



HAL
open science

A renewal theory approach to periodic copolymers with adsorption

Francesco Caravenna, Giambattista Giacomin, Lorenzo Zambotti

► **To cite this version:**

Francesco Caravenna, Giambattista Giacomin, Lorenzo Zambotti. A renewal theory approach to periodic copolymers with adsorption. 2005. hal-00007575

HAL Id: hal-00007575

<https://hal.science/hal-00007575>

Preprint submitted on 18 Jul 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A RENEWAL THEORY APPROACH TO PERIODIC COPOLYMERS WITH ADSORPTION

FRANCESCO CARAVENNA, GIAMBATTISTA GIACOMIN, AND LORENZO ZAMBOTTI

ABSTRACT. We consider a general model of an heterogeneous polymer chain fluctuating in the proximity of an interface between two selective solvents. The heterogeneous character of the model comes from the fact that *monomer units* interact with the solvents and with the interface according to some *charges* that they carry. The charges repeat themselves along the chain in a periodic fashion. The main question on this model is whether the polymer remains tightly close to the interface, a phenomenon called *localization*, or there is a marked preference for one of the two solvents yielding thus a *delocalization* phenomenon.

We propose an approach to this model, based on renewal theory, that yields sharp estimates on the partition function of the model in all the regimes (localized, delocalized and critical). This in turn allows to get a very precise description of the polymer measure, both in a local sense (*thermodynamic limit*) and in a global sense (*scaling limits*). A key point, but also a byproduct, of our analysis is the closeness of the polymer measure to suitable *Markov Renewal Processes*.

2000 *Mathematics Subject Classification*: 60K35, 82B41, 82B44

Keywords: Random Walks, Renewal Theory, Markov Renewal Theory, Scaling limits, Polymer models, Wetting Models.

1. INTRODUCTION AND MAIN RESULTS

1.1. Two motivating models. Let $S := \{S_n\}_{n=0,1,\dots}$ be a random walk, $S_0 = 0$ and $S_n = \sum_{j=1}^n X_j$, with IID symmetric increments taking values in $\{-1, 0, +1\}$. Hence the law of the walk is identified by $p := \mathbf{P}(X_1 = 1)$ ($= \mathbf{P}(X_1 = -1)$), and we assume that $p \in (0, 1/2)$. Note that we have excluded the case $p = 1/2$ and this has been done in order to lighten the exposition: all the results we present have a close analog in the case $p = 1/2$, however the statements require a minimum of notational care because of the periodicity of the walk. We also consider a sequence $\omega := \{\omega_n\}_{n \in \mathbb{N} = \{1, 2, \dots\}}$ of real numbers with the property that $\omega_n = \omega_{n+T}$ for some $T \in \mathbb{N}$ and for every n : we denote by $T(\omega)$ the minimal value of T .

Consider the following two families of modifications of the law of the walk, both indexed by a parameter $N \in \mathbb{N}$:

- (1) *Pinning and wetting models.* For $\lambda \geq 0$ consider the probability measure $\mathbf{P}_{N,\omega}$ defined by

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^N \omega_n \mathbf{1}_{\{S_n=0\}}\right). \quad (1.1)$$

The walk receives a *pinning reward*, which may be negative or positive, each time it visits the origin. By considering the directed walk viewpoint, that is $\{(n, S_n)\}_n$, one may interpret this model in terms of a directed linear chain receiving an energetic contribution when it touches an interface. In this context it is natural to introduce the asymmetry parameter $h := \sum_{n=1}^T \omega_n / T$, so that one isolates a constant drift term from the *fluctuating* behavior of ω . The question is whether for large N the measure $\mathbf{P}_{N,\omega}$ is rather attracted or repelled by the interface (there is in principle the possibility for the walk to be essentially indifferent of such a change of measure, but we anticipate that this happens only in trivially degenerate cases while in *critical* situations a more subtle scenario shows up).

By multiplying the right-hand side of (1.1) by $\mathbf{1}_{\{S_n \geq 0: n=1, \dots, N\}}$ one gets to a so called *wetting model*, that is the model of an interface interacting with an impenetrable wall. The *hard-wall* condition induces a repulsion effect of purely entropic origin which is in competition with attractive energy effects: one expects that in this case h needs to be positive for the energy term to overcome the entropic repulsion effect, but quantitative estimates are not a priori obvious.

There is an extensive literature on periodic pinning and wetting models, the majority of which is restricted to the $T = 2$ case, we mention for example [10, 17].

(2) *Copolymer near a selective interface.* Much in the same way we introduce

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) \propto \exp\left(\lambda \sum_{n=1}^N \omega_n \text{sign}(S_n)\right), \quad (1.2)$$

where if $S_n = 0$ we set $\text{sign}(S_n) := \text{sign}(S_{n-1}) \mathbf{1}_{\{S_{n-1} \neq 0\}}$. This convention for defining $\text{sign}(0)$, that will be kept throughout the paper, has the following simple interpretation: $\text{sign}(S_n) = +1, 0, -1$ according to whether the bond joining S_{n-1} and S_n lies above, on, or below the x -axis.

Also in this case we take a directed walk viewpoint and then $\mathbf{P}_{N,\omega}$ may be interpreted as a polymeric chain in which the monomer units, the bonds of the walk, are charged. An interface, the x -axis, separates two solvents, say oil above and water below: positively charged monomers are hydrophobic and negatively charged ones are instead hydrophilic. In this case one expects a competition between three possible scenarios: polymer preferring water, preferring oil or undecided between the two and choosing to fluctuate in the proximity of the interface. We will therefore talk of delocalization in water (or oil) or of localization at the interface. Critical cases are of course of particular interest.

We select [21, 24] from the physical literature on periodic copolymers, keeping however in mind that periodic copolymer modeling has a central role in applied chemistry and material science.

1.2. A general model. We point out that the models presented in § 1.1 are particular examples of the polymer measure with Hamiltonian

$$\mathcal{H}_N(S) = \sum_{i=\pm 1} \sum_{n=1}^N \omega_n^{(i)} \mathbf{1}_{\{\text{sign}(S_n)=i\}} + \sum_{n=1}^N \omega_n^{(0)} \mathbf{1}_{\{S_n=0\}} + \sum_{n=1}^N \tilde{\omega}_n^{(0)} \mathbf{1}_{\{\text{sign}(S_n)=0\}}, \quad (1.3)$$

where $\omega^{(\pm 1)}$, $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$ are periodic sequences of real numbers. Observe that, by our conventions on $\text{sign}(0)$, the last term gives an energetic contribution (of pinning/depinning type) to the bonds lying on the interface.

Besides being a natural model, generalizing and interpolating between pinning and copolymer models, the general model we consider is the one considered at several instances, see e.g. [26] and references therein.

Remark 1.1. The *copolymer* case corresponds to $\omega^{(+1)} = -\omega^{(-1)} = \lambda\omega$ and $\omega^{(0)} = \tilde{\omega}^{(0)} = 0$, while the *pinning* case corresponds to $\omega^{(0)} = \lambda\omega$ and $\omega^{(+1)} = \omega^{(-1)} = \tilde{\omega}^{(0)} = 0$. We stress that the *wetting* case can be included too, with the choice $\omega^{(0)} = \lambda\omega$, $\omega_n^{(-1)} = -\infty$ for every n and $\omega^{(+1)} = \tilde{\omega}^{(0)} = 0$. Of course plugging $\omega_n^{(-1)} = -\infty$ into the Hamiltonian (1.3) is a bit formal, but it simply corresponds to a constraint on S in the polymer measure associated to \mathcal{H}_N , see (1.4) below. For ease of exposition we will restrict to finite values of the charges ω , but the generalization is straightforward.

Remark 1.2. We take this occasion for stressing that, from an applied viewpoint, the interest in periodic models of the type we consider appears to be at least two-fold. On one hand periodic models are often chosen as caricatures of the *quenched disordered* models, like the ones in which the charges are a typical realization of a sequence of independent random variables (e.g. [1, 4, 11, 26] and references therein). In this respect and taking a mathematical standpoint, the relevance of periodic models, which may be viewed as *weakly inhomogeneous*, for understanding the strongly inhomogeneous quenched set-up is at least questionable and the approximation of quenched models with periodic ones, in the limit of large period, poses very interesting and challenging questions. In any case, the precise description of the periodic case that we have obtained in this work highlights limitations and perspectives of periodic modeling for strongly inhomogeneous systems. On the other hand, as already mentioned above, periodic models are absolutely natural and of direct relevance for application, for example when dealing with *molecularly engineered* polymers [16, 24].

Starting from the Hamiltonian (1.3), for $a = c$ (*constrained*) or $a = f$ (*free*) we introduce the *polymer measure* $\mathbf{P}_{N,\omega}^a$ on $\mathbb{Z}^{\mathbb{N}}$, defined by

$$\frac{d\mathbf{P}_{N,\omega}^a(S)}{d\mathbf{P}} = \frac{\exp(\mathcal{H}_N(S))}{\tilde{Z}_{N,\omega}^a} (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}), \quad (1.4)$$

where $\tilde{Z}_{N,\omega}^a := \mathbf{E}[\exp(\mathcal{H}_N) (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}})]$ is the *partition function*, that is the normalization constant. Here ω is a shorthand for the four periodic sequences appearing in the definition (1.3) of \mathcal{H}_N , and we will use $T = T(\omega)$ to denote the smallest common period of the sequences.

The Laplace asymptotic behavior of $\tilde{Z}_{N,\omega}$ plays an important role and the quantity

$$f_\omega := \lim_{N \rightarrow \infty} \frac{1}{N} \log \tilde{Z}_{N,\omega}^c, \quad (1.5)$$

is usually called *free energy*. The existence of the limit above follows from a direct super-additivity argument, and it is easy to check that $\tilde{Z}_{N,\omega}^c$ can be replaced by $\tilde{Z}_{N,\omega}^f$ without changing the value of f_ω , see e.g. [11]. The standard free energy approach to this type of

models starts from the observation that

$$\begin{aligned} f_\omega &\geq \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbf{E} \left[\exp(\mathcal{H}_N(S)); S_n > 0 \text{ for } n = 1, \dots, N \right] \\ &= \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} \omega_n^{(+1)} + \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbf{P}(S_n > 0 \text{ for } n = 1, \dots, N). \end{aligned} \quad (1.6)$$

It is a classical result [8, Ch. XII.7] that $\mathbf{P}(S_n > 0 \text{ for } n = 1, \dots, N) \sim cN^{-1/2}$, as $N \rightarrow \infty$, for some $c \in (0, \infty)$ (by $a_N \sim b_N$ we mean $a_N/b_N \rightarrow 1$). Hence the limit of the last term of (1.6) is zero and one easily concludes that

$$f_\omega \geq f_\omega^{\mathcal{D}} := \max_{i=\pm 1} h_\omega(i), \quad h_\omega(i) := \frac{1}{T(\omega)} \sum_{n=1}^{T(\omega)} \omega_n^{(i)}. \quad (1.7)$$

Having in mind the steps in (1.6), one is led to the following basic

Definition 1.3. *The polymer chain defined by (1.4) is said to be:*

- *localized (at the interface) if $f_\omega > f_\omega^{\mathcal{D}}$;*
- *delocalized above the interface if $f_\omega = h_\omega(+1)$;*
- *delocalized below the interface if $f_\omega = h_\omega(-1)$.*

Notice that, with this definition, if $h_\omega(+1) = h_\omega(-1)$ and the polymer is delocalized, it is delocalized both above and below the interface.

Remark 1.4. Observe that the polymer measure $\mathbf{P}_{N,\omega}^a$ is invariant under the joint transformation $S \rightarrow -S$, $\omega^{(+1)} \rightarrow \omega^{(-1)}$, hence by symmetry we may (and will) assume that

$$h_\omega := h_\omega(+1) - h_\omega(-1) \geq 0. \quad (1.8)$$

It is also clear that we can add to the Hamiltonian \mathcal{H}_N a constant term (with respect to S) without changing the polymer measure. Then we set

$$\mathcal{H}'_N(S) := \mathcal{H}_N(S) - \sum_{n=1}^N \omega_n^{(+1)},$$

which amounts to redefining $\omega_n^{(+1)} \rightarrow 0$, $\omega_n^{(-1)} \rightarrow (\omega_n^{(-1)} - \omega_n^{(+1)})$ and $\tilde{\omega}_n^{(0)} \rightarrow (\tilde{\omega}_n^{(0)} - \omega_n^{(+1)})$, and we can write

$$\frac{d\mathbf{P}_{N,\omega}^a}{d\mathbf{P}}(S) = \frac{\exp(\mathcal{H}'_N(S))}{Z_{N,\omega}^a} (\mathbf{1}_{\{a=f\}} + \mathbf{1}_{\{a=c\}} \mathbf{1}_{\{S_N=0\}}), \quad (1.9)$$

where $Z_{N,\omega}^a$ is a new partition function which coincides with $\tilde{Z}_{N,\omega}^a \exp(-\sum_{n=1}^N \omega_n^{(+1)})$. The corresponding free energy F_ω is given by

$$F_\omega := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,\omega}^a = f_\omega - f_\omega^{\mathcal{D}}, \quad (1.10)$$

and notice that in terms of F_ω the condition for localization (resp. delocalization) becomes $F_\omega > 0$ (resp. $F_\omega = 0$). From now on, speaking of partition function and free energy we will always mean $Z_{N,\omega}^a$ and F_ω .

1.3. From free energy to path behavior. In order to understand the spirit of the paper, let us briefly outline our results (complete results are given in § 1.5 below).

Our first goal is to give necessary and sufficient *explicit conditions* in terms of the charges ω for the (de)localization of the polymer chain, see Theorem 1.5. We point out that the content of this theorem is in fact much richer, as it gives the *sharp asymptotic behavior* (and not only the Laplace one [5]) as $N \rightarrow \infty$ of the constrained partition function $Z_{N,\omega}^c$. In particular we show that when the polymer is delocalized ($F_\omega = 0$) the constrained partition function $Z_{N,\omega}^c$ is actually vanishing as $N \rightarrow \infty$. Moreover the rate of the decay induces a further distinction in the delocalized regime between a *strictly delocalized regime* ($Z_{N,\omega}^c \sim c_1 N^{-3/2}$, $c_1 \in (0, \infty)$) and a *critical regime* ($Z_{N,\omega}^c \sim c_2 N^{-1/2}$, $c_2 \in (0, \infty)$).

These asymptotic results are important because they allow to address further interesting issues. For example, it has to be admitted that defining (de)localization in terms of the free energy is not completely satisfactory, because one would like to characterize the polymer path properties. In different terms, given a polymer measure which is (de)localized according to Definition 1.3, to what extent are its typical paths really (de)localized? Some partial answers to this question are known, at least in some particular instances: we mention here the case of $T(\omega) = 2$ copolymers [21] and the case of homogeneous pinning and wetting models [6, 18, 25].

The main purpose of this paper is to show that, for the whole class of models we are considering, free energy (de)localization does correspond to a *strong form* of path (de)localization. More precisely, we look at path behavior from two different viewpoints.

- *Thermodynamic limit.* We show that the measure $\mathbf{P}_{N,\omega}^a$ converges weakly as $N \rightarrow \infty$ toward a measure \mathbf{P}_ω on $\mathbb{Z}^{\mathbb{N}}$, of which we give an explicit construction, see Section 3. It turns out that the properties of \mathbf{P}_ω are radically different in the three regimes (localized, strictly delocalized and critical), see Theorem 1.7. It is natural to look at these results as those characterizing the *local* structure of the polymer chain.
- *Brownian scaling limits.* We prove that the diffusive rescaling of the polymer measure $\mathbf{P}_{N,\omega}^a$ converges weakly in $C([0, 1])$ as $N \rightarrow \infty$. Again the properties of the limit process, explicitly described in Theorem 1.8, differ considerably in the three regimes. Moreover we stress that scaling limits describe *global* properties of the chain.

We insist on the fact that the path analysis just outlined has been obtained exploiting heavily the sharp asymptotic behavior of $Z_{N,\omega}^c$ as $N \rightarrow \infty$. In this sense our results are the direct sharpening of the Large Deviations approach taken in [5], where a formula for F_ω was obtained for periodic copolymers (but the method of course directly extends to the general case considered here). Such a formula (see § 2.3), that reduces the problem of computing the free energy to a finite dimensional problem connected to a suitable Perron–Frobenius matrix, in itself suggests the new approach taken here since it makes rather apparent the link between periodic copolymers and the class of *Markov renewal processes* [2]. On the other hand, with respect to [5], we leave aside any issue concerning the phase diagram (except for § 1.6 below).

1.4. The order parameter δ^ω . It is a remarkable fact that the dependence of our results on the charges ω is essentially encoded in one single parameter δ^ω , that can be regarded as the *order parameter* of our models. For the definition of this parameter, we need some

preliminary notation. We start with the law of the first return to zero of the original walk:

$$\tau_1 := \inf\{n > 0 : S_n = 0\} \quad K(n) := \mathbf{P}(\tau_1 = n). \quad (1.11)$$

It is a classical result [8, Ch. XII.7] that

$$\exists \lim_{n \rightarrow \infty} n^{3/2} K(n) =: c_K \in (0, \infty). \quad (1.12)$$

Then we introduce the Abelian group $\mathbb{S} := \mathbb{Z}/(T\mathbb{Z})$ and to indicate that an integer n is in the equivalence class $\beta \in \mathbb{S}$ we write equivalently $[n] = \beta$ or $n \in \beta$. Notice that the charges ω_n are functions of $[n]$, and with some abuse of notation we can write $\omega_{[n]} := \omega_n$. The key observation is that, by the T -periodicity of the charges ω and by the definition (1.8) of h_ω , we can write

$$\sum_{n=n_1+1}^{n_2} (\omega_n^{(-1)} - \omega_n^{(+1)}) = -(n_2 - n_1) h_\omega + \Sigma_{[n_1], [n_2]}.$$

Thus we have decomposed the above sum into a drift term and a more fluctuating term, where the latter has the remarkable property of depending on n_1 and n_2 only through their equivalence classes $[n_1]$ and $[n_2]$. Now we can define three basic objects:

- for $\alpha, \beta \in \mathbb{S}$ and $\ell \in \mathbb{N}$ we set

$$\Phi_{\alpha, \beta}^\omega(\ell) := \begin{cases} \omega_\beta^{(0)} + (\tilde{\omega}_\beta^{(0)} - \omega_\beta^{(+1)}) & \text{if } \ell = 1, \ell \in \beta - \alpha \\ \omega_\beta^{(0)} + \log \left[\frac{1}{2} \left(1 + \exp(-\ell h_\omega + \Sigma_{\alpha, \beta}) \right) \right] & \text{if } \ell > 1, \ell \in \beta - \alpha \\ 0 & \text{otherwise} \end{cases}, \quad (1.13)$$

which is a sort of integrated version of our Hamiltonian;

- for $x \in \mathbb{N}$ we introduce the $\mathbb{S} \times \mathbb{S}$ matrix $M_{\alpha, \beta}^\omega(x)$ defined by

$$M_{\alpha, \beta}^\omega(x) := e^{\Phi_{\alpha, \beta}^\omega(x)} K(x) \mathbf{1}_{(x \in \beta - \alpha)}; \quad (1.14)$$

- summing the entries of M^ω over x we get a $\mathbb{S} \times \mathbb{S}$ matrix that we call B^ω :

$$B_{\alpha, \beta}^\omega := \sum_{x \in \mathbb{N}} M_{\alpha, \beta}^\omega(x). \quad (1.15)$$

The meaning and motivation of these definitions, that at this point might appear artificial, are explained in detail in § 2.2. For the moment we only stress that the above quantities are *explicit functions* of the charges ω and of the law of the underlying random walk (to lighten the notation, the ω -dependence of these quantity will be often dropped in the following).

We can now define our order parameter δ^ω . Observe that $B_{\alpha, \beta}$ is a finite dimensional matrix with nonnegative entries, hence the Perron–Frobenius (P–F) Theorem (see e.g. [2]) entails that $B_{\alpha, \beta}$ has a unique real positive eigenvalue, called the Perron–Frobenius eigenvalue, with the property that it is a simple root of the characteristic polynomial and that it coincides with the spectral radius of the matrix. This is exactly our parameter:

$$\delta^\omega := \text{Perron–Frobenius eigenvalue of } B^\omega. \quad (1.16)$$

1.5. **The main results.** Now we are ready to state our results. We start characterizing the (de)localization of the polymer chain in terms of δ^ω .

Theorem 1.5 (Sharp asymptotics). *The polymer chain is localized if and only if $\delta^\omega > 1$. More precisely, the asymptotic behavior of $Z_{N,\omega}^c$ as $N \rightarrow \infty$, $[N] = \eta$ is given by*

- (1) for $\delta^\omega > 1$ (localized regime) $Z_{N,\omega}^c \sim C_{\omega,\eta}^{>} \exp(\mathbb{F}_\omega N)$;
- (2) for $\delta^\omega < 1$ (strictly delocalized regime) $Z_{N,\omega}^c \sim C_{\omega,\eta}^{<} / N^{3/2}$;
- (3) for $\delta^\omega = 1$ (critical regime) $Z_{N,\omega}^c \sim C_{\omega,\eta}^= / \sqrt{N}$,

where $\mathbb{F}_\omega > 0$ is the free energy and its explicit definition in terms of ω is given in § 2.3, while $C_{\omega,\eta}^{>}$, $C_{\omega,\eta}^{<}$ and $C_{\omega,\eta}^=$ are explicit positive constants, depending on ω and η , whose value is given in Section 2.

