of  $kQ$ , let  $\varphi$  be an automorphism of  $kQ$  and set  $J = \varphi(I)$ . If  $\varphi$  is a dilatation, then  $\sim_I$  and  $\sim_J$  coincide. Assume now that  $\varphi = \varphi_{\alpha, u, \tau}$  is a transvection.*

- a) *if  $\alpha \sim_I u$  and  $\alpha \sim_J u$  then  $\sim_I$  and  $\sim_J$  coincide.*
- b) *if  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$  then  $\sim_J$  is generated by  $\sim_I$  and  $\alpha \sim_J u$ .*
- c) *if  $\alpha \not\sim_I u$  and  $\alpha \not\sim_J u$  then  $I = J$  and  $\sim_I$  and  $\sim_J$  coincide.*

**Remark 3.** *The following implication (symmetric to b)):*

*if  $\alpha \sim_I u$  and  $\alpha \not\sim_J u$  then  $\sim_I$  is generated by  $\sim_J$  and  $\alpha \sim_I u$*

*is also satisfied since  $\varphi_{\alpha, u, \tau}^{-1} = \varphi_{\alpha, u, -\tau}$*

**Proof of Proposition 2.5:** If  $\varphi$  is a dilatation, then  $\sim_I$  and  $\sim_J$  coincide because for any  $r \in {}_y kQ_x$  we have  $\text{supp}(r) = \text{supp}(\varphi(r))$  and because  $r$  is a minimal relation of  $I$  if and only if the same holds for  $\varphi(r)$  in  $J$ . Let us assume that  $\varphi = \varphi_{\alpha, u, \tau}$  is a

transvection.

a) Lemma 2.4 applied to  $I, J, \varphi$  (resp.  $J, I, \varphi^{-1} = \varphi_{\alpha, u, -\tau}$ ) shows that any two paths appearing in a same minimal relation of  $I$  (resp.  $J$ ) are  $\sim_J$ -equivalent (resp.  $\sim_I$ -equivalent). Hence  $\sim_I$  and  $\sim_J$  coincide.

b) Let  $\equiv$  be the equivalence relation generated by:  $(v \sim_I w \Rightarrow v \equiv w)$  and  $\alpha \equiv u$ . Our aim is to show that  $\sim_J$  and  $\equiv$  coincide. Thanks to Lemma 2.4 we have:  $v \equiv w \Rightarrow v \sim_J w$ . Let  $Min(I)$  be the set of the minimal relations of  $I$ . For each  $r \in Min(I)$  let us fix a normal form  $r = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b$  satisfying the hypotheses of Lemma 2.3. Hence there exists  $B' \subseteq B$  and a minimal relation  $r_1$  of  $J$  with normal form  $r_1 = \sum_C \lambda_c \theta_c + \sum_B \lambda_b v_b \alpha u_b + \sum_{B'} \lambda_{b'} \tau v_{b'} u_{b'}$ . Thus  $\varphi(r) - r_1 = \sum_{B \setminus B'} \tau \lambda_b v_b u_b \in J$  can be written as a sum  $r_2 + \dots + r_{n_r}$  of minimal relations of  $J$  with pairwise disjoint supports. In particular,  $\varphi(r) = r_1 + \dots + r_{n_r}$  where each  $r_i \in J$  is a minimal relation. Notice that any two paths appearing in  $r_1$  are  $\equiv$ -equivalent because of the normal form of  $r_1$  and because of the definition of  $\equiv$ . With these notations, the set  $\{r_i \mid r \in Min(I) \text{ and } 1 \leq i \leq n_r\}$  is made of minimal relations of  $J$  and generates the ideal  $J$ . Thus, in order to show that  $\sim_J$  and  $\equiv$  coincide, it is enough to show that any two paths appearing in some  $r_i$  are  $\equiv$ -equivalent. Let  $r \in Min(I)$ , let  $i \in \{1, \dots, n_r\}$ , and let  $v, w \in supp(r_i)$ . We have already proved that if  $i = 1$  then  $v \equiv w$ , thus we may assume that  $i \geq 2$ . Keeping the above notations for the normal form of  $r$ , there exist  $b, b' \in B$  such that  $v = v_b u_b$  and  $w = v_{b'} u_{b'}$ . Since  $\alpha \equiv u$  and since any two paths appearing in  $r_1$  are  $\equiv$ -equivalent we get  $v = v_b u_b \equiv v_b \alpha u_b \equiv v_{b'} \alpha u_{b'} \equiv v_{b'} u_{b'} = w$ . Hence any two paths appearing in some  $r_i$  are  $\equiv$ -equivalent. This implies that  $\sim_J$  and  $\equiv$  coincide. Therefore,  $\sim_J$  is generated by  $\sim_I$  and  $\alpha \sim_J u$ .

c) Let  $r \in I$  be a minimal relation of  $I$  and apply Lemma 2.3 to  $r$ . Since  $\alpha \not\sim_J u$ , we infer that  $r \in J$ . Since  $I$  is generated by its minimal relations we get  $I \subseteq J$ . Finally,  $I = J$  because  $I$  and  $J$  have the same dimension.  $\square$

**Remark 4.** *In the situation b) of Proposition 2.5, the identity map on the walks of  $Q$  induces a surjective group morphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ .*

Proposition 2.5 allows us to prove the following result which has already been proved in [5]. Recall that the algebra  $kQ/I$ , where  $I$  is admissible, is called **constricted** if  $dim_y(kQ/I)_x = 1$  for any arrow  $x \rightarrow y$  of  $Q$ .

**Proposition 2.6 (see also [5]).** *Assume that  $A$  is constricted. Then different admissible presentations of  $A$  yield the same homotopy relation. In particular, they have isomorphic fundamental groups.*

**Proof:** Let  $kQ/I \simeq A$  be any admissible presentation. If  $(\alpha, u)$  is a bypass in  $Q$  then  $u \in I$  because  $A$  is constricted and  $I$  is admissible. In particular, for any  $\tau \in k$  and any  $r \in kQ$  we have:  $\varphi_{\alpha, u, \tau}(r) - r$  belongs to the ideal of  $kQ$  generated by  $u \in I$  and therefore  $\varphi_{\alpha, u, \tau}(r) - r \in I$ . This shows that  $\varphi(I) \subseteq I$  and that  $\varphi(I) = I$  ( $I$  is finite dimensional because  $Q$  has no oriented cycle) for any transvection  $\varphi$ . Let  $kQ/J \simeq A$  be another admissible presentation. From Remark 2 we know that there exist a dilatation  $D$  and transvections  $\varphi_1, \dots, \varphi_n$  such that  $J = D\varphi_n \dots \varphi_1(I)$ . We deduce from what we have proved above that  $\varphi_n \dots \varphi_1(I) = I$ . Hence,  $J = D(I)$ . Proposition 2.5 implies that  $\sim_I$  and  $\sim_J$  coincide. Therefore,  $\pi_1(Q, I)$  and  $\pi_1(Q, J)$  are isomorphic.  $\square$

If  $\sim$  and  $\sim'$  are homotopy relations, we will say that  $\sim'$  is a **direct successor** (see also [16, Sect. 3]) of  $\sim$  if there exist admissible ideals  $I$  and  $J$  of  $kQ$ , together with

a transvection  $\varphi = \varphi_{\alpha, u, \tau}$  such that  $\sim = \sim_I$ ,  $\sim' = \sim_J$ ,  $J = \varphi(I)$ ,  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$ . Notice that  $I, J, \varphi$  need not be unique.

### 2.3 The quiver $\Gamma$ of the homotopy relations of the presentations of the algebra

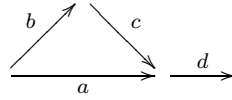
**Definition 2.7.** [16, 4.1] We define the quiver  $\Gamma$  as follows:

.  $\Gamma_0$  is the set of homotopy relations of the admissible presentations of  $A$ :

$$\Gamma_0 = \{ \sim_I \mid I \text{ is admissible and } kQ/I \simeq A \}$$

. there is an arrow  $\sim \rightarrow \sim'$  if and only if  $\sim'$  is a direct successor of  $\sim$ .

**Example 2.** Assume that  $A = kQ/I$  where  $Q$  is



and  $I = \langle da \rangle$ . Let  $J = \langle da - dc \rangle$ . Using Proposition 2.5 one can show that  $\Gamma$  is equal to:  $\sim_I \longrightarrow \sim_J$ . Notice that the identity map on walks induces a surjective group morphism  $\mathbb{Z} \simeq \pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J) \simeq 1$ .

