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# Le Cam spacings theorem in dimension two

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## Abstract

The definition of spacings associated to a sequence of random variables is extended to the case of random vectors in  $[0, 1]^2$ . Beirlant & *al.* (1991) give an alternative proof of the Le Cam (1958) theorem concerning asymptotic normality of additive functions of uniform spacings in  $[0, 1]$ . I adapt their technique to the two-dimensional case, leading the way to new directions in the domain of Complete Spatial Randomness (CSR) testing.

## 1 Introduction

Testing the uniformity of real random variables (r.v.) can be done in several ways : using Chi-square tests, tests based on the empirical distribution function (e.d.f.), tests based on spacings ... The latter ones have been extensively studied (Pyke, 1965) and recommended, for example, for the analysis of the local renewal structure of a point process (Deheuvels, 1983a).

In higher dimensions, when dealing with a spatial point pattern  $U \in S \subset \mathbb{R}^d$ , one first wishes to know whether it satisfies the CSR hypothesis : is the spatial process governing  $U$  a homogeneous Poisson process? This question is equivalent to the following : given the number of points in the pattern (also called events), are these points uniformly and independently distributed in  $S$  (Moller & Waagepetersen, 2004) ?

I concentrate here on point patterns distributed in rectangles in  $\mathbb{R}^2$ , which is similar, after linear transformation of the coordinates, to testing the uniformity in  $[0, 1]^2$ .

Most of two-dimensional uniformity tests are either Chi-square tests or distance-based methods (Cressie, 1993). The first ones depend on the number and location of the quadrats (cells in which events are counted), whereas the last ones require numerous simulations. More recently, there has been some interest in e.d.f.-based methods and extensions of the Cramer-Von Mises test (Zimmerman, 1993) and the Kolmogorov-Smirnov test (Justel & *al*, 1997) to the  $[0, 1]^2$  case have been established.

On the other hand, spacings theory, so useful for testing uniformity on  $\mathbb{R}$ , remains almost unworked in higher dimensions even if one may think, as Zimmerman (1993) does, that distances from events to their nearest neighbours can be viewed as two-dimensional analogues of spacings. We shall just mention the results of Deheuvels (1983b) and Janson (1987) concerning the asymptotic distribution of the maximal multidimensional spacing, i.e. the volume of the largest square (or ball) contained in  $[0, 1]^2$  and avoiding every point of the pattern.

A first application of spacings theory to CSR testing would be to test both x- and y-coordinates' uniformity using a spacings-based method. The rejection of either leads to refuse the two-dimensional uniformity hypothesis. But we can never accept it as a bivariate distribution with uniform marginals need not be uniform. This makes necessary to take into account the joint distribution of the x- and y-coordinates.

In this paper, following this idea, I introduce a new notion of two-dimensional spacings which is related to spacings based on x- and y-coordinates. This relationship then allows me to derive the limiting distribution, under the uniformity hypothesis, of a wide family of statistics based on these spacings. This is done by a direct decomposition method similar to the one Beirlant & *al* (1991) used for one-dimensional spacings.

An application of this asymptotic result is developed by Cucala & Thomas-Agnan (2005). Two of these statistics, the variance and the absolute mean deviation of the two-dimensional spacings, are selected and used in practice to test for CSR. A multiple procedure is adopted to generalize the tests to point processes in any domain (not necessarily rectangular). Then the power of these spacings-based tests is compared to the power of existing tests using real and simulated data sets : they appear to be inferior for detecting regularity or clustering but more powerful for detecting certain types of heterogeneity.

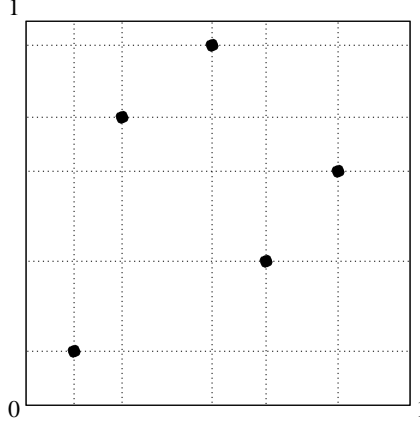


FIG. 1 – Two-dimensional spacings

## 2 Spacings in $[0, 1]^2$

### 2.1 Definition

Let  $U = ((U_1^x, U_1^y), \dots, (U_{n-1}^x, U_{n-1}^y))$  be a point pattern in  $[0, 1]^2$ .  
 Let  $U^x = (U_1^x, \dots, U_{n-1}^x)$  and  $U^y = (U_1^y, \dots, U_{n-1}^y)$ .

$U_{(1)}^x \leq \dots \leq U_{(n-1)}^x$  are the order statistics corresponding to  $U^x$ .

$U_{(1)}^y \leq \dots \leq U_{(n-1)}^y$  are the order statistics corresponding to  $U^y$ .

Set  $U_0^x = U_0^y = U_{(0)}^x = U_{(0)}^y = 0$  and  $U_n^x = U_n^y = U_{(n)}^x = U_{(n)}^y = 1$ .