Remark 1.6. Theorem 1.5 is the building block of all the path analysis that follows. It is therefore important to stress that, in the quenched disordered case, cf. Remark 1.2, such a strong statement in general does not hold, see [12, Section 4].

Next we investigate the thermodynamic limit, that is the weak limit as $N \rightarrow \infty$ of the sequence of measures $\mathbf{P}_{N,\omega}^a$ on $\mathbb{Z}^{\mathbb{N}}$ (endowed with the standard product topology). The next theorem provides a first connection between free energy (de)localization and the corresponding path properties.

Before stating the result, we need a notation: we denote by \mathcal{P} the set of ω such that:

$$\begin{aligned} \mathcal{P} &:= \{\omega : \delta^\omega \leq 1, \quad h_\omega = 0, \quad \exists \alpha, \beta : \Sigma_{\alpha,\beta} \neq 0\}, & (1.17) \\ \mathcal{P}^{<} &:= \mathcal{P} \cap \{\delta^\omega < 1\}, \quad \mathcal{P}^= := \mathcal{P} \cap \{\delta^\omega = 1\}. \end{aligned}$$

Here \mathcal{P} stands for *problematic*, or *pathologic*. Indeed, we shall see that for $\omega \in \mathcal{P}$ the results are weaker and more involved than for $\omega \notin \mathcal{P}$. We stress however that these restrictions do not concern localized regime, because $\mathcal{P} \subset \{\omega : \delta^\omega \leq 1\}$. We also notice that for the two motivating models of § 1.1, the pinning and the copolymer models, ω *never* belongs to \mathcal{P} . This is clear for the pinning case, where by definition $\Sigma \equiv 0$. On the other hand, in the copolymer case it is known that if $h_\omega = 0$ and $\exists \alpha, \beta : \Sigma_{\alpha,\beta} \neq 0$ then $\delta^\omega > 1$: see Appendix D or [5]. In reality the *pathological* aspects observed for $\omega \in \mathcal{P}$ may be understood in statistical mechanics terms and we sketch an interpretation in § 1.6 below: this goes rather far from the spirit of this paper, since it is an issue tightly entangled with the analysis of the free energy. It will therefore be taken up in a further work.

Theorem 1.7 (Thermodynamic limit). *If $\omega \notin \mathcal{P}^{<}$, then both the polymer measures $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge as $N \rightarrow \infty$ to the same limit \mathbf{P}_ω , law of an irreducible Markov chain on \mathbb{Z} which is:*

- (1) positive recurrent if $\delta^\omega > 1$ (localized regime) ;
- (2) transient if $\delta^\omega < 1$ (strictly delocalized regime) ;
- (3) null recurrent if $\delta^\omega = 1$ (critical regime).

If $\omega \in \mathcal{P}^{<}$ (in particular $\delta^\omega < 1$), for all $\eta \in \mathbb{S}$ and $a = f, c$ the measure $\mathbf{P}_{N,\omega}^a$ converges as $N \rightarrow \infty$, $[N] = \eta$ to $\mathbf{P}_\omega^{a,\eta}$, law of an irreducible transient Markov chain on \mathbb{Z} .

We stress that in all regimes the limit law \mathbf{P}_ω or $\mathbf{P}_\omega^{a,\eta}$ has an explicit construction in terms of $M_{\alpha,\beta}^\omega(x)$, see Section 3 for details.

We finally turn to the analysis of the *diffusive rescaling* of the polymer measure $\mathbf{P}_{N,\omega}^a$. More precisely, let us define the map $X^N : \mathbb{R}^N \mapsto C([0, 1])$:

$$X_t^N(x) = \frac{x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}} + (Nt - \lfloor Nt \rfloor) \frac{x_{\lfloor Nt \rfloor + 1} - x_{\lfloor Nt \rfloor}}{\sigma N^{1/2}}, \quad t \in [0, 1],$$

where $\lfloor \cdot \rfloor$ denotes the integer part and $\sigma^2 := 2p$ is the variance of X_1 under the original random walk measure \mathbf{P} . Notice that $X_t^N(x)$ is nothing but the linear interpolation of $\{x_{\lfloor Nt \rfloor} / (\sigma \sqrt{N})\}_{t \in \frac{\mathbb{N}}{N} \cap [0, 1]}$. For $a = \text{f, c}$ we set:

$$Q_{N,\omega}^a := \mathbf{P}_{N,\omega}^a \circ (X^N)^{-1},$$

Then $Q_{N,\omega}^a$ is a measure on $C([0, 1])$, the space of real continuous functions defined on the interval $[0, 1]$, and we want to study the behavior as $N \rightarrow \infty$ of this sequence of measures.

We start fixing a notation for the following standard processes:

- the Brownian motion $\{B_\tau\}_{\tau \in [0, 1]}$;
- the Brownian bridge $\{\beta_\tau\}_{\tau \in [0, 1]}$ between 0 and 0;
- the Brownian motion *conditioned to stay non-negative on $[0, 1]$* or, more precisely, the Brownian meander $\{m_\tau\}_{\tau \in [0, 1]}$, see [23];
- the Brownian bridge *conditioned to stay non-negative on $[0, 1]$* or, more precisely, the normalized Brownian excursion $\{e_\tau\}_{\tau \in [0, 1]}$, also known as the Bessel bridge of dimension 3 between 0 and 0, see [23].

Then we introduce a modification of the above processes labeled by a parameter $p \in [0, 1]$:

- the process $\{B_\tau^{(p)}\}_{\tau \in [0, 1]}$ is the so-called *skew Brownian motion of parameter p* , cf. [23]. More explicitly, $B^{(p)}$ is a process such that $|B^{(p)}| = |B|$ in distribution, but in which the sign of each excursion is chosen to be +1 (resp. -1) with probability p (resp. $1 - p$) instead of 1/2. In the same way, the process $\{\beta_\tau^{(p)}\}_{\tau \in [0, 1]}$ is the skew Brownian bridge of parameter p . Observe that for $p = 1$ we have $B^{(1)} = |B|$ and $\beta^{(1)} = |\beta|$ in distribution.
- the process $\{m_\tau^{(p)}\}_{\tau \in [0, 1]}$ is defined by

$$\mathbb{P}(m^{(p)} \in dw) := p \mathbb{P}(m \in dw) + (1 - p) \mathbb{P}(-m \in dw),$$

i.e. $m^{(p)} = \sigma m$, where $\mathbb{P}(\sigma = 1) = 1 - \mathbb{P}(\sigma = -1) = p$ and (m, σ) are independent. The process $\{e_\tau^{(p)}\}_{\tau \in [0, 1]}$ is defined in exactly the same manner. For $p = 1$ we have $m^{(1)} = m$ and $e^{(1)} = e$.

Finally, we introduce a last process, labeled by two parameters $p, q \in [0, 1]$:

- consider a r.v. $U \mapsto [0, 1]$ with the arcsin law: $\mathbb{P}(U \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}$, and processes $\beta^{(p)}, m^{(q)}$ as defined above, with $(U, \beta^{(p)}, m^{(q)})$ independent triple. Then

we denote by $\{B_\tau^{(p,q)}\}_{\tau \in [0,1]}$ the process defined by:

$$B_\tau^{(p,q)} := \begin{cases} \sqrt{U} \beta_{\frac{\tau}{U}}^{(p)} & \text{if } \tau \leq U \\ \sqrt{1-U} m_{\frac{\tau-U}{1-U}}^{(q)} & \text{if } \tau > U \end{cases}.$$

Notice that the process $B^{(p,q)}$ differs from the p -skew Brownian motion $B^{(p)}$ only for the last excursion in $[0, 1]$, whose sign is $+1$ with probability q instead of p .

We are going to show that the sequence $\{Q_{N,\omega}^a\}$ has a weak limit as $N \rightarrow \infty$ (with a weaker statement if $\omega \in \mathcal{P}$). Again the properties of the limit process differ considerably in the three regimes $\delta^\omega > 1$, $\delta^\omega < 1$ and $\delta^\omega = 1$. However for the precise description of the limit processes, for the regimes $\delta^\omega = 1$ and $\delta^\omega < 1$ we need to distinguish between $a \in \{f, c\}$ and to introduce further parameters $\mathbf{p}_\omega, \mathbf{q}_\omega$, defined as follows:

- case $\delta^\omega = 1$:
 - $\mathbf{p}_\omega := \mathbf{p}_\omega^\bar{}$, defined in (4.19). We point out two special cases: if $h_\omega > 0$ then $\mathbf{p}_\omega^\bar{=} = 1$, while if $h_\omega = 0$ and $\Sigma \equiv 0$ then $\mathbf{p}_\omega^\bar{=} = 1/2$;
 - for each $\eta \in \mathbb{S}$, $\mathbf{q}_\omega := \mathbf{q}_{\omega,\eta}^\bar{}$, defined by (4.21).
- case $\delta^\omega < 1$:
 - $\omega \notin \mathcal{P}^<$: if $h_\omega > 0$ we set $\mathbf{p}_\omega := \mathbf{p}_\omega^< = 1$ while if $h_\omega = 0$ we set $\mathbf{p}_\omega := \mathbf{p}_\omega^< = 1/2$;
 - $\omega \in \mathcal{P}^<$: for each $\eta \in \mathbb{S}$ and $a = f, c$, $\mathbf{p}_\omega := \mathbf{p}_{\omega,\eta}^{<,a}$ is defined in (4.8) and (4.10).

Theorem 1.8 (Scaling limits). *If $\omega \notin \mathcal{P}$, then the sequence of measures $\{Q_{N,\omega}^a\}$ on $C([0, 1])$ converges weakly as $N \rightarrow \infty$. More precisely:*

- (1) for $\delta^\omega > 1$ (localized regime) $Q_{N,\omega}^a$ converges to the measure concentrated on the constant function taking the value zero;
- (2) for $\delta^\omega < 1$ (strictly delocalized regime):
 - $Q_{N,\omega}^f$ converges to the law of $m^{(\mathbf{p}_\omega^<)}$;
 - $Q_{N,\omega}^c$ converges to the law of $e^{(\mathbf{p}_\omega^<)}$;
- (3) for $\delta^\omega = 1$ (critical regime):
 - $Q_{N,\omega}^f$ converges to the law of $B^{(\mathbf{p}_\omega^\bar{})}$;
 - $Q_{N,\omega}^c$ converges to the law of $\beta^{(\mathbf{p}_\omega^\bar{})}$.

If $\omega \in \mathcal{P}$, then for all $\eta \in \mathbb{S}$ the measures $Q_{N,\omega}^c$ and $Q_{N,\omega}^f$ converge as $N \rightarrow \infty$, $[N] = \eta$ to, respectively:

- (1) for $\delta^\omega < 1$, the law of $e^{(\mathbf{p}_{\omega,\eta}^{<,c})}$ and $m^{(\mathbf{p}_{\omega,\eta}^{<,f})}$.
- (2) for $\delta^\omega = 1$, the law of $\beta^{(\mathbf{p}_\omega^\bar{})}$ and $B^{(\mathbf{p}_\omega^\bar{}, \mathbf{q}_{\omega,\eta}^\bar{})}$.

Results on thermodynamic limits in the direction of Theorem 1.7 have been obtained in the physical literature by exact computations either for homogeneous polymers or for $T = 2$ pinning models and copolymers, see e.g. [21], while, in some cases, Brownian scaling limits have been conjectured on the base of heuristic arguments, see e.g. [25]. Rigorous results corresponding to our three main theorems have been obtained for homogeneous pinning/wetting models in [6, 18]. We would like to stress the very much richer variety of limit processes that we have obtained in our general context.

1.6. **About the regime \mathcal{P} .** We have seen, cf. Theorem 1.7, that if $\omega \in \mathcal{P}^<$ the infinite volume limit (in particular the probability that the walk escapes either to $+\infty$ or to $-\infty$) depends on $a = c$ or f and on the subsequence $[N] = \eta \in \mathbb{S}$. This reflects directly into Theorem 1.8 and in this case also the $\mathcal{P}^=$ regime is affected, but only for $a = f$ and the change is restricted to the sign of the very last excursion of the process. It is helpful to keep in mind that $\omega \in \mathcal{P}$ if and only if there is a non trivial unbiased copolymer part, that is $h_\omega = 0$ but the matrix Σ is non trivial, and at the same time the polymer is delocalized. It is known (Appendix D and [5]) that in absence of pinning terms, that is $\omega_n^{(0)} = \tilde{\omega}_n^{(0)} = 0$ for every n , the polymer is localized. However if the pinning rewards are sufficiently large and negative, one easily sees that (de)pinning takes over and the polymer delocalizes. This is the phenomenon that characterizes the regime \mathcal{P} and its lack of uniqueness of limit measures.

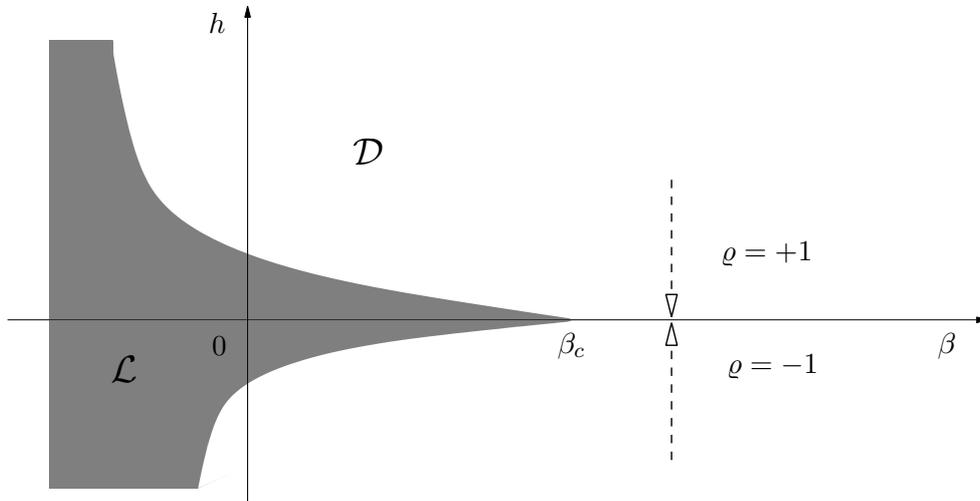


FIGURE 1. A sketch of the phase diagram for the model (1.18). In this case, with abuse of notation, $\mathcal{P} = \{(h, \beta) : h = 0, \beta \geq \beta_c\}$. Approaching \mathcal{P} in the sense of the dashed arrowed lines one observes the two sharply different behaviors of paths completely delocalized above ($\varrho = +1$) or below ($\varrho = -1$) the interface.

Lack of uniqueness of infinite volume measures and dependence on boundary conditions do not come as a surprise if one takes a statistical mechanics viewpoint and if one notices that the system undergoes a *first order* phase transition exactly at \mathcal{P} . In order to be more precise let us consider the particular case of

$$\frac{d\mathbf{P}_{N,\omega}}{d\mathbf{P}}(S) \propto \exp \left(\sum_{n=1}^N (\omega_n + h) \text{sign}(S_n) - \beta \sum_{n=1}^N \mathbf{1}_{\{S_n=0\}} \right), \quad (1.18)$$

with h and β two real parameters and ω a fixed non trivial centered ($\sum_{n=1}^T \omega_n = 0$) periodic configuration of charges. The phase diagram of such a model is sketched in Figure 1. In particular it is easy to show that for $h = 0$ and for β large and positive the polymer is delocalized and, recalling that for $\beta = 0$ the polymer is localized, by monotonicity of the free energy in β one immediately infers that there exists $\beta_c > 0$ such that localization

prevails for $\beta < \beta_c$, while the polymer is delocalized (both above and below the interface) if $\beta \geq \beta_c$. However the two regimes of delocalization above or below the interface, appearing for example as soon as h is either positive or negative and $\beta \geq \beta_c$, are characterized by opposite values (± 1) of $\varrho = \varrho(h, \beta) := \lim_{N \rightarrow \infty} \mathbf{E}_{N, \omega} \left[N^{-1} \sum_{n=1}^N \text{sign}(S_n) \right]$ and of course ϱ is the derivative of the free energy with respect to h . Therefore the free energy is not differentiable at $h = 0$ and we say that there is a *first order phase transition*. First order phase transitions are usually associated to multiple infinite volume limits (*phase coexistence*). A detailed analysis of this interesting phenomenon will be given elsewhere.

1.7. Outline of the paper. In Section 2 we study the asymptotic behavior of $Z_{N, \omega}^c$, proving Theorem 1.5. In Section 3 we compute the thermodynamic limits of $\mathbf{P}_{N, \omega}^a$, proving Theorem 1.7. In Section 4 we compute the scaling limits of $\mathbf{P}_{N, \omega}^a$, proving Theorem 1.8. Finally, in the Appendices we give the proof of some technical results and some additional material.

2. SHARP ASYMPTOTICS FOR THE PARTITION FUNCTION

In this section we are going to derive the precise asymptotic behavior of $Z_{N, \omega}^c$, in particular proving Theorem 1.5. The key observation is that the study of the partition function for the models we are considering can be set into the framework of the theory of *Markov renewal processes*, see [2, Ch. VII.4]. We start recalling the basic notions of this theory and setting the relative notation.

2.1. Markov Renewal Theory. Given a finite set \mathbb{S} (for us it will always be $\mathbb{Z}/(T\mathbb{Z})$), by a *kernel* we mean a family of nonnegative $\mathbb{S} \times \mathbb{S}$ matrices $F_{\alpha, \beta}(x)$ depending on a parameter $x \in \mathbb{N}$. We say that the kernel $F_{\alpha, \beta}(x)$ is *semi-Markov* if $F_{\alpha, \cdot}(\cdot)$ is a probability mass function on $\mathbb{S} \times \mathbb{N}$ for every $\alpha \in \mathbb{S}$, that is if $\sum_{\beta, x} F_{\alpha, \beta}(x) = 1$.

A semi-Markov kernel $F_{\alpha, \beta}(x)$ has a simple probabilistic interpretation: it defines a Markov chain $\{(J_k, T_k)\}$ on $\mathbb{S} \times \mathbb{N}$ through the transition kernel given by

$$\mathbb{P}[(J_{k+1}, T_{k+1}) = (\beta, x) \mid (J_k, T_k) = (\alpha, y)] = F_{\alpha, \beta}(x). \quad (2.1)$$

In this case we say that the process $\{J_k, T_k\}$ is a (discrete) *Markov-renewal process*, the $\{T_k\}$ being thought of as interarrival times. This provides a generalization of classical renewal processes, since the $\{T_k\}$ are no longer IID but their laws are rather *modulated* by the process $\{J_k\}$. Since the r.h.s. of (2.1) does not depend on y , it follows that $\{J_k\}$ is a Markov chain, and it is called the *modulating chain* of the Markov renewal process (observe that in general the process $\{T_k\}$ is *not* a Markov chain). The transition kernel of $\{J_k\}$ is given by $\sum_{x \in \mathbb{N}} F_{\alpha, \beta}(x)$. We will assume that this chain is irreducible (therefore positive recurrent, since \mathbb{S} is finite) and we denote by $\{\nu_\alpha\}_{\alpha \in \mathbb{S}}$ its invariant measure.

Given two kernels F and G , their convolution $F * G$ is the kernel defined by

$$(F * G)_{\alpha, \beta}(x) := \sum_{y \in \mathbb{N}} \sum_{\gamma \in \mathbb{S}} F_{\alpha, \gamma}(y) G_{\gamma, \beta}(x - y) = \sum_{y \in \mathbb{N}} [F(y) \cdot G(x - y)]_{\alpha, \beta}, \quad (2.2)$$

where \cdot denotes matrix product. Observe that if F and G are semi-Markov kernels, then $F * G$ is semi-Markov too. With standard notation, the n -fold convolution of a kernel F with itself will be denoted by F^{*n} , the $n = 0$ case being by definition the identity kernel $[F^{*0}]_{\alpha, \beta}(x) := \mathbf{1}_{(\beta=\alpha)} \mathbf{1}_{(x=0)}$.

A fundamental object associated to a semi-Markov kernel F the so-called *Markov-Green function* (or Markov-renewal kernel), which is the kernel \mathbf{U} defined by

$$\mathbf{U}_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} [F^{*k}]_{\alpha,\beta}(x). \quad (2.3)$$

Of course the kernel \mathbf{U} is the analog of the Green function of a classical renewal process, and it has a similar probabilistic interpretation in terms of the associated Markov renewal process $\{(J_k, T_k)\}$:

$$\mathbf{U}_{\alpha,\beta}(x) = \mathbb{P}_{\alpha}[\exists k \geq 0 : T_0 + \dots + T_k = x, J_k = \beta], \quad (2.4)$$

where \mathbb{P}_{α} is the law of $\{(T_k, J_k)\}$ conditioned on $\{J_0 = \alpha, T_0 = 0\}$.