The author thanks Mariano Suárez-Alvarez for the following remark:

**Remark 5.** A homotopy relation is determined by its restriction to the paths in  $Q$  with length at most the radical length of  $A$ . Thus there are only finitely many homotopy relations. This argument shows that  $\Gamma$  is finite.

The following proposition states some additional properties of  $\Gamma$  and is a direct consequence of Remark 2 and Proposition 2.5.

**Proposition 2.8.** Assume that  $Q$  has no oriented cycle and let  $m$  be the number of bypasses in  $Q$ . Then  $\Gamma$  is connected and has no oriented cycle. Any vertex of  $\Gamma$  is the source of at most  $m$  arrows and any oriented path in  $\Gamma$  has length at most  $m$ .

**Remark 6.** According to Remark 4, if there is a path in  $\Gamma$  with source  $\sim_I$  and target  $\sim_J$ , then the identity map on the walks in  $Q$  induces a surjective group morphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ . Moreover, since  $\Gamma$  is finite, any vertex of  $\Gamma$  is the target of a (finite) path whose source is a source of  $\Gamma$  (i.e. a vertex with no arrow ending at it). As a consequence, if  $\Gamma$  has a unique source  $\sim_{I_0}$ , then the fundamental group of any admissible presentation of  $A$  is a quotient of  $\pi_1(Q, I_0)$ .

### 2.4 The uniqueness of the source of $\Gamma$ and the proof of Theorem 1

Notice that until now we have used neither the characteristic of  $k$  nor the possible non existence of a double bypass in  $Q$ . These hypotheses will be needed in order to prove the uniqueness of the source of  $\Gamma$ . The complete proof of the uniqueness of the source of  $\Gamma$  is somewhat technical. For this reason we deal with the technical considerations in the two lemmas that follow.

**Lemma 2.9.** *Let  $E$  be a finite dimensional  $k$ -vector space endowed with a totally ordered basis  $e_1 < \dots < e_n$ . Assume that  $k$  has characteristic zero. Let  $\nu: E \rightarrow E$  be a linear map such that  $\nu(e_i) \in \text{Span}(e_j ; j < i)$  for any  $i \in \{1, \dots, n\}$ , and let  $I$  and  $J$  be two subspaces of  $E$  such that the following conditions are satisfied:*

- a)  $\psi(I) = J$  where  $\psi: E \rightarrow E$  is equal to  $\exp(\nu)$ .
- b) if  $e_i \in \text{supp}(\nu(e_j))$  then  $e_i \not\equiv_I e_j$  and  $e_i \not\equiv_J e_j$ .

*Then  $I$  and  $J$  have the same Gröbner basis and  $I = J$ .*

**Proof:** Let us prove Lemma 2.9 by induction on  $n$ . If  $n = 1$  the equality is obvious so let us assume that  $n > 1$  and that the conclusion of Lemma 2.9 holds for dimensions less than  $n$ . We will denote by  $r_1 < \dots < r_p$  (resp.  $r'_1 < \dots < r'_p$ ) the Gröbner basis of  $I$  (resp. of  $J$ ) and we will write  $i_1, \dots, i_p$  (resp.  $i'_1, \dots, i'_p$ ) for the integers such that  $r_j \in e_{i_j} + \text{Span}(e_i ; i < i_j)$  (resp.  $r'_j \in e_{i'_j} + \text{Span}(e_i ; i < i'_j)$ ). In order to prove that  $I = J$  we will prove the four following facts:

- a) the two sequences  $i_1 < \dots < i_p$  and  $i'_1 < \dots < i'_p$  coincide,
- b)  $\psi(r_1) = r'_1$ ,
- c)  $r_1 = r'_1$  and  $\nu(r_1) = 0$  (using the induction hypothesis on  $E/k.e_1$ ),
- d)  $r_2 = r'_2, \dots, r_p = r'_p$  (using the induction hypothesis on  $E/k.r_1$ ).

a) For simplicity let us set  $E_i = \text{Span}(e_j ; j \leq i)$ . Since  $\nu(e_j) \in E_{j-1}$  and  $r_j \in e_{i_j} + E_{i_j-1}$ , and since  $\psi = \exp(\nu)$ , we get  $\psi(r_j) \in J \cap (e_{i_j} + E_{i_j-1})$  for any  $j$ . Hence, the definition of the Gröbner basis of  $J$  forces  $\{i_1, \dots, i_p\} \subseteq \{i'_1, \dots, i'_p\}$  and the cardinality and the ordering on these two sets imply that  $i_1 = i'_1, \dots, i_p = i'_p$ .

b) Since  $i_1 = i'_1$  we infer that  $\psi(r_1) - r'_1 \in J \cap E_{i_1-1}$ . Then, the definition of the Gröbner basis of  $J$  forces  $\psi(r_1) - r'_1 = 0$ .

c) Let us prove that  $r_1 = r'_1$ . Notice that the definition of a Gröbner basis and the equalities  $\psi(r_1) = r'_1$  and  $\psi(e_1) = e_1$  force:  $e_1 \in I \Leftrightarrow r_1 = e_1 \Leftrightarrow r'_1 = e_1 \Leftrightarrow e_1 \in J$ . Hence we may assume that  $e_1 \notin I$  and  $e_1 \notin J$ .

Let  $\tilde{E} = E/k.e_1$  and let  $\pi: E \rightarrow \tilde{E}$  be the natural projection. We will write  $\tilde{x}$  for  $\pi(x)$ . Similarly we set  $\tilde{I} = \pi(I)$  and  $\tilde{J} = \pi(J)$ . In particular  $\tilde{E}$  has a totally ordered basis:  $\tilde{e}_2 < \dots < \tilde{e}_n$ . Since  $\nu(e_1) = 0$  and since  $\psi(e_1) = e_1$ , the mappings  $\nu$  and  $\psi$  induce linear mappings  $\tilde{\nu}, \tilde{\psi}: \tilde{E} \rightarrow \tilde{E}$ . It follows from the properties of  $\nu$  and  $\psi$  that  $\tilde{\psi}(\tilde{I}) = \tilde{J}$ , that  $\tilde{\nu}(\tilde{e}_i) \in \text{Span}(\tilde{e}_j ; 2 \leq j < i)$  for any  $i \geq 2$ , that  $\tilde{\psi} = \exp(\tilde{\nu})$ , and that  $\text{supp}(\tilde{\nu}(\tilde{e}_i)) = \{\tilde{e}_j \mid j \geq 2 \text{ and } e_j \in \text{supp}(\nu(e_i))\}$  for any  $i \geq 2$ . Moreover, with the definition of the Gröbner basis of  $I$  we get:

- .  $\tilde{r}_j \in \tilde{e}_{i_j} + \text{Span}(\tilde{e}_i ; i < i_j)$  for any  $j$  (recall that  $e_1 \notin I$ ),
- .  $\text{supp}(\tilde{r}_j) = \{\tilde{e}_i \mid i \geq 2 \text{ and } e_i \in \text{supp}(r_j)\}$  for any  $j$ .

Therefore  $\tilde{r}_1 < \dots < \tilde{r}_p$  is the Gröbner basis of  $\tilde{I}$  and:  $\tilde{e}_i \equiv_{\tilde{I}} \tilde{e}_j \Rightarrow e_i \equiv_I e_j$ . Similarly  $\tilde{r}'_1 < \dots < \tilde{r}'_p$  is the Gröbner basis of  $\tilde{J}$  and:  $\tilde{e}_i \equiv_{\tilde{J}} \tilde{e}_j \Rightarrow e_i \equiv_J e_j$ . Using the above description of  $\text{supp}(\tilde{\nu}(\tilde{e}_i))$  together with the above link between  $\equiv_I$  (resp.  $\equiv_J$ ) and  $\equiv_{\tilde{I}}$  (resp.  $\equiv_{\tilde{J}}$ ) we infer that:

$$\tilde{e}_i \not\equiv_{\tilde{I}} \tilde{e}_j \text{ and } \tilde{e}_i \not\equiv_{\tilde{J}} \tilde{e}_j \text{ as soon as } \tilde{e}_j \in \text{supp}(\tilde{\nu}(\tilde{e}_i))$$

For this reason we may apply the induction hypothesis to  $\tilde{E}, \tilde{I}$  and  $\tilde{J}$ . Hence  $\tilde{I}$  and  $\tilde{J}$  have the same Gröbner basis and  $\tilde{r}_1 = \tilde{r}'_1$  i.e.  $r'_1 = r_1 + \lambda e_1$  with  $\lambda \in k$ . Therefore

