The universal cover of an algebra without double bypasses

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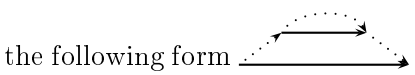
Abstract

Let A be a finite dimensional connected algebra over a field k of characteristic zero. We show that if the quiver of A has no double bypasses then the fundamental group (as defined in [14]) of any presentation of A by quiver and relations is the quotient of the fundamental group of a privileged presentation of A. We then show that the Galois covering of A associated with this privileged presentation satisfies a universal property with respect to the connected Galois coverings of A in a similar fashion to the universal cover of a topological space.

Introduction

Let A be a finite dimensional algebra over a field k. In order to study left A-modules we may assume that A is basic and connected, where basic means that A is the direct sum of pairwise non isomorphic projective left A-modules. For such an algebra, the study of the Galois coverings of A gives some information on the representation theory of A (see [6], [10] and [14]) and is a particular case of the covering techniques introduced in [5], [9] and [15]. Recall that in order to manipulate coverings of A we consider (and we always will unless otherwise stated) A as a k-category with set of objects a complete set $\{e_i\}_i$ of primitive pairwise orthogonal idempotents and with morphisms space $e_i \to e_j$ the vector space e_jAe_i. The covering techniques have led to the definition (see [10] and [14]) of a fundamental group associated with any presentation of A by quiver and admissible relations, and which satisfies many topological flavoured properties (see [1], [10] and [14]). This construction and its associated properties depend on the choice of a presentation of A. In particular, one can find algebras for which there exist different presentations giving rise to non isomorphic fundamental groups. In this text we compare the fundamental groups of the presentations of A as defined in [14], and we study the coverings of A with the following question in mind: does A have a universal Galois covering? i.e. does A admit a Galois covering which is factorised by any other Galois covering? This question has been successfully treated in the case A is representation finite (see [5] and [9]). The present study will involve quivers "without double bypasses". In simple terms, a quiver without double bypasses is a quiver which has no distinct parallel arrows, no oriented cycles and has no subquiver of the following form  where continued (resp. dotted) arrows represent arrows (resp. oriented paths) of the quiver. Assuming that k is a characteristic zero field and that the ordinary quiver Q of A has no double bypasses, we prove the following result announced in [13]:

Theorem 1. *Assuming the above conditions, there exists a presentation $kQ/I_0 \simeq A$ by quiver and relations such that for any other presentation $kQ/I \simeq A$, the identity map on the walks of Q induces a surjective group morphism $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, I)$.*

The proof of the above Theorem allows us to recover the following fact that was proven in [4]: if A is a basic triangular connected and constricted finite dimensional k-algebra, then different presentations of A give rise to isomorphic fundamental groups. Under the hypotheses made before stating Theorem 1 and with the same notations, if $k\tilde{Q}/\tilde{I}_0 \xrightarrow{F_0} kQ/I_0$ is the Galois covering with group $\pi_1(Q, I_0)$ induced by the universal Galois covering of (Q, I_0) (see [14]), we show the following result.

Theorem 2. *For any connected Galois covering $F\colon \mathcal{C}' \to A$ with group G there exist an isomorphism $kQ/I_0 \xrightarrow{\sim} A$, a Galois covering $p\colon k\tilde{Q}/\tilde{I}_0 \to \mathcal{C}'$ with group a normal subgroup N of $\pi_1(Q, I_0)$ and a commutative diagram:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I}_0 & \xrightarrow{p} & \mathcal{C}' \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F} \\ kQ/I_0 & \xrightarrow{\sim} & A \end{array}$$

together with an exact sequence of groups: $1 \to N \to \pi_1(Q, I_0) \to G \to 1$

Hence the Theorem 2 partially answers the question concerning the existence of a universal Galois covering. The text is organised as follows: in section 1 we define the notions we will use, in section 2 we prove Theorem 1, in section 3 we give useful results on covering functors, these results will be used in the proof of Theorem 2 to which section 4 is devoted. The section 2 gives the proofs of all the results that have been announced by the author in [13]. This text is part of the author's thesis made at Université Montpellier 2 under the supervision of Claude Cibils.

1 Basic definitions

k-categories, covering functors, Galois coverings

A **k-category** is a category $\mathcal{C}$ such that the objects class $\mathcal{C}_0$ of $\mathcal{C}$ is a non empty set and each set ${}_y\mathcal{C}_x$ of morphisms $x \to y$ of $\mathcal{C}$ is a k-vector space with k-bilinear composition. Let $\mathcal{C}$ be a k-category. We will say that $\mathcal{C}$ is **locally bounded** if the following properties are satisfied:

a) distinct objects are not isomorphic,
b) for each $x \in \mathcal{C}_0$, the k-algebra ${}_x\mathcal{C}_x$ is local,
c) $\oplus_{y \in \mathcal{C}_0}\ {}_y\mathcal{C}_x$ is finite dimensional for any $x \in \mathcal{C}_0$,
d) $\oplus_{x \in \mathcal{C}_0}\ {}_y\mathcal{C}_x$ is finite dimensional for any $y \in \mathcal{C}_0$.

Unless otherwise stated, all the k-categories we will introduce will be locally bounded. As an example, let A be a basic finite dimensional k-algebra, where basic means that A is the direct sum of pairwise non isomorphic projective left A-modules. If $1 = \sum_{i=1}^{n} e_i$ is a decomposition of the unit into a sum of pairwise orthogonal primitive idempotents, then $A = \oplus_{i,j} e_j A e_i$ and A is a locally bounded k-category with set of objects $\{e_1, \ldots, e_n\}$ and with morphisms space $e_i \to e_j$ equal to $e_j A e_i$. We will say that the k-category $\mathcal{C}$ is **connected** if for any $x, y \in \mathcal{C}_0$ there exists a sequence $x_0 = x, \ldots, x_n = y$ in $\mathcal{C}_0$ such that ${}_{x_i}\mathcal{C}_{x_{i+1}} \neq 0$ or ${}_{x_{i+1}}\mathcal{C}_{x_i} \neq 0$ for any i. Recall that an **ideal** I of $\mathcal{C}$ is the data of vector subspaces ${}_yI_x \subseteq {}_y\mathcal{C}_x$ for each $x, y \in \mathcal{C}_0$, such that the composition of a morphism

in I with any morphism of $\mathcal{C}$ lies in I. The **radical** (see [5]) of $\mathcal{C}$ is the ideal $\mathcal{RC}$ of $\mathcal{C}$ such that ${}_y\mathcal{RC}_x$ is the set of non invertible morphisms $x \to y$ for any $x, y \in \mathcal{C}_0$. If $n \geqslant 2$ we set $\mathcal{R}^n\mathcal{C} = (\mathcal{RC})^n$. The **ordinary quiver** of $\mathcal{C}$ has set of vertices $\mathcal{C}_0$, and for $x, y \in \mathcal{C}_0$ the number of arrows $x \to y$ is exactly $dim_k\ {}_y\mathcal{RC}_x/{}_y\mathcal{R}^2\mathcal{C}_x$. Finally, we say $\mathcal{C}$ is **triangular** if Q has no oriented cycles. All functors are assumed to be k-linear functors between k-categories.

A functor $F\colon \mathcal{E} \to \mathcal{B}$ is called a **covering functor** (see [5]) if the following properties are satisfied:

a) $F^{-1}(x) \neq \emptyset$ for any $x \in \mathcal{B}_0$,

b) for any $x_0, y_0 \in \mathcal{C}$ and any $\hat{x}_0, \hat{y}_0 \in \mathcal{E}_0$ such that $F(\hat{x}_0) = x_0$ and $F(\hat{y}_0) = y_0$, the following maps induced by F are isomorphisms:

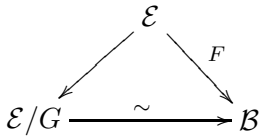
$$\bigoplus_{F(\hat{y})=y} {}_{\hat{y}}\mathcal{E}_{\hat{x}_0} \to {}_{y_0}\mathcal{B}_{x_0} \text{ and } \bigoplus_{F(\hat{x})=x} {}_{\hat{y}_0}\mathcal{E}_{\hat{x}} \to {}_{y_0}\mathcal{B}_{x_0}\ .$$

In particular, if $u \in {}_{y_0}\mathcal{B}_{x_0}$, the inverse images of u by these isomorphisms will be called **the lifting of** u (w.r.t. F) with source (resp. target) $\hat{x}_0$ (resp. $\hat{y}_0$). Recall that if $\mathcal{B}$ is locally bounded then $\mathcal{E}$ is locally bounded as well, recall also that if $\mathcal{E}$ is connected then so is $\mathcal{B}$.

A **G-category** is a k-category $\mathcal{C}$ endowed with $G \to Aut(\mathcal{C})$ a group morphism. If moreover the induced action of G on $\mathcal{C}_0$ is free, then $\mathcal{C}$ is called a **free G-category**. The **quotient category** $\mathcal{C}/G$ of a free G-category $\mathcal{C}$ (see [6] for instance) has set of objects $\mathcal{C}_0/G$. For any $\alpha, \beta \in C_0/G$ we set:

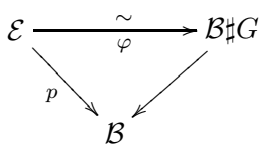
$${}_\beta(\mathcal{C}/G)_\alpha = \left(\bigoplus_{x\in\alpha, y\in\beta} {}_y\mathcal{C}_x \right) /G$$

and the composition is induced by the composition in $\mathcal{C}$. The natural projection $\mathcal{C} \to \mathcal{C}/G$ is a covering functor. A **Galois covering with group** G is a functor $F\colon \mathcal{E} \to \mathcal{B}$ with $\mathcal{E}$ a free G-category and such that there exists a commutative diagram:



where $\mathcal{E} \to \mathcal{E}/G$ is the natural projection and the horizontal arrow is an isomorphism. In particular a Galois covering is a covering functor. A **connected Galois covering** is a Galois covering $\mathcal{E} \to \mathcal{B}$ where $\mathcal{E}$ is connected.

A **G-graded category** is a k-category $\mathcal{C}$ such that each morphism space has a decomposition ${}_y\mathcal{C}_x = \oplus_{g\in G}\ {}_y\mathcal{C}_x^g$ satisfying ${}_z\mathcal{C}_y^g \cdot {}_y\mathcal{C}_x^h \subseteq {}_z\mathcal{C}_x^{gh}$. The **smash-product** category (see [6]) $\mathcal{C}\sharp G$ has set of objects $(\mathcal{C}\sharp G)_0 = \mathcal{C}_0 \times G$, and ${}_{(y,t)}(\mathcal{C}\sharp G)_{(x,s)} = {}_y\mathcal{C}_x^{t^{-1}s}$ for (x, s) and (y, t) in $(\mathcal{C}\sharp G)_0$. The composition in $\mathcal{C}\sharp G$ is induced by the composition in $\mathcal{C}$. The natural projection $F\colon \mathcal{C}\sharp G \to \mathcal{C}$, defined by $F(x, s) = x$ and $F(u) = u$ for $u \in {}_{(y,t)}(\mathcal{C}\sharp G)_{(x,s)} \subseteq {}_y\mathcal{C}_x$, is a Galois covering with group G. It has been shown in [6] that if $p\colon \mathcal{E} \to \mathcal{B}$ is a Galois covering with group G, then $\mathcal{B}$ is a G-graded category and there exists a commutative diagram:



where $\mathcal{B}\sharp G \to \mathcal{B}$ is the natural projection and φ is an isomorphism.