One may define the spacings related to the pattern

$$\text{the x-spacings } D_i^x = U_{(i)}^x - U_{(i-1)}^x, \quad i = 1, \dots, n,$$

$$\text{the y-spacings } D_j^y = U_{(j)}^y - U_{(j-1)}^y, \quad j = 1, \dots, n.$$

A way to take account of how x- and y-spacings vary jointly is then to define the two-dimensional spacings as the areas  $A_{ij}$  formed by the grid in Figure 1

$$\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, n\}, \quad A_{ij} = D_i^x D_j^y.$$

### 2.2 Uniformity hypothesis

From now on, I will assume the uniformity hypothesis  $H_0$  : the point pattern  $U$  is uniformly distributed in  $[0, 1]^2$ .

Let  $(D_1, \dots, D_n)$  be the spacings corresponding to a  $(n-1)$  uniform sample on  $[0, 1]$ . Then it is easy to see that

$$\begin{cases} \mathcal{L}(D_1^x, \dots, D_n^x) = \mathcal{L}(D_1, \dots, D_n), \\ \mathcal{L}(D_1^y, \dots, D_n^y) = \mathcal{L}(D_1, \dots, D_n), \\ (D_1^x, \dots, D_n^x) \perp\!\!\!\perp (D_1^y, \dots, D_n^y). \end{cases}$$

### 3 Asymptotic normality of additive functions of spacings

#### 3.1 Main result

Many statistics based on one-dimensional spacings are additive functions

$$V_n = \sum_{i=1}^n g(nD_i)$$

for a measurable function  $g$ .

For the two-dimensional case consider

$$V_n^{(2)} = \sum_{i=1}^n \sum_{j=1}^n g(n^2 A_{ij}).$$

The asymptotic normality of  $V_n$  was proved by Beirlant & *al.* (1991) using the following distributional equivalence (Moran, 1947)

$$(nD_1, \dots, nD_n) \sim \left( \frac{E_1}{\bar{E}}, \dots, \frac{E_n}{\bar{E}} \right) \quad (1)$$

where  $(E_1, \dots, E_n)$  are independent exponentially distributed r.v.'s with mean 1

and 
$$\bar{E} = \frac{1}{n} \sum_{i=1}^n E_i.$$

I will use the same technique to prove the asymptotic normality of  $V_n^{(2)}$  under  $H_0$ .

Introduce  $(X_1, \dots, X_n)$  and  $(Y_1, \dots, Y_n)$  two samples of independent exponentially distributed r.v.'s with mean 1. Denote  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  and

$\bar{Y} = \frac{1}{n} \sum_{j=1}^n Y_j$ . Then define the statistic

$$G_n = \sum_{i=1}^n \sum_{j=1}^n g\left(\frac{X_i Y_j}{\bar{X} \bar{Y}}\right).$$

From (1),  $V_n^{(2)}$  and  $G_n$  have same distribution.

From now on I will assume  $g$  satisfies the following, where  $\varphi$  and  $\psi$  are measurable functions

$$g \text{ continuous on } \mathbb{R}^+, \quad (2)$$

$$\mathbb{E}g^2(X_1Y_1) < \infty, \quad (3)$$

$$\begin{aligned} \forall t_0 \in \mathbb{R}^+, \exists \varphi : \mathbb{R} \rightarrow \mathbb{R}, \forall n \in \mathbb{N}, \mathbb{E}\varphi(\bar{X}\bar{Y}) < \infty \\ \text{and } \forall t < t_0, \forall x \in \mathbb{R}^+, |g(tx)| < \varphi(x), \end{aligned} \quad (4)$$

$$\begin{aligned} \forall t_0 \in \mathbb{R}^+, \exists \psi : \mathbb{R} \rightarrow \mathbb{R}, \forall n \in \mathbb{N}, \mathbb{E}\psi(\bar{X}\bar{Y}) < \infty \\ \text{and } \forall t > t_0, \forall x \in \mathbb{R}^+, \left| \frac{g(tx)}{g(t)} \right| < \psi(x). \end{aligned} \quad (5)$$

Denote

$$\begin{aligned} \mu &= \mathbb{E}[g(X_1Y_1)], \\ \eta &= \text{Cov}(g(X_1Y_1), g(X_1Y_2)) = \text{Cov}(g(X_1Y_1), g(X_2Y_1)), \\ c &= \text{Cov}(g(X_1Y_1), X_1) = \text{Cov}(g(X_1Y_1), Y_1). \end{aligned}$$

To justify my decomposition, I make the following argument.