$(\psi - Id)(r_1) = \lambda e_1$ , and since  $\psi(e_1) = e_1$  we get  $\nu(r_1) = \log(\psi)(r_1) = \lambda e_1$ . Assume that  $\lambda \neq 0$  i.e.  $e_1 \in \text{supp}(\nu(r_1))$ . Thus there exists  $e_i \in \text{supp}(r_1)$  such that  $e_1 \in \text{supp}(\nu(e_i))$ . This implies that  $e_1 \not\equiv_I e_i$ , and since any two elements in  $\text{supp}(r_1)$  are  $\equiv_I$ -equivalent, this forces  $e_1 \notin \text{supp}(r_1)$ . Hence  $e_i, e_1 \in \text{supp}(r'_1) = \text{supp}(r_1) \sqcup \{e_1\}$  and therefore  $e_i \equiv_J e_1$ . This contradicts  $e_1 \in \text{supp}(\nu(e_i))$  and shows that  $\lambda = 0$ , that  $r_1 = r'_1$  and that  $\nu(r_1) = 0$ .

d) Let us show that  $r_2 = r'_2, \dots, r_p = r'_p$ . For this purpose we will apply the induction hypothesis to  $\bar{E} = E/k.r_1$ . Let  $q: E \twoheadrightarrow \bar{E}$  be the natural projection. We will write  $\bar{e}_i$  (resp.  $\bar{I}, \bar{J}, \bar{r}_j, \bar{r}'_j$ ) for  $q(e_i)$  (resp.  $q(I), q(J), q(r_j), q(r'_j)$ ). Hence  $\bar{E}$  has a totally ordered basis:  $\bar{e}_1 < \dots < \bar{e}_{i_1-1} < \bar{e}_{i_1+1} < \dots < \bar{e}_n$ . Since  $\nu(r_1) = 0$  and since  $\psi(r_1) = r_1$ , the mappings  $\nu$  and  $\psi$  induce linear mappings  $\bar{\nu}, \bar{\psi}: \bar{E} \rightarrow \bar{E}$ . These mappings obviously satisfy  $\bar{\psi}(\bar{I}) = \bar{J}$ ,  $\bar{\nu}(\bar{e}_i) \in \text{Span}(\bar{e}_j; j \neq i_1 \text{ and } j < i)$  for any  $i \neq i_1$ , and  $\bar{\psi} = \text{exp}(\bar{\nu})$ . Moreover, our choice for the basis of  $\bar{E}$  and the definition of the Gröbner basis of  $I$  imply that:

- $\text{supp}(\bar{r}_j) = \{\bar{e}_i \mid e_i \in \text{supp}(r_j)\}$  for any  $j \geq 2$ ,
- $\bar{r}_2 < \dots < \bar{r}_p$  is the Gröbner basis of  $\bar{I}$ .

These two properties imply in particular that:  $\bar{e}_i \equiv_{\bar{I}} \bar{e}_j \Rightarrow e_i \equiv_I e_j$  for any  $i, j \neq i_1$ . The corresponding properties hold for  $\bar{J}$  (replace  $r_j$  by  $r'_j$ ,  $I$  by  $J$  and  $\bar{I}$  by  $\bar{J}$ ). Thus, in order to apply the induction hypothesis to  $\bar{E}$  it only remains to prove that:  $\bar{e}_j \in \text{supp}(\bar{\nu}(\bar{e}_i)) \Rightarrow \bar{e}_i \not\equiv_{\bar{I}} \bar{e}_j$  and  $\bar{e}_i \not\equiv_{\bar{J}} \bar{e}_j$  for any  $i, j \neq i_1$ . Assume that  $i, j \neq i_1$  satisfy  $\bar{e}_j \in \text{supp}(\bar{\nu}(\bar{e}_i))$ . From the definition of  $\bar{E}$  and  $\bar{\nu}$  we know that:

- $\text{supp}(\bar{\nu}(\bar{e}_i)) = \{\bar{e}_l \mid e_l \in \text{supp}(\nu(e_i))\}$  if  $e_{i_1} \notin \text{supp}(\nu(e_i))$ ,
- $\text{supp}(\bar{\nu}(\bar{e}_i)) \subseteq \{\bar{e}_l \mid e_l \in \text{supp}(\nu(e_i)) \text{ and } l \neq i_1\} \cup \{\bar{e}_l \mid l < i_1 \text{ and } e_l \in \text{supp}(r_1)\}$  if  $e_{i_1} \in \text{supp}(\nu(e_i))$ .

Let us distinguish the cases  $e_j \in \text{supp}(\nu(e_i))$  and  $e_j \notin \text{supp}(\nu(e_i))$ :

- if  $e_j \in \text{supp}(\nu(e_i))$  then  $e_i \not\equiv_I e_j$  and  $e_i \not\equiv_J e_j$  and the above comparison between  $\equiv_I$  (resp.  $\equiv_J$ ) and  $\equiv_{\bar{I}}$  (resp.  $\equiv_{\bar{J}}$ ) yields  $\bar{e}_i \not\equiv_{\bar{I}} \bar{e}_j$  and  $\bar{e}_i \not\equiv_{\bar{J}} \bar{e}_j$ .
- if  $e_j \notin \text{supp}(\nu(e_i))$  then necessarily  $e_{i_1} \in \text{supp}(\nu(e_i))$  and  $e_j \in \text{supp}(r_1)$ . Since  $r_1 = r'_1$ , the property  $e_j \in \text{supp}(r_1)$  implies that  $e_j \equiv_I e_{i_1}$  and  $e_j \equiv_J e_{i_1}$ . On the other hand, the property  $e_{i_1} \in \text{supp}(\nu(e_i))$  implies that  $e_{i_1} \not\equiv_I e_i$  and  $e_{i_1} \not\equiv_J e_i$ . Therefore  $e_j \not\equiv_I e_i$  and  $e_j \not\equiv_J e_i$  and finally  $\bar{e}_j \not\equiv_{\bar{I}} \bar{e}_i$  and  $\bar{e}_j \not\equiv_{\bar{J}} \bar{e}_i$ .

Thus all the conditions of Lemma 2.9 are satisfied for  $\bar{E}, \bar{I}, \bar{J}, \bar{\nu}$ . For this reason we can apply the induction hypothesis which gives:  $\bar{I}$  and  $\bar{J}$  have the same Gröbner basis. We infer that  $q(r_i) = q(r'_i)$  for each  $i = 2, \dots, p$ . Hence for each  $i \geq 2$  there exists  $\lambda_i \in k$  such that  $r_i = r'_i + \lambda_i r_1$ , and  $\lambda_i$  is necessarily zero because  $e_{i_1}^*(r_i) = e_{i_1}^*(r'_i) = 0$  (cf the definition of a Gröbner basis). Therefore  $r_i = r'_i$  for each  $i = 1, \dots, p$  and  $I = J$  as announced.  $\square$

**Lemma 2.10.** *Let  $\varphi: kQ \rightarrow kQ$  be an automorphism extending the identity map on  $Q_0$ . Let  $I$  be an admissible ideal of  $kQ$  and set  $J = \varphi(I)$ . Suppose that  $k$  has characteristic zero. Suppose that for any arrow  $\alpha$  there is a normal form  $\varphi(\alpha) = \alpha + \sum_i \lambda_i u_i$  where each  $u_i$  satisfies:  $\alpha \not\sim_I u_i$ ,  $\alpha \not\sim_J u_i$  and  $\varphi(a) = a$  for any arrow appearing in  $u_i$  (in particular  $\varphi(u_i) = u_i$ ). Then  $I$  and  $J$  coincide.*

**Proof:** Let  $E$  be the vector space  $kQ = \bigoplus_{x,y} ykQ_x$ . Hence  $E$  is finite dimensional since  $Q$  has no oriented cycle, and  $I$  and  $J$  can be considered as subspaces of  $E$ . In order to

apply Lemma 2.9 to  $E, I, J$ , we need to exhibit a totally ordered basis of  $E$  together with a mapping  $\nu: E \rightarrow E$ . Let us take the family of paths in  $Q$  for the basis of  $E$ . The following construction of a total order  $<$  on this basis is taken from [8]. Let us fix a total order on  $Q_0 \cup Q_1$  (which is finite) and let  $\prec$  be the induced lexicographical order on the paths in  $Q$  ( $e_x \prec u$  if  $u$  is non trivial). If  $u$  is a path we let  $W(u)$  be the number of arrows  $\alpha \in Q_1$  appearing in  $u$  and such that  $\varphi(\alpha) \neq \alpha$ . Hence, for any  $\alpha \in Q_1$ , we have  $W(\alpha) = 0$  if  $\varphi(\alpha) = \alpha$  and  $W(\alpha) = 1$  if  $\varphi(\alpha) \neq \alpha$ . The total order  $<$  is then defined as follows:

$$u < v \Leftrightarrow \begin{cases} W(u) < W(v) \\ \text{or} \\ W(u) = W(v) \text{ and } u \prec v \end{cases}$$