Quivers with admissible relations

Let Q be a **locally finite** quiver with set of vertices Q_0, set of arrows Q_1 and source and target map $s, t\colon Q_1 \to Q_0$ respectively. Recall that locally finite means that $s^{-1}(x)$ and $t^{-1}(x)$ are finite sets for any $x \in Q_0$. For simplicity we will write x^+ (resp. x^-) for the set $s^{-1}(x)$ (resp. $t^{-1}(x)$). A (non trivial) **oriented path** in Q is a non empty sequence $u_1, \dots, u_n$ of arrows of Q such that $s(u_{i+1}) = t(u_i)$ for any $1 \leqslant i \leqslant n-1$. Such a path is written $\alpha_n \dots \alpha_1$, its source (resp. target) is $s(\alpha_1)$ (resp. $t(\alpha_n)$). For each $x \in Q_0$ we will write e_x for the (trivial) path of length 0 and with source and target equal to x. The **path category** kQ has set of objects Q_0, the morphism space ${}_ykQ_x$ is the vector space with basis the set of oriented paths in Q with source x and target y (including e_x in case $x = y$). The composition of morphisms in kQ is induced by the concatenation of paths. Notice that kQ is a free k-category in the following sense: for any k-category $\mathcal{C}$, a functor $kQ \xrightarrow{F} \mathcal{C}$ is uniquely determined by by the family of morphisms $\{F(\alpha) \in {}_{F(y)}\mathcal{C}_{F(x)} \mid x \xrightarrow{\alpha} y \in Q_1\}$. We will denote by kQ^+ the ideal of kQ generated by Q_1. Notice also that if Q_0 is finite then kQ is also a k-algebra, $kQ = \oplus_{x,y}\, {}_ykQ_x$, with unit $1 = \sum_{x \in Q_0} e_x$, and kQ^+ becomes an ideal of this k-algebra. If $r \in {}_ykQ_x$ we call **support of** r (denoted by $supp(r)$) the set of paths in Q which appear in r with a non zero coefficient, and we call a **normal form of** r an equality of the type $r = \sum_i \lambda_i\, u_i$ such that $\lambda_i \in k^*$ for any i and where the paths u_i are pairwise distinct. An **admissible ideal** of kQ is an ideal $I \subseteq kQ$ such that $I \subseteq (kQ^+)^2$ and such that for any $x \in Q_0$ there exists $n \geqslant 2$ such that I contains all the paths with length at least n and with source or target x. The couple (Q, I) is then called a **quiver with admissible relations** and the quotient category kQ/I is locally bounded. When Q_0 is finite, an admissible ideal I of kQ is exactly an ideal I of the k-algebra kQ such that $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$ for some integer $n \geqslant 2$. Recall from [5] that if $\mathcal{C}$ is a locally bounded k-category then there exists an admissible ideal I for the ordinary quiver Q of $\mathcal{C}$ and there exists an isomorphism $kQ/I \xrightarrow{\sim} \mathcal{C}$. Such an isomorphism is called a **presentation of $\mathcal{C}$ with quiver and (admissible) relations** (or an **admissible presentation** for short). Similarly, if A is a finite dimensional and basic k-algebra, an admissible presentation of A is an isomorphism of k-algebras $kQ/I \xrightarrow{\sim} A$ where (Q, I) is a bound quiver.

Transvections, dilatations

A **bypass** (see [3]) of kQ is a couple (α, u) where $\alpha \in Q_1$ and $u \neq \alpha$ is a path in Q parallel to α (this means that α and u share the same source and the same target). A **double bypass** is a 4-tuple (α, u, β, v) such that (α, u) and (β, v) are bypasses and such that the arrow β appears in the path u. Notice that if α, β are distinct parallel arrows of Q, then $(\alpha, \beta, \beta, \alpha)$ is a double bypass. Notice also that if $u = va$ is an oriented cycle in Q with first arrow a, then (a, au, a, au) is a double bypass. Hence, if Q has no double bypassses, then Q has no distinct parallel arrows and no oriented cycles. If A is a basic k-algebra with quiver Q, we will say for short that A **has no double bypasses** if Q has no double bypasses. A **transvection** is an automorphism $\varphi_{\alpha,u,\tau}$ of the k-category kQ where (α, u) is a bypass, $\tau \in k$ and $\varphi_{\alpha,u,\tau}$ is given by $\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau\, u$ and $\varphi_{\alpha,u,\tau}(\beta) = \beta$ for any arrow $\beta \neq \alpha$ (this uniquely defines $\varphi_{\alpha,u,\tau}$ since kQ is a free k-category). Notice that Q has no double bypasses if and only if any two transvections commute. A **dilatation** is an automorphism $D\colon kQ \xrightarrow{\sim} kQ$ such that $D(\alpha) \in k^*\alpha$ for any arrow α. Notice that the definition of transvections and dilatations

are analogous to the one of transvection and dilatation matrices (see [12, Chap. XIII, § 9] for instance). Recall that a dilatation matrix of $Gl_n(k)$ is a diagonal invertible matrix and a transvection matrix is a matrix with diagonal entries equal to 1 and which has at most one non diagonal entry different from 0.

Fundamental group, coverings of quivers with relations

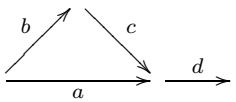
Let (Q, I) be a quiver with admissible relations. For each arrow $\alpha \in Q_1$ we will write α^{-1} for its formal inverse with source (resp. target) $s(\alpha^{-1}) = t(\alpha)$ (resp. $t(\alpha^{-1}) = s(\alpha)$). A walk is an unoriented path in Q, more precisely it is a formal product $u_n \dots u_1$ of arrows and of formal inverse of arrows such that $s(u_{i+1}) = t(u_i)$ for any $1 \leqslant i \leqslant n-1$. Let $r = t_1u_1 + \ldots + t_nu_n \in {}_yI_x$ where $t_i \in k^*$ and the u_i's are distinct paths. Then r is called a **minimal relation** if $n \geqslant 1$ and if for any non empty proper subset E of $\{1, \dots, n\}$ we have $\sum_{i\in E} t_iu_i \notin {}_yI_x$. With this definition, any $r \in I$ can be written as the sum of minimal relations with pairwise disjoint supports. Notice that in this definition we do not ask that $n \geqslant 2$ as done usually (see [14]). This change is done for simplicity and does not affect the constructions which follow. The **homotopy relation** of (Q, I) is the smallest equivalence relation $\sim_I$ on the set of walks (of Q) which is compatible with the concatenation of walks and such that:

. $\alpha\alpha^{-1} \sim_I e_y$ and $\alpha^{-1}\alpha \sim_I e_x$ for any arrow $x \xrightarrow{\alpha} y$,

. $u_1 \sim_I u_2$ for any minimal relation $t_1u_1 + \ldots + t_nu_n$.

Notice that in order to compute $\sim_I$ we may restrict ourselves to any set of minimal relations generating the ideal I (see [7]). Assume Q is connected (i.e. Q is connected as an unoriented graph) and let $x_0 \in Q_0$. The **fundamental group** (see [14]) $\pi_1(Q, I, x_0)$ of (Q, I) at x_0 is the set of $\sim_I$-classes of walks starting and ending at x_0. The composition is induced by the concatenation of walks and the unit is the $\sim_I$-class of e_{x_0}. Since different choices for x_0 give rise to isomorphic fundamental groups (since Q is connected) we will write $\pi_1(Q, I)$ for short.

Example 1. *(see [1]) Assume Q is the following quiver:*



and set $I =< da >$ and $J =< da - dcb >$. Then $kQ/I \simeq kQ/J$ whereas $\pi_1(Q, I) \simeq \mathbb{Z}$ and $\pi_1(Q, J) = 0$.

A **covering** $(Q', I') \xrightarrow{p} (Q, I)$ of quivers with admissible relations (see [14]) is a quiver morphism $Q' \xrightarrow{p} Q$ such that $p(I') \subseteq I$ and such that:

a) $p^{-1}(x) \neq \emptyset$ for any $x \in Q_0$,

b) $x^+ \xrightarrow{p} p(x)^+$ and $x^- \xrightarrow{p} p(x)^-$ are bijective for any $x \in Q'_0$,

c) for any minimal relation $r \in {}_yI_x$ and for any $x' \in p^{-1}(x)$ there exist $y' \in p^{-1}(y)$ and $r' \in {}_{y'}I'_{x'}$ such that $p(r') = r$,

d) same statement as c) after interchanging x and y.

Recall that the automorphism group $Aut(Q, I)$ of a bound quiver (Q, I) is the group of automorphisms $g\colon Q \xrightarrow{\sim} Q$ of the quiver Q such that $g(I) \subseteq I$. Assume that $p\colon (Q', I') \to (Q, I)$ is a covering, then the **group of automorphisms** $Aut(p)$ of p is defined by $Aut(p) = \{g \in Aut(Q', I') \mid p \circ g = p\}$. With this definition, if $p\colon (Q', I') \to (Q, I)$ is a covering and if G is a subgroup of $Aut(p)$, then p is called a **Galois covering with group** G if Q and Q' are connected and if G acts transitively on $p^{-1}(x)$ for any $x \in Q_0$. If $p\colon (Q', I') \to (Q, I)$ is a covering (resp. a Galois

covering with group G) then the induced functor $kQ'/I' \xrightarrow{\bar{p}} kQ/I$ is a covering functor (resp. a Galois covering with group G). Let (Q, I) be a connected quiver with admissible relations and let $x_0 \in Q_0$. The **universal cover of** (Q, I) is a Galois covering $(\tilde{Q}, \tilde{I}) \xrightarrow{\pi} (Q, I)$ with group $\pi_1(Q, I, x_0)$ defined in [14]. One can describe it as follows: $\tilde{Q}_0$ is the set of $\sim_I$-classes $[w]$ of walks w starting from x_0. The arrows of $\tilde{Q}$ are the couples $(\alpha, [w])$ where $\alpha \in Q_1$ and $[w] \in \tilde{Q}_0$ are such that $s(\alpha) = t(w)$. The source (resp. target) of the arrow $(\alpha, [w])$ is $[w]$ (resp. $[\alpha w]$). The map $\tilde{Q} \xrightarrow{p} Q$ is defined by $p([w]) = t(w)$ and $p(\alpha, [w]) = \alpha$. The ideal $\tilde{I}$ is equal to $p^{-1}(I)$. Finally, the action of $\pi_1(Q, I)$ on $(\tilde{Q}, \tilde{I})$ is the following: if $g \in \pi_1(Q, I)$ we may write $g = [\gamma]$ with γ some walk with source and target equal to x_0. Then for any $[w] \in \tilde{Q}_0$ (resp. $(\alpha, [w]) \in \tilde{Q}_1$) we have $g.[w] = [w\gamma^{-1}]$ (resp. $g.(\alpha, [w]) = (\alpha, [w\gamma^{-1}])$).

Some linear algebra

We introduce here some notions that will be useful in the sequel and freely used without reference. Let E be a finite dimensional k-vector space with a basis $(e_1, \ldots, e_n)$. We will denote by $(e_1^*, \ldots, e_n^*)$ the basis of E^* dual of $(e_1, \ldots, e_n)$ (i.e. $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ if $j \neq i$). If $\{r_t\}_{t \in T} \in E^T$ is a family in E, then $Span(r_t \ ; \ t \in T)$ will the denote the subspace of E generated by this family. If $r \in E$ we will write $supp(r)$ (the support of r) for the set of those $e_i's$ appearing in r with a non zero coefficient. Therefore $e_i \in supp(r)$ is equivalent to $e_i^*(r) \neq 0$. Let $F \subseteq E$ be a subspace. A non zero element $r \in F$ is called minimal if it cannot be written as the sum of two non zero elements of F with pairwise disjoint supports. We will denote by $\equiv_F$ the smallest equivalence relation on $\{e_1, \ldots, e_n\}$ such that $e_i \equiv_F e_j$ for any $r \in F$ minimal and any $e_i, e_j \in supp(r)$. Like in the situation of the homotopy relation of a bound quiver, the equivalence relation $\equiv_F$ is determined by set of the supports of a generating family of F. Notice that if E is the vector space with basis the set of oriented paths in a finite quiver Q and if I is an admissible ideal of kQ, then for any paths u and v we have: $u \equiv_I v \Rightarrow u \sim_I v$. The converse is usually false as one can see in Example 1 where $a \sim_J cb$ and $a \not\equiv_J cb$. Assume now that the basis of E is totally ordered: $e_1 < \ldots < e_n$. A **Groebner basis** of F is a basis $(r_1, \ldots, r_t)$ of F such that:

. for any j there is some i_j such that $r_j \in e_{i_j} + Vect(e_i \ ; \ i < i_j)$.

. $e_{i_j} \notin supp(r_{j'})$ unless $j = j'$.

. if $r = e_l + \sum_{i<l} \tau_i \ e_i \in F$ then $e_l = e_{i_j}$ for some j.

With this definition, F has a unique Groebner basis which has a natural total order: $r_1 < \ldots < r_t$ if we assume that $i_1 < \ldots < i_t$. Moreover, if $e_i, e_j \in supp(r_l)$ for some l then $e_i \equiv_I e_j$. This last property implies in particular that $e_i \equiv_F e_j$ if and only if there exists a sequence of integers $m_1, \ldots, m_p$ such that $e_i \in supp(r_{m_1})$, $e_j \in supp(r_{m_p})$ and $supp(r_{m_j}) \cap supp(r_{m_{j+1}}) \neq \emptyset$ for each j. Notice that our definition of the Groebner basis is weaker than the usual one since we do not assume that E has a multiplicative structure. For a general introduction to Groebner bases we refer the reader to [2]. Notice also that a study of Groebner bases in path algebras of quivers has been made in [8].