We need some notation to treat our periodic setting: we say that a kernel $F_{\alpha,\beta}(x)$ has period $T \in \mathbb{N}$ if the set $\{x : \mathbf{U}_{\alpha,\beta}(x) \neq 0\}$ is contained in $T\mathbb{Z}$, for the least such T (this definition does not depend on α because the chain $\{J_k\}$ is supposed to be irreducible, see the discussion at p. 208 of [2]). It follows that the set $\{x : \mathbf{U}_{\alpha,\beta}(x) \neq 0\}$ is contained in the translated lattice $\gamma(\alpha, \beta) + T\mathbb{N}$, where $\gamma(\alpha, \beta) \in \{0, \dots, T-1\}$ (for us it will be $\gamma(\alpha, \beta) = [\beta - \alpha]$).

In analogy to the classical case, the asymptotic behavior of $\mathbf{U}_{\alpha,\beta}(x)$ as $x \rightarrow \infty$ is of particular interest. Let us define the (possibly infinite) *mean* μ of a semi-Markov kernel $F_{\alpha,\beta}(x)$ as

$$\mu := \sum_{\alpha, \beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} x \nu_{\alpha} F_{\alpha,\beta}(x). \quad (2.5)$$

Then we have an analog of Blackwell's Renewal Theorem, that in our periodic setting reads as

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} \mathbf{U}_{\alpha,\beta}(x) = T \frac{\nu_{\beta}}{\mu}, \quad (2.6)$$

cf. Corollary 2.3 p. 10 of [2] for the classical case.

We will see that determining the asymptotic behavior of $\mathbf{U}_{\alpha,\beta}(x)$ when the kernel $F_{\alpha,\beta}(x)$ is no more semi-Markov is the key to get the asymptotic behavior of the partition function $Z_{N,\omega}^c$.

2.2. A random walk excursion viewpoint. Now we are ready to make explicit the link between the partition function for our model and the Theory of Markov Renewal Processes. Let us look back to our Hamiltonian (1.3): its specificity comes from the fact that it can be decomposed in an efficient way by considering the return times to the origin of S . More precisely we set for $j \in \mathbb{N}$

$$\tau_0 = 0 \quad \tau_{j+1} = \inf\{n > \tau_j : S_n = 0\},$$

and for \mathbf{P} -typical trajectories of S one has an infinite sequence $\tau := \{\tau_j\}_j$ of stopping times. We set $T_j = \tau_j - \tau_{j-1}$ and of course $\{T_j\}_{j=1,2,\dots}$ is, under \mathbf{P} , an IID sequence. By conditioning on τ and integrating on the up-down symmetry of the random walk excursions one easily obtains the following expression for the constrained partition function:

$$Z_{N,\omega}^c = \mathbf{E} \left[\prod_{j=1}^{\iota_N} \exp(\Psi^{\omega}(\tau_{j-1}, \tau_j)); \tau_{\iota_N} = N \right], \quad (2.7)$$

where $\iota_N = \sup\{k : \tau_k \leq N\}$ and we have introduced the *integrated Hamiltonian* $\Psi^\omega(n_1, n_2)$, which gives the energetic contribution of an excursion from n_1 to n_2 :

$$\Psi^\omega(n_1, n_2) = \begin{cases} \omega_{n_2}^{(0)} + (\tilde{\omega}_{n_2}^{(0)} - \omega_{n_2}^{(+1)}) & \text{if } n_2 = n_1 + 1 \\ \omega_{n_2}^{(0)} + \log \left[\frac{1}{2} \left(1 + \exp \sum_{n=n_1+1}^{n_2} (\omega_n^{(-1)} - \omega_n^{(+1)}) \right) \right] & \text{if } n_2 > n_1 + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Now we are going to use in an essential way the fact that our charges are T -periodic. In fact a look at (2.8) shows that the energy $\Psi^\omega(n_1, n_2)$ of an excursion from n_1 to n_2 is a function only of $(n_2 - n_1)$, $[n_1]$ and $[n_2]$, where by $[\cdot]$ we mean the equivalence class modulo T , see § 1.4. More precisely for $n_1 \in \alpha$, $n_2 \in \beta$ and $\ell = n_2 - n_1$ we have $\Psi^\omega(n_1, n_2) = \Phi_{\alpha, \beta}^\omega(\ell)$, where Φ^ω was defined in (1.13). Then recalling the law $K(n)$ of the first return, introduced in (1.11), we can rewrite (2.7) as

$$Z_{N, \omega}^c = \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \\ 0 = t_0 < t_1 < \dots < t_k := N}} \prod_{j=1}^k K(t_j - t_{j-1}) \exp(\Phi_{[t_{j-1}], [t_j]}^\omega(t_j - t_{j-1})). \quad (2.9)$$

This decomposition of $Z_{N, \omega}^c$ according to the random walk excursions makes explicit the link with Markov Renewal Theory. In fact using the kernel $M_{\alpha, \beta}(x)$ introduced in (1.14) we can rewrite it as

$$\begin{aligned} Z_{N, \omega}^c &= \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \\ 0 = t_0 < t_1 < \dots < t_k := N}} \prod_{j=1}^k M_{[t_{j-1}], [t_j]}(t_j - t_{j-1}) \\ &= \sum_{k=1}^N \sum_{\substack{t_0, \dots, t_k \in \mathbb{N} \\ 0 = t_0 < t_1 < \dots < t_k := N}} [M(t_1) \cdot M(t_2 - t_1) \cdot \dots \cdot M(N - t_{k-1})]_{0, [N]} \\ &= \sum_{k=0}^{\infty} [M^{*k}]_{[0], [N]}(N). \end{aligned} \quad (2.10)$$

Therefore it is natural to introduce the kernel $\mathcal{Z}_{\alpha, \beta}(x)$ defined by

$$\mathcal{Z}_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} [M^{*k}]_{\alpha, \beta}(x), \quad (2.11)$$

so that $Z_{N, \omega}^c = \mathcal{Z}_{[0], [N]}(N)$. More generally $\mathcal{Z}_{\alpha, \beta}(x)$ for $[x] = \beta - \alpha$ can be interpreted as the partition function of a directed polymer of size x that starts at a site $(M, 0)$, with $[M] = \alpha$, and which is pinned at the site $(M + x, 0)$.

Our purpose is to get the precise asymptotic behavior of $\mathcal{Z}_{\alpha, \beta}(x)$ as $x \rightarrow \infty$, from which we will obtain the asymptotic behavior of $Z_{N, \omega}^c$ and hence the proof of Theorem 1.5. It is clear that equation (2.11) is the same as equation (2.3), except for the fact that in general the kernel M has no reason to be semi-Markov. Nevertheless we will see that with some transformations one can reduce the problem to a semi-Markov setting.

It turns out that for the derivation of the asymptotic behavior of $\mathcal{Z}_{\alpha, \beta}(x)$ it is not necessary to use the specific form (1.14) of the kernel $M_{\alpha, \beta}(x)$, the computations being more transparent if carried out in a general setting. For these reasons, in the following

we will assume that $M_{\alpha,\beta}(x)$ is a generic T -periodic kernel such that the matrix $B_{\alpha,\beta}$ defined by (1.15) is finite. While these assumption are sufficient to yield the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ when $\delta^\omega > 1$, for the cases $\delta^\omega < 1$ and $\delta^\omega = 1$ it is necessary to know the asymptotic behavior as $x \rightarrow \infty$ of $M_{\alpha,\beta}(x)$ itself. Notice that our setting is an *heavy-tailed* one: more precisely we will assume that for every $\alpha, \beta \in \mathbb{S}$:

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} M_{\alpha,\beta}(x) =: L_{\alpha,\beta} \in (0, \infty). \quad (2.12)$$

From equation (1.13) it is easy to check that the kernel $M_{\alpha,\beta}(x)$ defined by (1.14) does satisfy (2.12) (see Section 3 for more details on this issue).

For ease of exposition, we will treat separately the three cases $\delta^\omega > 1$, $\delta^\omega < 1$ and $\delta^\omega = 1$.

2.3. The localized regime ($\delta^\omega > 1$). The key idea is to introduce the following exponential perturbation of the kernel M (cf. [2, Theorem 4.6]), depending on the positive real parameter b :

$$A_{\alpha,\beta}^b(x) := M_{\alpha,\beta}(x) e^{-bx}.$$

Let us denote by $\Delta(b)$ the Perron–Frobenius eigenvalue of the matrix $\sum_x A_{\alpha,\beta}^b(x)$. As the entries of this matrix are analytic and nonincreasing functions of b , $\Delta(b)$ is analytic and nonincreasing too, hence strictly decreasing because $\Delta(0) = \delta^\omega > 1$ and $\Delta(\infty) = 0$. Therefore there exists a single value $F_\omega > 0$ such that $\Delta(F_\omega) = 1$, and we denote by $\{\zeta_\alpha\}_\alpha$, $\{\xi_\alpha\}_\alpha$ the Perron–Frobenius left and right eigenvectors of $\sum_x A_{\alpha,\beta}^{F_\omega}(x)$, chosen to have (strictly) positive components and normalized in such a way that $\sum_\alpha \zeta_\alpha \xi_\alpha = 1$ (of course there is still a degree of freedom in the normalization, which however is immaterial).

Now we set

$$\Gamma_{\alpha,\beta}^>(x) := A_{\alpha,\beta}^{F_\omega}(x) \frac{\xi_\beta}{\xi_\alpha} = M_{\alpha,\beta}(x) e^{-F_\omega x} \frac{\xi_\beta}{\xi_\alpha}, \quad (2.13)$$

and it is immediate to check that $\Gamma^>$ is a semi–Markov kernel. Furthermore, we can rewrite (2.11) as

$$\mathcal{Z}_{\alpha,\beta}(x) := e^{F_\omega x} \frac{\xi_\alpha}{\xi_\beta} \sum_{k=0}^{\infty} [(\Gamma^>)^{*k}]_{\alpha,\beta}(x) = e^{F_\omega x} \frac{\xi_\alpha}{\xi_\beta} \mathcal{U}_{\alpha,\beta}(x), \quad (2.14)$$

where $\mathcal{U}_{\alpha,\beta}(x)$ is nothing but the Markov–Green function associated to the semi–Markov kernel $\Gamma_{\alpha,\beta}^>(x)$. Therefore the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ is easily obtained applying Blackwell’s Renewal Theorem (2.6). To this end, let us compute the mean μ of the semi–Markov kernel $\Gamma^>$: it is easily seen that the invariant measure of the associated modulating chain is given by $\{\zeta_\alpha \xi_\alpha\}_\alpha$, therefore

$$\begin{aligned} \mu &= \sum_{\alpha,\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} x \zeta_\alpha \xi_\alpha \Gamma_{\alpha,\beta}^>(x) = \sum_{\alpha,\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} x e^{-F_\omega x} \zeta_\alpha M_{\alpha,\beta}(x) \xi_\beta \\ &= - \left(\frac{\partial}{\partial b} \Delta(b) \right) \Big|_{b=F_\omega} \in (0, \infty), \end{aligned}$$

(for the last equality see for example [5, Lemma 2.1]). Coming back to (2.14), we can now apply Blackwell’s Renewal Theorem (2.6) obtaining the desired asymptotic behavior:

$$\mathcal{Z}_{\alpha,\beta}(x) \sim \xi_\alpha \zeta_\beta \frac{T}{\mu} \exp(F_\omega x) \quad x \rightarrow \infty, \quad [x] = \beta - \alpha. \quad (2.15)$$

In particular, for $\alpha = [0]$ and $\beta = \eta$ we have part (1) of Theorem 1.5, where $C_{\omega,\eta}^> = \xi_0 \zeta_\eta T / \mu$.

2.4. The strictly delocalized case ($\delta^\omega < 1$). We prove that the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ when $\delta^\omega < 1$ is given by

$$\mathcal{Z}_{\alpha,\beta}(x) \sim \left([(1-B)^{-1}L(1-B)^{-1}]_{\alpha,\beta} \right) \frac{1}{x^{3/2}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha, \quad (2.16)$$

where the matrixes L and B have been defined in (2.12) and (1.15). In particular, taking $\alpha = [0]$ and $\beta = \eta$, (2.16) proves part (2) of Theorem 1.5 with

$$C_{\omega,\eta}^< := [(1-B)^{-1}L(1-B)^{-1}]_{0,\eta}.$$

To start with, we prove by induction that for every $n \in \mathbb{N}$

$$\sum_{x \in \mathbb{N}} [M^{*n}]_{\alpha,\beta}(x) = [B^n]_{\alpha,\beta}. \quad (2.17)$$

The $n = 1$ case is the definition of B , while for $n \geq 1$

$$\begin{aligned} \sum_{x \in \mathbb{N}} M^{*(n+1)}(x) &= \sum_{x \in \mathbb{N}} \sum_{z \leq x} M^{*n}(z) \cdot M(x-z) = \sum_{z \in \mathbb{N}} M^{*n}(z) \cdot \sum_{x \geq z} M(x-z) \\ &= \sum_{z \in \mathbb{N}} M^{*n}(z) \cdot B = B^n \cdot B = B^{n+1}. \end{aligned}$$

Next we claim that, if (2.12) holds, then for every $\alpha, \beta \in \mathbb{S}$

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} [M^{*k}]_{\alpha,\beta}(x) = \sum_{i=0}^{k-1} [B^i \cdot L \cdot B^{(k-1)-i}]_{\alpha,\beta}. \quad (2.18)$$

We proceed by induction on k . The $k = 1$ case is given by (2.12), and we have that

$$M^{*(n+1)}(x) = \sum_{y=1}^{x/2} \left(M(y) \cdot M^{*n}(x-y) + M(x-y) \cdot M^{*n}(y) \right)$$

(strictly speaking this formula is true only when x is even, however the odd x case is analogous). By the inductive hypothesis equation (2.18) holds for every $k \leq n$, and in particular this implies that $\{x^{3/2}[M^{*k}]_{\alpha,\beta}(x)\}_{x \in \mathbb{N}}$ is a bounded sequence. Therefore we can apply Dominated Convergence and (2.17), getting

$$\begin{aligned} &\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} [M^{*(n+1)}]_{\alpha,\beta}(x) \\ &= \sum_{\gamma} \sum_{y=1}^{\infty} \left(M_{\alpha,\gamma}(y) \sum_{i=0}^{n-1} [B^i \cdot L \cdot B^{(n-1)-i}]_{\gamma,\beta} + L_{\alpha,\gamma} [M^{*n}]_{\gamma,\beta}(y) \right) \\ &= \sum_{\gamma} \left(B_{\alpha,\gamma} \sum_{i=0}^{n-1} [B^i \cdot L \cdot B^{(n-1)-i}]_{\gamma,\beta} + L_{\alpha,\gamma} [B^{*n}]_{\gamma,\beta} \right) \\ &= \sum_{i=0}^n [B^i \cdot L \cdot B^{n-i}]_{\alpha,\beta}. \end{aligned}$$

Our purpose is to apply the asymptotic result (2.18) to the terms of (2.11), hence we need a bound to apply Dominated Convergence. What we are going to show is that

$$x^{3/2} [M^{*k}]_{\alpha,\beta}(x) \leq C k^3 [B^k]_{\alpha,\beta} \quad (2.19)$$

for some positive constant C and for all $\alpha, \beta \in \mathbb{S}$ and $x, k \in \mathbb{N}$. Observe that the r.h.s. above, as a function of k , is a summable sequence because the matrix B has spectral radius $\delta^\omega < 1$. We proceed again by induction: for the $k = 1$ case, thanks to (2.12), it is possible to find C such that (2.19) holds true (this fixes C once for all). Now assuming that (2.19) holds for all $k < n$ we show that it does also for $k = n$ (we suppose for simplicity that $n = 2m$ is even, the odd n case being analogous). Then we have (assuming that also x is even for simplicity)

$$\begin{aligned} x^{3/2} [M^{*2m}]_{\alpha,\beta}(x) &= 2 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} [M^{*m}]_{\alpha,\gamma}(y) x^{3/2} [M^{*m}]_{\gamma,\beta}(x-y) \\ &\leq 2 \cdot 2^{3/2} C m^3 \sum_{y=1}^{x/2} \sum_{\gamma \in \mathbb{S}} [M^{*m}]_{\alpha,\gamma}(y) [B^m]_{\gamma,\beta} \\ &\leq C (2m)^3 [B^{2m}]_{\alpha,\beta}, \end{aligned}$$

where we have applied (2.17), and (2.19) is proven.

We can finally obtain the asymptotic behavior of $\mathcal{Z}_{\alpha,\beta}(x)$ applying the bound (2.18) to (2.11), using Dominated Convergence thanks to (2.19). In this way we get

$$\begin{aligned} \exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} \mathcal{Z}_{\alpha,\beta}(x) &= \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} [B^i \cdot L \cdot B^{(k-1)-i}]_{\alpha,\beta} \\ &= \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} [B^i \cdot L \cdot B^{(k-1)-i}]_{\alpha,\beta} = \sum_{i=0}^{\infty} [B^i \cdot L \cdot (1-B)^{-1}]_{\alpha,\beta} \\ &= [(1-B)^{-1} \cdot L \cdot (1-B)^{-1}]_{\alpha,\beta}, \end{aligned}$$

and equation (2.16) is proven.

2.5. The critical case ($\delta^\omega = 1$). In the critical case the matrix B defined in (1.15) has Perron–Frobenius eigenvalue equal to 1. Let $\{\zeta_\alpha\}_\alpha, \{\xi_\alpha\}_\alpha$ denote its corresponding left and right eigenvectors, always chosen to have positive components and normalized so that $\sum_\alpha \zeta_\alpha \xi_\alpha = 1$. Then it is immediate to check that the kernel

$$\Gamma_{\alpha,\beta}^= (x) := M_{\alpha,\beta}(x) \frac{\xi_\beta}{\xi_\alpha} \quad (2.20)$$

is semi–Markov, and the corresponding Markov–Green function $U_{\alpha,\beta}(x)$ is given by

$$U_{\alpha,\beta}(x) := \sum_{k=0}^{\infty} [(\Gamma^=)^{*k}]_{\alpha,\beta}(x) = \frac{\xi_\beta}{\xi_\alpha} \mathcal{Z}_{\alpha,\beta}(x), \quad (2.21)$$

where the last equality follows easily from (2.11). We are going to derive the asymptotic behavior of $U_{\alpha,\beta}(x)$, and from the above relation we will get the analogous result for $\mathcal{Z}_{\alpha,\beta}(x)$.

Denoting by $\{(T_k, J_k)\}$ under \mathbb{P} the Markov–renewal process generated by the semi–Markov kernel $\Gamma_{\alpha, \beta}^-(x)$, for $U_{\alpha, \beta}(x)$ we have the probabilistic interpretation (2.4), that we rewrite for convenience

$$U_{\alpha, \beta}(x) = \mathbb{P}_\alpha[\exists k \geq 0 : T_0 + \dots + T_k = x, J_k = \beta]. \quad (2.22)$$

For $\beta \in \mathbb{S}$ we introduce the sequence of stopping times $\{\kappa_n^{(\beta)}\}_{n \geq 0}$ corresponding to the visit of the chain $\{J_k\}$ to the state β :

$$\kappa_0^{(\beta)} := \inf\{k \geq 0 : J_k = \beta\} \quad \kappa_{n+1}^{(\beta)} := \inf\{k > \kappa_n^{(\beta)} : J_k = \beta\}, \quad (2.23)$$

and we define the process $\{T_n^{(\beta)}\}_{n \geq 0}$ by setting

$$T_0^{(\beta)} := T_0 + \dots + T_{\kappa_0^{(\beta)}} \quad T_n^{(\beta)} := T_{\kappa_{n-1}^{(\beta)}+1} + \dots + T_{\kappa_n^{(\beta)}}. \quad (2.24)$$

The key point is that under \mathbb{P}_α the random variables $\{T_n^{(\beta)}\}$ are the interarrival times of a (possibly delayed) *classical renewal process*, equivalently the sequence $\{T_n^{(\beta)}\}_{n \geq 1}$ is IID and independent of $T_0^{(\beta)}$. We denote for $x \in \mathbb{N}$ by $q^{(\beta)}(x)$ the (mass function of the) law of $T_n^{(\beta)}$ for $n \geq 1$, while the law of $T_0^{(\beta)}$ under \mathbb{P}_α is denoted by $q^{(\alpha; \beta)}(x)$. Since clearly

$$\{\exists k \geq 0 : T_0 + \dots + T_k = x, J_k = \beta\} \iff \{\exists n \geq 0 : T_0^{(\beta)} + \dots + T_n^{(\beta)} = x\},$$

from (2.22) we get

$$U_{\alpha, \beta}(x) = \mathbb{P}_\alpha[\exists n \geq 0 : T_0^{(\beta)} + \dots + T_n^{(\beta)} = x] = \left(q^{(\alpha; \beta)} * \sum_{n=0}^{\infty} (q^{(\beta)})^{*n} \right)(x), \quad (2.25)$$

which shows that $U_{\alpha, \beta}(x)$ is indeed the Green function of the classical renewal process whose interarrival times are the $\{T_n^{(\beta)}\}_{n \geq 0}$.