This yields:  $e_1 < \dots < e_n$  a totally ordered basis of  $E$  made of the paths in  $Q$ . Notice that with this basis, the equivalence relations  $\equiv_I$  and  $\sim_I$  (resp.  $\equiv_J$  and  $\sim_J$ ) satisfy the following property:  $e_i \equiv_I e_j \Rightarrow e_i \sim_I e_j$  (resp.  $e_i \equiv_J e_j \Rightarrow e_i \sim_J e_j$ ). Let  $\nu: kQ \rightarrow kQ$  be the derivation (i.e. the  $k$ -linear map such that  $\nu(vu) = \nu(v)u + v\nu(u)$  for any  $u$  and  $v$ ) such that  $\nu(e_x) = 0$  for any  $x \in Q_0$  and  $\nu(\alpha) = \varphi(\alpha) - \alpha$  for any arrow  $\alpha \in Q_1$ . Thus, for any path  $u$  and any  $v \in \text{supp}(\nu(u))$  there exist an arrow  $\alpha \in Q_1$  together with paths  $u_1, u_2, u_3$  such that  $u = u_3\alpha u_1$ ,  $v = u_3u_2u_1$  and  $u_2 \in \text{supp}(\nu(\alpha))$ . Notice that with the assumptions made on  $\varphi$ , this implies that  $e_i \not\equiv_I e_j$  and  $e_i \not\equiv_J e_j$  as soon as  $e_j \in \text{supp}(\nu(e_i))$ . Moreover, for any  $\alpha \in Q_1$  and any  $u \in \text{supp}(\nu(\alpha))$  we have  $W(u) = 0$ , hence  $\nu \circ \nu(\alpha) = 0$ . Since  $\nu: kQ \rightarrow kQ$  is a derivation, we infer that:  $e_j \in \text{supp}(\nu(e_i)) \Rightarrow W(e_j) < W(e_i) \Rightarrow e_j < e_i$ . Hence  $\nu(e_i) \in \text{Span}(e_j; j < i)$  for any  $i$ . In order to apply Lemma 2.9, it only remains to prove that  $J = \text{exp}(\nu)(I)$ . To do this it suffices to prove that  $\varphi = \text{exp}(\nu)$ . Since  $\nu$  is a derivation,  $\text{exp}(\nu): kQ \rightarrow kQ$  is an automorphism such that  $\text{exp}(\nu)(e_x) = e_x$  for any  $x \in Q_0$  (recall that  $\nu(e_x) = 0$ ). Moreover, if  $\alpha \in Q_1$  then  $\nu^2(\alpha) = 0$  and  $\nu(\alpha) = \varphi(\alpha) - \alpha$ , therefore  $\text{exp}(\nu)(\alpha) = \varphi(\alpha)$ . Hence  $\varphi$  and  $\text{exp}(\nu)$  are automorphisms of  $kQ$  and they coincide on  $Q_0 \cup Q_1$ . This implies that  $\varphi = \text{exp}(\nu)$ . Hence, the data  $E, I, J, \nu$  together with the ordered basis  $e_1 < \dots < e_n$  satisfy the hypotheses of Lemma 2.9 which implies that  $I = J$ .  $\square$

The uniqueness of the source of  $\Gamma$  is given by the following result.

**Proposition 2.11.** [16, 4.3] *Assume that  $A$  satisfies the hypotheses made before stating Theorem 1, then  $\Gamma$  has a unique source.*

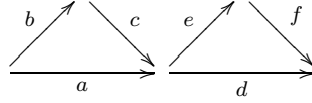
**Proof:** Notice that any two transvections of  $kQ$  commute since  $Q$  has no double bypass. Let  $\sim$  and  $\sim'$  be sources of  $\Gamma$ . Let  $I$  and  $J$  be admissible ideals of  $kQ$  such that  $kQ/I \simeq A \simeq kQ/J$  and such that  $\sim = \sim_I$  and  $\sim' = \sim_J$ . According to Remark 2 there exist a sequence of transvections  $\varphi_1 = \varphi_{\alpha_1, u_1, \tau_1}, \dots, \varphi_n = \varphi_{\alpha_n, u_n, \tau_n}$  of  $kQ$  and a dilatation  $D$  such that  $J = \varphi_n \dots \varphi_1 D(I)$ . Thanks to Proposition 2.5 we know that  $\sim_I = \sim_{D(I)}$ . Thus, in order to prove that  $\sim = \sim'$ , we may assume that  $D = \text{Id}_{kQ}$  and  $J = \varphi_n \dots \varphi_1(I)$ . Moreover we may assume that  $n$  is the smallest non negative integer such that there exist  $I, J$  and a sequence of transvections  $\varphi_1, \dots, \varphi_n$  satisfying  $\sim = \sim_I$ ,  $\sim' = \sim_J$  and  $J = \varphi_n \dots \varphi_1(I)$ . Let us prove that  $\alpha_i \not\sim_I u_i$  for any  $i \in \{1, \dots, n\}$ . If  $i$  is such that  $\alpha_i \sim_I u_i$  then Proposition 2.5 implies that  $\sim_I = \sim_{\varphi_i(I)}$  since  $\sim_I$  is a source of  $\Gamma$ . Hence  $\sim = \sim_{\varphi_i(I)}$ ,  $\sim' = \sim_J$  and  $J = \varphi_n \dots \varphi_{i+1} \varphi_{i-1} \dots \varphi_1(\varphi_i(I))$  which contradicts the minimality of  $n$ . Thus  $\alpha_i \not\sim_I u_i$  for any  $i$  and the same arguments apply to  $J$  since  $I = \varphi_1^{-1} \dots \varphi_n^{-1}(J)$  and since  $\sim_J$  is a source of  $\Gamma$ . Hence  $\alpha_i \not\sim_J u_i$  for any  $i$ . This shows that the data  $I, J, \varphi_n \dots \varphi_1$  satisfy the hypotheses of Lemma 2.10. We infer that  $I = J$  and that  $\sim = \sim'$ . This shows that  $\Gamma$  has a unique source.  $\square$

Proposition 2.11 and Remark 6 prove Theorem 1:

**Theorem 1.** (see also [16, Thm 1.1]) *Let  $A$  be a basic connected finite dimensional algebra over a field  $k$  of characteristic zero. If the quiver  $Q$  of  $A$  has no double bypass, then there exists a presentation  $kQ/I_0 \simeq A$  with quiver and admissible relations such that for any other admissible presentation  $kQ/I \simeq A$ , the identity map on walks induces a surjective group morphism  $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, I)$ .*

The following example shows that one cannot remove the hypothesis on the characteristic of  $k$  in Proposition 2.11:

**Example 3.** *Let  $Q$  be the following quiver without double bypass:*



Set  $u = cb$  and  $v = fe$ . Set  $A = kQ/I_0$  where  $I_0 = \langle da + vu, va + du \rangle$ . Then  $\pi_1(Q, I_1) = \mathbb{Z}/2$ . Let  $I_1$  and  $I_2$  be the ideals defined below:

- $I_1 = \varphi_{a,u,1}(I_0) = \langle da + du + vu, va + du + vu \rangle$ ,
- $I_2 = \varphi_{a,u,-1} \circ \varphi_{d,v,-1}(I_0) = \langle da, va + du - 2vu \rangle$ .

Hence  $A \simeq kQ/I_1 \simeq kQ/I_2$ . If  $\text{car}(k) = 0$ , then  $\pi_1(Q, I_1) = \pi_1(Q, I_2) = 1$  and  $\Gamma$  is equal to  $\sim_{I_0} \longrightarrow \sim_{I_1}$ . Suppose now that  $\text{car}(k) = 2$ . Then  $I_2 = \langle da, va + du \rangle$ ,  $\pi_1(Q, I_0) \simeq \mathbb{Z}/2$ ,  $\pi_1(Q, I_1) = 1$ ,  $\pi_1(Q, I_2) \simeq \mathbb{Z}$  and  $\Gamma$  is equal to  $\sim_{I_0} \rightarrow \sim_{I_1} \leftarrow \sim_{I_2}$ . Hence  $\Gamma$  has two sources. Notice that the identity map on walks induces a surjective group morphism  $\pi_1(Q, I_2) \twoheadrightarrow \pi_1(Q, I_0)$ . Notice also that one can build similar examples for any non zero value  $p$  of  $\text{car}(k)$  by taking for  $Q$  a sequence of  $p$  bypasses.

### 3 Preliminaries on covering functors

In this section we give some useful facts on covering functors.