We end this paragraph with a reminder on the exponential and on the logarithm of an endomorphism. If $u \colon E \to E$ is a nilpotent endomorphism, we define the **exponential** of u to be $exp(u) = \sum_{l \geqslant 0} \frac{1}{l!} \ u^l$. Thus, $exp(u) \colon E \to E$ is a well defined linear isomorphism such that $exp(u) - Id$ is nilpotent. If $v \colon E \to E$ is an isomorphism such that $v - Id$ is nilpotent, we define the **logarithm** of v to be $log(v) = \sum_{l \geqslant 0} (-1)^{l+1} \frac{1}{l} \ (v - Id)^l$. Recall that if $u \colon E \to E$ is a nilpotent endomorphism, then $log(exp(u)) = u$.

2 Proof of Theorem 1

In this section we provide the proof of Theorem 1 (see also [13, thm 1.1]). We fix A a finite dimensional basic and connected k-algebra with quiver Q. **Until the end of the section we will assume that Q has no oriented cycles**. The proof of Theorem 1 decomposes into 4 steps as follows, and we will devote a subsection to each step:

a) If kQ/I and kQ/J are isomorphic to A as k-algebras, then there exists $\varphi\colon kQ \xrightarrow{\sim} kQ$ a product of transvections and of a dilatation such that $\varphi(I) = J$.

b) If $\varphi(I) = J$ and if φ is a dilatation then $\pi_1(Q, I) \simeq \pi_1(Q, J)$. If φ is a transvection, then there exists a surjective group morphism $\pi_1(Q, I) \to \pi_1(Q, J)$ or $\pi_1(Q, J) \to \pi_1(Q, I)$, induced by the identity map on the walks of Q.

c) The homotopy relations $\sim_I$ of the admissible presentations kQ/I of A can be displayed as the vertices of a quiver Γ such that for any arrow $\sim_I \to \sim_J$ the identity map on walks induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$.

d) If k has characteristic zero and if Q has no double bypasses, then the quiver Γ has a unique source, and if $\sim_{I_0}$ is the source of Γ then I_0 fits Theorem 1.

2.1 Different presentations of an algebra are linked by products of transvections and dilatations

In order to consider A as a k-category we need to choose a decomposition of the unit into a sum of pairwise orthogonal primitive idempotents. The following Proposition shows that for the study the presentations of A, this choice is irrelevant and that we may fix these idempotents once and for all. We will omit the proof which is basic linear algebra.

Proposition 2.1. *[13, 3.1] Let I and J be admissible ideals of kQ. If $kQ/I \simeq kQ/J$ as k-algebras then there exists $\varphi\colon kQ \xrightarrow{\sim} kQ$ an automorphism equal to the identity map on Q_0 and such that $\varphi(I) = J$.*

Recall that $GL_n(k)$ is generated by transvections and dilatations matrices. The following Proposition states an analogous result for the group of automorphisms of kQ that are equal to the identity map on Q_0.

Proposition 2.2. *Let $\mathcal{G}$ be the group of automorphisms of kQ that equal the identity map on Q_0. Let $\mathcal{D} \subseteq \mathcal{G}$ be the subgroup of the dilatations of kQ and let $\mathcal{T} \subseteq \mathcal{G}$ be the subgroup generated by the transvections. Then $\mathcal{T}$ is a normal subgroup and $\mathcal{G}$ is the semi-direct product $\mathcal{T} \rtimes \mathcal{D}$.*

Remark 1. *The group of automorphism of an algebra was already studied. More precisely the reader can find in [11], [17] and [18] a study of the group of outer automorphisms of an algebra.*

Proof of Proposition 2.2: Obviously we have $\mathcal{T} \cap \mathcal{D} = 1$. Moreover, for any transvection $\varphi = \varphi_{\alpha,u,\tau}$ and any dilatation D we have $D\varphi D^{-1} = \varphi_{\alpha,u,\frac{\tau\lambda}{\mu}}$ where $\lambda \in k^*$ and $\mu \in k^*$ are such that $D(u) = \lambda\ u$ and $D(\alpha) = \mu\ \alpha$. Hence, in order to prove the Proposition, it is enough to prove that $\mathcal{G} = \mathcal{T}\mathcal{D}$. To do this, let us fix some notation: for any $\psi \in \mathcal{G}$ we set $n(\psi)$ to be the number of arrows $\alpha \in Q_1$ such that $\psi(\alpha) \notin k^*\alpha$. Notice that $n(\psi) = 0$ if and only if $\psi \in \mathcal{D}$. Let us prove by induction on $n \geqslant 0$ that R_n is true where $R_n =$"*if $\psi \in \mathcal{G}$ and $n(\psi) \leqslant n$ then there exists $g \in \mathcal{T}$ such that*

$g\psi \in \mathcal{D}$". Obviously R_0 is true. Let $n \geqslant 1$, assume that R_{n-1} is true, and let $\psi \in \mathcal{G}$ such that $n(\psi) = n$. Hence there exists $x \xrightarrow{\alpha_1} y \in Q_1$ such that $\psi(\alpha_1) \notin k^*\alpha_1$. Let $\alpha_1, \ldots, \alpha_d$ be the arrows $x \to y$ of Q and let $E = Vect(\alpha_1, \ldots, \alpha_d)$. Since $kQ \xrightarrow{\psi} kQ$ is an automorphism, ψ induces an automorphism of $kQ^+/(kQ^+)^2$ and the composition $f\colon E \hookrightarrow {}_ykQ_x \xrightarrow{\psi} {}_ykQ_x \twoheadrightarrow E$ of ψ with the natural inclusion $E \hookrightarrow {}_ykQ_x$ (resp. the natural projection ${}_ykQ_x \twoheadrightarrow E$) is a k-linear isomorphism hence an element of $GL_d(k)$. Recall (see [12, Chap. XIII Prop. 9.1]) that the group $GL_d(k)$ is generated by transvections and dilatations matrices. Thus there exist transvections $f_1, \ldots, f_l$ of $GL_d(k)$ such that $f_1 \ldots f_l f(\alpha_i) \in k^*\alpha_i$ for each $i \in \{1, \ldots, d\}$. For each f_j, let $\bar{f}_j\colon kQ \to kQ$ be the automorphism such that $\bar{f}_j(\alpha_i) = f_j(\alpha_i)$ for each $i \in \{1, \ldots, d\}$ and such that $\bar{f}_j(\beta) = \beta$ for any arrow β not parallel to α_1. In particular, $\bar{f}_j$ is a transvection with respect to some α_{i_j}. Let $g_1 = \bar{f}_1 \ldots \bar{f}_d$, then $g_1\psi(\alpha_i) \in k^*\alpha_i + (kQ^+)^2$, and if $\beta \in Q_1$ is not parallel to α_1 and satisfies $\psi(\beta) \in k^*\beta$ then $g_1\psi(\beta) \in k^*\beta$. Let $\psi_1 = g_1\psi$. By construction, for each $i \in \{1, \ldots, d\}$, we have $\psi_1(\alpha_i) = \lambda_i\ \alpha_i + \sum_{j=1}^{n_i}\ \tau_{i,j}\ u_{i,j}$ with $u_{i,j}$ paths of length at least 2. Let $\varphi_{i,j}$ be the transvection $\varphi_{\alpha_i, u_{i,j}, -\tau_{i,j}/\lambda_i}$ for each $i \in \{1, \ldots, d\}$ and each $j \in \{1, \ldots, n_i\}$, and let $g_2 \in \mathcal{T}$ be the product of the $\varphi_{i,j}$'s (for any $i \in \{1, \ldots, d\}$ and any $j \in \{1, \ldots, n_i\}$). It is easy to check that the $\varphi_{i,j}$'s are pairwise commuting hence the definition of g_2 is unambiguous. Since Q has no oriented cycles, for each i we have $g_2\psi_1(\alpha_i) = \lambda_i\alpha_i$ and $g_2\psi_1(\beta) \in k^*\beta$ if $\beta \in Q_1$ is not parallel to α_1 and satisfies $\psi_1(\beta) \in k^*\beta$. In particular $n(g_2g_1\psi) < n(\psi) = n$. Since R_{n-1} is true, there exists $g_3 \in \mathcal{T}$ such that $g_3g_2g_1\psi \in \mathcal{D}$. Hence R_n is true for any $n \geqslant 0$. This achieves the proof of Proposition 2.2. □

Remark 2. *Propositions 2.1 and 2.2 imply that if I and J are admissible ideals of kQ such that $kQ/I \simeq kQ/J$ as k-algebras, then there exist $\varphi_1, \ldots, \varphi_n$ (resp. $\varphi'_1, \ldots, \varphi'_n$) a sequence of transvections of kQ, together with D a dilatation such that $J = D\varphi_n \ldots \varphi_1(I)$ (resp. $J = \varphi'_n \ldots \varphi'_1 D(I)$).*

2.2 Comparison of the fundamental group of two presentations of an algebra linked by a transvection or a dilatation

If I is an ideal and φ is a dilatation or a transvection, then I and $\varphi(I)$ are close enough in order to compare the associated homotopy relations. Before stating this comparison we prove two useful Lemmas.

Lemma 2.1. *Let I be an admissible ideal of kQ, let $\varphi = \varphi_{\alpha,u,\tau}$ be a transvection and set $J = \varphi(I)$. Assume that $\alpha \not\sim_I u$ and let $r \in {}_yI_x$ be a minimal relation with normal form $r = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b$ such that α does not appear in the path θ_c for any $c \in C$. Then there exists a minimal relation $r' \in {}_yJ_x$ with normal form $r' = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b + \sum_{B'} \lambda_b\tau\ v_buu_b$ where $B' \subseteq B$.*

Proof: Let us assume $B \neq \emptyset$ (if $B = \emptyset$, the conclusion is immediate). Since Q has no oriented cycles, the paths v_b and u_b do not contain α. Since r is a minimal relation of I and since $\alpha \not\sim_I u$, we have $\theta_c \neq v_buu_b$ for any $c \in C, b \in B$. Therefore, $\varphi(r)$ has a normal form $\varphi(r) = \sum_C\ \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b + \sum_B \lambda_b\tau\ v_buu_b \in {}_yJ_x\backslash\{0\}$. Thus there exists a minimal relation $r' \in {}_yJ_x$ with normal form $r' = \sum_{C'} \lambda_c\ \theta_c + \sum_{B'_1} \lambda_b\ v_b\alpha u_b + \sum_{B'} \lambda_b\tau\ v_buu_b$ such that $\emptyset \neq B'_1 \subseteq B$, $C' \subseteq C$ and $B' \subseteq B$. Hence $\varphi^{-1}(r')$ has a normal form $\varphi^{-1}(r') = \sum_{C'} \lambda_c\ \theta_c + \sum_{B'_1} \lambda_b\ v_b\alpha u_b + \sum_{B'\backslash B'_1} \lambda_b\tau\ v_buu_b - \sum_{B'_1\backslash B'} \lambda_b\tau\ v_buu_b \in$

${}_yI_x\backslash\{0\}$. Since $r \in {}_yI_x$ is a minimal relation and since $\alpha \not\sim_I u$ we infer that there exists a minimal relation $r'' \in {}_yI_x$ with normal form $r'' = \sum_{C''} \lambda_c\ \theta_c + \sum_{B''} \lambda_b\ v_b\alpha u_b$ such that $C'' \subseteq C' \subseteq C$ and $\emptyset \neq B'' \subseteq B'_1$. This forces $C'' = C$ and $B'' = B$ because $r \in {}_yI_x$ is a minimal relation. Thus $C' = C$ and $B'_1 = B$. Hence we have a minimal relation $r' \in {}_yJ_x$ with normal form $r' = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b + \sum_{B'} \lambda_b\tau\ v_b u u_b$ as announced. □

Lemma 2.2. *Let I be an admissible ideal of kQ, let $\varphi = \varphi_{\alpha,u,\tau}$ be a transvection and set $J = \varphi(I)$. Assume that $\alpha \sim_J u$ and let $r \in {}_yI_x$ be a minimal relation. Then $v \sim_J w$ for any $v, w \in supp(r)$.*