Now we claim that the asymptotic behavior of $q^{(\beta)}(x)$ as $x \rightarrow \infty$, $x \in \beta$, is given by

$$q^{(\beta)}(x) \sim \frac{c_\beta}{x^{3/2}} \quad c_\beta := \frac{1}{\zeta_\beta \xi_\beta} \sum_{\alpha, \gamma} \zeta_\alpha L_{\alpha, \gamma} \xi_\gamma > 0, \quad (2.26)$$

see Appendix A for a proof of this relation. Then the asymptotic behavior of (2.25) is given by

$$U_{\alpha, \beta}(x) \sim \frac{T^2}{2\pi c_\beta} \frac{1}{\sqrt{x}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha, \quad (2.27)$$

as it follows by [7, Th. B] (the factor T^2 is due to our periodic setting). Combining equations (2.21), (2.26) and (2.27) we finally get the asymptotic behavior of $\mathcal{Z}_{\alpha, \beta}(x)$:

$$\mathcal{Z}_{\alpha, \beta}(x) \sim \frac{T^2}{2\pi} \frac{\xi_\alpha \zeta_\beta}{\sum_{\gamma, \gamma'} \zeta_\gamma L_{\gamma, \gamma'} \xi_{\gamma'}} \frac{1}{\sqrt{x}} \quad x \rightarrow \infty, \quad [x] = \beta - \alpha. \quad (2.28)$$

Taking $\alpha = [0]$ and $\beta = \eta$, we have the proof of part (3) of Theorem 1.5.

3. THERMODYNAMIC LIMITS

In this section we study the limit as $N \rightarrow \infty$ of the polymer measure $\mathbf{P}_{N,\omega}^a$, using the sharp asymptotics for the partition function obtained in the previous section. We recall that $\mathbf{P}_{N,\omega}^c$ is a probability measure on $\mathbb{Z}^{\mathbb{N}}$, which we endow with the product topology. In particular, weak convergence on $\mathbb{Z}^{\mathbb{N}}$ means convergence of all finite dimensional marginals.

We start giving a very useful decomposition of $\mathbf{P}_{N,\omega}^a$. The intuitive idea is that a path $(S_n)_{n \leq N}$ can be split into two main ingredients:

- the family $(\tau_k)_{k=0,1,\dots}$ of *returns to zero* of S (defined in § 2.2);
- the family of *excursions from zero* $(S_{i+\tau_{k-1}} : 0 \leq i \leq \tau_k - \tau_{k-1})_{k=1,2,\dots}$

Moreover, since each excursion can be either positive or negative, it is also useful to consider separately the signs of the excursions $\sigma_k := \text{sign}(S_{\tau_{k-1}+1})$ and the absolute values $(e_k(i) := |S_{i+\tau_{k-1}}| : i = 1, \dots, \tau_k - \tau_{k-1})$. Observe that these are trivial for an excursion with length 1: in fact if $\tau_k = \tau_{k-1} + 1$ then $\sigma_k = 0$ and $e_k(0) = e_k(1) = 0$.

Let us first consider the returns $(\tau_k)_{k \leq \iota_N}$ under $\mathbf{P}_{N,\omega}^a$, where $\iota_N = \sup\{k : \tau_k \leq N\}$. The law of this process can be viewed as a probability measure $p_{N,\omega}^a$ on the class \mathcal{A}_N of subsets of $\{1, \dots, N\}$: indeed for $A \in \mathcal{A}_N$, writing

$$A = \{t_1, \dots, t_{|A|}\}, \quad 0 =: t_0 < t_1 < \dots < t_{|A|} \leq N, \quad (3.1)$$

we can set

$$p_{N,\omega}^a(A) := \mathbf{P}_{N,\omega}^a(\tau_i = t_i, i \leq \iota_N). \quad (3.2)$$

The measure $p_{N,\omega}^a$ describes the zero set of the polymer of size N , and it is analyzed in detail below. From the inclusion of \mathcal{A}_N into $\{0, 1\}^{\mathbb{N}}$, the family of all subsets of \mathbb{N} , $p_{N,\omega}^a$ can be viewed as a measure on $\{0, 1\}^{\mathbb{N}}$ (this observation will be useful in the following).

Now we pass to the signs: we can see that, given $(\tau_j)_{j \leq \iota_N}$, under $\mathbf{P}_{N,\omega}^a$ the signs $(\sigma_k)_{k \leq \iota_N}$ form an independent family. Conditionally on $(\tau_j)_{j \leq \iota_N}$, the law of σ_k is specified by:

- if $\tau_k = 1 + \tau_{k-1}$, then $\sigma_k = 0$;
- if $\tau_k > 1 + \tau_{k-1}$, then σ_k can take the two values ± 1 with

$$\mathbf{P}_{N,\omega}^a(\sigma_k = +1 \mid (\tau_j)_{j \leq \iota_N}) = \frac{1}{1 + \exp\{-(\tau_k - \tau_{k-1})h_\omega + \sum_{[\tau_{k-1}], [\tau_k]}\}}. \quad (3.3)$$

Observe that when $\tau_{\iota_N} < N$ (which can happen only for $a = f$) there is a last (incomplete) excursion in the interval $\{0, \dots, N\}$, and the sign of this excursion is also expressed by (3.3) for $k = \iota_N + 1$, provided we set $\tau_{\iota_N+1} := N$.

Finally we have the moduli: again, once $(\tau_{k-1}, \sigma_k)_{1 \leq k \leq \iota_N+1}$ are given, the excursions $(e_k)_{k=1, \dots, \iota_N+1}$ form an independent family. The conditional law of $e_k(\cdot)$ on the event $\{\tau_{k-1} = \ell_0, \tau_k = \ell_1\}$ and for $f = (f_i)_{i=1, \dots, \ell_1 - \ell_0}$ is, for $k \leq \iota_N$, given by

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a(e_k(\cdot) = f \mid (\tau_{j-1}, \sigma_j)_{1 \leq j \leq \iota_N+1}) \\ &= \mathbf{P}(S_i = f_i : i = 1, \dots, \ell_1 - \ell_0 \mid S_i > 0 : i = 1, \dots, \ell_1 - \ell_0 - 1, S_{\ell_1 - \ell_0} = 0). \end{aligned} \quad (3.4)$$

In the case $\tau_{\ell_N} < N$ we have a last excursion $e_{\ell_N+1}(\cdot)$: its conditional law, on the event $\{\tau_{\ell_N} = \ell < N\}$ and for $f = (f_i)_{i=1, \dots, N-\ell}$, is given by

$$\begin{aligned} & \mathbf{P}_{N,\omega}^a \left(e_{\ell_N+1}(\cdot) = f \mid (\tau_{j-1}, \sigma_j)_{1 \leq j \leq \ell_N+1} \right) \\ &= \mathbf{P} \left(S_i = f_i : i = 1, \dots, N - \ell \mid S_i > 0 : i = 1, \dots, N - \ell \right), \end{aligned} \quad (3.5)$$

We would like to stress that the above relations fully characterize the polymer measure $\mathbf{P}_{N,\omega}^a$. A remarkable fact is that, conditionally on $(\tau_k)_{k \in \mathbb{N}}$, the joint distribution of $(\sigma_j, e_j)_{j \leq \ell_N}$ *does not depend on N* : in this sense, all the N -dependence is contained in the measure $p_{N,\omega}^a$.

For this reason, this section is mainly devoted to the study of the asymptotic behavior of the zero set measures $p_{N,\omega}^a$ as $N \rightarrow \infty$. The main result is that $p_{N,\omega}^c$ and $p_{N,\omega}^f$ have the same weak limit p_ω on $\{0, 1\}^{\mathbb{N}}$ as $N \rightarrow \infty$ (with some restrictions when $\omega \in \mathcal{P}^<$). Once this is proven, it follows easily that also the polymer measure $\mathbf{P}_{N,\omega}^a$ converges to a limit measure \mathbf{P}_ω on $\mathbb{Z}^{\mathbb{N}}$, constructed by pasting the excursion over the limit zero set. More precisely, \mathbf{P}_ω is the measure under which the processes (τ_j) , (σ_j) and (e_j) have the following laws:

- the law of the $(\tau_j)_{j \in \mathbb{N}}$ is determined in an obvious way by the limiting zero set measure p_ω ;
- conditionally on $(\tau_j)_{j \in \mathbb{N}}$, the variables $(\sigma_j)_{j \in \mathbb{N}}$ are independent with marginal laws given by (3.3);
- conditionally on $(\tau_j, \sigma_j)_{j \in \mathbb{N}}$, the variables $(e_j(\cdot))_{j \in \mathbb{N}}$ are independent. On the event $\{\tau_{k-1} = \ell_0, \tau_k = \ell_1\}$ with $\ell_0 < \ell_1 < \infty$ the law of e_k is given by the r.h.s. of (3.4). We have to consider also the case $\ell_0 < \infty, \ell_1 = \infty$, because in the regime $\delta^\omega < 1$ it turns out that $\mathbf{P}_\omega(\tau_k = \infty) > 0$ (see below and Appendix B): in this case the law of e_k is given for any $n \in \mathbb{N}$ and for $f = (f_i)_{i=1, \dots, n}$ by:

$$\begin{aligned} \mathbf{P}_\omega \left(e_k(i) = f_i : i = 1, \dots, n \mid (\tau_j, \sigma_j)_{j \in \mathbb{N}} \right) &= \mathbf{P}^+ \left(S_i = f_i : i = 1, \dots, n \right) \\ &:= \lim_{N \rightarrow \infty} \mathbf{P} \left(S_i = f_i : i = 1, \dots, n \mid S_i > 0 : i = 1, \dots, N \right), \end{aligned} \quad (3.6)$$

where the existence of such limit is well known: see e.g. [11].

3.1. Law of the zero level set in the free and constrained cases. Let us describe more explicitly $p_{N,\omega}^a(A)$, using the (strong) Markov property of $\mathbf{P}_{N,\omega}^a$. We use throughout the paper the notation (3.1). Recalling the definition (1.14) of $M_{\alpha,\beta}(t)$, we have:

- for $a = c$ and $A \in \mathcal{A}_N$: $p_{N,\omega}^c(A) \neq 0$ if and only if $t_{|A|} = N$, and in this case:

$$p_{N,\omega}^c(A) = \frac{1}{Z_{N,\omega}^c} \prod_{i=1}^{|A|} M_{[t_{i-1}], [t_i]}(t_i - t_{i-1})$$

- for $a = f$ and $A \in \mathcal{A}_N$:

$$p_{N,\omega}^f(A) = \frac{1}{Z_{N,\omega}^f} \left[\prod_{i=1}^{|A|} M_{[t_{i-1}], [t_i]}(t_i - t_{i-1}) \right] P(N - t_{|A|}) \exp \left(\tilde{\Phi}_{[t_{|A|}], [N]}(N - t_{|A|}) \right).$$

where $P(n) := \sum_{k=n+1}^{\infty} K(k) = \sum_{k=n+1}^{\infty} \mathbf{P}(\tau_1 = k)$ and we have introduced

$$\tilde{\Phi}_{\alpha,\beta}(\ell) := \log \left[\frac{1}{2} \left(1 + \exp(-\ell h_\omega + \Sigma_{\alpha,\beta}) \right) \right] \mathbf{1}_{(\ell > 1)} \mathbf{1}_{(\ell \in \beta - \alpha)}, \quad (3.7)$$

which differs from Φ in not having the terms of interaction with the interface, cf. (1.13).

We are going to show that, for any value of δ^ω , the measure $p_{N,\omega}^a$ on $\{0, 1\}^{\mathbb{N}}$ converges as $N \rightarrow \infty$ (with some restrictions if $\omega \in \mathcal{P}^<$) to a limit measure under which the process $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a *Markov renewal process*. Moreover, we will compute explicitly the corresponding semi-Markov kernel, showing that the returns to zero are

- (1) integrable if $\delta^\omega > 1$ (localized regime);
- (2) defective if $\delta^\omega < 1$ (strictly delocalized regime);
- (3) non integrable if $\delta^\omega = 1$ (critical regime).

Thanks to the preceding observations, this will complete the proof of Theorem 1.7. We stress that the key result in our derivation is given by the sharp asymptotics of the partition function $Z_{N,\omega}^c$ obtained in the previous section.

Before going into the proof, we give some preliminary material which is useful for all values of δ^ω . For $k \in \mathbb{N}$ we define the shift operator:

$$\theta_k : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}, \quad \theta_k \zeta := \zeta_{[k+.]},$$

and it is easy to check that the following relations hold true:

$$Z_{N-k, \theta_k \omega}^c = Z_{[k],[N]}(N-k), \quad k \leq N. \quad (3.8)$$

$$Z_{N,\omega}^f = \sum_{t=0}^N Z_{t,\omega}^c P(N-t) \exp\left(\tilde{\Phi}_{[t],[N]}(N-t)\right), \quad (3.9)$$

$$\mathbf{P}_{N,\omega}^a(\tau_1 = k) = M_{0,[k]}(k) \frac{Z_{N-k, \theta_k \omega}^a}{Z_{N,\omega}^a}, \quad 1 \leq k \leq N, \quad a = c, f. \quad (3.10)$$

Finally, using (1.12), (1.14) and (1.13) it is easy to see that (2.12) holds true, namely

$$\exists \lim_{\substack{x \rightarrow \infty \\ [x] = \beta - \alpha}} x^{3/2} M_{\alpha,\beta}(x) = L_{\alpha,\beta}, \quad (3.11)$$

where:

$$L_{\alpha,\beta} = \begin{cases} c_K \frac{1}{2} \left(1 + \exp(\Sigma_{\alpha,\beta}) \right) \exp(\omega_\beta^{(0)}) & \text{if } h_\omega = 0 \\ c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) & \text{if } h_\omega > 0 \end{cases}. \quad (3.12)$$

Since also the asymptotic behavior of $P(\ell) \exp(\tilde{\Phi}_{\alpha,\beta}(\ell))$ will be needed, we set

$$\tilde{L}_{\alpha,\beta} := \lim_{\ell \rightarrow \infty, \ell \in \beta - \alpha} \sqrt{\ell} P(\ell) e^{\tilde{\Phi}_{\alpha,\beta}(\ell)} = \begin{cases} c_K (1 + \exp(\Sigma_{\alpha,\beta})) & \text{if } h_\omega = 0 \\ c_K & \text{if } h_\omega > 0 \end{cases}, \quad (3.13)$$

as it follows easily from (3.7) and from the fact that $P(\ell) \sim 2c_K/\sqrt{\ell}$ as $\ell \rightarrow \infty$.

3.2. The localized regime ($\delta^\omega > 1$). We prove point (1) of Theorem 1.7. More precisely, we prove the following:

Proposition 3.1. *If $\delta^\omega > 1$ then the polymer measures $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge as $N \rightarrow \infty$ to the same limit \mathbf{P}_ω , under which $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma_{\alpha,\beta}^>(x) : \alpha, \beta \in \mathbb{S}, x \in \mathbb{N})$.*

For the definition of $\Gamma^>$ see (2.13).

Proof of Proposition 3.1. We prove first the case $a = c$. By (3.8), (3.10) and by the asymptotics of \mathcal{Z} in (2.15) above, we have for all $\alpha, \beta, \gamma \in \mathbb{S}$ and $\ell \in \alpha, m \in \beta$

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \gamma}} \frac{Z_{N-m, \theta_m \omega}^c}{Z_{N-\ell, \theta_\ell \omega}^c} = \lim_{\substack{N \rightarrow \infty \\ N \in \gamma}} \frac{\mathcal{Z}_{\beta, \gamma}(N-m)}{\mathcal{Z}_{\alpha, \gamma}(N-\ell)} = e^{-F_\omega k} \frac{\xi_\beta}{\xi_\alpha},$$

and since the right hand side does not depend on γ , then the limit exists as $N \rightarrow \infty$. It follows that for $\ell \in \alpha, k + \ell \in \beta$:

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N-\ell, \theta_\ell \omega}^c(\tau_1 = k) = M_{\alpha, \beta}(k) e^{-F_\omega k} \frac{\xi_\beta}{\xi_\alpha} = \Gamma_{\alpha, \beta}^>(k).$$

By the Markov property of $\mathbf{P}_{N,\omega}^c$ this yields

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c(\tau_1 = k_1, \dots, \tau_j = k_j) = \prod_{i=1}^j \Gamma_{[k_{i-1}], [k_i]}^>(k_i - k_{i-1}), \quad k_0 := 0.$$

The argument for $\mathbf{P}_{N,\omega}^f$ goes along the very same line: by (3.9),

$$\begin{aligned} e^{-F_\omega N} Z_{N-k, \theta_k \omega}^f &= e^{-F_\omega N} \sum_{t=0}^{N-k} \mathcal{Z}_{[k], [N-t]}(N-k-t) P(t) \exp\left(\tilde{\Phi}_{[N-t], [N]}(t)\right) \\ &= e^{-F_\omega k} \sum_{\eta \in \mathbb{S}} \sum_{t=0}^{N-k} e^{-F_\omega t} P(t) \left[\exp\left(\tilde{\Phi}_{\eta, [N]}(t)\right) e^{-F_\omega(N-k-t)} \mathcal{Z}_{[k], \eta}(N-k-t) \right]. \end{aligned}$$

Since by (2.15) the expression in brackets converges as $N \rightarrow \infty$ and $N \in [t] + \eta$, we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \gamma}} e^{-F_\omega N} Z_{N-k, \theta_k \omega}^f = \xi_{[k]} e^{-F_\omega k} \left(\frac{T}{\mu} \sum_{\eta \in \mathbb{S}} \sum_{\substack{t \in \mathbb{N} \\ [t] = \gamma - \eta}} e^{-F_\omega t} P(t) \exp\left(\tilde{\Phi}_{\eta, \gamma}(t)\right) \zeta_\eta \right).$$

Observe that the term in parenthesis is just a function of γ . Having found the precise asymptotics of $Z_{N,\omega}^f$, we can argue as for $\mathbf{P}_{N,\omega}^c$ to conclude the proof. \square

3.3. The critical regime ($\delta^\omega = 1$). We prove point (3) of Theorem 1.7. More precisely, we prove the following:

Proposition 3.2. *If $\delta^\omega = 1$ then the polymer measures $\mathbf{P}_{N,\omega}^f$ and $\mathbf{P}_{N,\omega}^c$ converge as $N \rightarrow \infty$ to the same limit \mathbf{P}_ω , under which $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma_{\alpha,\beta}^=(x) : \alpha, \beta \in \mathbb{S}, x \in \mathbb{N})$.*

For the definition of $\Gamma^=$ see (2.20).

Proof of Proposition 3.2. We prove first the case $a = c$. By (3.11) and by the asymptotics of \mathcal{Z} in (2.28) above, we obtain for all $k \in \alpha$:

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \beta}} N^{1/2} \mathcal{Z}_{\alpha, \beta}(N - k) = \frac{T^2}{2\pi} \frac{\xi_\alpha \zeta_\beta}{\sum_{\gamma, \gamma'} \zeta_\gamma L_{\gamma, \gamma'} \xi_{\gamma'}}.$$

It follows for all $\alpha, \beta, \gamma \in \mathbb{S}$ and $\ell \in \alpha, m \in \beta$

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \gamma}} \frac{Z_{N-m, \theta_m \omega}^c}{Z_{N-\ell, \theta_\ell \omega}^c} = \lim_{\substack{N \rightarrow \infty \\ N \in \gamma}} \frac{\mathcal{Z}_{\beta, \gamma}(N - m)}{\mathcal{Z}_{\alpha, \gamma}(N)} = \frac{\xi_\beta}{\xi_\alpha},$$

and since the right hand side does not depend on γ , then the limit exists as $N \rightarrow \infty$. It follows for $\ell \in \alpha, k + \ell \in \beta$:

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N-\ell, \theta_\ell \omega}^c(\tau_1 = k) = M_{\alpha, \beta}(k) \frac{\xi_\beta}{\xi_\alpha} = \Gamma_{\alpha, \beta}^-(k).$$

By the Markov property of $\mathbf{P}_{N, \omega}^c$ this yields

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N, \omega}^c(\tau_1 = k_1, \dots, \tau_j = k_j) = \prod_{i=1}^j \Gamma_{[k_{i-1}], [k_i]}^-(k_i - k_{i-1}), \quad k_0 := 0.$$

For $\mathbf{P}_{N, \omega}^f$, by (3.9) we have for $N \in \beta$ and $k \leq N$:

$$Z_{N-k, \theta_k \omega}^f = \sum_{\gamma} \sum_{t=0}^{N-k} \mathcal{Z}_{[k], \gamma}(t) P(N - k - t) \exp\left(\tilde{\Phi}_{\gamma, \beta}(N - k - t)\right).$$

By the previous results and using (3.13) we obtain that for every $k \in \mathbb{N}$

$$\begin{aligned} \exists \lim_{N \rightarrow \infty, N \in \beta} Z_{N-k, \theta_k \omega}^f &= \xi_{[k]} \frac{T}{2\pi} \frac{\sum_{\eta} \zeta_{\eta} \tilde{L}_{\eta, \beta}}{\sum_{\eta, \eta'} \zeta_{\eta} L_{\eta, \eta'} \xi_{\eta'}} \int_0^1 \frac{dt}{t^{\frac{1}{2}} (1-t)^{\frac{1}{2}}} \\ &= \xi_{[k]} \left(\frac{T}{2} \frac{\sum_{\eta} \zeta_{\eta} \tilde{L}_{\eta, \beta}}{\sum_{\eta, \eta'} \zeta_{\eta} L_{\eta, \eta'} \xi_{\eta'}} \right). \end{aligned} \quad (3.14)$$

To conclude it suffices to argue as in the constrained case. \square

3.4. The strictly delocalized regime ($\delta^\omega < 1$). We prove point (2) and the last assertion of Theorem 1.7. In this case the result is different according to whether $\omega \in \mathcal{P}^<$ or $\omega \notin \mathcal{P}^<$ (recall the definition (1.17)). To be more precise, there is first a weak formulation for *all* ω which gives a thermodynamic limit of $\mathbf{P}_{N, \omega}^a$ depending on the sequence $\{N : [N] = \eta\}$ and on $a = f, c$; secondly, there is a stronger formulation only for $\omega \notin \mathcal{P}^<$, which says that such limits coincide for all $\eta \in \mathbb{S}$ and $a = f, c$.