**Lemma 3.1.** *Let  $p: \mathcal{E} \rightarrow \mathcal{B}$  and  $q: \mathcal{E}' \rightarrow \mathcal{B}$  be covering functors where  $\mathcal{E}$  is connected. Let  $r, r': \mathcal{E} \rightarrow \mathcal{E}'$  be such that  $q \circ r = q \circ r' = p$ . If there exists  $x_0 \in \mathcal{E}_0$  such that  $r(x_0) = r'(x_0)$  then  $r = r'$ .*

**Proof:** Since  $q$  is a covering functor, for any  $u \in {}_y\mathcal{E}_x \setminus \{0\}$  we have:

$$(r(x) = r'(x) \text{ or } r(y) = r'(y)) \Rightarrow (r(u) = r'(u), r(x) = r'(x) \text{ and } r(y) = r'(y)) \quad (\star)$$

Assume that there exists  $x_0 \in \mathcal{E}_0$  such that  $r(x_0) = r'(x_0)$ . Since  $\mathcal{E}$  is connected, for any  $x \in \mathcal{E}_0$  there exists a sequence  $x_0, \dots, x_n = x$  of objects of  $\mathcal{E}$  together with a non zero morphism between  $x_i$  and  $x_{i+1}$  for any  $i$ . This implies (thanks to  $(\star)$ ) that  $r(x) = r'(x)$ . Thus  $r$  and  $r'$  coincide on  $\mathcal{E}_0$ , and  $(\star)$  implies  $r = r'$ .  $\square$

The following proposition generalises the result [17, Prop. 3.3]. Using Lemma 3.1 its proof is immediate.

**Proposition 3.2.** *Let  $F: \mathcal{E} \rightarrow \mathcal{B}$  be a covering functor where  $\mathcal{E}$  is connected. Then  $\mathcal{E}$  is an  $\text{Aut}(F)$ -category. Moreover,  $F$  is a Galois covering if and only if  $\text{Aut}(F)$  acts transitively on each  $F^{-1}(x)$ . Finally, if  $F$  is Galois covering with group  $G$ , then  $G = \text{Aut}(F)$ .*



**Proposition 3.3.** *Let  $p: \mathcal{E} \rightarrow \mathcal{B}$  and  $q: \mathcal{F} \rightarrow \mathcal{E}$  be functors where  $\mathcal{E}$  is connected and set  $r = p \circ q: \mathcal{F} \rightarrow \mathcal{B}$ . Then  $p, q, r$  are covering functors as soon as two of them are so.*

**Proof:** Using basic linear algebra arguments, it is easy to verify the two following facts: if  $p$  and  $q$  (resp.  $q$  and  $r$ ) are covering functors then so is  $r$  (resp.  $p$ ); if  $p$  and  $r$  are covering functors, then  $q$  satisfies the condition b) in the definition of a covering functor. Assume that  $p$  and  $r$  are covering functors. We only need to prove that  $q^{-1}(x) \neq \emptyset$  for any  $x \in \mathcal{E}_0$ . The condition b) satisfied by  $q$  implies that  $q^{-1}(x) \neq \emptyset \Leftrightarrow q^{-1}(y) \neq \emptyset$  as soon as  ${}_y\mathcal{E}_x \neq 0$ . Since  $\mathcal{E}$  is connected and  $q^{-1}(q(x)) \neq \emptyset$  for any  $x \in \mathcal{F}_0$  we deduce that  $q^{-1}(x) \neq \emptyset$  for any  $x \in \mathcal{E}_0$ .  $\square$

**Proposition 3.4.** *Let  $p: \mathcal{C} \rightarrow \mathcal{B}$  (resp.  $q: \mathcal{C}' \rightarrow \mathcal{B}$ ) be a connected Galois covering with group  $G$  (resp.  $G'$ ) and assume there exists a commutative diagram of  $k$ -categories and  $k$ -linear functors where  $\varphi$  is an isomorphism extending the identity map on  $\mathcal{B}_0$ :*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r} & \mathcal{C}' \\ p \downarrow & & \downarrow q \\ \mathcal{B} & \xrightarrow[\varphi]{\sim} & \mathcal{B} \end{array}$$

*Then there exists a unique mapping  $\lambda: G \rightarrow G'$  such that  $r \circ g = \lambda(g) \circ r$  for any  $g \in G$ . Moreover  $\lambda$  is a surjective morphism of groups and  $r$  is a Galois covering with group  $\text{Ker}(\lambda)$ .*

**Proof:** Thanks to Proposition 3.3,  $r$  is a covering functor. Fix  $\hat{x}_0 \in \mathcal{C}$  and set  $x_0 = p(\hat{x}_0)$ . For any  $g \in \text{Aut}(p)$  we have  $q(r(\hat{x}_0)) = x_0 = q(r(g(\hat{x}_0)))$ . Since  $q$  is Galois with group  $G'$ , there exists a unique  $\lambda(g) \in G'$  such that  $\lambda(g)(r(\hat{x}_0)) = r(g(\hat{x}_0))$ , and Lemma 3.1 yields  $\lambda(g) \circ r = r \circ g$ . Hence:  $(\forall g \in G) (\exists! \lambda(g) \in G') \lambda(g) \circ r = r \circ g$ . This last property shows the existence and the uniqueness of  $\lambda$ . It also shows that  $\lambda: G \rightarrow G'$  is a group morphism and that  $\text{Aut}(r) = \text{Ker}(\lambda)$ . Moreover,  $\lambda$  is surjective because of its definition and because  $p$  is Galois with group  $G$ . Finally, Proposition 3.2 shows that  $r$  is a Galois covering with group  $\text{Ker}(\lambda)$ .  $\square$

## 4 The universal cover of an algebra

In this section we will prove Theorem 2. Let  $Q$  be a connected quiver without oriented cycle and fix  $x_0 \in Q_0$  for the computation of the groups  $\pi_1(Q, I)$ . If there is no ambiguity we shall write  $[w]$  for the homotopy class of a walk  $w$ .

**Lemma 4.1.** *Let  $I$  be an admissible ideal of  $kQ$ , let  $D$  be a dilatation of  $kQ$  and set  $J = D(I)$ . Let  $\lambda: \pi_1(Q, I) \xrightarrow{\sim} \pi_1(Q, J)$  be the isomorphism given by Proposition 2.5. Let  $p: (\tilde{Q}, \tilde{I}) \rightarrow (Q, I)$  (resp.  $q: (\hat{Q}, \hat{J}) \rightarrow (Q, J)$ ) be the universal Galois covering with group  $\pi_1(Q, I)$  (resp.  $\pi_1(Q, J)$ ). Then there exists an isomorphism  $\psi: k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$  such that the following diagram commutes:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I} & \xrightarrow{\psi} & k\hat{Q}/\hat{J} \\ \bar{p} \downarrow & & \downarrow \bar{q} \\ kQ/I & \xrightarrow{\bar{D}} & kQ/J \end{array}$$

*where  $\bar{D}, \bar{p}$  and  $\bar{q}$  are induced by  $D, p$  and  $q$  respectively. Moreover,  $\psi$  satisfies:  $\psi \circ g = \lambda(g) \circ \psi$  for any  $g \in \pi_1(Q, I)$ .*

**Proof:** We have  $\hat{Q} = \tilde{Q}$  since  $\sim_I$  and  $\sim_J$  coincide (see Proposition 2.5). Set  $\hat{D}: k\tilde{Q} \rightarrow k\hat{Q}$  to be defined by:  $\hat{D}(a, [w]) = (D(a), [w])$  for any arrow  $(a, [w]) \in \tilde{Q}_1$ . By construction  $\hat{D}$  is an automorphism of  $kQ$  and  $\hat{D}(I) = \hat{J}$ . Set  $\psi: k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$  to be induced by  $\hat{D}$ . It is then easy to check all announced properties.  $\square$

**Lemma 4.2.** *Let  $I$  be an admissible ideal of  $kQ$ , let  $\varphi = \varphi_{\alpha, u, \tau}$  be a transvection, set  $J = \varphi(I)$  and assume that  $\alpha \sim_J u$ . Let  $\lambda: \pi_1(Q, I) \rightarrow \pi_1(Q, J)$  be the surjection given by Proposition 2.5. Denote by  $p: (\tilde{Q}, \tilde{I}) \rightarrow (Q, I)$  (resp. by  $q: (\hat{Q}, \hat{J}) \rightarrow (Q, J)$ ) the universal Galois covering with group  $\pi_1(Q, I)$  (resp.  $\pi_1(Q, J)$ ). Then there exists a Galois covering  $\psi: k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$  with group  $\text{Ker}(\lambda)$  and such that the following diagram commutes:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I} & \xrightarrow{\psi} & k\hat{Q}/\hat{J} \\ \bar{p} \downarrow & & \downarrow \bar{q} \\ kQ/I & \xrightarrow{\bar{\varphi}} & kQ/J \end{array}$$

where  $\bar{\varphi}, \bar{p}$  and  $\bar{q}$  are induced by  $\varphi, p$  and  $q$  respectively.