Proof: We may write $r = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b + \mu_b\ v_b u u_b$ so as:
. $\lambda_c, \lambda_b \in k^*$ and $\mu_b \in k$ for any $c \in C$ and $b \in B$,
. the paths θ_c, $v_b\alpha u_b$, $v_{b'} u u_{b'}$ $(c \in C, b, b' \in B)$ are pairwise distinct,
. for any $c \in C$, the path θ_c does not contain α.
Hence $\varphi(r) = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b\alpha u_b + (\mu_b + \tau\lambda_b)\ v_b u u_b \in {}_yJ_x$ and there exists a decomposition $\varphi(r) = r_1 + \ldots + r_n$ where $r_i \in {}_yJ_x$ is a minimal relation and $supp(r_i) \cap supp(r_j) = \emptyset$ if $i \neq j$. If $B = \emptyset$ then $\varphi(r) = r \in {}_yJ_x$ is a minimal relation and the Lemma is proved. Hence we may assume that $B \neq \emptyset$. This implies that for any $i \in \{1, \ldots, n\}$ there exists $b \in B$ such that $v_b\alpha u_b \in supp(r_i)$ or $v_b u u_b \in supp(r_i)$ (if this is not the case then $r_i = \sum_{C'} \lambda_c\ \theta_c$ for some non empty subset C' of C, thus $\varphi^{-1}(r_i) = \sum_{C'} \lambda_c \theta_c \in {}_yI_x$ which contradicts the minimality of r). Let $\equiv$ be the smallest equivalence relation on the set $\{1, \ldots, n\}$ such that: $i \equiv j$ if there exists $b \in B$ such that $v_b\alpha u_b \in supp(r_i)$ and $v_b u u_b \in supp(r_j)$. Since the r_i's are minimal relations of J and since $\alpha \sim_J u$, we get: if $i \equiv j$ then $v \sim_J w$ for any $v, w \in supp(r_i) \sqcup supp(r_j)$. Let $\mathcal{O} \subseteq \{1, \ldots, n\}$ be a $\equiv$-orbit and let $r' = \sum_{i \in \mathcal{O}} r_i \in {}_yJ_x$. Hence $r' = \sum_{C'} \lambda_c\ \theta_c + \sum_{B'} \lambda_b\ v_b\alpha u_b + (\mu_b + \tau\lambda_b)\ v_b u u_b$ where $C' \subseteq C$ and $\emptyset \neq B' \subseteq B$. This implies that $\varphi^{-1}(r') = \sum_{C'} \lambda_c\ \theta_c + \sum_{B'} \lambda_b\ v_b\alpha u_b + \mu_b\ v_b u u_b \in {}_yI_x$ and the minimality of r yields $C' = C$, $B' = B$, $r' = \varphi(r)$ and $\mathcal{O} = \{1, \ldots, n\}$. Hence $\{1, \ldots, n\}$ is an $\equiv$-orbit. Therefore $v \sim_J w$ for any $v, w \in supp(\varphi(r))$. And since $\alpha \sim_J u$ we infer that $v \sim_J w$ for any $v, w \in supp(r)$. □

We can now state the announced comparison. For the sake of simplicity, the word *generated* for an equivalence relation on the set of walks in Q stands for: *generated as an equivalence relation which is compatible with the concatenation of walks and which satisfies $e_x \sim \alpha^{-1}\alpha$ and $e_y \sim \alpha\alpha^{-1}$ for any arrow $x \xrightarrow{\alpha} y$.*

Proposition 2.3. *[13, 3.2] Let I be an admissible ideal of kQ, let φ be an automorphism of kQ and set $J = \varphi(I)$. If φ is a dilatation, then $\sim_I$ and $\sim_J$ coincide. Assume now that $\varphi = \varphi_{\alpha,u,\tau}$ is a transvection.*

a) if $\alpha \sim_I u$ and $\alpha \sim_J u$ then $\sim_I$ and $\sim_J$ coincide.

b) if $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then $\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$.

c) if $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $I = J$ and $\sim_I$ and $\sim_J$ coincide.

Remark 3. *The following implication (symmetric to b)):*

if $\alpha \sim_I u$ and $\alpha \not\sim_J u$ then $\sim_I$ is generated by $\sim_J$ and $\alpha \sim_I u$

is also satisfied since $\varphi^{-1}_{\alpha,u,\tau} = \varphi_{\alpha,u,-\tau}$

Proof of Proposition 2.3: If φ is a dilatation, then $\sim_I$ and $\sim_J$ coincide because for any $r \in {}_ykQ_x$ we have $supp(r) = supp(\varphi(r))$ and because r is a minimal relation of I

if and only if the same holds in J. Let us assume that $\varphi = \varphi_{\alpha,u,\tau}$ is a transvection.
a) The Lemma 2.2 applied to I, J, φ (resp. $J, I, \varphi^{-1} = \varphi_{\alpha,u,-\tau}$) shows that any two paths appearing in a same minimal relation of I (resp. J) are $\sim_J$-equivalent (resp. $\sim_I$-equivalent). Hence $\sim_I$ and $\sim_J$ coincide.
b) Let $\equiv$ be the equivalence relation generated by: $(v \sim_I w \Rightarrow v \equiv w)$ and $\alpha \equiv u$. Our aim is to show that $\sim_J$ and $\equiv$ coincide. Thanks to Lemma 2.2 we have: $v \equiv w \Rightarrow v \sim_J w$. Let $Min(I)$ be the set of the minimal relations of I. For each $r \in Min(I)$ let us fix a normal form $r = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b \alpha u_b$ satisfying the hypotheses of Lemma 2.1. Hence there exists $B' \subseteq B$ and a minimal relation r_1 of J with normal form $r_1 = \sum_C \lambda_c\ \theta_c + \sum_B \lambda_b\ v_b \alpha u_b + \sum_{B'} \lambda_b \tau\ v_b u u_b$. Thus $\varphi(r) - r_1 = \sum_{B \setminus B'} \tau \lambda_b\ v_b u u_b \in J$ can be written as a sum $r_2 + \ldots + r_{n_r}$ of minimal relations of J with pairwise disjoint supports. In particular, $\varphi(r) = r_1 + \ldots + r_{n_r}$ where each $r_i \in J$ is a minimal relation. Notice that any two paths appearing in r_1 are $\equiv$-equivalent because of the normal form of r_1 and because of the definition of $\equiv$. With these notations, the set $\{r_i \mid r \in Min(I)\ and\ 1 \leqslant i \leqslant n_r\}$ is made of minimal relations of J and generates the ideal J. Thus, in order to show that $\sim_J$ and $\equiv$ coincide, it is enough to show that any two paths appearing in some r_i are $\equiv$-equivalent. Let $r \in Min(I)$, let $i \in \{1, \ldots, n_r\}$, and let v, w be two paths appearing in r_i. We have already proved that if $i = 1$ then $v \equiv w$, thus we may assume that $i \geqslant 2$. Keeping the above notations for the normal form of r, there exist $b, b' \in B$ such that $v = v_b u u_b$ and $w = v_{b'} u u_{b'}$. Since $\alpha \equiv u$ and since any two paths appearing in r_1 are $\equiv$-equivalent we get $v = v_b u u_b \equiv v_b \alpha u_b \equiv v_{b'} \alpha u_{b'} \equiv v_{b'} u u_{b'} = w$. Hence any two paths appearing in some r_i are $\equiv$-equivalent. This implies that $\sim_J$ and $\equiv$ coincide. Therefore, $\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$.
c) Let $r \in I$ be a minimal relation of I and apply Lemma 2.1 to r. Since $\alpha \not\sim_J u$, we infer that $r \in J$. Since I is generated by its minimal relations we get $I \subseteq J$. Finally, $I = J$ because I and J have the same dimension. □

Remark 4. *In the situation b) of Proposition 2.3 the identity map on the walks of Q induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$.*

The Proposition 2.3 allows us to prove the following result which has already been proved in [4]. Recall that the algebra kQ/I, where I is admissible, is called **constricted** if $dim\ {}_y(kQ/I)_x = 1$ for any arrow $x \to y$ of Q.

Proposition 2.4 (see also [4]). *Assume that A is constricted. Then different admissible presentations of A yield the same homotopy relation. In particular, they have isomorphic fundamental groups.*

Proof: Notice that if $\nu\colon kQ/J \xrightarrow{\sim} A$ is any admissible presentation and if (α, u) is a bypass in Q then $u \in J$ because A is constricted and J is admissible. Let us assume that the conclusion of the Proposition 2.4 is false. From Remark 2 and Proposition 2.3 we deduce that there exist two presentations $kQ/I \simeq A$ and $kQ/J \simeq A$, and a transvection $\varphi_{\alpha,u,\tau}$ such that: $J = \varphi_{\alpha,u,\tau}(I)$, $\alpha \not\sim_I u$ and $\alpha \sim_J u$. Let $r \in I$ be a minimal relation. Thanks to Lemma 2.1 we know that there exist paths $v_b u u_b$ and scalars $\lambda_b \in k^*$ $(b \in B)$ such that $r + \sum_{b \in B} \lambda_b\ v_b u u_b \in J$. Since $u \in J$ we get $r \in J$ for any minimal relation r of I. Hence $I \subseteq J$, and $I = J$ because $dim_k\ I = dim_k\ J$. This contradicts the property: $\alpha \not\sim_I u$ and $\alpha \sim_J u$. Hence the homotopy relation does not depend on the presentation of A. □

If $\sim$ and $\sim'$ are homotopy relations, we will say that $\sim'$ is a **direct successor** (see also [13, sect. 3]) of $\sim$ if there exist admissible ideals I and J of kQ, together with

a transvection $\varphi = \varphi_{\alpha,u,\tau}$ such that $\sim = \sim_I$, $\sim' = \sim_J$, $J = \varphi(I)$, $\alpha \not\sim_I u$ and $\alpha \sim_J u$. Notice that I, J, φ need not be unique.

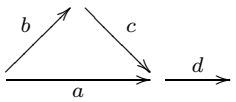
2.3 The quiver Γ of the homotopy relations of the presentations of the algebra

Definition 2.1. *[13, 4.1] We define the quiver Γ as follows:*
. Γ_0 is the set of homotopy relations of the admissible presentations of A:

$$\Gamma_0 = \{ \sim_I \mid I \text{ is admissible and } kQ/I \simeq A\}$$

. there is an arrow $\sim \to \sim'$ if and only if $\sim'$ is a direct successor of $\sim$.

Example 2. *Assume that $A = kQ/I$ where Q is*



and $I =< da >$. Let $J =< da - dcb >$. Using Proposition 2.3 one can show that Γ is equal to: $\sim_I \longrightarrow \sim_J$. Notice that the identity map on walks induces a surjective group morphism $\mathbb{Z} \simeq \pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J) \simeq 1$.

The author thanks Mariano Suárez-Alvarez for the following remark:

Remark 5. *A homotopy relation is determined by its restriction to the paths in Q with length at most the radical length of A. Thus there are only finitely many homotopy relations. This argument shows that Γ is finite.*

The following Proposition states some additional properties of Γ and is a direct consequence of Remark 2 and Proposition 2.3.

Proposition 2.5. *Assume Q has no oriented cycles and let m be the number of bypasses in Q. Then Γ is connected and has no oriented cycles. Any vertex of Γ is the source of at most m arrows and any oriented path in Γ has length at most m.*

Remark 6. *According to Remark 4, if there is a path in Γ with source $\sim_I$ and target $\sim_J$, then the identity map on the walks in Q induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$. Moreover, since Γ is finite, any vertex of Γ is the target of a (finite) path the source of which is a source of Γ (i.e. a vertex with no arrow arriving at it). As a consequence, if Γ has a unique source $\sim_{I_0}$ then the fundamental group of any admissible presentation of A is a quotient of $\pi_1(Q, I_0)$.*

2.4 The unicity of the source of Γ and the proof of Theorem 1

Notice that up to now we have used neither the characteristic of k nor the possible non existence of double bypasses in Q. These hypotheses will be needed in order to prove the uniqueness of the source of Γ. The complete proof of the unicity of the source of Γ is somewhat technical. For this reason we deal with the technical considerations in the two Lemmas that follow.

Lemma 2.3. *Let E be a finite dimensional k-vector space endowed with a totally ordered basis $e_1 < \ldots < e_n$. Assume that k has characteristic zero. Let $\nu\colon E \to E$ be a linear map such that $\nu(e_i) \in Span(e_j \ ; \ j < i)$ for any $i \in \{1, \ldots, n\}$, and let I and J be two subspaces of E such that the following conditions are satisfied:*

a) $\psi(I) = J$ where $\psi\colon E \to E$ is equal to $exp(\nu)$.

b) if $e_i \in supp(\nu(e_j))$ then $e_i \not\equiv_I e_j$ and $e_i \not\equiv_J e_j$.

Then I and J have the same Groebner basis and $I = J$.