It will turn out that in the strictly delocalized regime there exists a.s. a last return to zero, i.e. the process $(\tau_k)_{k \in \mathbb{N}}$ is defective. In order to express this with the language of Markov renewal processes, we introduce the sets $\bar{\mathbb{S}} := \mathbb{S} \cup \{\infty\}$ and $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, extending the equivalence relation to $\bar{\mathbb{N}}$ by $[\infty] = \infty$. Finally we set for all $\alpha, \eta \in \mathbb{S}$:

$$\begin{aligned} \Lambda_{\alpha, \eta}^c &:= [(1 - B)^{-1} L (1 - B)^{-1}]_{\alpha, \eta}, & \mu_{\alpha, \eta}^c &:= [L (1 - B)^{-1}]_{\alpha, \eta}, \\ \Lambda_{\alpha, \eta}^f &:= [(1 - B)^{-1} \tilde{L}]_{\alpha, \eta}, & \mu_{\alpha, \eta}^f &:= \tilde{L}_{\alpha, \eta}, \end{aligned}$$

and for all $\eta \in \mathbb{S}$ and $a = f, c$ we introduce the semi-Markov kernel on $\overline{\mathbb{S}} \times \overline{\mathbb{N}}$:

$$\Gamma_{\alpha, \beta}^{\eta, a}(x) := \begin{cases} M_{\alpha, \beta}(k) \Lambda_{\beta, \eta}^a / \Lambda_{\alpha, \eta}^a & \alpha \in \mathbb{S}, x \in \mathbb{N}, \beta = [x] \in \mathbb{S} \\ \mu_{\alpha, \eta}^a / \Lambda_{\alpha, \eta}^a & \alpha \in \mathbb{S}, x = \infty, \beta = [\infty] \\ 1 & \alpha = \beta = [\infty], x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\Gamma^{\eta, a}$ is really a semi-Markov kernel, since for $\alpha \in \mathbb{S}$:

$$\begin{aligned} \sum_{\beta \in \overline{\mathbb{S}}} \sum_{x \in \overline{\mathbb{N}}} \Gamma_{\alpha, \beta}^{\eta, a}(x) &= \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \sum_{\beta \in \mathbb{S}} \sum_{x \in \mathbb{N}} \frac{M_{\alpha, \beta}(x) \Lambda_{\beta, \eta}^a}{\Lambda_{\alpha, \eta}^a} = \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \frac{1}{\Lambda_{\alpha, \eta}^a} [B \cdot \Lambda^a]_{\alpha, \eta} \\ &= \frac{\mu_{\alpha, \eta}^a}{\Lambda_{\alpha, \eta}^a} + \frac{1}{\Lambda_{\alpha, \eta}^a} (\Lambda_{\alpha, \eta}^a - \mu_{\alpha, \eta}^a) = 1. \end{aligned}$$

We are going to prove the following:

Proposition 3.3. *Let $\delta^\omega < 1$. Then:*

- (1) *for $a = f, c$, $\mathbf{P}_{N, \omega}^a$ converges as $N \rightarrow \infty$, $[N] = \eta$ to a measure $\mathbf{P}_\omega^{a, \eta}$, under which $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $(\Gamma_{\alpha, \beta}^{\eta, a}(x) : \alpha, \beta \in \overline{\mathbb{S}}, x \in \overline{\mathbb{N}})$.*
- (2) *if $\omega \notin \mathcal{P}^<$, then $\mathbf{P}_\omega^{a, \eta} =: \mathbf{P}_\omega$ and $\Gamma^{\eta, a} =: \Gamma^<$ depend neither on η nor on a , and both $\mathbf{P}_{N, \omega}^f$ and $\mathbf{P}_{N, \omega}^c$ converge as $N \rightarrow \infty$ to \mathbf{P}_ω , under which $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a Markov renewal process with semi-Markov kernel $\Gamma^<$.*

Remark 3.4. Part (2) of Proposition 3.3 is an easy consequence of part (1). In fact from equations (3.12) and (3.13) it follows immediately that when $\omega \notin \mathcal{P}^<$ then both matrices $(L_{\alpha, \beta})$ and $(\tilde{L}_{\alpha, \beta})$ are constant in α , and therefore Λ^a factorizes into a tensor product, i.e.

$$\Lambda_{\alpha, \eta}^a = \lambda_\alpha^a \nu_\eta^a, \quad \alpha, \eta \in \mathbb{S},$$

where $(\lambda_\alpha^a)_{\alpha \in \mathbb{S}}$ and $(\nu_\alpha^a)_{\alpha \in \mathbb{S}}$ are easily computed. But then it is immediate to check that the semi-Markov kernel $\Gamma^{\eta, a} =: \Gamma^<$ depends neither on η nor on a .

Proof of Proposition 3.3. By the preceding Remark it suffices to prove part (1). For all $k \in \alpha$, by (2.16) we have

$$\exists \lim_{\substack{N \rightarrow \infty \\ [N] = \beta}} N^{3/2} \mathcal{Z}_{\alpha, \beta}(N - k) = [(1 - B)^{-1} L (1 - B)^{-1}]_{\alpha, \beta} = \Lambda_{\alpha, \beta}^c. \quad (3.15)$$

In particular, we have for all $\alpha, \beta, \eta \in \mathbb{S}$ and $\ell \in \alpha, m \in \beta$:

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \frac{Z_{N-m, \theta_m \omega}^c}{Z_{N-\ell, \theta_\ell \omega}^c} = \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \frac{\mathcal{Z}_{\beta, \eta}(N - m)}{\mathcal{Z}_{\alpha, \eta}(N)} = \frac{\Lambda_{\beta, \eta}^c}{\Lambda_{\alpha, \eta}^c},$$

Then by (3.10) we get

$$\lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N, \omega}^c(\tau_1 = k) = \frac{M_{0, [k]}(k) \Lambda_{[k], \eta}^c}{\Lambda_{0, \eta}} = \Gamma_{0, [k]}^{\eta, c}(k).$$

By the Markov property of $\mathbf{P}_{N,\omega}^c$ this generalizes to

$$\lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(\tau_1 = k_1, \dots, \tau_j = k_j) = \prod_{i=1}^j \Gamma_{[k_{i-1}], [k_i]}^{\eta, c}(k_i - k_{i-1}), \quad k_0 := 0.$$

We prove now the case $a = f$. Recalling (3.9) above, we see here that

$$N^{1/2} Z_{N-k, \theta_k \omega}^f = \sum_{t=0}^{N-k} \mathcal{Z}_{[k], [t+k]}(t) N^{1/2} P(N - k - t) \exp\left(\tilde{\Phi}_{[t+k], [N]}(N - k - t)\right).$$

Then by (3.13) we obtain

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{1/2} Z_{N-k, \theta_k \omega}^f = \sum_{t=0}^{\infty} \mathcal{Z}_{[k], [t+k]}(t) \tilde{L}_{[t+k], \eta} = [(1 - B)^{-1} \tilde{L}]_{[k], \eta} = \Lambda_{[k], \eta}^f, \quad (3.16)$$

since

$$\sum_{t=0}^{\infty} \mathcal{Z}_{\alpha, \gamma}(t) = \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} M_{\alpha, \gamma}^{*k}(t) = \sum_{k=0}^{\infty} B_{\alpha, \gamma}^{*k} = [(I - B)^{-1}]_{\alpha, \gamma}. \quad (3.17)$$

Arguing as for $\mathbf{P}_{N,\omega}^c$, we conclude the proof. \square

4. SCALING LIMITS

In this section we prove that the measures $\mathbf{P}_{N,\omega}^a$ converge under Brownian rescaling. The results and proofs follow closely those of [6] and we shall refer to this paper for several technical lemmas.

The first step is tightness of $(Q_{N,\omega}^a)_{N \in \mathbb{N}}$ in $C([0, 1])$.

Lemma 4.1. *For any ω and $a = c, f$ the sequence $(Q_{N,\omega}^a)_{N \in \mathbb{N}}$ is tight in $C([0, 1])$.*

For the standard proof we refer to Lemma 4 in [6].

In the rest of the section we prove Theorem 1.8.

4.1. The localized regime ($\delta^\omega > 1$). We prove point (1) of Theorem 1.8. By Lemma 4.1 it is enough to prove that $\mathbf{P}_{N,\omega}^a(|X_t^N| > \varepsilon) \rightarrow 0$ for all $\varepsilon > 0$ and $t \in [0, 1]$ and one can obtain this estimate explicitly. We point out however that in this regime one can avoid using the compactness lemma and one can obtain a stronger result by elementary means: observe that for any $k, n \in \mathbb{N}$ such that $n > 1$ and $k + n \leq N$, we have

$$\begin{aligned} \mathbf{P}_{N,\omega}^a(S_k = S_{k+n} = 0, S_{k+i} \neq 0 \text{ for } i = 1, \dots, n-1) \\ \leq \frac{\frac{1}{2} \left(1 + \exp\left(\sum_{i=1}^n \left(\omega_{k+i}^{(-1)} - \omega_{k+i}^{(+1)}\right)\right)\right)}{Z_{n, \theta_k \omega}^c} =: \widehat{K}_k(n), \end{aligned} \quad (4.1)$$

and this holds both for $a = c$ and $a = f$. Inequality (4.1) is obtained by using the Markov property of S both in the numerator and the denominator of the expression (1.9) defining $\mathbf{P}_{N,\omega}^a(\cdot)$ after having bounded $Z_{N,\omega}^a$ from below by inserting the event $S_k = S_{k+n} = 0$. Of course $\lim_{n \rightarrow \infty} (1/n) \log \widehat{K}_k(n) = -F_\omega$ uniformly in k (notice that $\widehat{K}_{k+T}(n) = \widehat{K}_k(n)$).

Therefore if we fix $\varepsilon > 0$ by the union bound we obtain (we recall that $\{\tau_j\}_j$ and ι_N were defined in Section 3)

$$\begin{aligned} \mathbf{P}_{N,\omega}^a \left(\max_{j=1,2,\dots,\iota_N} \tau_j - \tau_{j-1} > (1 + \varepsilon) \log N / F_\omega \right) \\ \leq \sum_{k \leq N - (1 + \varepsilon) \log N / F_\omega} \sum_{n > (1 + \varepsilon) \log N / F_\omega} \widehat{K}_k(n) \\ \leq N \sum_{n > (1 + \varepsilon) \log N / F_\omega} \max_{k=0,\dots,T-1} \widehat{K}_k(n) \leq \frac{c}{N^\varepsilon}, \end{aligned}$$

for some $c > 0$.

Let us start with the constrained case: notice that $\mathbf{P}_{N,\omega}^c$ (dS)–a.s. we have $\tau_{\iota_N} = N$ and hence $\max_{j \leq \iota_N} \tau_j - \tau_{j-1} \geq \max_{n=1,\dots,N} |S_n|$, since $|S_{n+1} - S_n| \leq 1$. Then we immediately obtain that for any $C > 1/F_\omega$

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c \left(\max_{n=1,\dots,N} |S_n| > C \log N \right) = 0, \quad (4.2)$$

which is of course a much stronger statement than the scaling limit of point (1) of Theorem 1.8. If we consider instead the measure $\mathbf{P}_{N,\omega}^f$, the length of the last excursion has to be taken into account too: however, an argument very close to the one used in (4.1) yields also that the last excursion is exponentially bounded (with the same exponent) and the proof of point (1) of Theorem 1.8 is complete.

4.2. The strictly delocalized regime ($\delta^\omega < 1$). We prove point (2) of Theorem 1.8. We set for $t \in \{1, \dots, N\}$:

$$D_t := \inf\{k = 1, \dots, N : k > t, S_k = 0\}, \quad G_t := \sup\{k = 1, \dots, N : k \leq t, S_k = 0\}.$$

The following result shows that in the strictly delocalized regime, as $N \rightarrow \infty$, the visits to zero under $\mathbf{P}_{N,\omega}^a$ tend to be very few and concentrated at a finite distance from the origin if $a = f$ and from 0 or N if $a = c$.

Lemma 4.2. *If $\delta^\omega < 1$ there exists a constant $C > 0$ such that for all $L > 0$:*

$$\limsup_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^f (G_N \geq L) \leq C L^{-1/2}, \quad (4.3)$$

$$\limsup_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c (G_{N/2} \geq L) \leq C L^{-1/2}, \quad (4.4)$$

$$\limsup_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c (D_{N/2} \leq N - L) \leq C L^{-1/2}. \quad (4.5)$$

Lemma 4.2 is a quantitative version of point (2) of Theorem 1.7 and it is a rather straightforward complement: the proof is sketched in Appendix B, in particular (B.3).

The signs. In order to prove point (2) of Theorem 1.8, it is now enough to argue as in the proof of Theorem 9 in [6], with the difference that now the excursions are not necessarily in the upper half plane, i.e. the signs are not necessarily positive. So the proof is complete if we can show that there exists the limit (as $N \rightarrow \infty$ along $[N] = \eta$) of the probability that the process (away from $\{0, 1\}$) lives in the upper half plane. In analogy with Section 3.4, in the general case we have different limits depending on the sequence $[N] = \eta$ and on $a = f, c$, while if $\omega \notin \mathcal{P}^<$ all such limits coincide.

We start with the constrained case: given Lemma 4.2, it is sufficient to show that

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(S_{N/2} > 0) =: \mathbf{p}_{\omega,\eta}^{<,c}. \quad (4.6)$$

Formula (4.6) follows from the fact that

$$\mathbf{P}_{N,\omega}^c(S_{N/2} > 0) = \sum_{\alpha,\beta} \sum_{x < N/2} \sum_{y > N/2} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)},$$

where for all $z \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{S}$:

$$\rho_{\alpha,\beta}^+(z) := \frac{1}{1 + \exp(-z h_\omega + \Sigma_{\alpha,\beta})}, \quad (4.7)$$

cf. (3.3). By Dominated Convergence and by (3.12) and (3.17):

$$\begin{aligned} \exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{3/2} \sum_{x < N/2} \sum_{y > N/2} \mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,\eta}(N-y) \\ = [(1-B)^{-1}]_{0,\alpha} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) [(1-B)^{-1}]_{\beta,\eta}. \end{aligned}$$

By (2.16) we obtain (4.6) with

$$\mathbf{p}_{\omega,\eta}^{<,c} := \frac{\sum_{\alpha,\beta} [(1-B)^{-1}]_{0,\alpha} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) [(1-B)^{-1}]_{\beta,\eta}}{[(1-B)^{-1} L (1-B)^{-1}]_{0,\eta}}. \quad (4.8)$$

Observe that by (3.12):

- if $h_\omega > 0$ then in (4.8) the denominator is equal to the numerator, so that $\mathbf{p}_{\omega,\eta}^{<,c} = 1$ for all η .
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.8) the denominator is equal to *twice* the numerator, so that $\mathbf{p}_{\omega,\eta}^{<,c} = 1/2$ for all η .
- in the remaining case, i.e. if $\omega \in \mathcal{P}^<$, in general $\mathbf{p}_{\omega,\eta}^{<,c}$ depends on η .

Now let us consider the free case. This time it is sufficient to show that

$$\exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^f(S_N > 0) =: \mathbf{p}_{\omega,\eta}^{<,f}. \quad (4.9)$$

Formula (4.9) follows from the fact that

$$\mathbf{P}_{N,\omega}^f(S_N > 0) = \sum_{\alpha} \sum_{x < N} \frac{\mathcal{Z}_{0,\alpha}(x) \cdot \frac{1}{2} P(N-k)}{Z_{N,\omega}^f},$$

and using (3.9), (3.17) and (3.13) we obtain that (4.9) holds with

$$\mathbf{p}_{\omega,\eta}^{<,f} = \frac{\sum_{\alpha} [(1-B)^{-1}]_{0,\alpha} c_K}{[(1-B)^{-1} \tilde{L}]_{0,\eta}}. \quad (4.10)$$

Again, observe that by (3.13):

- if $h_\omega > 0$ then in (4.10) the denominator is equal to the numerator and $\mathbf{p}_{\omega,\eta}^{<,f} = 1$ for all η .
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.10) the denominator is equal to *twice* the numerator, so that $\mathbf{p}_{\omega,\eta}^{<,f} = 1/2$ for all η .

- in the remaining case, i.e. if $\omega \in \mathcal{P}^<$, in general $\mathbf{p}_{\omega,\eta}^{<,\mathbf{f}}$ depends on η and is different from $\mathbf{p}_{\omega,\eta}^{<,\mathbf{c}}$.

4.3. The critical regime ($\delta^\omega = 1$). In this section we prove point (3) of Theorem 1.8. As in the previous section, we first determine the asymptotic behavior of the zero level set of the copolymer and then we pass to the study of the signs of the excursions.

We introduce the random closed subset \mathcal{A}_N^a of $[0, 1]$, describing the zero set of the polymer of size N rescaled by a factor $1/N$:

$$\mathbb{P}(\mathcal{A}_N^a = A/N) = p_{N,\omega}^a(A), \quad A \subseteq \{0, \dots, N\},$$

where we recall that $p_{N,\omega}^a(\cdot)$ has been defined in § 3.1. Let us denote by \mathcal{F} the class of *all closed subsets* of $\mathbb{R}^+ := [0, +\infty)$. We are going to put on \mathcal{F} a topological and measurable structure, so that we can view the law of \mathcal{A}_N^a as a probability measure on (a suitable σ -field of) \mathcal{F} and we can study the weak convergence of \mathcal{A}_N^a .

We endow \mathcal{F} with the topology of Matheron, cf. [19] and [9, § 3], which is a metrizable topology. To define it, to a closed subset $F \subseteq \mathbb{R}^+$ we associate the closed *nonempty* subset \tilde{F} of the compact interval $[0, \pi/2]$ defined by $\tilde{F} := \arctan(F \cup \{+\infty\})$. Then the metric $\rho(\cdot, \cdot)$ we take on \mathcal{F} is

$$\rho(F, F') := \max \left\{ \sup_{t \in \tilde{F}} d(t, \tilde{F}'), \sup_{t' \in \tilde{F}'} d(t', \tilde{F}) \right\} \quad F, F' \in \mathcal{F}, \quad (4.11)$$

where $d(s, A) := \inf\{|t - s|, t \in A\}$ is the standard distance between a point and a set. We point out that the r.h.s. of (4.11) is the so-called Hausdorff metric between the compact sets \tilde{F}, \tilde{F}' . Thus given a sequence $\{F_n\}_n \subset \mathcal{F}$ and $F \in \mathcal{F}$, we say that $F_n \rightarrow F$ in \mathcal{F} if and only if $\rho(F_n, F) \rightarrow 0$. We observe that this is equivalent to requiring that for each open set G and each compact K

$$\begin{aligned} F \cap G \neq \emptyset &\implies F_n \cap G \neq \emptyset \text{ eventually} \\ F \cap K = \emptyset &\implies F_n \cap K = \emptyset \text{ eventually} \end{aligned} \quad (4.12)$$

Another necessary and sufficient condition for $F_n \rightarrow F$ is that $d(t, F_n) \rightarrow d(t, F)$ for every $t \in \mathbb{R}^+$.

This topology makes \mathcal{F} a separable and compact metric space [19, Th. 1-2-1], in particular a Polish space. We endow \mathcal{F} with the Borel σ -field, and by standard theorems on weak convergence we have that also the space $\mathcal{M}_1(\mathcal{F})$ of probability measures on \mathcal{F} is compact.