Moreover,  $\psi$  satisfies:  $\psi \circ g = \lambda(g) \circ \psi$  for any  $g \in \pi_1(Q, I)$ .

**Proof:** Let  $\varphi': k\tilde{Q} \rightarrow k\hat{Q}$  be defined by:  $\varphi'([w]) = [w]$  for any  $[w] \in \tilde{Q}_0$ ,  $\varphi'(\beta, [w]) = (\beta, [w])$  for any  $(\beta, [w]) \in \tilde{Q}_1$  such that  $\beta \neq \alpha$ , and  $\varphi'(\alpha, [w]) = (\alpha, [w]) + \tau(u, [w])$  for any  $(\alpha, [w]) \in \tilde{Q}_1$ . Then  $\varphi'$  is well defined since  $\alpha \sim_J u$ . Moreover,  $\varphi \circ p(a) = q \circ \varphi'(a)$  for any  $a \in \tilde{Q}_1$ , and  $\varphi'(\tilde{I}) \subseteq \hat{J}$ . Let  $\psi: k\tilde{Q}/\tilde{I} \rightarrow k\hat{Q}/\hat{J}$  be induced by  $\varphi'$ . Thus  $\bar{q} \circ \psi = \bar{\varphi} \circ \bar{p}$ . Let  $g = [\gamma] \in \pi_1(Q, I)$  and let  $[w] \in \tilde{Q}_0$ . Then  $\psi \circ g([w]) = \psi([w\gamma^{-1}]) = [w\gamma^{-1}] = \lambda(g)([w]) = \lambda(g) \circ \psi([w])$ . The Lemma 3.1 implies that  $\psi \circ g = \lambda(g) \circ \psi$  for any  $g \in \pi_1(Q, I)$ . Finally, Proposition 3.4 gives:  $\psi$  is a Galois covering with group  $\text{Ker}(\lambda)$ .  $\square$

**Lemma 4.3.** *Let  $A$  be a basic and connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$ . Assume that  $k$  has characteristic zero and that  $Q$  has no double bypass. Let  $\sim_{I_0}$  be the unique source of  $\Gamma$  and  $\sim_I$  be a vertex of  $\Gamma$ . Then there exist a sequence  $\varphi_1, \dots, \varphi_n$  ( $\varphi_i = \varphi_{\alpha_i, u_i, \tau_i}$ ) of transvections and a dilatation  $D$  such that:*

- a)  $I = D\varphi_n \dots \varphi_1(I_0)$ ,
- b) if  $I_i$  is the ideal  $\varphi_i \dots \varphi_1(I_0)$  then  $\alpha_i \sim_{I_i} u_i$ .

**Proof:** We shall write  $[n]$  for the set  $\{1, \dots, n\}$ . Remark 2 implies that  $I = D\psi_1 \dots \psi_m(I_0)$  where the  $\psi_i$ 's are transvections and  $D$  is a dilatation. Set  $J = D^{-1}(I) = \psi_1 \dots \psi_m(I_0)$ . Thus we only need to prove that the conclusion of Lemma 4.3 holds for  $J$ . Let  $R_m$  be the property: "If  $J$  is the image of  $I_0$  by a product of  $m$  transvections, then there exists a sequence  $\varphi_1, \dots, \varphi_n$  of transvections such that  $J = \varphi_n \dots \varphi_1(I_0)$  and which satisfies the property b) of Lemma 4.3". Let us prove that  $R_m$  is true by induction on  $m \geq 0$ . Obviously  $R_0$  is true, let  $m \geq 1$  and let us assume that  $R_{m-1}$  is true. Let  $J = \psi_1 \dots \psi_m(I_0)$  where  $\psi_i = \varphi_{a_i, v_i, t_i}$ . Assume first that there exists  $i_0 \in [m]$  such that  $a_{i_0} \sim_J v_{i_0}$ . Set  $J' = \psi_1 \dots \psi_{i_0-1} \psi_{i_0+1} \dots \psi_m(I_0)$ . Thanks to  $R_{m-1}$ , there exists a sequence  $\varphi_1, \dots, \varphi_n$  of transvections such that  $J' = \varphi_n \dots \varphi_1(I_0)$  and which satisfies the property b) of Lemma 4.3. The sequence  $\varphi_1, \dots, \varphi_n, \psi_{i_0}$  shows that  $R_m$  is true when such an  $i_0$  exists. Assume now that for any  $i \in [m]$  we have  $a_i \not\sim_J v_i$ . Let  $\varphi = \psi_m \dots \psi_1$ . Lemma 2.10, applied to the data  $I_0, J, \varphi$ , shows that  $J = I_0$ . Hence  $R_m$  is true (with  $n = 0$ ) in this situation as well. This achieves the proof of Lemma 4.3.  $\square$

The following proposition shows how a Galois covering of locally bounded  $k$ -category is induced by a covering of quivers with relations. Notice that this proposition makes no assumption on the ordinary quiver of the involded  $k$ -categories (in particular, the quiver may have loops, oriented cycles, multiple arrows...). It generalises [17, prop 3.4, 3.5]. The proof uses the ideas presented in [11, sect. 3].

**Proposition 4.4.** *Let  $F: \hat{\mathcal{C}} \rightarrow \mathcal{C}$  be a Galois covering with group  $G$  where  $\mathcal{C}$  and  $\hat{\mathcal{C}}$  are locally bounded. Then, there exist admissible presentations  $\varphi: kQ/I' \xrightarrow{\sim} \mathcal{C}$  and  $\psi: k\hat{Q}/\hat{I} \xrightarrow{\sim} \hat{\mathcal{C}}$  and a covering of quiver with relations  $p: (\hat{Q}, \hat{I}) \rightarrow (Q, I')$ , such that  $\varphi$  restricts to the identity map  $Q_0 = \mathcal{C}_0 \rightarrow \mathcal{C}_0$  on  $\mathcal{C}_0$  and such that the following diagram is commutative:*

$$\begin{array}{ccc} k\hat{Q}/\hat{I} & \xrightarrow{\psi} & \hat{\mathcal{C}} \\ \bar{p} \downarrow & & \downarrow F \\ kQ/I' & \xrightarrow{\varphi} & \mathcal{C} \end{array}$$

where  $\bar{p}$  is induced by  $p$ . If  $\hat{\mathcal{C}}$  is connected, then  $p$  is Galois with group  $G$ .