Proof: Let us prove Lemma 2.3 by induction on n. If $n = 1$ the equality is obvious so let us assume that $n > 1$ and that the conclusion of Lemma 2.3 holds for dimensions less than n. We will denote by $r_1 < \ldots < r_p$ (resp. $r'_1 < \ldots < r'_p$) the Groebner basis of I (resp. of J) and we will write $i_1, \ldots, i_p$ (resp. $i'_1, \ldots, i'_p$) for the integers such that $r_j \in e_{i_j} + Span(e_i \ ; \ i < i_j)$ (resp. $r'_j \in e_{i'_j} + Span(e_i \ ; \ i < i'_j)$). In order to prove that $I = J$ we will prove the following four facts:

a) the two sequences $i_1 < \ldots < i_p$ and $i'_1 < \ldots < i'_p$ coincide,

b) $\psi(r_1) = r'_1$,

c) $r_1 = r'_1$ and $\nu(r_1) = 0$ (using the induction hypothesis on $E/k.e_1$),

d) $r_2 = r'_2, \ldots, r_p = r'_p$ (using the induction hypothesis on $E/k.r_1$).

a) For simplicity let us set $E_i = Span(e_j \ ; \ j \leqslant i)$. Since $\nu(e_j) \in E_{j-1}$ and $r_j \in e_{i_j} + E_{i_j - 1}$, and since $\psi = exp(\nu)$, we get $\psi(r_j) \in J \cap \big(e_{i_j} + E_{i_j - 1}\big)$ for any j. Hence, the definition of the Groebner basis of J forces $\{i_1, \ldots, i_p\} \subseteq \{i'_1, \ldots, i'_p\}$ and the cardinality and the ordering on these two sets imply that $i_1 = i'_1, \ldots, i_p = i'_p$

b) Since $i_1 = i'_1$ we infer that $\psi(r_1) - r'_1 \in J \cap E_{i_1 - 1}$. The definition of the Groebner basis of J then forces $\psi(r_1) - r'_1 = 0$.

c) Let us prove that $r_1 = r'_1$. Notice that the definition of a Groebner basis and the equalities $\psi(r_1) = r'_1$ and $\psi(e_1) = e_1$ force: $e_1 \in I \Leftrightarrow r_1 = e_1 \Leftrightarrow r'_1 = e_1 \Leftrightarrow e_1 \in J$. Hence we may assume that $e_1 \notin I$ and $e_1 \notin J$.
Let $\tilde{E} = E/k.e_1$ and let $\pi\colon E \twoheadrightarrow \tilde{E}$ be the natural projection. We will write $\tilde{x}$ for $\pi(x)$. Similarly we set $\tilde{I} = \pi(I)$ and $\tilde{J} = \pi(J)$. In particular $\tilde{E}$ has a totally ordered basis: $\tilde{e}_2 < \ldots < \tilde{e}_n$. Since $\nu(e_1) = 0$ and since $\psi(e_1) = e_1$, the mappings ν and ψ induce linear mappings $\tilde{\nu}, \tilde{\psi}\colon \tilde{E} \to \tilde{E}$. It follows from the properties of ν and ψ that $\tilde{\psi}(\tilde{I}) = \tilde{J}$, that $\tilde{\nu}(\tilde{e}_i) \in Span(\tilde{e}_j \ ; \ 2 \leqslant j < i)$ for any $i \geqslant 2$, that $\tilde{\psi} = exp(\tilde{\nu})$, and that $supp(\tilde{\nu}(\tilde{e}_i)) = \{ \ \tilde{e}_j | \ j \geqslant 2 \ and \ e_j \in supp(\nu(e_i))\}$ for any $i \geqslant 2$. Moreover, with the definition of the Groebner basis of I we get:

- $\tilde{r}_j \in \tilde{e}_{i_j} + Span(\tilde{e}_i \ ; \ i < i_j)$ for any j (recall that $e_1 \notin I$),
- $supp(\tilde{r}_j) = \{\tilde{e}_i \mid i \geqslant 2 \ and \ e_i \in supp(r_j)\}$ for any j.

Therefore $\tilde{r}_1 < \ldots < \tilde{r}_p$ is the Groebner basis of $\tilde{I}$ and: $\tilde{e}_i \equiv_{\tilde{I}} \tilde{e}_j \Rightarrow e_i \equiv_I e_j$. Similarly $\tilde{r}'_1 < \ldots < \tilde{r}'_p$ is the Groebner basis of $\tilde{J}$ and: $\tilde{e}_i \equiv_{\tilde{J}} \tilde{e}_j \Rightarrow e_i \equiv_J e_j$. Using the above description of $supp(\tilde{\nu}(\tilde{e}_i))$ together with the above link between $\equiv_I$ (resp. $\equiv_J$) and $\equiv_{\tilde{I}}$ (resp. $\equiv_{\tilde{J}}$) we infer that:

$$\tilde{e}_i \not\equiv_{\tilde{I}} \tilde{e}_j \text{ and } \tilde{e}_i \not\equiv_{\tilde{J}} \tilde{e}_j \text{ as soon as } \tilde{e}_j \in supp(\tilde{\nu}(\tilde{e}_i))$$

For this reason we may apply the induction hypothesis to $\tilde{E}$, $\tilde{I}$ and $\tilde{J}$. Hence $\tilde{I}$ and $\tilde{J}$ have the same Groebner basis and $\tilde{r}_1 = \tilde{r}'_1$ i.e. $r'_1 = r_1 + \lambda \ e_1$ with $\lambda \in k$. Therefore

$(\psi - Id)(r_1) = \lambda\, e_1$, and since $\psi(e_1) = e_1$ we get $\nu(r_1) = log(\psi)(r_1) = \lambda\, e_1$. Assume that $\lambda \neq 0$ i.e. $e_1 \in supp(\nu(r_1))$. Thus there exists $e_i \in supp(r_1)$ such that $e_1 \in supp(\nu(e_i))$. This implies that $e_1 \not\equiv_I e_i$, and since any two elements in $supp(r_1)$ are $\equiv_I$-equivalent, this forces $e_1 \notin supp(r_1)$. Hence $e_i, e_1 \in supp(r'_1) = supp(r_1) \sqcup \{e_1\}$ and therefore $e_i \equiv_J e_1$. This contradicts $e_1 \in supp(\nu(e_i))$ and shows that $\lambda = 0$, that $r_1 = r'_1$ and that $\nu(r_1) = 0$.

d) Let us show that $r_2 = r'_2, \ldots, r_p = r'_p$. For this purpose we will apply the induction hypothesis to $\bar{E} = E/k.r_1$. Let $q\colon E \twoheadrightarrow \bar{E}$ be the natural projection. We will write $\bar{e}_i$ (resp. $\bar{I}, \bar{J}, \bar{r}_j, \bar{r}'_j$) for $q(e_i)$ (resp. $q(I), q(J), q(r_j), q(r'_j)$). Hence $\bar{E}$ has a totally ordered basis: $\bar{e}_1 < \ldots < \bar{e}_{i_1-1} < \bar{e}_{i_1+1} < \ldots < \bar{e}_n$. Since $\nu(r_1) = 0$ and since $\psi(r_1) = r_1$, the mappings ν and ψ induce linear mappings $\bar{\nu}, \bar{\psi}\colon \bar{E} \to \bar{E}$. These mappings obviously satisfy $\bar{\psi}(\bar{I}) = \bar{I}$, $\bar{\nu}(\bar{e}_i) \in Span(\bar{e}_j \ ;\ j \neq i_1 \text{ and } j < i)$ for any $i \neq i_1$, and $\bar{\psi} = exp(\bar{\nu})$. Moreover, our choice for the basis of $\bar{E}$ and the definition of the Groebner basis of I imply that:

- $supp(\bar{r}_j) = \{\bar{e}_i \mid e_i \in supp(r_j)\}$ for any $j \geqslant 2$,
- $\bar{r}_2 < \ldots < \bar{r}_p$ is the Groebner basis of $\bar{I}$.

These two properties imply in particular that: $\bar{e}_i \equiv_{\bar{I}} \bar{e}_j \Rightarrow e_i \equiv_I e_j$ for any $i, j \neq i_1$. The corresponding properties hold for $\bar{J}$ (replace r_j by r'_j, I by J and $\bar{I}$ by $\bar{J}$). Thus, in order to apply the induction hypothesis to $\bar{E}$ it only remains to prove that: $\bar{e}_j \in supp(\bar{\nu}(\bar{e}_i)) \Rightarrow \bar{e}_i \not\equiv_{\bar{I}} \bar{e}_j$ and $\bar{e}_i \not\equiv_{\bar{J}} \bar{e}_j$ for any $i, j \neq i_1$. Assume that $i, j \neq i_1$ satisfy $\bar{e}_j \in supp(\bar{\nu}(\bar{e}_i))$. From the definition of $\bar{E}$ and $\bar{\nu}$ we know that:

- $supp(\bar{\nu}(\bar{e}_i)) = \{\bar{e}_l \mid e_l \in supp(\nu(e_i))\}$ if $e_{i_1} \notin supp(\nu(e_i))$,
- $supp(\bar{\nu}(\bar{e}_i)) \subseteq \{\bar{e}_l \mid e_l \in supp(\nu(e_i)) \text{ and } l \neq i_1\} \cup \{\bar{e}_l \mid l < i_1 \text{ and } e_l \in supp(r_1)\}$ if $e_{i_1} \in supp(\nu(e_i))$.

Let us distinguish the cases $e_j \in supp(\nu(e_i))$ and $e_j \notin supp(\nu(e_i))$:

· if $e_j \in supp(\nu(e_i))$ then $e_i \not\equiv_I e_j$ and $e_i \not\equiv_J e_j$ and the above comparison between $\equiv_I$ (resp. $\equiv_J$) and $\equiv_{\bar{I}}$ (resp. $\equiv_{\bar{J}}$) yields $\bar{e}_i \not\equiv_{\bar{I}} \bar{e}_j$ and $\bar{e}_i \not\equiv_{\bar{J}} \bar{e}_j$.

· if $e_j \notin supp(\nu(e_i))$ then necessarily $e_{i_1} \in supp(\nu(e_i))$ and $e_j \in supp(r_1)$. Since $r_1 = r'_1$, the property $e_j \in supp(r_1)$ implies that $e_j \equiv_I e_{i_1}$ and $e_j \equiv_J e_{i_1}$. On the other hand, the property $e_{i_1} \in supp(\nu(e_i))$ implies that $e_{i_1} \not\equiv_I e_i$ and $e_{i_1} \not\equiv_J e_i$. Therefore $e_j \not\equiv_I e_i$ and $e_j \not\equiv_J e_i$ and finally $\bar{e}_j \not\equiv_{\bar{I}} \bar{e}_i$ and $\bar{e}_j \not\equiv_{\bar{J}} \bar{e}_i$.

Thus all the conditions of Lemma 2.3 are satisfied for $\bar{E}, \bar{I}, \bar{J}, \bar{\nu}$. For this reason we can apply the induction hypothesis which gives: $\bar{I}$ and $\bar{J}$ have the same Groebner basis. We infer that $q(r_i) = q(r'_i)$ for each $i = 2, \ldots, p$. Hence for each $i \geqslant 2$ there exists $\lambda_i \in k$ such that $r_i = r'_i + \lambda_i\, r_1$, and λ_i is necessarily zero because $e^*_{i_1}(r_i) = e^*_{i_1}(r_i) = 0$ (cf the definition of a Groebner basis). Therefore $r_i = r'_i$ for each $i = 1, \ldots, p$ and $I = J$ as announced. □

Lemma 2.4. *Let $\varphi\colon kQ \to kQ$ be an automorphism equal to the identity map on Q_0. Let I be an admissible ideal of kQ and set $J = \varphi(I)$. Suppose that k has characteristic zero. Suppose that for any arrow α there is a normal form $\varphi(\alpha) = \alpha + \sum_i \lambda_i\, u_i$ where each u_i satisfies: $\alpha \not\sim_I u_i$ and $\alpha \not\sim_J u_i$, and $\varphi(a) = a$ for any arrow appearing in u_i (in particular $\varphi(u_i) = u_i$). Then I and J coincide.*

Proof: Let E be the vector space $\oplus_{x \neq y}\ {}_ykQ_x$. Hence E is finite dimensional since Q has no oriented cycles, and I and J can be considered as subspaces of E. In order to

apply Lemma 2.3 to E, I, J, we need to exhibit a totally ordered basis of E together with a mapping $\nu\colon E \to E$. Let us take the non trivial paths in Q for the basis of E. For short, we will write *path* for *non trivial path*. The following construction of a total order $<$ on this basis is taken from [8]. Let us fix a total order on Q_1 (which is finite) and let $\prec$ be the induced lexicographical order on the paths in Q. If u is a path we let $W(u)$ be the number of arrows $\alpha \in Q_1$ appearing in u and such that $\varphi(\alpha) \neq \alpha$. Hence, for any $\alpha \in Q_1$, we have $W(\alpha) = 0$ if $\varphi(\alpha) = \alpha$ and $W(\alpha) = 1$ if $\varphi(\alpha) \neq \alpha$. The total order $<$ is then defined as follows:

$$u < v \;\Leftrightarrow \begin{cases} W(u) < W(v) \\ or \\ W(u) = W(v) \text{ and } u \prec v \end{cases}$$