The main result of this section is to show that the law of the random set $\mathcal{A}_N^a \in \mathcal{M}_1(\mathcal{F})$ converges as $N \rightarrow \infty$ to the law of the zero set of a Brownian motion $\{B(t)\}_{t \in [0,1]}$ for $a = \mathbf{f}$ or of a Brownian bridge $\{\beta(t)\}_{t \in [0,1]}$ for $a = \mathbf{c}$.

Proposition 4.3. *If $\delta^\omega = 1$ then as $N \rightarrow \infty$*

$$\mathcal{A}_N^{\mathbf{f}} \implies \{t \in [0, 1] : B(t) = 0\}, \quad (4.13)$$

$$\mathcal{A}_N^{\mathbf{c}} \implies \{t \in [0, 1] : \beta(t) = 0\}. \quad (4.14)$$

The proof of Proposition 4.3 is achieved comparing the law of $\mathcal{A}_N^{\mathbf{f}}$ and $\mathcal{A}_N^{\mathbf{c}}$ with the law of a random set \mathcal{R}_N defined as follows: recalling that $\{\tau_k\}_{k \in \mathbb{N}}$ denotes the sequence of return times of S to zero, we set

$$\mathcal{R}_N := \text{range } \{\tau_i/N, i \geq 0\}$$

and we look at the law \mathcal{R}_N under the critical infinite volume measure \mathbf{P}_ω of Proposition 3.2. Observe that under \mathbf{P}_ω the process $([\tau_k], \tau_k - \tau_{k-1})_{k \in \mathbb{N}}$ is a Markov renewal process, whose semi-Markov kernel is given by Γ^β . The key point of the proof is given by the following result:

Lemma 4.4. *The law of $\{\mathcal{R}_N\}_N$ under \mathbf{P}_ω converges weakly to the law of the random set $\{t \geq 0 : B(t) = 0\}$.*

The core of the proof (see Step 1 below) uses the theory of *regenerative sets* and their connection with the concept of *subordinator*, see [9]. However we point out that it is also possible to give a more standard proof, using tightness and checking “convergence of the finite dimensional distributions”: this approach is outlined in Appendix C.

Proof of Lemma 4.4 We introduce the random set

$$\mathcal{R}_N^{(\beta)} := \text{range}\{\tau_k/N : k \geq 0, [\tau_k] = \beta\} \quad \beta \in \mathbb{S}.$$

Notice that $\mathcal{R}_N = \cup_\beta \mathcal{R}_N^{(\beta)}$. Let us also recall the definitions (2.23) and (2.24):

$$\begin{aligned} \kappa_0^{(\beta)} &:= \inf\{k \geq 0 : [\tau_k] = \beta\}, & \kappa_{i+1}^{(\beta)} &:= \inf\{k > \kappa_i^{(\beta)} : [\tau_k] = \beta\}, \\ T_0^{(\beta)} &:= \tau_{\kappa_0^{(\beta)}}, & T_i^{(\beta)} &:= \tau_{\kappa_i^{(\beta)}} - \tau_{\kappa_{i-1}^{(\beta)}}, \quad i \geq 1. \end{aligned}$$

Then $(T_i^{(\beta)})_{i \geq 1}$ is under \mathbf{P}_ω an IID sequence, independent of $T_0^{(\beta)}$: see the discussion before (2.25). We divide the rest of the proof in two steps.

Step 1. This is the main step: we prove that the law of $\mathcal{R}_N^{(\beta)}$ under \mathbf{P}_ω converges to the law of $\{t \geq 0 : B(t) = 0\}$. For this we follow the proof of Lemma 5 in [6].

Let $\{P(t)\}_{t \geq 0}$ be a Poisson process with rate $\gamma > 0$, independent of $(T_i^{(\beta)})_{i \geq 0}$. Then $\sigma_t = [T_1^{(\beta)} + \dots + T_{P(t)}^{(\beta)}]/N$ forms a non decreasing CAD process with independent stationary increments and $\sigma_0 = 0$: in other words $\sigma = (\sigma_t)_{t \geq 0}$ is a subordinator. As for any Levy process, the law of σ is characterized by the Laplace transform of the one-time distributions:

$$\mathbb{E}[\exp(-\lambda \sigma_t)] = \exp(-t \phi_N(\lambda)), \quad \lambda \geq 0, t \geq 0,$$

for a suitable function $\phi_N : [0, \infty) \mapsto [0, \infty)$, called Lévy exponent, which has a canonical representation, the Lévy–Khintchin formula (see e.g. (1.15) in [9]):

$$\phi_N(\lambda) = \int_{(0, \infty)} (1 - e^{-\lambda s}) \gamma \mathbb{P}(T_1^{(\beta)}/N \in ds) = \gamma \sum_{n=1}^{\infty} (1 - \exp(-\lambda n/N)) q^{(\beta)}(n).$$

We denote the closed range $\{\sigma_t : t \geq 0\}$ of the subordinator σ by $\widehat{\mathcal{R}}_N^{(\beta)}$. Then, following [9], $\widehat{\mathcal{R}}_N^{(\beta)}$ is a regenerative set. Moreover $\mathcal{R}_N^{(\beta)} = T_0^{(\beta)}/N + \widehat{\mathcal{R}}_N^{(\beta)}$.

Notice now that the law of the regenerative set $\widehat{\mathcal{R}}_N^{(\beta)}$ is invariant under the change of time scale $\sigma_t \rightarrow \sigma_{ct}$, for $c > 0$, and in particular independent of $\gamma > 0$. Since $\phi_N \rightarrow c \phi_N$ under this change of scale, we can fix $\gamma = \gamma_N$ such that $\phi_N(1) = 1$ and this will be implicitly assumed from now on. By Proposition (1.14) of [9], the law of $\widehat{\mathcal{R}}_N^{(\beta)}$ is uniquely determined by ϕ_N .

By the asymptotics of $q^{(\beta)}$ given in (2.26), one directly obtains that $\phi_N(\lambda) \rightarrow \lambda^{1/2} =: \Phi_{BM}(\lambda)$ as $N \rightarrow \infty$. It is now a matter of applying the result in [9, §3] to obtain that $\widehat{\mathcal{R}}_N^{(\beta)}$ converges in law to the regenerative set corresponding to Φ_{BM} . However by direct computation one obtains that the latter is nothing but the zero level set of a Brownian

motion, hence $\widehat{\mathcal{R}}_N^{(\beta)} \Rightarrow \{t \in [0, 1] : B(t) = 0\}$. From the fact that $T_0^{(\beta)}/N$ tends to 0 a.s., the same weak convergence for $\mathcal{R}_N^{(\beta)}$ follows immediately.

Step 2. We notice now that $\mathcal{R}_N = \cup_{\beta} \mathcal{R}_N^{(\beta)}$ is the union of non independent sets. Therefore, although we know that each $\mathcal{R}_N^{(\beta)}$ converges in law to $\{t \geq 0 : B(t) = 0\}$, it is not trivial that \mathcal{R}_N converges to the same limit. We start showing that for every positive $t \geq 0$, the distance between the first point in $\mathcal{R}_N^{(\alpha)}$ after t and the first point in $\mathcal{R}_N^{(\beta)}$ after t converges to zero in probability. More precisely, for any closed set $F \subset [0, \infty)$ we set:

$$d_t(F) := \inf(F \cap (t, \infty)). \quad (4.15)$$

and we claim that for all $\alpha, \beta \in \mathbb{S}$ and $t \geq 0$, $|d_t(\mathcal{R}_N^{(\alpha)}) - d_t(\mathcal{R}_N^{(\beta)})| \rightarrow 0$ in probability.

Recalling (2.25) and setting $q^{(\alpha; \beta)}(t) = \mathbf{P}_{\theta_{\alpha\omega}}(T_0^{(\beta)} = t)$, for all $\epsilon > 0$:

$$\begin{aligned} & \mathbf{P}_{\omega} \left(d_t(\mathcal{R}_N^{(\alpha)}) \geq d_t(\mathcal{R}_N^{(\beta)}) + \epsilon \right) \\ &= \sum_k \sum_{\gamma} \sum_{y=0}^{\lfloor Nt \rfloor} \mathbf{P}_{\omega}(\tau_k = y, [\tau_k] = \gamma) \sum_{z=\lfloor Nt \rfloor - y + 1}^{\infty} \mathbf{P}_{\theta_{\gamma\omega}}(T_0^{(\beta)} = z) \mathbf{P}_{\theta_{\beta\omega}}(T_0^{(\alpha)} \geq \lfloor N\epsilon \rfloor) \\ &= \sum_{\gamma} \sum_{y=0}^{\lfloor Nt \rfloor} U_{0,\gamma}(y) \sum_{z=\lfloor Nt \rfloor - y + 1}^{\infty} q^{(\gamma; \beta)}(z) \sum_{w=\lfloor N\epsilon \rfloor}^{\infty} q^{(\beta; \alpha)}(w). \end{aligned}$$

Arguing as in the proof of (2.26), it is easy to obtain the bound: $q^{(\beta; \alpha)}(w) \leq C_1 w^{-3/2}$, and by (2.27): $U_{0,\gamma}(y) \leq C_2 y^{-1/2}$, where C_1, C_2 are positive constants. Then asymptotically

$$\mathbf{P}_{\omega} \left(d_t(\mathcal{R}_N^{(\alpha)}) \geq d_t(\mathcal{R}_N^{(\beta)}) + \epsilon \right) \leq \frac{C_3}{N^{1/2}} \left(\int_0^{t/T} dy \int_{(t-y)/T}^{\infty} dz \int_{\epsilon/T}^{\infty} dw \frac{1}{y^{1/2} z^{3/2} w^{3/2}} \right)$$

for some positive constant C_3 , having used the convergence of the Riemann sums to the corresponding integral. The very same computations can be performed exchanging α with β , hence the claim is proven.

Now notice that $d_t(\mathcal{R}_N) = \min_{\alpha \in \mathbb{S}} d_t(\mathcal{R}_N^{(\alpha)})$, and since \mathbb{S} is a finite set we have that also $|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^{(\beta)})| \rightarrow 0$ in probability for any fixed $\beta \in \mathbb{S}$. Since we already know that $\mathcal{R}_N^{(\beta)}$ converges weakly to the law of $\{t \geq 0 : B(t) = 0\}$, the analogous statement for \mathcal{R}_N follows by standard arguments. More precisely, let us look at $(\mathcal{R}_N, \mathcal{R}_N^{(\beta)})$ as a random element of the space $\mathcal{F} \times \mathcal{F}$: by the compactness of \mathcal{F} it suffices to take any convergent subsequence $(\mathcal{R}_{k_n}, \mathcal{R}_{k_n}^{(\beta)}) \Rightarrow (\mathfrak{B}, \mathfrak{C})$ and to show that $\mathbb{P}(\mathfrak{B} \neq \mathfrak{C}) = 0$. By the Portmanteau Theorem it is sufficient to prove that $\lim_{N \rightarrow \infty} \mathbf{P}_{\omega}(\mathcal{R}_N \neq \mathcal{R}_N^{(\beta)}) = 0$, and this is an immediate consequence of the decomposition

$$\{\mathcal{R}_N \neq \mathcal{R}_N^{(\beta)}\} = \bigcup_{t \in \mathbb{Q}^+} \bigcup_{n \in \mathbb{N}} \{|d_t(\mathcal{R}_N) - d_t(\mathcal{R}_N^{(\beta)})| > 1/n\},$$

which holds by the right-continuity of $t \mapsto d_t$. \square

Proof of (4.14). First, we compute the Radon-Nykodim density of the law of $\mathcal{A}_N^c \cap [0, 1/2]$ with respect to the law of $\mathcal{R}_N^{1/2} := \mathcal{R}_N \cap [0, 1/2]$: for $F = \{t_1/N, \dots, t_k/N\} \subset [0, 1/2]$ with

$0 =: t_0 < t_1 < \dots < t_k$ integer numbers, the Radon–Nykodim derivative of the law of $\mathcal{A}_N^c \cap [0, 1/2]$ with respect to the law of $\mathcal{R}_N^{1/2}$ for $\mathcal{R}_N^{1/2} = F$ is:

$$f_N^c(g_{1/2}(F)) = f_N^c(t_k/N) = \frac{\sum_{n=N/2}^N M_{[t_k],[n]}(n-t_k) \mathcal{Z}_{[n],[N]}(N-n)}{\mathcal{Z}_{0,[N]}(N) Q_{[t_k]}(N/2-t_k)} \frac{\xi_0}{\xi_{[t_k]}},$$

where $Q_\alpha(t) := \sum_\beta \sum_{s=t+1}^\infty \Gamma_{\alpha,\beta}^-(s)$ and for any closed set $F \subset [0, \infty)$ we set:

$$g_t(F) := \sup(F \cap [0, t]). \quad (4.16)$$

By (2.28), for all $\varepsilon > 0$ and uniformly in $g \in [0, 1/2 - \varepsilon]$:

$$\begin{aligned} f_N^c(g) &\sim \frac{\sum_\gamma L_{[Ng],\gamma} \frac{T^2}{2\pi} \frac{\xi_\gamma \zeta_{[N]}}{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} T^{-1} \int_0^{1/2} y^{-1/2} (1-y-g)^{-3/2} dy}{\frac{T^2}{2\pi} \frac{\xi_0 \zeta_{[N]}}{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} T^{-1} \sum_\gamma L_{[Ng],\gamma} \xi_\gamma / \xi_{[Ng]} 2(1/2-g)^{-1/2}} \frac{\xi_0}{\xi_{[Ng]}} \\ &= \frac{\sqrt{1/2}}{1-g} =: r(g). \end{aligned}$$

If Ψ is a bounded continuous functional on \mathcal{F} such that $\Psi(F) = \Psi(F \cap [0, 1/2])$ for all $F \in \mathcal{F}$, then, setting $Z_B := \{t \in [0, 1] : B(t) = 0\}$ and $Z_\beta := \{t \in [0, 1] : \beta(t) = 0\}$, we get:

$$\mathbb{E}[\Psi(Z_\beta)] = \mathbb{E}[\Psi(Z_B) r(g_{1/2}(Z_B))],$$

see formula (49) in [6]. By the asymptotics of f_N^c we obtain that

$$\mathbb{E}[\Psi(\mathcal{A}_N^c)] = \mathbb{E}[\Psi(\mathcal{R}_N^{1/2}) f_N^c(g_{1/2}(\mathcal{R}_N^{1/2}))] \rightarrow \mathbb{E}[\Psi(Z_B) r(g_{1/2}(Z_B))] = \mathbb{E}[\Psi(Z_\beta)]$$

i.e. $\mathcal{A}_N^c \cap [0, 1/2]$ converges to $Z_\beta \cap [0, 1/2]$. Notice now that the distribution of the random set $\{1-t : t \in \mathcal{A}_N^c \cap [1/2, 1]\}$ under $\mathbf{P}_{N,\omega}^c$ is the same as the distribution of $\mathcal{A}_N^c \cap [0, 1/2]$ under $\mathbf{P}_{N,\bar{\omega}}^c$, where $\bar{\omega}_{[i]} := \omega_{[N-i]}$. Therefore we obtain that $\mathcal{A}_N^c \cap [1/2, 1]$ converges to $Z_\beta \cap [0, 1/2]$ and the proof is complete.

Proof of (4.13). By conditioning on the last zero, we see that if Ψ is a bounded continuous functional on \mathcal{F} then:

$$\mathbb{E}[\Psi(\mathcal{A}_N^f)] = \sum_{t=0}^N \mathbb{E}[\Psi(\mathcal{A}_t^c)] \frac{Z_{t,\omega}^c}{Z_{N,\omega}^f} P(N-t) \exp\left(\tilde{\Phi}_{[t],[N]}(N-t)\right).$$

We denote by β^t a Brownian bridge over the interval $[0, t]$, i.e. a Brownian motion over $[0, t]$ conditioned to be 0 at time t , and we set $Z_{\beta^t} := \{s \in [0, t] : \beta^t(s) = 0\}$. By (4.14), (2.28) and (3.14) we obtain as $N \rightarrow \infty$:

$$\begin{aligned} \mathbb{E}[\Psi(\mathcal{A}_N^f)] &= \sum_{t=0}^N \sum_\gamma 1_{(t \in \gamma)} \mathbb{E}[\Psi(\mathcal{A}_t^c)] \frac{Z_{t,\omega}^c}{Z_{N,\omega}^f} P(N-t) \exp\left(\tilde{\Phi}_{\gamma,[N]}(N-t)\right) \\ &\sim \int_0^1 \mathbb{E}[\Psi(Z_{\beta^t})] \frac{1}{t^{1/2}(1-t)^{1/2}} dt \cdot \sum_\gamma \frac{1}{T} \frac{T^2}{2\pi} \frac{\xi_0 \zeta_\gamma}{\sum_{\eta,\eta'} \zeta_\eta L_{\eta,\eta'} \xi_{\eta'}} \frac{\tilde{L}_{\gamma,[N]}}{\xi_0 \frac{T}{2} \frac{\sum_\eta \zeta_\eta \tilde{L}_{\eta,[N]}}{\sum_{\eta,\eta'} \zeta_\eta L_{\eta,\eta'} \xi_{\eta'}}} \\ &= \int_0^1 \mathbb{E}[\Psi(Z_{\beta^t})] \frac{1}{\pi t^{1/2}(1-t)^{1/2}} dt = \mathbb{E}[\Psi(Z_B)]. \quad \square \end{aligned}$$

The signs. To complete the proof of point (3) of Theorem 1.8 in the critical case ($\delta^\omega = 1$) we follow closely the proof given in Section 8 of [6]. We have already proven the convergence of the set of zeros and we have to “paste” the excursions. From Section 3 we know that, *conditionally on the zeros*:

- the signs $\{\sigma_k\}_k$ and the absolute values $\{e_k(\cdot)\}_k$ of the excursions are independent;
- the (conditional) law of $e_k(\cdot)$ is the same as under the original random walk measure \mathbf{P} .

The weak convergence under diffusive rescaling of $e_k(\cdot)$ for $k \leq \iota_N$ towards the Brownian excursion $e(\cdot)$ and of the last excursion $e_{\iota_N+1}(\cdot)$ for $a = f$ towards the Brownian meander $m(\cdot)$ has been proved in [14] and, respectively, in [3]. Then it only remains to concentrate on the signs.

We start with the constrained case: we are going to show that for all $t \in (0, 1)$

$$\exists \lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c(S_{[tN]} > 0) =: \mathbf{p}_\omega^{\bar{=}}, \quad (4.17)$$

and the limit is independent of t . We point out that actually we should fix the extremities of the excursion embracing t , that is we should rather prove that

$$\lim_{N \rightarrow \infty} \mathbf{P}_{N,\omega}^c(S_{[tN]} > 0 \mid G_{[tN]}/N \in (a - \varepsilon, a), D_{[tN]}/N \in (b, b + \varepsilon)) = \mathbf{p}_\omega^{\bar{=}}, \quad (4.18)$$

for $a < t < b$ and $\varepsilon > 0$ (recall the definition of G_t and D_t in § 4.2), but in order to lighten the exposition we will stick to (4.17), since proving (4.18) requires only minor changes.

We have, recalling (4.7):

$$\mathbf{P}_{N,\omega}^c(S_{[tN]} > 0) = \sum_{\alpha,\beta} \sum_{x < [tN]} \sum_{y > [tN]} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,[N]}(N-y)}{\mathcal{Z}_{0,[N]}(N)}.$$

By Dominated Convergence and by (2.28):

$$\begin{aligned} & \exists \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} N^{1/2} \sum_{x < [tN]} \sum_{y > [tN]} \mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) \mathcal{Z}_{\beta,\eta}(N-y) \\ &= \frac{1}{T^2} \int_0^t dx \int_t^1 dy [x(y-x)^3(1-y)]^{-\frac{1}{2}} \left(\frac{T^2}{2\pi} \right)^2 \frac{\xi_0 \zeta_\alpha \xi_\beta \zeta_\eta}{(\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'})^2} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \end{aligned}$$

see (3.12). We obtain (4.17) with

$$\mathbf{p}_\omega^{\bar{=}} := \frac{\sum_{\alpha,\beta} \zeta_\alpha c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \xi_\beta}{\sum_{\alpha,\beta} \zeta_\alpha L_{\alpha,\beta} \xi_\beta} \quad (4.19)$$

Observe the following: by (3.12),

- if $h_\omega > 0$ then in (4.19) the denominator is equal to the numerator, so that $\mathbf{p}_\omega^{\bar{=}} = 1$.
- if $h_\omega = 0$ and $\Sigma \equiv 0$ then in (4.19) the denominator is equal to *twice* the numerator, so that $\mathbf{p}_\omega^{\bar{=}} = 1/2$.