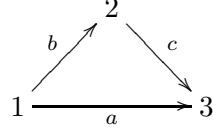
**Proof:** Using [7, Thm 3.8] we may assume that  $\mathcal{C}$  is  $G$ -graded, that  $\mathcal{C}' = \mathcal{C} \sharp G$  and that  $\mathcal{C}' = \mathcal{C} \sharp G \xrightarrow{F} \mathcal{C}$  is the natural projection. Since  $\hat{\mathcal{C}}$  and  $\mathcal{C}$  are locally bounded, [6, 3.3] implies that any morphism in  $\mathcal{RC}$  is the sum of images (under  $F$ ) of morphisms in  $\mathcal{R}\hat{\mathcal{C}}$ . Since the image under  $F$  of a morphism in  $\hat{\mathcal{C}}$  is a homogeneous morphism in  $\mathcal{C}$ , we deduce that the ideals  $\mathcal{RC}$  and  $\mathcal{R}^2\mathcal{C}$  are homogeneous. Thus, for any  $x \neq y \in Q_0$  there exist homogeneous elements  ${}_y u_x^{(1)}, \dots, {}_y u_x^{(y n_x)}$  of  ${}_y \mathcal{RC}_x$  giving rise to a basis of  ${}_y (\mathcal{RC}/\mathcal{R}^2\mathcal{C})_x$ . In particular,  ${}_y n_x$  is equal to the number of arrows  $x \rightarrow y$  in  $Q$ . Let  $\mu: kQ \rightarrow \mathcal{C}$  be defined as follows:  $\mu(x) = x$  for any  $x \in Q_0 = \mathcal{C}_0$ , and  $\mu$  induces a bijection between the set of arrows  $x \rightarrow y$  of  $Q$  and  $\{{}_y u_x^{(1)}, \dots, {}_y u_x^{(y n_x)}\}$  for any  $x \neq y \in Q_0$ . Set  $I' = \text{Ker}(\mu)$ . Hence  $I'$  is admissible and  $\mu$  induces an isomorphism  $\varphi: kQ/I' \xrightarrow{\sim} \mathcal{C}$ . The following construction of  $p$  uses the ideas of Green in [11, Sect. 3]. The  $k$ -category  $kQ$  is a  $G$ -graded as follows: a path  $u$  in  $Q$  is homogeneous of degree the degree of  $\mu(u)$ . By using the  $G$ -grading on  $\mathcal{C}$ , it is easy to check that  $I'$  is homogeneous and that  $\varphi: kQ/I' \rightarrow \mathcal{C}$  is homogeneous of degree  $1_G$ . Let  $\hat{Q}$  be the quiver as follows:  $\hat{Q}_0 = Q_0 \times G$ , and the arrows  $(x, s) \xrightarrow{\alpha} (y, t)$  in  $\hat{Q}_1$  are exactly the arrows  $x \xrightarrow{\alpha} y$  in  $Q_1$  with degree  $t^{-1}s$ . Let  $p: \hat{Q} \rightarrow Q$  be defined by:  $p(x, s) = x$  and  $p((x, s) \xrightarrow{\alpha} (y, t)) = \alpha$  for any  $(x, s) \in \hat{Q}_0$  and any  $(x, s) \xrightarrow{\alpha} (y, t) \in \hat{Q}_1$ . Let  $\hat{I} \subseteq \hat{Q}$  be the admissible ideal  $p^{-1}(I')$  of  $k\hat{Q}$ . According to [11, Sect. 3],  $p$  is a covering, and if  $\hat{Q}$  is connected then  $p$  is Galois with group  $G$ . In particular  $\bar{p}: k\hat{Q}/\hat{I} \rightarrow kQ/I'$  is a covering functor. Let  $\nu: k\hat{Q} \rightarrow \mathcal{C}' = \mathcal{C} \sharp G$  be as follows:  $\nu(x, s) = (\varphi(x), s)$  for any  $(x, s) \in \hat{Q}_0$ , and if  $(x, s) \xrightarrow{\alpha} (y, t) \in \hat{Q}_1$  then  $\nu(\alpha) = \mu(p(\alpha)) \in {}_{\varphi(y)} \mathcal{C}'_{\varphi(x)} = {}_{\varphi(y), t} \mathcal{C}'_{\varphi(x), s}$ . Therefore  $F \circ \nu = \varphi \circ p$ , and since  $\varphi$  is an isomorphism, we have  $\hat{I} = \text{Ker}(\nu)$ . Let  $\psi: k\hat{Q}/\hat{I} \rightarrow \mathcal{C}'$  be induced by  $\nu$ . Hence  $\psi: \hat{Q}_0 \rightarrow \hat{\mathcal{C}}_0$  is bijective,  $\psi$  is faithful and  $\varphi \circ \bar{p} = F \circ \psi$ . Moreover  $\psi$  is full because  $\bar{p}$  and  $F$  are covering functors. Thus,  $\psi$  is an isomorphism. Finally, if  $\mathcal{C}'$  is connected then  $\hat{Q}$  is connected and this implies that  $p$  is a Galois covering with group  $G$ .  $\square$

Recall from [4] that a triangular algebra is called simply connected if the fundamental group of any presentation of this algebra is trivial. It was proved in [19] that a triangular algebra is simply connected if and only if it has no proper connected Galois covering. The following corollary generalises this characterisation to non triangular algebras, it is a direct consequence of Proposition 4.4

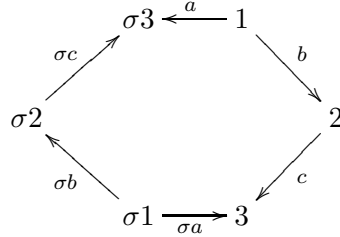
**Corollary 4.5.** *Let  $A$  be a finite dimensional basic and connected  $k$ -algebra. Then the following assertions are equivalent:*

- . for any admissible presentation  $kQ/I \simeq A$  we have  $\pi_1(Q, I) = 1$ ,
- .  $A$  has no proper Galois covering  $\mathcal{C} \rightarrow A$  with  $\mathcal{C}$  connected and locally bounded.

**Remark 7.** *Proposition 4.4 does not necessarily hold when  $F$  is a covering functor and not a Galois covering. As an example, set  $\mathcal{C} = kQ$  where  $Q$  is equal to:*



set  $G = \mathbb{Z}/2 = \langle \sigma | \sigma^2 \rangle$  and set  $\mathcal{C}' = kQ'$  where  $Q'$  is the quiver:



Set  $F: \mathcal{C}' \rightarrow \mathcal{C}$  to be defined by:  $F(b) = F(\sigma b) = b$ ,  $F(c) = F(\sigma c) = c$ ,  $F(a) = a$  and  $F(\sigma a) = a + cb$ . Then  $F$  is a covering functor. The group  $\text{Aut}(F)$  is trivial therefore  $F$  is not Galois, and  $F$  cannot be induced by any covering of bound quivers. Notice that if  $F: \mathcal{C}' \rightarrow \mathcal{C}$  is a covering functor and if the ordinary quiver of  $\mathcal{C}$  has no bypass, then  $F$  is induced by a covering of bound quivers.

**Theorem 2.** *Assume that  $A$  satisfies the hypotheses made before stating Theorem 1. Let  $\varphi_0: kQ/I_0 \simeq A$  be an admissible presentation such that  $\sim_{I_0}$  is the source of  $\Gamma$ . Let  $(\tilde{Q}, \tilde{I}_0) \xrightarrow{p_0} (Q, I_0)$  be the universal Galois covering with group  $\pi_1(Q, I_0)$  and let  $k\tilde{Q}/\tilde{I}_0 \xrightarrow{\tilde{p}_0} kQ/I$  be induced by  $p_0$ . For any connected Galois covering  $F: \mathcal{C}' \rightarrow A$  with group  $G$  there exist an isomorphism  $kQ/I_0 \xrightarrow{\sim} A$  equal to  $\varphi_0$  on objects, a Galois covering  $F': k\tilde{Q}/\tilde{I}_0 \rightarrow \mathcal{C}'$  with group  $N$  a normal subgroup of  $\pi_1(Q, I_0)$  such that the following diagram commutes:*

$$\begin{array}{ccc}
 k\tilde{Q}/\tilde{I}_0 & \xrightarrow{F'} & \mathcal{C}' \\
 \tilde{p}_0 \downarrow & & \downarrow F \\
 kQ/I_0 & \xrightarrow{\sim} & A
 \end{array}$$

Moreover, there is an exact sequence of groups:  $1 \rightarrow N \rightarrow \pi_1(Q, I_0) \rightarrow G \rightarrow 1$ .

**Proof:** Let  $\mathcal{C}' \xrightarrow{F} A$  be a connected Galois covering with group  $G$ . Thanks to Proposition 4.4 we may assume that there exists a Galois covering (with group  $G$ ) of bound quivers  $(Q', I') \xrightarrow{q} (Q, I)$  such that:  $A = kQ/I$ ,  $\mathcal{C}' = kQ'/I'$  and  $F: \mathcal{C}' \rightarrow A$  is induced by  $q$ . Let  $(\hat{Q}, \hat{I}) \xrightarrow{p} (Q, I)$  be the universal Galois covering with group  $\pi_1(Q, I)$ . Thus (see [17]) there exists a Galois covering  $(\hat{Q}, \hat{I}) \xrightarrow{r} (Q', I')$  such that  $q \circ r = p$ . Hence we have a commutative diagram (denoted by  $\mathcal{D}$ ):

$$\begin{array}{ccccc}
 k\hat{Q}/\hat{I} & \xrightarrow{\tilde{r}} & kQ'/I' & \xrightarrow{\sim} & \mathcal{C}' \\
 \tilde{p} \downarrow & & \tilde{q} \downarrow & & \downarrow F \\
 kQ/I & \xrightarrow{Id} & kQ/I & \xrightarrow{\sim} & A
 \end{array}$$