This yields: $e_1 < \ldots < e_n$ a totally ordered basis of E made of the non trivial paths in Q. Notice that with this basis, the equivalence relations $\equiv_I$ and $\sim_I$ (resp. $\equiv_J$ and $\sim_J$) satisfy the following property: $e_i \equiv_I e_j \Rightarrow e_i \sim_I e_j$ (resp. $e_i \equiv_J e_j \Rightarrow e_i \sim_J e_j$). Let $\nu\colon kQ \to kQ$ be the derivation (i.e. the functor such that $\nu(vu) = \nu(v)u + v\nu(u)$ for any u and v) such that $\nu(\alpha) = \varphi(\alpha) - \alpha$ for any arrow $\alpha \in Q_1$. We will write $\nu\colon E \to E$ for the induced map on E. Thus, for any path u and any $v \in supp(\nu(u))$ there exist an arrow $\alpha \in Q_1$ together with paths u_1, u_2, u_3 such that $u = u_3\alpha u_1$, $v = u_3u_2u_1$ and $u_2 \in supp(\nu(\alpha))$. Notice that with the assumptions made on φ, this implies that $e_i \not\equiv_I e_j$ and $e_i \not\equiv_J e_j$ as soon as $e_j \in supp(\nu(e_i))$. Moreover, for any $\alpha \in Q_1$ and any $u \in supp(\nu(\alpha))$ we have $W(u) = 0$ hence $\nu \circ \nu(\alpha) = 0$. Since $\nu\colon kQ \to kQ$ is a derivation, we infer that: $e_j \in supp(\nu(e_i)) \Rightarrow W(e_j) < W(e_i) \Rightarrow e_j < e_i$. Hence $\nu(e_i) \in Span(e_j \ ; \ j < i)$ for any i. In order to apply Lemma 2.3, it only remains to prove that $J = exp(\nu)(I)$. To do this it suffices to prove that $\varphi = exp(\nu)$. This equality is easily checked on any path in Q by induction on the length of the path using the fact that ν is a derivation, that $\varphi(\alpha) = \alpha + \nu(\alpha)$ and that $\nu \circ \nu(\alpha) = 0$ for any arrow α. Hence, the data E, I, J, ν together with the ordered basis $e_1 < \ldots < e_n$ satisfy the hypotheses of Lemma 2.3 which implies that $I = J$. □

The uniqueness of the source of Γ is given by the following result.

Proposition 2.6. *[13, 4.3] Assume that A satisfies the hypotheses made before stating Theorem 1, then Γ has a unique source.*

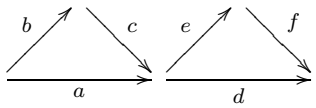
Proof: Notice that any two transvections of kQ commute since Q has no double bypasses. Let $\sim$ and $\sim'$ be sources of Γ. Let I and J be admissible ideals of kQ such that $kQ/I \simeq A \simeq kQ/J$ and such that $\sim = \sim_I$ and $\sim' = \sim_J$. According to Remark 2 there exist a sequence of transvections $\varphi_1 = \varphi_{\alpha_1,u_1,\tau_1}, \ldots, \varphi_n = \varphi_{\alpha_n,u_n,\tau_n}$ of kQ and a dilatation D such that $J = \varphi_n \ldots \varphi_1 D(I)$. Thanks to Lemma 2.3 we know that $\sim_I = \sim_{D(I)}$. Thus, in order to prove that $\sim = \sim'$, we may assume that $D = Id_{kQ}$ and $J = \varphi_n \ldots \varphi_1(I)$. Moreover we may assume that n is the smallest non negative integer such that there exist I, J and a sequence of transvections $\varphi_1, \ldots, \varphi_n$ satisfying $\sim = \sim_I$, $\sim' = \sim_J$ and $J = \varphi_n \ldots \varphi_1(I)$. Let us prove that $\alpha_i \not\sim_I u_i$ for any $i \in \{1, \ldots, n\}$. If i is such that $\alpha_i \sim_I u_i$ then Proposition 2.3 implies that $\sim_I = \sim_{\varphi_i(I)}$ since $\sim_I$ is a source of Γ. Hence $\sim = \sim_{\varphi_i(I)}$, $\sim' = \sim_J$ and $J = \varphi_n \ldots \varphi_{i+1}\varphi_{i-1} \ldots \varphi_1(\varphi_i(I))$ which contradicts the minimality of n. Thus $\alpha_i \not\sim_I u_i$ for any i and the same arguments apply to J since $I = \varphi_1^{-1} \ldots \varphi_n^{-1}(J)$ and $\sim_J$ is a source of Γ. Hence $\alpha_i \not\sim_J u_i$ for any i. This shows that the data $I, J, \varphi_n \ldots \varphi_1$ satisfy the hypotheses of Lemma 2.4. We infer that $I = J$ and that $\sim = \sim'$ coincide. This shows that Γ has a unique source. □

The Proposition 2.6 and the Remark 6 prove the Theorem 1:

Theorem 1. *(see also [13, thm 1.1]) Let A be a basic connected finite dimensional algebra over a field of characteristic zero. If the quiver Q of A has no double bypasses, then there exists a presentation $kQ/I_0 \simeq A$ with quiver and admissible relations such that for any other admissible presentation $kQ/I \simeq A$, the identity map on walks induces a surjective group morphism $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, I)$.*

The following example shows that one cannot remove the hypothesis on the characteristic of k in Proposition 2.6:

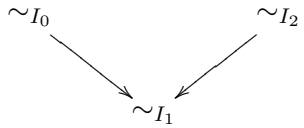
Example 3. *Let Q be the following quiver without double bypasses:*



Set $u = cb$ and $v = fe$. Set $A = kQ/I_0$ where $I_0 =< da + vu, va + du >$. Then $\pi_1(Q, I_1) = \mathbb{Z}/2$. Let I_1 and I_2 be the ideals defined below:

- $I_1 = \varphi_{a,u,1}(I_0) =< da + du + vu, va + du + vu >$,
- $I_2 = \varphi_{a,u,-1} \circ \varphi_{d,v,-1}(I_1) =< da, va + du - 2vu >$.

Hence $A \simeq kQ/I_1 \simeq kQ/I_2$. If $car(k) = 0$, then $\pi_1(Q, I_1) = \pi_1(Q, I_2) = 1$ and Γ is equal to $\sim_{I_0} \longrightarrow \sim_{I_1}$. Suppose now that $car(k) = 2$. Then $I_2 =< da, va + du >$, $\pi_1(Q, I_0) \simeq \mathbb{Z}/2$, $\pi_1(Q, I_1) = 1$, $\pi_1(Q, I_2) \simeq \mathbb{Z}$ and Γ is equal to:



Hence Γ has two sources. Notice that the identity map on walks induces a surjective group morphism $\pi_1(Q, I_2) \twoheadrightarrow \pi_1(Q, I_0)$. Notice also that one can build similar examples for any non zero value p of $car(k)$ by taking for Q a sequence of p bypasses.

3 Preliminaries on covering functors

In this section we give some useful facts on covering functors.

Lemma 3.1. *Let $p\colon \mathcal{E} \to \mathcal{B}$ and $q\colon \mathcal{E}' \to \mathcal{B}$ be functors where $\mathcal{E}$ is connected. Let $r, r'\colon \mathcal{E} \to \mathcal{E}'$ be such that $q \circ r = q \circ r' = p$. If there exists $x_0 \in \mathcal{E}_0$ such that $r(x_0) = r'(x_0)$ then $r = r'$.*

Proof: Since q is a covering functor, for any $u \in {}_y\mathcal{E}_x \backslash \{0\}$ we have:

$(r(x) = r'(x)$ or $r(y) = r'(y)) \Rightarrow (r(u) = r'(u),\ r(x) = r'(x)$ and $r(y) = r'(y))$ $\quad (\star)$

Assume that there exists $x_0 \in \mathcal{E}_0$ such that $r(x_0) = r'(x_0)$. Since $\mathcal{E}$ is connected, for any $x \in \mathcal{E}_0$ there exists a sequence $x_0, \ldots, x_n = x$ of objects of $\mathcal{E}$ together with a non zero morphism between x_i and x_{i+1} for any i. This implies (thanks to $(\star)$) that $r(x) = r'(x)$. Thus r and r' coincide on $\mathcal{E}_0$ and $(\star)$ implies $r = r'$. □

The following Proposition generalises the result [14, Prop. 3.3]. Using Lemma 3.1 its proof is immediate.

Proposition 3.1. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a covering functor where $\mathcal{E}$ is connected. Then $\mathcal{E}$ is an $Aut(F)$-category. Moreover, F is a Galois covering if and only if $Aut(F)$*

acts transitively on each $F^{-1}(x)$. Finally if F is Galois covering with group G, then $G = Aut(F)$.

Proposition 3.2. *Let $p\colon \mathcal{E} \to \mathcal{B}$ and $q\colon \mathcal{F} \to \mathcal{E}$ be functors where $\mathcal{E}$ is connected and set $r = p \circ q\colon \mathcal{F} \to \mathcal{B}$. Then p, q, r are covering functors as soon as two of them are so.*

Proof: We only need to prove that if p and r are covering functors then $q^{-1}(x) \neq \emptyset$ for any $x \in \mathcal{E}_0$ (the other properties are basic linear algebra). Assume that p and r are covering functors. Since q is a covering functor, for any $u \in {}_y\mathcal{E}_x \backslash \{0\}$ we have: $q^{-1}(x) \neq \emptyset \Leftrightarrow q^{-1}(y) \neq \emptyset$ $(\star)$. Fix $\hat{x}_0 \in \mathcal{F}_0$ and set $x_0 = q(\hat{x}_0)$, thus $q^{-1}(x_0) \neq \emptyset$. The connectedness of $\mathcal{E}$ and $(\star)$ imply that $q^{-1}(x) \neq \emptyset$ for any $x \in \mathcal{E}_0$. □

Proposition 3.3. *Let $p\colon \mathcal{C} \to \mathcal{B}$ (resp. $q\colon \mathcal{C}' \to \mathcal{B}$) be a connected Galois covering with group G (resp. G') and assume there exists a commutative diagram of k-categories and k-linear functors where φ is an isomorphism equal to the identity map on $\mathcal{B}_0$:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r} & \mathcal{C}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{B} & \xrightarrow[\varphi]{\sim} & \mathcal{B} \end{array}$$

Then there exists a unique mapping $\lambda\colon G \to G'$ such that $r \circ g = \lambda(g) \circ r$ for any $g \in G$. Moreover λ is a surjective morphism of groups and r is a Galois covering with group $Ker(\lambda)$.

Proof: Thanks to Proposition 3.2, r is a covering functor. Fix $\hat{x}_0 \in \mathcal{C}$ and set $x_0 = p(\hat{x}_0)$. For any $g \in Aut(p)$ we have $q(r(\hat{x}_0)) = x_0 = q(r(g(\hat{x}_0)))$. Since q is Galois with group G', there exists a unique $\lambda(g) \in G'$ such that $\lambda(g)(r(\hat{x}_0)) = r(g(\hat{x}_0))$, and Lemma 3.1 yields $\lambda(g) \circ r = r \circ g$. Hence: $(\forall g \in G)\ (\exists! \lambda(g) \in G')\ \lambda(g) \circ r = r \circ g$. This last property shows the existence and the uniqueness of λ. It also shows that $\lambda\colon G \to G'$ is a group morphism and that $Aut(r) = Ker(\lambda)$. Moreover, λ is surjective because of its definition and because p is Galois with group G. Finally Proposition 3.1 shows that r is a Galois covering with group $Ker(\lambda)$. □

4 The universal cover of an algebra

In this section we will prove Theorem 2. Let Q be a connected quiver without oriented cycles and let $x_0 \in Q_0$ for the computation of the groups $\pi_1(Q, I)$. If there is no ambiguity we will write $[w]$ for the homotopy class of a walk w.

Lemma 4.1. *Let I be an admissible ideal of kQ, let D be a dilatation of kQ and set $J = D(I)$. Let $\lambda\colon \pi_1(Q, I) \xrightarrow{\sim} \pi_1(Q, J)$ be the isomorphism given by Proposition 2.3. Let $p\colon (\tilde{Q}, \tilde{I}) \to (Q, I)$ (resp. $q\colon (\hat{Q}, \hat{J}) \to (Q, J)$) be the universal Galois covering with group $\pi_1(Q, I)$ (resp. $\pi_1(Q, J)$). Then there exists an isomorphism $\psi\colon k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$ such that the following diagram commutes:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I} & \xrightarrow{\psi} & k\hat{Q}/\hat{J} \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\ kQ/I & \xrightarrow{\bar{D}} & kQ/J \end{array}$$

where $\bar{D}, \bar{p}$ and $\bar{q}$ are induced by D, p and q respectively.
Moreover, ψ satisfies: $\psi \circ g = \lambda(g) \circ \psi$ for any $g \in \pi_1(Q, I)$.