Now let us consider the free case. We are going to show that for all $t \in (0, 1]$:

$$\exists \lim_{\substack{N \rightarrow \infty \\ [N] = \eta}} \mathbf{P}_{N,\omega}^f(S_{[tN]} > 0) = \left(1 - \frac{2 \arcsin \sqrt{t}}{\pi} \right) \mathbf{p}_\omega^{\bar{=}} + \frac{2 \arcsin \sqrt{t}}{\pi} \mathbf{q}_{\omega,\eta}^{\bar{=}} =: \mathbf{p}_{\omega,\eta}^{\bar{=},f}(t), \quad (4.20)$$

where $\mathbf{p}_\omega^{\bar{=}}$ is the same as above, see (4.19), while $\mathbf{q}_{\omega,\eta}^{\bar{=}}$ is defined in (4.21) below. We stress again that we should actually fix the values of $G_{[tN]}$ and $D_{[tN]}$ like in (4.18), proving that

the limiting probability is either \mathbf{p}_ω^- or $\mathbf{q}_{\omega,\eta}^-$ according to whether $D_{[tN]} \leq N$ or $D_{[tN]} > N$, but this will be clear from the steps below. Formula (4.20) follows from the fact that

$$\begin{aligned} \mathbf{P}_{N,\omega}^f(S_{[tN]} > 0) &= \sum_{\alpha,\beta} \sum_{x < [tN]} \sum_{y > [tN]} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,\beta}^+(y-x) M_{\alpha,\beta}(y-x) Z_{N-y,\theta_{[y]}\omega}^f}{Z_{N,\omega}^f} \\ &\quad + \sum_{\alpha} \sum_{x < [tN]} \frac{\mathcal{Z}_{0,\alpha}(x) \rho_{\alpha,[N]}^+(N-x) P(N-x) \exp\left(\tilde{\Phi}_{[x],[N]}(N-x)\right)}{Z_{N,\omega}^f}. \end{aligned}$$

Letting $N \rightarrow \infty$ with $[N] = \eta$, by (3.14) the first term in the r.h.s. converges to:

$$\begin{aligned} &\int_0^t \frac{dx}{x^{\frac{1}{2}}} \int_t^1 \frac{dy}{(y-x)^{\frac{3}{2}}} \cdot \\ &\quad \cdot \sum_{\alpha,\beta} \frac{1}{T^2} \frac{T^2 \xi_0 \zeta_\alpha}{2\pi \sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} c_K \frac{1}{2} \exp(\omega_\beta^{(0)}) \frac{\xi_\beta \frac{T}{2} \sum_\gamma \zeta_\gamma \tilde{L}_{\gamma,\eta}}{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} \cdot \frac{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}}{\xi_0 \frac{T}{2} \sum_\gamma \zeta_\gamma \tilde{L}_{\gamma,\eta}} \\ &= \left(1 - \frac{2 \arcsin \sqrt{t}}{\pi}\right) \cdot \mathbf{p}_\omega^- \end{aligned}$$

while the second term converges to

$$\begin{aligned} &\int_0^t \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \frac{1}{T} \sum_{\alpha} \frac{T^2 \xi_0 \zeta_\alpha}{2\pi \sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}} c_K \cdot \frac{\sum_{\gamma,\gamma'} \zeta_\gamma L_{\gamma,\gamma'} \xi_{\gamma'}}{\xi_0 \frac{T}{2} \sum_\gamma \zeta_\gamma \tilde{L}_{\gamma,\eta}} \\ &= \frac{2 \arcsin \sqrt{t}}{\pi} \cdot \frac{c_K \sum_\gamma \zeta_\gamma}{\sum_\gamma \zeta_\gamma \tilde{L}_{\gamma,\eta}}. \end{aligned}$$

Therefore we obtain (4.20) with:

$$\mathbf{q}_{\omega,\eta}^- = \frac{c_K \sum_\gamma \zeta_\gamma}{\sum_\gamma \zeta_\gamma \tilde{L}_{\gamma,\eta}}. \quad (4.21)$$

We observe that, by (3.13):

- if $h_\omega > 0$ or if $h_\omega = 0$ and $\Sigma \equiv 0$, then $\mathbf{p}_{\omega,\eta}^{-;f}(t) = \mathbf{q}_{\omega,\eta}^- = \mathbf{p}_\omega^-$ for all t and η
- in the remaining case, i.e. if $\omega \in \mathcal{P}^-$, in general $\mathbf{p}_{\omega,\eta}^{-;f}(t)$ depends on t and η .

Now that we have proven the convergence of the probabilities of the signs of the excursion, in order to conclude the proof of point (3) of Theorem 1.8 it is enough to argue as in the proof of Theorem 11 in [6].

APPENDIX A. AN ASYMPTOTIC RESULT

We are going to prove that equation (2.26) holds true. Let us first give some preliminary notation: given an *irreducible* $T \times T$ matrix $Q_{\alpha,\beta}$ with *nonnegative entries*, its Perron–Frobenius eigenvalue (= spectral radius) will be denoted by $Z = Z(Q)$ and the corresponding left and right eigenvectors (with any normalization) will be denoted by $\{\zeta_\alpha\}, \{\xi_\alpha\}$. We recall that $\zeta_\alpha, \xi_\alpha > 0$. Being a simple root of the characteristic polynomial, $Z(Q)$ is an *analytic* function of the entries of Q , and

$$\frac{\partial Z}{\partial Q_{\alpha,\beta}} = \frac{\zeta_\alpha \xi_\beta}{\left(\sum_\gamma \zeta_\gamma \xi_\gamma\right)}. \quad (\text{A.1})$$

Hence $Z(Q)$ is a strictly increasing function of each of the entries of Q .

Now, let Q denote the transition matrix of an irreducible, positive recurrent Markov chain, and let us introduce the matrix $Q^{(\gamma)}$ and the (column) vector $|\gamma\rangle$ defined by

$$[Q^{(\gamma)}]_{\alpha,\beta} := Q_{\alpha,\beta} \mathbf{1}_{(\beta \neq \gamma)} \quad [|\gamma\rangle]_{\alpha} := \mathbf{1}_{(\alpha = \gamma)}.$$

By monotonicity, $Z(Q^{(\gamma)}) < Z(Q) = 1$ for all γ . Then we can define the geometric series

$$(1 - Q^{(\gamma)})^{-1} := \sum_{k=0}^{\infty} (Q^{(\gamma)})^k.$$

The interesting point is that, for every γ , the row vector $\langle \gamma | \cdot (1 - Q^{(\gamma)})^{-1}$ is (a multiple of) the left Perron–Frobenius eigenvector of the matrix Q (by $\langle \gamma |$ we denote the transposed of $|\gamma\rangle$). Similarly the column vector $(1 - Q^{(\gamma)})^{-1} \cdot Q \cdot |\gamma\rangle$ is (a multiple of) the right Perron–Frobenius eigenvector of Q . More precisely we have

$$[\langle \gamma | \cdot (1 - Q^{(\gamma)})^{-1}]_{\alpha} = \frac{\nu_{\alpha}}{\nu_{\gamma}} \quad [(1 - Q^{(\gamma)})^{-1} \cdot Q \cdot |\gamma\rangle]_{\alpha} = 1, \quad (\text{A.2})$$

where $\{\nu_{\alpha}\}_{\alpha}$ is the invariant measure of the chain, that is $\sum_{\alpha} \nu_{\alpha} Q_{\alpha,\beta} = \nu_{\beta}$ and $\sum_{\alpha} \nu_{\alpha} = 1$. Equation (A.2) can be proved by exploiting its probabilistic interpretation in terms of expected number of visits to state α before the first return to site γ , see [2, § I.3].

Next we turn to the asymptotic behavior of $q^{(\beta)}(x)$, giving the law of $T_0^{(\beta)}$ under \mathbb{P}_{β} (recall the notations introduced in § 2.5). With a standard renewal argument, we can express it as

$$q^{(\beta)}(x) = \sum_{y=0}^{x-1} \sum_{\gamma \in \mathbb{S}} V_{\beta,\gamma}^{(\beta)}(y) \Gamma_{\gamma,\beta}^{\leftarrow}(x-y) = (V^{(\beta)} * \Gamma^{\leftarrow})_{\beta,\beta}(x), \quad (\text{A.3})$$

where the kernel $V^{(\beta)}$ is defined by

$$V_{\alpha,\gamma}^{(\beta)}(x) = \sum_{k=0}^{\infty} [(\Gamma^{(\beta)})^{*k}]_{\alpha,\gamma}(x),$$

and we have set $\Gamma_{\alpha,\gamma}^{(\beta)}(x) := \Gamma_{\alpha,\gamma}^{\leftarrow}(x) \mathbf{1}_{(\gamma \neq \beta)}$. Let us look more closely at both terms in the r.h.s. of (A.3).

- For the semi–Markov kernel Γ^{\leftarrow} , recall its definition (2.20), the asymptotic behavior as $x \rightarrow \infty$, $[x] = \beta - \gamma$ is given by

$$\Gamma_{\gamma,\beta}^{\leftarrow}(x) \sim \frac{\widehat{L}_{\gamma,\beta}}{x^{3/2}} \quad \widehat{L}_{\gamma,\beta} := L_{\gamma,\beta} \frac{\xi_{\beta}}{\xi_{\gamma}}. \quad (\text{A.4})$$

Moreover, we have that

$$\sum_{x \in \mathbb{N}} \Gamma_{\gamma,\beta}^{\leftarrow}(x) = B_{\gamma,\beta} \frac{\xi_{\beta}}{\xi_{\gamma}} =: \widehat{B}_{\gamma,\beta}. \quad (\text{A.5})$$

- On the other hand, for the kernel $V^{(\beta)}$ we can apply the theory developed in § 2.4 for the case $\delta^{\omega} < 1$, because the matrix

$$\sum_{x \in \mathbb{N}} \Gamma_{\alpha,\gamma}^{(\beta)}(x) = [\widehat{B}^{(\beta)}]_{\alpha,\gamma}$$

has Perron–Frobenius eigenvalue strictly smaller than 1 (we recall the convention $[Q^{(\beta)}]_{\alpha,\gamma} := Q_{\alpha,\gamma} \mathbf{1}_{(\gamma \neq \beta)}$ for any matrix Q). Since

$$\Gamma_{\alpha,\gamma}^{(\beta)}(x) \sim \frac{[\widehat{L}^{(\beta)}]_{\alpha,\gamma}}{x^{3/2}} \quad x \rightarrow \infty, \quad [x] = \gamma - \alpha,$$

we can apply (2.16) to get that as $x \rightarrow \infty$, $[x] = \alpha - \gamma$

$$V_{\alpha,\gamma}^{(\beta)}(x) \sim \left([(1 - \widehat{B}^{(\beta)})^{-1} \widehat{L}^{(\beta)} (1 - \widehat{B}^{(\beta)})^{-1}]_{\alpha,\gamma} \right) \frac{1}{x^{3/2}}. \quad (\text{A.6})$$

Moreover applying an analog of (2.17) we get that

$$\sum_{y \in \mathbb{N}} V_{\alpha,\gamma}^{(\beta)}(y) = \sum_{k=0}^{\infty} [(\widehat{B}^{(\beta)})^k]_{\alpha,\gamma} = [(1 - \widehat{B}^{(\beta)})^{-1}]_{\alpha,\gamma}. \quad (\text{A.7})$$

We are finally ready to get the asymptotic behavior of $q^{(\beta)}$. As both $V^{(\beta)}$ and $\Gamma^=$ have a $x^{-3/2}$ -like tail, it is easy to check from (A.3) that as $x \rightarrow \infty$, $x \in T\mathbb{N}$

$$q^{(\beta)}(x) \sim \sum_{\gamma \in \mathbb{S}} \left\{ \left(\sum_{y \in \mathbb{N}} V_{\beta,\gamma}^{(\beta)}(y) \right) \Gamma_{\gamma,\beta}^=(x) + V_{\beta,\gamma}^{(\beta)}(x) \left(\sum_{y \in \mathbb{N}} \Gamma_{\gamma,\beta}^=(y) \right) \right\},$$

and applying (A.7), (A.4), (A.6) and (A.5) we get that $q^{(\beta)}(x) \sim c_\beta/x^{3/2}$ as $x \rightarrow \infty$, $x \in T\mathbb{N}$, with

$$\begin{aligned} c_\beta &= \left[(1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L} \right]_{\beta,\beta} + \left[(1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L}^{(\beta)} \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta,\beta} \\ &= \left[(1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L} \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta,\beta} \\ &= \langle \beta | \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{L} \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \cdot | \beta \rangle. \end{aligned}$$

To obtain the second equality we have used the fact that

$$\left[(1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta,\beta} = \left[\langle \beta | \cdot (1 - \widehat{B}^{(\beta)})^{-1} \cdot \widehat{B} \right]_{\beta} = 1,$$

which follows from (A.2) applied to the matrix $Q = \widehat{B}$. Again from (A.2) we get

$$c_\beta = \frac{1}{\nu_\beta} \sum_{\alpha,\gamma \in \mathbb{S}} \nu_\alpha \widehat{L}_{\alpha,\gamma},$$

where $\{\nu_\alpha\}_\alpha$ is the invariant measure (that is the normalized left Perron–Frobenius eigenvector) of the matrix \widehat{B} . However from the definition (A.5) of \widehat{B} it is immediate to see that $\{\nu_\alpha\} = \{\zeta_\alpha \xi_\alpha\}$, and recalling the definition (A.4) of \widehat{L} we obtain the expression for c_β we were looking for:

$$c_\beta = \frac{1}{\zeta_\beta \xi_\beta} \sum_{\alpha,\gamma} \zeta_\alpha L_{\alpha,\gamma} \xi_\gamma. \quad (\text{A.8})$$

APPENDIX B. SOME COMPUTATIONS ON THE THERMODYNAMIC LIMIT MEASURE

We want now to give a description of the typical paths under $\mathbf{P}_\omega^{a,\eta}$ in the delocalization regime, i.e. when $\delta^\omega < 1$. We are going to compute the distribution of two interesting random variables under $\mathbf{P}_\omega^{a,\eta}$ in this case: the last return to zero and the total number of returns to zero. Other analogous computations are possible using the same procedure.

The last return to zero. We want to study the law under $\mathbf{P}_\omega^{a,\eta}$ of the last zero $\ell := \sup\{i \in \mathbb{N} : S_i = 0\}$ in the strictly delocalized regime. For simplicity we consider the case $a = c$, the case $a = f$ being completely analogous. We compute first the law of $\ell_k := \sup\{i \leq k : S_i = 0\}$ with $k \in \mathbb{N}$: for $x \leq k < N$ and $N \in \eta$:

$$\mathbf{P}_{N,\omega}^c(\ell_k = x) = \mathcal{Z}_{0,[x]}(x) \sum_{z=k+1}^N \frac{M_{[x],[z]}(z-x) \mathcal{Z}_{[z],\eta}(N-z)}{\mathcal{Z}_{0,\eta}(N)} \quad (\text{B.1})$$

By (3.11) and (3.15) we obtain:

$$\lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(\ell_k = x) = \mathcal{Z}_{0,[x]}(x) \left[\sum_{z=0}^{\infty} \frac{L_{[x],\eta-[z]} \mathcal{Z}_{\eta-[z],\eta}(z)}{\Lambda_{0,\eta}^c} + \sum_{z=k+1}^{\infty} M_{[x],[z]}(z-x) \frac{\Lambda_{[z],\eta}^c}{\Lambda_{0,\eta}^c} \right].$$

Notice now that, by (3.17):

$$\sum_{z=0}^{\infty} L_{[x],\eta-[z]} \mathcal{Z}_{\eta-[z],\eta}(z) = \sum_{\gamma} L_{[x],\gamma} \sum_{z=0}^{\infty} \mathcal{Z}_{\gamma,\eta}(z) = [L \cdot (I - B)^{-1}]_{[x],\eta} = \mu_{[x],\eta}^c. \quad (\text{B.2})$$

Therefore, we have proven that:

$$\mathbf{P}_\omega^{c,\eta}(\ell_k = x) = \lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(\ell_k = x) = \mathcal{Z}_{0,[x]}(x) \left[\frac{\mu_{[x],\eta}^c}{\Lambda_{0,\eta}^c} + \sum_{z=k+1}^{\infty} M_{[x],[z]}(z-x) \frac{\Lambda_{[z],\eta}^c}{\Lambda_{0,\eta}^c} \right]$$

and letting $k \rightarrow \infty$ we obtain:

$$\mathbf{P}_\omega^{c,\eta}(\ell = x) = \mathcal{Z}_{0,[x]}(x) \frac{\mu_{[x],\eta}^c}{\Lambda_{0,\eta}^c}.$$

For the proof of Lemma 4.2 above, notice for instance that by (B.1):

$$\begin{aligned} \mathbf{P}_{N,\omega}^c(G_{N/2} \geq L) &= \mathbf{P}_{N,\omega}^c(\ell_{N/2} \geq L) \\ &\leq C_1 N^{3/2} \sum_{t=L}^{\lfloor N/2 \rfloor} t^{-3/2} \sum_{k=\lfloor N/2 \rfloor + 1}^{N+1} (k-t)^{-3/2} (N+2-k)^{-3/2} \leq C_2 L^{-1/2}, \end{aligned} \quad (\text{B.3})$$

where C_1, C_2 are positive constants.

The number of returns to zero. Analogously, we want to study the law of the total number of returns to zero $\mathcal{N} := \#\{i \in \mathbb{N} : S_i = 0\}$ under $\mathbf{P}_\omega^{c,\eta}$. Let $\mathcal{N}_K := \#\{i : 1 \leq i \leq K : S_i = 0\}$ for $k \in \mathbb{N}$. For $k \leq K$ and $N \in \eta$:

$$\mathbf{P}_{N,\omega}^c(\mathcal{N}_K = k) = \sum_{x=1}^K M_{0,[x]}^{*k}(x) \sum_{y=K+1}^N \frac{M_{[x],[y]}(y-x) \mathcal{Z}_{[y],\eta}(N-y)}{\mathcal{Z}_{0,\eta}(N)}$$

Then by (3.11) and (3.15):

$$\begin{aligned} &\lim_{\substack{N \rightarrow \infty \\ N \in \eta}} \mathbf{P}_{N,\omega}^c(\mathcal{N}_K = k) \\ &= \sum_{x=0}^K M_{0,[x]}^{*k}(x) \left[\sum_{y=0}^{\infty} \frac{L_{[x],\eta-[y]} \mathcal{Z}_{\eta-[y],\eta}(y)}{\Lambda_{0,\eta}^c} + \sum_{z=K+1}^{\infty} M_{[x],[y]}(y-x) \frac{\Lambda_{[y],\eta}^c}{\Lambda_{0,\eta}^c} \right]. \end{aligned}$$

By (B.2), letting $K \rightarrow \infty$ we obtain:

$$\mathbf{P}_\omega^{c,\eta}(\mathcal{N} = k) = \frac{1}{\Lambda_{0,\eta}^c} \left[B^k \cdot \mu^c \right]_{0,\eta}.$$

APPENDIX C. ON THE WEAK CONVERGENCE OF THE CRITICAL ZERO SET

We are going to outline an alternative proof of Lemma 4.4, that is we are going to show that when $\delta^\omega = 1$ as $N \rightarrow \infty$

$$\mathcal{R}_N \text{ under } \mathbf{P}_\omega \implies \{t \geq 0 : B(t) = 0\}. \quad (\text{C.1})$$

To keep the notation transparent, it is convenient to denote by $\mathcal{G}_N \in \mathcal{M}_1(\mathcal{F})$ the image law of \mathcal{R}_N under \mathbf{P}_ω . That is \mathcal{G}_N is a probability law on \mathcal{F} (the class of all closed subsets of \mathbb{R}^+) defined for a measurable subset $A \subseteq \mathcal{F}$ by

$$\mathcal{G}_N(A) := \mathbf{P}_\omega(\mathcal{R}_N \in A).$$

In the same way the law of $\{t \geq 0 : B(t) = 0\}$ will be denoted by $\mathcal{G}^{(BM)}$. Then we can reexpress our goal (C.1) as

$$\mathcal{G}_N \implies \mathcal{G}^{(BM)}. \quad (\text{C.2})$$

Remember the definition (4.15) of the mapping $d_t : \mathcal{F} \mapsto \mathbb{R}^+ \cup \{+\infty\}$. We claim that to prove (C.2) it suffices to show that, for every $n \in \mathbb{N}$ and for all $t_1, \dots, t_n \in \mathbb{R}$, the law of the vector $(d_{t_1}, \dots, d_{t_n})$ under \mathcal{G}_N converges to the law of the same vector under $\mathcal{G}^{(BM)}$:

$$(d_{t_1}, \dots, d_{t_n}) \circ (\mathcal{G}_N)^{-1} \implies (d_{t_1}, \dots, d_{t_n}) \circ (\mathcal{G}^{(BM)})^{-1}. \quad (\text{C.3})$$

The intuitive explanation of why (C.3) should imply (C.2) is that an element $\xi \in \mathcal{F}$ can be *identified* with the process $\{d_t(\xi)\}_{t \in \mathbb{R}^+}$, since $\xi = \{t \in \mathbb{R}^+ : d_{t-}(\xi) = t\}$. Hence the convergence in $\mathcal{M}_1(\mathcal{F})$ can be read in terms of the random process $\{d_t(\cdot)\}_{t \in \mathbb{R}^+}$, and using the compactness of $\mathcal{M}_1(\mathcal{F})$ it turns out that (C.3) is indeed sufficient to ensure (C.2). Let us sketch more in detail these arguments.