Since  $\sim_{I_0}$  is the source of  $\Gamma$ , Lemma 4.3 implies that there exist both a sequence of transvections  $\varphi_1 = \varphi_{\alpha_1, u_1, \tau_1}, \dots, \varphi_n = \varphi_{\alpha_n, u_n, \tau_n}$  of  $kQ$  and a dilatation  $D$  such that  $I = D\varphi_n \dots \varphi_1(I_0)$  and such that  $\alpha_i \sim_{I_i} u_i$  if  $I_i = \varphi_i \dots \varphi_1(I_0)$  for any  $i$ . Lemma 4.1 and Lemma 4.2 applied to  $D, I, I_n$  and  $\varphi_i, I_{i-1}, I_i$  respectively yield the following commutative diagrams denoted by  $\mathcal{D}'$  and  $\mathcal{T}_i$  respectively:

$$\begin{array}{ccc} kQ^{(n)}/I^{(n)} & \longrightarrow & k\hat{Q}/\hat{I} & & kQ^{(i-1)}/I^{(i-1)} & \longrightarrow & kQ^{(i)}/I^{(i)} \\ \downarrow \bar{p}_n & & \downarrow \bar{p} & & \downarrow \bar{p}_{i-1} & & \downarrow \bar{p}_i \\ kQ/I_n & \xrightarrow{\bar{D}} & kQ/I & & kQ/I_{i-1} & \xrightarrow{\bar{\varphi}_i} & kQ/I_i \end{array}$$

where  $\bar{\varphi}_i$  (resp.  $\bar{D}$ ) is induced by  $\varphi_i$  (resp.  $D$ ) and  $kQ^{(i)}/I^{(i)} \xrightarrow{\bar{p}_i} kQ/I_i$  is induced by the universal Galois covering  $(Q^{(i)}, I^{(i)}) \xrightarrow{p_i} (Q, I_i)$  with group  $\pi_1(Q, I_i)$ . If we connect  $\mathcal{T}_1, \dots, \mathcal{T}_n, \mathcal{D}'$  and  $\mathcal{D}$  we get the announced commutative diagram. Finally the announced properties of  $F'$  are given by Proposition 3.4.  $\square$

**Remark 8.** Using the universal property in Theorem 2 it is easily verified that if there exists a Galois covering  $\mathcal{C}' \rightarrow A$  such that  $\mathcal{C}'$  is simply connected (i.e. the fundamental group of any presentation of  $\mathcal{C}'$  is trivial), then  $\mathcal{C}' \simeq k\tilde{Q}/\tilde{I}_0$ .

One may wish to use the more general framework of Galois categories (see [12]) in order to recover Theorem 1 and Theorem 2. Unfortunately this cannot be done in general because the category of covering functors with finite fibre of  $A$  may not be a Galois category as explained in the following example:

**Example 4.** Let  $A = kQ/I$  where  $Q$  is equal to

$$\begin{array}{ccccc} & & 2 & & 4 \\ & b \nearrow & & c \searrow & e \nearrow \\ 1 & \xrightarrow{a} & 3 & \xrightarrow{d} & 5 \\ & & & & f \searrow \end{array}$$

and  $I = \langle da, dcb + fea, fecb \rangle$ . Set  $G = \mathbb{Z}/2 = \langle \sigma | \sigma^2 \rangle$ . Let  $Q'$  be the quiver:

$$\begin{array}{ccccc} & & 1 & \xrightarrow{a} & \sigma 3 \\ & & \downarrow b & & \downarrow \sigma d \\ & & \sigma 1 & \xrightarrow{\sigma c} & \sigma 2 \\ & & \downarrow \sigma b & & \downarrow \sigma e \\ 2 & & \sigma 1 & \xrightarrow{\sigma a} & \sigma 2 \\ & & \downarrow \sigma a & & \downarrow \sigma f \\ & & 3 & \xrightarrow{d} & \sigma 5 \\ & & \downarrow c & & \downarrow \sigma f \\ & & \sigma 1 & \xrightarrow{\sigma a} & \sigma 2 \\ & & \downarrow \sigma a & & \downarrow \sigma f \\ & & 3 & \xrightarrow{d} & \sigma 5 \\ & & \downarrow c & & \downarrow \sigma f \\ & & 3 & \xrightarrow{e} & 4 \\ & & & & \downarrow f \end{array}$$

and set:

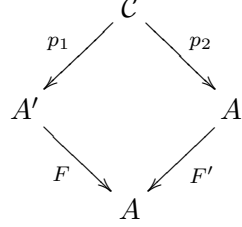
$$I' = \langle \sigma d a, d \sigma a, dcb + \sigma f \sigma e a, \sigma d \sigma c \sigma b + f e \sigma a, fecb, \sigma f \sigma e \sigma c \sigma b \rangle$$

Hence the natural mapping  $p: (Q', I') \rightarrow (Q, I)$  ( $x, \sigma x \mapsto x$ ) is a Galois covering with group  $G$ . Therefore, if we set  $A' = kQ'/I'$ , then  $p$  induces a Galois covering  $F: A' \rightarrow A$  with group  $G$ . If  $u$  is a path in  $Q'$  (resp. in  $Q$ ) we will write  $\tilde{u}$  (resp.  $\hat{u}$ ) for:  $u \text{ mod } I' \in A'$

(resp. for:  $u \bmod I \in A$ ). Let us set  $F': A' \rightarrow A$  to be the Galois covering with group  $G$  as well and defined as follows:

- $F'(\tilde{a}) = F'(\tilde{\sigma a}) = \hat{a} + \hat{c}b$ ,
- $F'(\tilde{x}) = F'(\tilde{\sigma x}) = \hat{x}$  for any  $x \in \{b, c, d, e, f\}$ .

Assume that the category of the coverings of  $A$  with finite fibre is a Galois category. Hence this category admits finite products and the product of  $F$  with  $F'$  gives rise to a diagram:



such that  $F'' = F \circ p_1 = F' \circ p_2$  is a covering functor with fibre the product of the fibres of  $F$  and  $F'$ . Hence  $\mathcal{C}_0 = Q'_0 \times_{Q_0} Q'_0 = \bigcup_{x \in Q_0} \{(x, x), (x, \sigma x), (\sigma x, x), (\sigma x, \sigma x)\}$ . Moreover, Proposition 3.3 implies that  $p_1$  and  $p_2$  are covering functors as well. Let us compute the lifting  $u$  of  $\hat{a} \in {}_3A_1$  w.r.t.  $F''$  and with source  $(1, 1)$ . Using the lifting property of  $p_1$  and  $p_2$  we get:

- $p_1(u_1) + p_1(u_2) = \tilde{a}$  where  $u_1 + u_2 \in {}_{(\sigma 3, 3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3, \sigma 3)}\mathcal{C}_{(1,1)}$ ,
- $p_2(v_1) + p_2(v_2) = \tilde{a}$  where  $v_1 + v_2 \in {}_{(3, \sigma 3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3, \sigma 3)}\mathcal{C}_{(1,1)}$ ,
- $p_2(v_3) + p_2(v_4) = \tilde{c}b$  where  $v_3 + v_4 \in {}_{(\sigma 3, 3)}\mathcal{C}_{(1,1)} \oplus {}_{(3, 3)}\mathcal{C}_{(1,1)}$ .

Since  $\hat{a} = F(\tilde{a}) = F'(\tilde{a} - \tilde{c}b)$ , we infer that  $u = u_1 + u_2 = v_1 + v_2 - v_3 - v_4$ . Therefore  $v_1 = v_4 = 0$ ,  $u_1 = -v_3$  and  $u_2 = v_2$ . Notice that  $v_3 \neq 0$  and  $v_2 \neq 0$  since  $\tilde{a} \neq 0$  and  $\tilde{c}b \neq 0$ . Therefore, the spaces  ${}_{(\sigma 3, 3)}\mathcal{C}_{(1,1)}$  and  ${}_{(\sigma 3, \sigma 3)}\mathcal{C}_{(1,1)}$  are non zero. Since  $p_1$  is a covering functor, we infer that  $p_1$  induces an inclusion  ${}_{(\sigma 3, \sigma 3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3, 3)}\mathcal{C}_{(1,1)} \hookrightarrow {}_{\sigma 3}A'_1$  of a space of dimension at least 2 in  ${}_{\sigma 3}A'_1 = k.\tilde{a}$ . This contradiction shows that  $F$  and  $F'$  do not have a product and that the category of the coverings of  $A$  with finite fibre need not be a Galois category.

We end this study with a final remark concerning monomial algebras. Recall that an algebra  $A$  is monomial if it admits a presentation  $kQ/I_0 \simeq A$  where  $I_0$  is generated by a set of paths. In such a case,  $\pi_1(Q, I_0) \simeq \pi_1(Q)$  (the fundamental group of  $Q$ ) and therefore the fundamental group of any other presentation of  $A$  is a quotient of  $\pi_1(Q, I_0)$ . Thus, Theorem 1 holds for  $A$  monomial without hypothesis on the characteristic of  $k$  or on the double bypasses in  $Q$ . Hence we can wonder if Theorem 2 holds for monomial algebras. This question will be studied in a subsequent text.

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