Proof: We have $\hat{Q} = \tilde{Q}$ since $\sim_I$ and $\sim_J$ coincide (see Proposition 2.3). Set $\hat{D}\colon k\tilde{Q} \to k\hat{Q}$ to be defined by: $\hat{D}(a, [w]) = (D(a), [w])$ for any arrow $(a, [w]) \in \tilde{Q}_1$. By construction $\hat{D}$ is an automorphism of kQ and $\hat{D}(\tilde{I}) = \hat{J}$. Set $\psi\colon k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$ to be induced by $\hat{D}$. It is then easy to check all announced properties. □

Lemma 4.2. *Let I be an admissible ideal of kQ, let $\varphi = \varphi_{\alpha,u,\tau}$ be a transvection, set $J = \varphi(I)$ and assume that $\alpha \sim_J u$. Let $\lambda\colon \pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ be the surjection given by Proposition 2.3. Denote by $p\colon (\tilde{Q}, \tilde{I}) \to (Q, I)$ (resp. by $q\colon (\hat{Q}, \hat{J}) \to (Q, J)$) the universal Galois covering with group $\pi_1(Q, I)$ (resp. $\pi_1(Q, J)$). Then there exists a Galois covering $\psi\colon k\tilde{Q}/\tilde{I} \xrightarrow{\sim} k\hat{Q}/\hat{J}$ with group $Ker(\lambda)$ and such that the following diagram commutes:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I} & \xrightarrow{\psi} & k\hat{Q}/\hat{J} \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\ kQ/I & \xrightarrow{\bar{\varphi}} & kQ/J \end{array}$$

where $\bar{\varphi}, \bar{p}$ and $\bar{q}$ are induced by φ, p and q respectively.
Moreover, ψ satisfies: $\psi \circ g = \lambda(g) \circ \psi$ for any $g \in \pi_1(Q, I)$.

Proof: Let $\varphi'\colon k\tilde{Q} \to k\hat{Q}$ be defined by: $\varphi'([w]) = [w]$ for any $[w] \in \tilde{Q}_0$, $\varphi'(\beta, [w]) = (\beta, [w])$ for any $(\beta, [w]) \in \tilde{Q}_1$ such that $\beta \neq \alpha$, and $\varphi'(\alpha, [w]) = (\alpha, [w]) + \tau(u, [w])$ for any $(\alpha, [w]) \in \tilde{Q}_1$. Then φ' is well defined since $\alpha \sim_J u$. Moreover, $\varphi \circ p(a) = q \circ \varphi'(a)$ for any $a \in \tilde{Q}_1$, and $\varphi'(\tilde{I}) \subseteq \hat{J}$. Let $\psi\colon k\tilde{Q}/\tilde{I} \to k\hat{Q}/\hat{J}$ be induced by φ'. Thus $\bar{q} \circ \psi = \bar{\varphi} \circ \bar{p}$. Let $g = [\gamma] \in \pi_1(Q, I)$ and let $[w] \in \tilde{Q}_0$. Then $\psi \circ g([w]) = \psi([w\gamma^{-1}]) = [w\gamma^{-1}] = \lambda(g)([w]) = \lambda(g) \circ \psi([w])$. The Lemma 3.1 implies that $\psi \circ g = \lambda(g) \circ \psi$ for any $g \in \pi_1(Q, I)$. Finally, Proposition 3.3 gives: ψ is a Galois covering with group $Ker(\lambda)$. □

Lemma 4.3. *Let A be a finite dimensional basic k-algebra with ordinary quiver Q. Assume that k has characteristic zero and that Q no double bypasses. Let $kQ/I_0 \simeq A$ and $kQ/I \simeq A$ be two presentations with quiver and relations such that $\sim_{I_0}$ is the unique source of Γ. Then there exist a sequence $\varphi_1, \ldots, \varphi_n$ of transvections of kQ (with $\varphi_i = \varphi_{\alpha_i, u_i, \tau_i}$) and a dilatation D such that:*
a) $I = D\varphi_n \ldots \varphi_1(I_0)$,
b) if I_i is the ideal $\varphi_i \ldots \varphi_1(I_0)$ then $\alpha_i \sim_{I_i} u_i$ for $0 \leqslant i \leqslant n$.

Proof: We will write $[n]$ for the set $\{1, \ldots, n\}$. The Remark 2 implies that there exist a dilatation D and a sequence $\varphi_1, \ldots, \varphi_n$ of transvections ($\varphi_i = \varphi_{\alpha_i, u_i, \tau_i}$) such that $I = D\varphi_n \ldots \varphi_1(I_0)$. Let us set $J = D^{-1}(I)$. For $n \geqslant 0$ let R_n be the following property: *"if $J = \varphi_n \ldots \varphi_1(I_0)$ with $\varphi_i = \varphi_{\alpha_i, u_i, \tau_i}$, there exist $r \geqslant 0$ and a bijection $[n] \xrightarrow{\sigma} [n]$ such that $I_0 = \varphi_{\sigma(r-1)} \ldots \varphi_{\sigma(1)}(I_0)$, and such that if I_i is the ideal $\varphi_{\sigma(i)} \ldots \varphi_{\sigma(1)}(I_0)$ then $\alpha_{\sigma(i)} \sim_{I_i} u_{\sigma(i)}$ for any $r \leqslant i \leqslant n$"*. Notice that if R_n is true, then the sequence $\varphi_r, \ldots, \varphi_n$ and the dilatation D satisfy the conclusion of Lemma 4.3. Hence we only need to show that R_n is true for $n \geqslant 0$. Obviously R_0 is true. Let $n \geqslant 1$ be such that R_{n-1} is true. Assume first that there exists $i_0 \in [n]$ such that $\alpha_{i_0} \sim_J u_{i_0}$. Let $[n] \xrightarrow{\nu} [n]$ be the transposition (i_0, n) and set $I' = \varphi_{\nu(n-1)} \ldots \varphi_{\nu(1)}(I_0)$. Applying R_{n-1} to this last equality gives rise to a bijection $[n-1] \xrightarrow{\mu} [n-1]$ together with $r \geqslant 0$, such

that $I_0 = \varphi_{\mu\nu(r-1)} \dots \varphi_{\mu\nu(1)}(I_0)$, and such that if I_i is the ideal $\varphi_{\mu\nu(i)} \dots \varphi_{\mu\nu(1)}(I_0)$ then $\alpha_{\mu\nu(i)} \sim_{I_i} u_{\mu\nu(i)}$ for $r \leqslant i \leqslant n-1$. Set $\mu(n) = n$, then $[n] \xrightarrow{\mu} [n]$ is a bijection and the couple $(\sigma = \mu\nu, r)$ shows that R_n is true when such an i_0 exists. Assume now that for any $i \in [n]$ we have $\alpha_i \not\sim_J u_i$. Let $\varphi = \varphi_n \dots \varphi_1$. The Lemma 2.4, applied to the data I_0, J, φ, shows that $J = I_0$. Hence R_n is true (with $r = n+1$) in this situation as well. This achieves the proof of the Lemma 4.3. □

The following Proposition shows how a Galois covering of k-categories is induced by a covering of quivers with relations. It generalises the results [14, prop 3.4, 3.5]. The proof uses the ideas presented in [10, sect. 3].

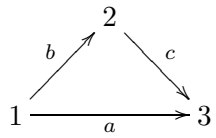
Proposition 4.1. *Let $F\colon \hat{\mathcal{C}} \to \mathcal{C}$ be a Galois covering with group G where $\mathcal{C}$ is a locally bounded and triangular k-category. Fix $\varphi_0\colon kQ/I \xrightarrow{\sim} \mathcal{C}$ be an admissible presentation. Then, there exist admissible presentations $\varphi\colon kQ/I' \xrightarrow{\sim} \mathcal{C}$ and $\psi\colon k\hat{Q}/\hat{I} \xrightarrow{\sim} \hat{\mathcal{C}}$ and a covering of quiver with relations $p\colon (\hat{Q}, \hat{I}) \to (Q, I')$, such that φ and φ_0 coincide on Q_0 and such that the following diagram is commutative:*

$$\begin{array}{ccc} k\hat{Q}/\hat{I} & \xrightarrow{\psi} & \hat{\mathcal{C}} \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle F} \\ kQ/I' & \xrightarrow{\varphi} & \mathcal{C} \end{array}$$

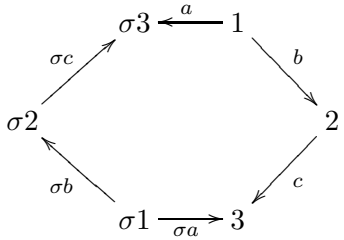
where $\bar{p}$ is induced by p. If $\hat{\mathcal{C}}$ is connected, then p is Galois with group G.

Proof: Using [6, thm 3.8] we may assume that $\mathcal{C}$ is G-graded, that $\mathcal{C}' = \mathcal{C}\sharp G$ and that $\mathcal{C}' = \mathcal{C}\sharp G \xrightarrow{F} \mathcal{C}$ is the natural projection. Since $\mathcal{C}$ is triangular, the ideals $\mathcal{RC}$ and $\mathcal{R}^2\mathcal{C}$ are homogeneous. Thus, for any $x \neq y \in Q_0$ there exist homogeneous elements ${}_yu_x^{(1)}, \dots, {}_yu_x^{({}_yn_x)}$ of ${}_{\varphi_0(y)}\mathcal{RC}_{\varphi_0(x)} = {}_{\varphi_0(y)}\mathcal{C}_{\varphi_0(x)}$ giving rise to a basis of ${}_{\varphi_0(y)}\left(\mathcal{RC}/\mathcal{R}^2\mathcal{C}\right)_{\varphi_0(x)}$. In particular, ${}_yn_x$ is equal to the number of arrows $x \to y$ in Q. Let $\mu\colon kQ \to \mathcal{C}$ be defined as follows: $\mu(x) = \varphi_0(x)$ for any $x \in Q_0$, and μ induces a bijection between the arrows $x \to y$ of Q and $\{{}_yu_x^{(1)}, \dots, {}_yu_x^{({}_yn_x)}\}$ for any $x \neq y \in Q_0$. Set $I' = Ker(\mu)$. Hence I' is admissible and μ induces an isomorphism $\varphi\colon kQ/I' \xrightarrow{\sim} \mathcal{C}$. The following construction of p uses the ideas of Green in [10, sect. 3]. The k-category kQ is a G-graded as follows: a path u in Q is homogeneous of degree the degree of $\mu(u)$. By using the G-grading on $\mathcal{C}$, it is easy to check that I' is homogeneous and that $\varphi\colon kQ/I' \to \mathcal{C}$ is homogeneous of degree 1_G. Let $\hat{Q}$ be the quiver as follows: $\hat{Q}_0 = Q_0 \times G$, and the arrows $(x,s) \xrightarrow{\alpha} (y,t)$ in $\hat{Q}_1$ are exactly the arrows $x \xrightarrow{\alpha} y$ in Q_1 with degree $t^{-1}s$. Let $p\colon \hat{Q} \to Q$ be defined by: $p(x,s) = x$ and $p((x,s) \xrightarrow{\alpha} (y,t)) = \alpha$ for any $(x,s) \in \hat{Q}_0$ and any $(x,s) \xrightarrow{\alpha} (y,t) \in \hat{Q}_1$. Let $\hat{I} \subseteq \hat{Q}$ be the admissible ideal $p^{-1}(I')$ of $k\hat{Q}$. According to [10, sect. 3], p is a covering, and if $\hat{Q}$ is connected then p is Galois with group G. In particular $\bar{p}\colon k\hat{Q}/\hat{I} \to kQ/I'$ is a covering functor. Let $\nu\colon k\hat{Q}/ \to \mathcal{C}' = \mathcal{C}\sharp G$ be as follows: $\nu(x,s) = (\varphi(x), s)$ for any $(x,s) \in \hat{Q}_0$, and if $(x,s) \xrightarrow{\alpha} (y,t) \in \hat{Q}_1$ then $\nu(\alpha) = \mu(p(\alpha)) \in {}_{\varphi(y)}\mathcal{C}'^{t^{-1}s}_{\varphi(x)} = {}_{(\varphi(y),t)}\mathcal{C}'_{(\varphi(x),s)}$. Therefore $F \circ \nu = \varphi \circ p$, and since φ is an isomorphism, we have $\hat{I} = Ker(\nu)$. Let $\psi\colon k\hat{Q}/\hat{I} \to \mathcal{C}'$ be induced by ν. Hence $\psi\colon \hat{Q}_0 \to \hat{\mathcal{C}}_0$ is bijective, ψ is faithful and $\varphi \circ \bar{p} = F \circ \psi$. Moreover ψ is full because $\bar{p}$ and F are covering functors. Thus, ψ is an isomorphism. Finally, if $\mathcal{C}'$ is connected then $\hat{Q}$ is connected and this implies that p is a Galois covering with group G. □