- (1) The Borel σ -field of \mathcal{F} coincides with $\sigma(\{d_t\}_{t \in \mathbb{R}^+})$, i.e. with the σ -field generated by $\{d_t\}_{t \in \mathbb{R}^+}$, and also with $\sigma(\{d_t\}_{t \in I})$ where I is any dense subset of \mathbb{R}^+ .
- (2) Suppose that we are given $\{\nu_k\}$, $\nu \in \mathcal{M}_1(\mathcal{F})$ such that $\nu_k \Rightarrow \nu$: this fact does not entail the convergence of all the finite dimensional marginals of $\{d_t\}$, that is *it is not true* that the law of the vector $(d_{t_1}, \dots, d_{t_n})$ under ν_k converges to the law of the same vector under ν , because the mappings $d_t(\cdot)$ are not continuous on \mathcal{F} . Nevertheless one can show that this convergence does hold for *almost all* choices of the indexes t_1, \dots, t_n . More precisely, given any measure $\nu \in \mathcal{M}_1(\mathcal{F})$ there exists a subset $I_\nu \subseteq \mathbb{R}^+$ with $\text{Leb}(I_\nu^c) = 0$ with the following property: for any sequence $\{\nu_k\}$ with $\nu_k \Rightarrow \nu$, for any $n \in \mathbb{N}$ and for all $t_1, \dots, t_n \in I_\nu$, the law of the vector $(d_{t_1}, \dots, d_{t_n})$ under ν_k converges as $k \rightarrow \infty$ to the law of the same vector under ν . This is a well-known feature of processes whose discontinuity points form a negligible set, in particular CADLAG processes: in fact the set I_ν can be chosen as the set of $t \in \mathbb{R}^+$ such that $\nu\{\xi : d_{t-}(\xi) = d_t(\xi)\} = 1$, because $d_{t-}(\xi) = d_t(\xi)$ implies that $d_t(\cdot)$ is continuous at ξ .
- (3) Since $\mathcal{M}_1(\mathcal{F})$ is compact, to prove (C.2) it suffices to show that any convergent subsequence of $\{\mathcal{G}_N\}_N$ converges to $\mathcal{G}^{(BM)}$. Thus we take a convergent subsequence $\mathcal{G}_{k_n} \Rightarrow \nu$ for some $\nu \in \mathcal{M}_1(\mathcal{F})$ and we want to prove that $\nu = \mathcal{G}^{(BM)}$. By point (2) there exists a dense subset $I_\nu \subseteq \mathbb{R}^+$ such that for $t_1, \dots, t_n \in I_\nu$ the law of the

vector $(d_{t_1}, \dots, d_{t_n})$ under \mathcal{G}_{k_n} converges to the law of the same vector under ν , and since we are assuming that (C.3) holds this means that the vector $(d_{t_1}, \dots, d_{t_n})$ has the same law under ν and under $\mathcal{G}^{(BM)}$. This is equivalent to say that ν and $\mathcal{G}^{(BM)}$ coincide on the σ -field $\sigma(\{d_t\}_{t \in I_\nu})$, and by point (1) it follows that indeed $\nu = \mathcal{G}^{(BM)}$.

Thus it only remains to show that (C.3) holds, and this can be done by direct computation. For simplicity we consider only the case $n = 1$ of the one-time marginals, but everything can be extended to the case $n > 1$.

For any $t > 0$ the law of d_t under $\mathcal{G}^{(BM)}$ is given by

$$\mathcal{G}^{(BM)}(d_t \in dy) = \frac{t^{1/2}}{\pi y(y-t)^{1/2}} \mathbf{1}_{(y>t)} dy =: \rho_t(y) dy,$$

see [23]. Hence we have to show that for every $x \in \mathbb{R}^+$

$$\lim_{N \rightarrow \infty} \mathbf{P}_\omega(d_t(\mathcal{R}_N) > x) = \int_x^\infty \rho_t(y) dy.$$

We recall that $\mathcal{R}_N = \text{range}\{\tau_n/N : n \geq 0\}$ is the range of the process $\{\tau_n\}_{n \in \mathbb{N}}$ rescaled by a factor $1/N$, and that under \mathbf{P}_ω the process $\{\tau_n\}_{n \in \mathbb{N}}$ is a Markov-renewal process with semi-Markov kernel $\Gamma_{\alpha,\beta}^-(x)$ defined by (2.20). We also use the notation $U_{\alpha,\beta}(x)$ for the corresponding Markov-Green function, defined by (2.21). Then using the Markov property we get

$$\begin{aligned} \mathbf{P}_\omega(d_t(\mathcal{R}_N) > x) &= \sum_{k \in \mathbb{N}} \mathbf{P}_\omega(\tau_k \leq Nt, \tau_{k+1} > Nx) \\ &= \sum_{\alpha, \beta \in \mathbb{S}} \sum_{y=1}^{Nt} \sum_{w=Nx}^\infty \sum_{k \in \mathbb{N}} \mathbf{P}_\omega(\tau_k = y, [\tau_k] = \alpha) \mathbf{P}_{\theta_y \omega}(\tau_1 = w - y, [\tau_1] = \beta - \alpha) \\ &= \sum_{\alpha, \beta \in \mathbb{S}} \sum_{y=1}^{Nt} U_{0,\alpha}(y) \sum_{w=Nx}^\infty \Gamma_{\alpha,\beta}^-(w - y) \end{aligned}$$

The asymptotic behavior of the terms appearing in the expression can be extracted from (2.27) and (2.12): the net result is that as $z \rightarrow \infty$

$$\begin{aligned} \sqrt{z} U_{0,\alpha}(z) &\xrightarrow{[z]=\alpha} \frac{T^2}{2\pi} \frac{\zeta_\alpha \xi_\alpha}{\sum_{\gamma, \gamma'} \zeta_\gamma L_{\gamma, \gamma'} \xi_{\gamma'}} =: c_{0,\alpha}^U \\ z^{3/2} \Gamma_{\alpha,\beta}^-(z) &\xrightarrow{[z]=\beta-\alpha} \frac{\xi_\beta}{\xi_\alpha} L_{\alpha,\beta} =: c_{\alpha,\beta}^\Gamma. \end{aligned}$$

Therefore we have as $N \rightarrow \infty$

$$\begin{aligned} \mathbf{P}_\omega(d_t(\mathcal{R}_N) > x) &\sim \sum_{\alpha, \beta \in \mathbb{S}} c_{0,\alpha}^U c_{\alpha,\beta}^\Gamma \sum_{y=1}^{Nt} \frac{1}{\sqrt{y}} \mathbf{1}_{([y]=\alpha)} \sum_{w=Nx}^\infty \frac{1}{(w-y)^{3/2}} \mathbf{1}_{([w]=\beta)} \\ &\sim \frac{1}{T^2} \left(\sum_{\alpha, \beta \in \mathbb{S}} c_{0,\alpha}^U c_{\alpha,\beta}^\Gamma \right) \frac{1}{N^2} \sum_{s \in (0, \frac{t}{N}) \cap \frac{\mathbb{Z}}{N}} \frac{1}{\sqrt{s}} \sum_{u \in (\frac{x}{N}, \infty) \cap \frac{\mathbb{Z}}{N}} \frac{1}{(u-s)^{3/2}}, \end{aligned}$$

and from the explicit expressions for $c_{0,\alpha}^U, c_{\alpha,\beta}^\Gamma$ together with the convergence of the Riemann sums to the corresponding integral we get

$$\begin{aligned} \exists \lim_{N \rightarrow \infty} \mathbf{P}_\omega(d_t(\mathcal{R}_N) > x) &= \frac{1}{2\pi} \int_0^{t/T} ds \frac{1}{\sqrt{s}} \int_{x/T}^\infty du \frac{1}{(u-s)^{3/2}} \\ &= \frac{1}{\pi} \int_0^{t/T} ds \frac{1}{\sqrt{s}} \frac{1}{\sqrt{x/T-s}} = \frac{1}{\pi} \int_0^t dy \frac{1}{\sqrt{y}} \frac{1}{\sqrt{x-y}} = \int_x^\infty dz \rho_t(z), \end{aligned}$$

that is what was to be proven.

APPENDIX D. A LOCALIZATION ARGUMENT

Let us give a proof that for the *copolymer near a selective interface* model, described in § 1.1, the charge ω never belongs to \mathcal{P} (see (1.17) for the definition of \mathcal{P}). More precisely, we are going to show that if $h_\omega = 0$ and $\Sigma \neq 0$ then $\delta^\omega > 1$, that is the periodic copolymer with zero-mean, nontrivial charges is always localized. As a matter of fact this is an immediate consequence of the estimates on the critical line obtained in [5]. However we want to give here an explicit proof, both because it is more direct and because the model studied in [5] is built over the simple random walk measure, corresponding to $p = 1/2$ with the language of § 1, while we consider the case $p < 1/2$.

We recall that, by (A.1), the Perron-Frobenius eigenvalue $Z(Q)$ of an irreducible matrix Q is increasing in the entries of Q . We also point out a result proved by Kingman [15]: if the matrix $Q = Q(t)$ is a function of a real parameter t such that all the entries $Q_{\alpha,\beta}(t)$ are *log-convex* functions of t (that is $t \mapsto \log Q_{\alpha,\beta}(t)$ is convex for all α, β), then also $t \mapsto Z(Q(t))$ is a *log-convex* function of t .

Next we come to the *copolymer near a selective interface* model: with reference to the general Hamiltonian (1.3), we are assuming that $\omega_n^{(0)} = \tilde{\omega}_n^{(0)} = 0$ and $h_\omega = 0$ (where h_ω was defined in (1.8)). In this case the integrated Hamiltonian $\Phi_{\alpha,\beta}(\ell)$, see (1.13), is given by

$$\Phi_{\alpha,\beta}(\ell) = \begin{cases} 0 & \text{if } \ell = 1 \text{ or } \ell \notin \beta - \alpha \\ \log \left[\frac{1}{2} \left(1 + \exp(\Sigma_{\alpha,\beta}) \right) \right] & \text{if } \ell > 1 \text{ and } \ell \in \beta - \alpha \end{cases}.$$

We recall that the law of the first return to zero of the original walk is denoted by $K(\cdot)$, see (1.11), and we introduce the function $q : \mathbb{S} \rightarrow \mathbb{R}^+$ defined by

$$q(\gamma) := \sum_{x \in \mathbb{N}, [x]=\gamma} K(x)$$

(notice that $\sum_\gamma q(\gamma) = 1$). Then the matrix $B_{\alpha,\beta}$ defined by (1.15) becomes

$$B_{\alpha,\beta} = \begin{cases} \frac{1}{2} \left(1 + \exp(\Sigma_{\alpha,\beta}) \right) q(\beta - \alpha) & \text{if } \beta - \alpha \neq [1] \\ K(1) + \frac{1}{2} \left(1 + \exp(\Sigma_{\alpha,\alpha+[1]}) \right) \cdot (q([1]) - K(1)) & \text{if } \beta - \alpha = [1] \end{cases} \quad (\text{D.1})$$

By (1.16), to prove localization we have to show that the Perron-Frobenius eigenvalue of the matrix $(B_{\alpha,\beta})$ is strictly greater than 1, that is $Z(B) > 1$.

Applying the elementary convexity inequality $(1 + \exp(x))/2 \geq \exp(x/2)$ to (D.1) we get

$$B_{\alpha,\beta} \geq \tilde{B}_{\alpha,\beta} := \begin{cases} \exp(\Sigma_{\alpha,\beta}/2) q(\beta - \alpha) & \text{if } \beta - \alpha \neq [1] \\ K(1) + \exp(\Sigma_{\alpha,\alpha+[1]}/2) \cdot (q([1]) - K(1)) & \text{if } \beta - \alpha = [1] \end{cases}. \quad (\text{D.2})$$

By hypothesis $\Sigma_{\alpha_0,\beta_0} \neq 0$ for some α_0, β_0 , therefore the inequality above is strict for $\alpha = \alpha_0, \beta = \beta_0$. We have already observed that the P–F eigenvalue is a strictly increasing function of the entries of the matrix, hence $Z(B) > Z(\tilde{B})$. Therefore it only remains to show that $Z(\tilde{B}) \geq 1$, and the proof will be completed.

Again an elementary convexity inequality applied to the second line of (D.2) yields

$$\tilde{B}_{\alpha,\beta} \geq \hat{B}_{\alpha,\beta} := \exp(c(\beta - \alpha) \Sigma_{\alpha,\beta}/2) \cdot q(\beta - \alpha) \quad (\text{D.3})$$

where

$$c(\gamma) := \begin{cases} 1 & \text{if } \gamma \neq [1] \\ \frac{q([1]) - K(1)}{q([1])} & \text{if } \gamma = [1] \end{cases}.$$

We are going to prove that $Z(\hat{B}) \geq 1$. Observe that setting $v_\alpha := \Sigma_{[0],\alpha}$ we can write

$$\Sigma_{\alpha,\beta} = \Sigma_{[0],\beta} - \Sigma_{[0],\alpha} = v_\beta - v_\alpha.$$

Then we make a similarity transformation via the matrix $L_{\alpha,\beta} := \exp(v_\beta/2) \mathbf{1}_{(\beta=\alpha)}$, getting

$$\begin{aligned} C_{\alpha,\beta} &:= [L \cdot \hat{B} \cdot L^{-1}]_{\alpha,\beta} = \exp\left((c(\beta - \alpha) - 1) \Sigma_{\alpha,\beta}/2\right) \cdot q(\beta - \alpha) \\ &= \exp\left(d \Sigma_{\alpha,\alpha+[1]} \mathbf{1}_{(\beta-\alpha=1)}\right) \cdot q(\beta - \alpha), \end{aligned}$$

where we have introduced the constant $d := -K(1)/(2q([1]))$. Of course $Z(\hat{B}) = Z(C)$. Also notice that by the very definition of $\Sigma_{\alpha,\beta}$ we have $\Sigma_{\alpha,\alpha+[1]} = \omega_{\alpha+[1]}^{(-1)} - \omega_{\alpha+[1]}^{(+1)}$, hence the hypothesis $h_\omega = 0$ yields $\sum_{\alpha \in \mathbb{S}} (\Sigma_{\alpha,\alpha+[1]}) = 0$.

Thus we are finally left with showing that $Z(C) \geq 1$ where $C_{\alpha,\beta}$ is an $\mathbb{S} \times \mathbb{S}$ matrix of the form

$$C_{\alpha,\beta} = \exp(w_\alpha \mathbf{1}_{(\beta-\alpha=1)}) \cdot q(\beta - \alpha) \quad \text{where} \quad \sum_{\alpha} w_\alpha = 0 \quad \sum_{\gamma} q(\gamma) = 1.$$

To this end, we introduce an interpolation matrix

$$C_{\alpha,\beta}(t) := \exp(t \cdot w_\alpha \mathbf{1}_{(\beta-\alpha=1)}) \cdot q(\beta - \alpha),$$

defined for $t \in \mathbb{R}$, and notice that $C(1) = C$. Let us denote by $\eta(t) := Z(C(t))$ the Perron–Frobenius eigenvalue of $C(t)$: as the entries of $C(t)$ are log–convex functions of t , it follows that also $\eta(t)$ is log–convex, therefore in particular convex. Moreover $\eta(0) = 1$ (the matrix $C(0)$ is bistochastic) and using (A.1) one easily checks that $\frac{d}{dt} \eta(t)|_{t=0} = 0$. Since clearly $\eta(t) \geq 0$ for all $t \in \mathbb{R}$, by convexity it follows that indeed $\eta(t) \geq 1$ for all $t \in \mathbb{R}$, and the proof is complete.

REFERENCES

- [1] K. S. Alexander and V. Sidoravicius, *Pinning of polymers and interfaces by random potentials*, preprint (2005). Available on: arXiv.org e-Print archive: math.PR/0501028
- [2] S. Asmussen *Applied Probability and Queues*, Second Edition, Application of Mathematics **51**, Springer-Verlag, New York (2003).
- [3] E. Bolthausen, *On a functional central limit theorem for random walks conditioned to stay positive*, Ann. Probability **4** (1976), no. 3, 480–485.
- [4] E. Bolthausen and F. den Hollander, *Localization transition for a polymer near an interface*, Ann. Probab. **25** (1997), 1334–1366.
- [5] E. Bolthausen and G. Giacomin, *Periodic copolymers at selective interfaces: a large deviations approach*, Ann. Appl. Probab. **15** (2005), 963–983.
- [6] J.-D. Deuschel, G. Giacomin, L. Zambotti, *Scaling limits of equilibrium wetting models in (1+1)-dimension*, Probab. Theory Rel. Fields (2005), published online (December 2004).
- [7] R. A. Doney (1997), *One-sided local large deviation and renewal theorems in the case of infinite mean*, Probab. Theory Rel. Fields **107**, 451–465
- [8] W. Feller, *An introduction to probability theory and its applications*, Vol. II, Second edition, John Wiley & Sons, Inc., New York–London–Sydney, 1971.
- [9] P. J. Fitzsimmons, B. Fristedt and B. Maisonneuve (1985), *Intersections and limits of regenerative sets*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **70**, 157–173.
- [10] S. Galluccio and R. Graber, *Depinning transition of a directed polymer by a periodic potential: a d -dimensional solution*, Phys. Rev. E **53** (1996), R5584–R5587.
- [11] G. Giacomin, *Localization phenomena in random polymer models*, preprint (2004), available on the web page of the author.
- [12] G. Giacomin and F. L. Toninelli, *Estimates on path delocalization for copolymers at selective interfaces*, Probab. Theor. Rel. Fields (2005), published online (May 2005).
- [13] T. Garel, D. A. Huse, S. Leibler and H. Orland, *Localization transition of random chains at interfaces*, Europhys. Lett. **8** (1989), 9–13.
- [14] W.D. Kaigh, *An invariance principle for random walk conditioned by a late return to zero*, Ann. Probability **4** (1976), no. 1, 115–121.
- [15] J. F. C. Kingman *A convexity property of positive matrices*, Quart. J. Math. (1961), Oxford, **12**, 283–284.
- [16] A. Naidedov and S. Nechaev, *Adsorption of a random heteropolymer at a potential well revisited: location of transition point and design of sequences*, J. Phys. A: Math. Gen. **34** (2001), 5625–5634.
- [17] S. Nechaev and Y.-C. Zhang, *Exact solution of the 2D wetting problem in a periodic potential*, Phys. Rev. Lett. **74** (1995), 1815–1818.
- [18] Y. Iozaki and N. Yoshida, *Weakly pinned random walk on the wall: pathwise descriptions of the phase transition*, Stochastic Process. Appl. **96** (2001), 261–284.
- [19] G. Matheron, *Random sets and integral geometry*, John Wiley & Sons (1975)
- [20] H. Minc, *Nonnegative matrices*, Wiley interscience series in discrete mathematics and optimization, John Wiley & Sons, New York, 1988.
- [21] C. Monthus, T. Garel and H. Orland, *Copolymer at a selective interface and two dimensional wetting: a grand canonical approach*, Eur. Phys. J. B **17** (2000), 121–130.
- [22] A. Rechnitzer and E. J. Janse van Rensburg, *Exchange relations, Dyck paths and copolymer adsorption*, Discrete Appl. Math. **140** (2004), 49–71.
- [23] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, New York, Heidelberg (1991).
- [24] J.-U. Sommer and M. Daoud, *Copolymers at selective interfaces*, Europhys. Lett., **32(5)** (1995), 407–412.
- [25] P. J. Upton, (1999), *Exact interface model for wetting in the planar Ising model*, Phys. Rev. E **60**, no. 4, R3475–R3478.
- [26] C. E. Soteris and S. G. Whittington, *The statistical mechanics of random copolymers*, J. Phys. A: Math. Gen. **37** (2004), R279–R325.

UNIVERSITÀ DI MILANO-BICOCCA, DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, EDIFICIO U5,
VIA COZZI 53, 20125 MILANO, ITALY

and

LABORATOIRE DE PROBABILITÉS DE P 6 & 7 (CNRS U.M.R. 7599) AND UNIVERSITÉ PARIS 7 – DENIS
DIDEROT, U.F.R. MATHÉMATIQUES, CASE 7012, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE

E-mail address: f.caravenna@sns.it

URL: <http://www.matapp.unimib.it/~fcaraven/>

LABORATOIRE DE PROBABILITÉS DE P 6 & 7 (CNRS U.M.R. 7599) AND UNIVERSITÉ PARIS 7 – DENIS
DIDEROT, U.F.R. MATHÉMATIQUES, CASE 7012, 2 PLACE JUSSIEU, 75251 PARIS CEDEX 05, FRANCE

E-mail address: giacomini@math.jussieu.fr

URL: <http://www.proba.jussieu.fr/pageperso/giacomini/GBpage.html>

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI 32, 20133
MILANO, ITALY

E-mail address: zambotti@mate.polimi.it