Remark 7. *The Proposition 4.1 does not necessarily hold when F is a covering functor and not a Galois covering. As an example, set $\mathcal{C} = kQ$ where Q is equal to:*



set $G = \mathbb{Z}/2 =< \sigma | \sigma^2 >$ *and set* $\mathcal{C}' = kQ'$ *where* Q' *is the quiver:*



Set $F\colon \mathcal{C}' \to \mathcal{C}$ *to be defined by:* $F(b) = F(\sigma b) = b$, $F(c) = F(\sigma c) = c$, $F(a) = a$ *and* $F(\sigma a) = a + cb$. *Then* F *is a covering functor. The group* $Aut(F)$ *is trivial therefore* F *is not Galois, and* F *cannot be induced by any covering of bound quivers. Notice that if* $F\colon \mathcal{C}' \to \mathcal{C}$ *is a covering functor and if the ordinary quiver of* $\mathcal{C}$ *has no bypasses, then* F *is induced by a covering of bound quivers.*

Theorem 2. *Assume that* A *satisfies the hypotheses made before stating Theorem 1. Let* $\varphi_0\colon kQ/I_0 \simeq A$ *be an admissible presentation such that* $\sim_{I_0}$ *is the source of* Γ. *Let* $(\tilde{Q}, \tilde{I}_0) \xrightarrow{p_0} (Q, I_0)$ *be the universal Galois covering with group* $\pi_1(Q, I_0)$ *and let* $k\tilde{Q}/\tilde{I}_0 \xrightarrow{\bar{p}_0} kQ/I$ *be induced by* p_0. *For any connected Galois covering* $F\colon \mathcal{C}' \to A$ *with group* G *there exist an isomorphism* $kQ/I_0 \xrightarrow{\sim} A$ *equal to* φ_0 *on objects, a Galois covering* $F'\colon k\tilde{Q}/\tilde{I}_0 \to \mathcal{C}'$ *with group* N *a normal subgroup of* $\pi_1(Q, I_0)$ *such that the following diagram commutes:*

$$\begin{array}{ccc} k\tilde{Q}/\tilde{I}_0 & \xrightarrow{F'} & \mathcal{C}' \\ {\scriptstyle \bar{p}_0}\downarrow & & \downarrow{\scriptstyle F} \\ kQ/I_0 & \xrightarrow{\sim} & A \end{array}$$

Moreover, there is exact sequence of groups: $1 \to N \to \pi_1(Q, I_0) \to G \to 1$.

Proof: Let $\mathcal{C}' \xrightarrow{F} A$ be a connected Galois covering with group G. The Proposition 4.1 implies there exist admissible presentations $\varphi\colon kQ/I \xrightarrow{\sim} A$ and $\psi\colon kQ'/I' \xrightarrow{\sim} \mathcal{C}'$ together with $(Q', I') \xrightarrow{q} (Q, I)$ a Galois covering with group G such that φ and φ_0 coincide on objects and such that $F \circ \psi = \varphi \circ \bar{q}$ (where $kQ'/I' \xrightarrow{\bar{q}} kQ/I$ is induced by q). Let $(\hat{Q}, \hat{I}) \xrightarrow{p} (Q, I)$ be the universal Galois covering with group $\pi_1(Q, I)$. Thus (see [14]) there exists a Galois covering $(\hat{Q}, \hat{I}) \xrightarrow{r} (Q', I')$ with group $\pi_1(Q', I')$ and such that $q \circ r = p$. Hence we have a commutative diagram (denoted by $\mathcal{D}$):

$$\begin{array}{ccccc} k\hat{Q}/\hat{I} & \xrightarrow{\bar{r}} & kQ'/I' & \xrightarrow[\psi]{\sim} & \mathcal{C}' \\ {\scriptstyle \bar{p}}\downarrow & & {\scriptstyle \bar{q}}\downarrow & & \downarrow{\scriptstyle F} \\ kQ/I & \xrightarrow{Id} & kQ/I & \xrightarrow[\varphi]{\sim} & A \end{array}$$

Since $\sim_{I_0}$ is the source of Γ, the Lemma 4.3 implies that there exist a sequence of transvections $\varphi_1 = \varphi_{\alpha_1, u_1, \tau_1}, \ldots, \varphi_n = \varphi_{\alpha_b, u_n, \tau_n}$ of kQ and a dilatation D such that $I = D\varphi_n \ldots \varphi_1(I_0)$ and such that $\alpha_i \sim_{I_i} u_i$ if $I_i = \varphi_i \ldots \varphi_1(I_0)$ for any i. The Lemmas 4.1 and 4.2 applied to D, I, I_n and φ_i, I_{i-1}, I_i respectively yield the following commutative diagrams denoted by $\mathcal{D}'$ and $\mathcal{T}_i$ respectively:

$$\begin{array}{ccc} kQ^{(n)}/I^{(n)} & \longrightarrow & k\hat{Q}/\hat{I} \\ \downarrow{\scriptstyle \bar{p}_n} & & \downarrow{\scriptstyle \bar{p}} \\ kQ/I_n & \xrightarrow{\bar{D}} & kQ/I \end{array} \qquad \begin{array}{ccc} kQ^{(i-1)}/I^{(i-1)} & \longrightarrow & kQ^{(i)}/I^{(i)} \\ \downarrow{\scriptstyle \bar{p}_{i-1}} & & \downarrow{\scriptstyle \bar{p}_i} \\ kQ/I_{i-1} & \xrightarrow{\bar{\varphi}_i} & kQ/I_i \end{array}$$

where $\bar{\varphi}_i$ (resp. $\bar{D}$) is induced by φ_i (resp. D) and $kQ^{(i)}/I^{(i)} \xrightarrow{\bar{p}_i} kQ/I_i$ is induced by the universal Galois covering $(Q^{(i)}, I^{(i)}) \xrightarrow{p_i} (Q, I_i)$ with group $\pi_1(Q, I_i)$. If we connect $\mathcal{T}_1, \ldots, \mathcal{T}_n, \mathcal{D}'$ and $\mathcal{D}$ we get the announced commutative diagram:

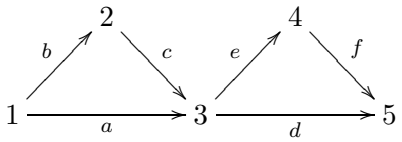
$$\begin{array}{ccc} k\tilde{Q}/\tilde{I}_0 & \xrightarrow{F'} & \mathcal{C}' \\ \downarrow{\scriptstyle \bar{p}_0} & & \downarrow{\scriptstyle F} \\ kQ/I_0 & \xrightarrow{\sim} & A \end{array}$$

where the bottom arrow is an isomorphism equal to φ_0 on objects. Finally the announced properties of F' are given by Proposition 3.3. □

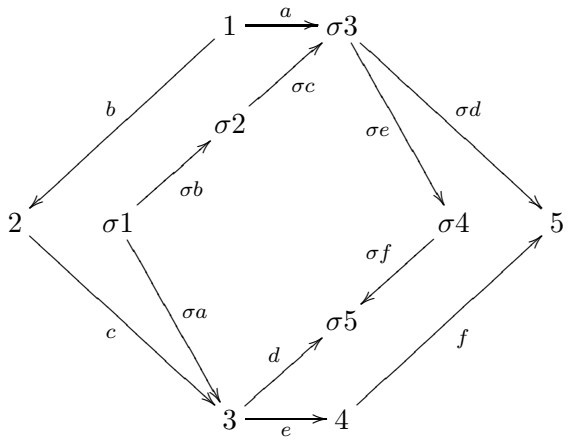
Remark 8. *Using the universal property in Theorem 2 it is quickly checked that if there exists a Galois covering $\mathcal{C}' \to \mathcal{C}$ such that $\mathcal{C}'$ is simply connected (i.e. the fundamental group of any presentation of $\mathcal{C}'$ is trivial), then $\mathcal{C}' \simeq k\tilde{Q}/\tilde{I}_0$.*

One may wish to use the more general framework of Galois categories (see [16]) in order to recover Theorem 1 and Theorem 2. Unfortunately this cannot be done in general because the category of covering functors with finite fibre of A may not have products as explained in the following example:

Example 4. *Let $A = kQ/I$ where Q is equal to*



and $I =< da,\ dcb + fea,\ fecb >$. Set $G = \mathbb{Z}/2 =< \sigma | \sigma^2 >$. Let Q' be the quiver:

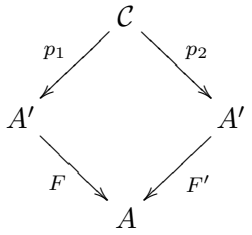


and set $I' =< \sigma d\ a, d\ \sigma a, dcb + \sigma f\ \sigma e\ a, \sigma d\ \sigma e\ \sigma b + fea, fecb, \sigma f\ \sigma e\ \sigma c\ \sigma b >$. Hence the natural mapping $p\colon (Q', I') \to (Q, I)$ $(x, \sigma x \mapsto x)$ is a Galois covering with group G. Therefore, if we set $A' = kQ'/I'$, then p induces a Galois covering $F\colon A' \to A$ with group G. Let us set $F'\colon A' \to A$ to be the Galois covering with group G as well and defined as follows:

. $F'(a \ mod\ I') = F'(\sigma a \ mod\ I') = a + cb \ mod\ I$,

. $F'(x \bmod I') = F'(\sigma x \bmod I') = x \bmod I$ *for any arrow* $x \neq a$.

Assume that the category of the coverings of A *with finite fibre is a Galois category. Hence this category admits finite products and the product of* F *with* F' *gives rise to a diagram:*



such that $F'' = F \circ p_1 = F' \circ p_2$ *is a covering functor with fibre the product of the fibres of* F *and* F'. *In particular, we may assume that* $\mathcal{C}_0 = Q'_0 \times_{Q_0} Q'_0 = \bigcup_{x \in Q_0} \{(x,x),(x,\sigma x),(\sigma x,x),(\sigma x,\sigma x)\}$. *Moreover, the Proposition 3.2 implies that* p_1 *and* p_2 *are covering functors as well. Let us compute the lifting* u *of* $a \bmod I \in {}_3A_1$ *w.r.t.* F'' *and with source* $(1,1)$. *Using the lifting property of* p_1 *and* p_2 *we get:*

- $u_1 + u_2 \in {}_{(\sigma 3,3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3,\sigma 3)}\mathcal{C}_{(1,1)}$ *the lifting of* $a \bmod I'$ *w.r.t.* p_1 *and with source* $(1,1)$,
- $v_1 + v_2 \in {}_{(3,\sigma 3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3,\sigma 3)}\mathcal{C}_{(1,1)}$ *the lifting of* $a \bmod I'$ *w.r.t.* p_2 *and with source* $(1,1)$,
- $v_3 + v_4 \in {}_{(\sigma 3,3)}\mathcal{C}_{(1,1)} \oplus {}_{(3,3)}\mathcal{C}_{(1,1)}$ *the lifting of* $cb \bmod I'$ *w.r.t.* p_2 *and with source* $(1,1)$,

Since $a \bmod I = F(a \bmod I')$ *and* $a \bmod I = F'(a \bmod I' - cb \bmod I')$, *we infer that* $u_1 + u_2$ *and* $v_1 + v_2 - v_3 - v_4$ *both lift* $a \bmod I$ *w.r.t.* $F \circ p_1 = F' \circ p_2$ *and have their source equal to* $(1,1)$. *Therefore* $u = u_1 + u_2 = v_1 + v_2 - v_3 - v_4$. *Using the target of the involved morphims we get* $v_1 = v_4 = 0$, $u_1 = v_3$ *and* $u_2 = v_2$. *Notice that* $v_3 \neq 0$ *and* $v_2 \neq 0$ *since* $a \bmod I' \neq 0$ *and* $cb \bmod I' \neq 0$. *Hence* ${}_{(\sigma 3,\sigma 3)}\mathcal{C}_{(1,1)}$ *and* ${}_{(\sigma 3,3)}\mathcal{C}_{(1,1)}$ *are non zero spaces. Moreover,* p_1 *induces an inclusion* ${}_{(\sigma 3,\sigma 3)}\mathcal{C}_{(1,1)} \oplus {}_{(\sigma 3,3)}\mathcal{C}_{(1,1)} \hookrightarrow {}_{\sigma 3}A'_1$. *Therefore* ${}_{\sigma 3}A'_1 = k.(a \bmod I')$ *contains a 2 dimensional subspace. This contradiction shows that* F *and* F' *do not have a product and that the category of the coverings of* A *with finite fibre need not be a Galois category.*

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