Using the same Taylor-expansion as Proschan & Pyke (1964), it appears that  $n^{-3/2}(G_n - n^2\mu)$  could be asymptotically equivalent in distribution to

$$\begin{aligned} &n^{-3/2} \sum_{i=1}^n \sum_{j=1}^n \{g(X_iY_j) - \mu - (\bar{X} - 1)g'(X_iY_j)(X_iY_j) - (\bar{Y} - 1)g'(X_iY_j)(X_iY_j)\} \\ &= n^{-3/2} \sum_{i=1}^n \sum_{j=1}^n \left\{ g(X_iY_j) - \mu - (X_i - 1 + Y_j - 1) \left( \frac{1}{n^2} \sum_{k=1}^n \sum_{l=1}^n g'(X_kY_l)(X_kY_l) \right) \right\}. \end{aligned}$$

By partial integration it follows that

$$\mathbb{E}g'(X_1Y_1)(X_1Y_1) = c \quad \Rightarrow \quad \frac{1}{n^2} \sum_{k=1}^n \sum_{l=1}^n g'(X_kY_l)(X_kY_l) \xrightarrow[n \rightarrow \infty]{P} c.$$

That is why it seems useful to decompose  $G_n$  as follows

$$\begin{aligned} \frac{1}{n^{3/2}}(G_n - n^2\mu) &= \frac{S_n}{n^{3/2}} + \frac{R_n}{n^{3/2}} \\ \text{where } S_n &= \sum_{i=1}^n \sum_{j=1}^n [g(X_iY_j) - \mu - c(X_i - 1) - c(Y_j - 1)] \\ \text{and } R_n &= \sum_{i=1}^n \sum_{j=1}^n \left[ g\left(\frac{X_i Y_j}{\bar{X} \bar{Y}}\right) - g(X_iY_j) \right] + cn^2(\bar{X} - 1) + cn^2(\bar{Y} - 1). \end{aligned}$$

As  $\frac{S_n}{n^{3/2}}$  is a two-sample U-statistic with mean 0 and limiting variance  $\sigma^2 = 2(\eta - c^2)$ , one has from Van Der Vaart (1998)

$$\frac{S_n}{n^{3/2}} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \sigma^2).$$

I will prove in the next section that

$$\frac{\mathbb{E}(R_n^2)}{n^3} \xrightarrow[n \rightarrow \infty]{} 0,$$

which will yield the following result.

**Theorem 1** *Assume  $g$  satisfies (2), (3), (4) and (5). Then*

$$\frac{1}{n^{3/2}} \{G_n - n^2 \mu\} \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \sigma^2).$$

### 3.2 Behaviour of the remainder term

Using the independence of  $\frac{X_i}{\bar{X}}$  and  $\bar{X}$ , as well as the independence of  $\frac{Y_j}{\bar{Y}}$  and  $\bar{Y}$ , one gets

$$\frac{\mathbb{E}(R_n^2)}{n^3} = T_{1,n} + 2T_{2,n} + T_{3,n}$$

$$\text{where } T_{1,n} = \frac{1}{n} \left[ \mathbb{E} \left\{ g^2 \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) \right\} - 2\mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g(X_1 Y_1) \right\} + \mathbb{E} g^2(X_1 Y_1) \right], \quad (6)$$

$$T_{2,n} = \frac{n-1}{n} \left[ \mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g \left( \frac{X_1 Y_2}{\bar{X} \bar{Y}} \right) \right\} - 2\mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g(X_1 Y_2) \right\} + \mathbb{E} g(X_1 Y_1) g(X_1 Y_2) \right], \quad (7)$$

$$\text{and } T_{3,n} = \frac{(n-1)^2}{n} \left[ \mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g \left( \frac{X_2 Y_2}{\bar{X} \bar{Y}} \right) \right\} - 2\mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g(X_2 Y_2) \right\} + \mu^2 \right] - 2c^2. \quad (8)$$

#### 3.2.1 Preliminary results

The marginal and bivariate densities of the spacings  $D_i$  are given by Pyke (1965) and lead to

$$\mathbb{E} g^2 \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) = \frac{(n-1)^2}{n^2} \int_0^n \int_0^n g^2(xy) \left(1 - \frac{x}{n}\right)^{n-2} \left(1 - \frac{y}{n}\right)^{n-2} dy dx, \quad (9)$$

$$\mathbb{E}\left\{g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g\left(\frac{X_1 Y_2}{\bar{X} \bar{Y}}\right)\right\} = \frac{(n-1)^2(n-2)}{n^3} \int_0^n \int_0^n \int_0^{n-y} g(xy)g(xv) \left(1 - \frac{x}{n}\right)^{n-2} \left(1 - \frac{y+v}{n}\right)^{n-3} dv dy dx, \quad (10)$$

$$\mathbb{E}\left\{g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g\left(\frac{X_2 Y_2}{\bar{X} \bar{Y}}\right)\right\} = \frac{(n-1)^2(n-2)^2}{n^4} \int_0^n \int_0^n \int_0^{n-x} \int_0^{n-y} g(xy)g(uv) \left(1 - \frac{x+u}{n}\right)^{n-3} \left(1 - \frac{y+v}{n}\right)^{n-3} dv du dy dx. \quad (11)$$

Using the independence of  $\frac{X_i}{\bar{X}}$  and  $\bar{X}$ , one finds

$$\begin{aligned} \mathbb{E}\left\{g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g(X_1 Y_1)\right\} &= \mathbb{E}\left[g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)\mathbb{E}\left\{g\left(\frac{X_1 \bar{X}}{\bar{X} \bar{Y}} \frac{Y_1 \bar{Y}}{\bar{Y} \bar{Y}}\right) \middle| \frac{X_1}{\bar{X}}, \frac{Y_1}{\bar{Y}}\right\}\right] \\ &= \frac{(n-1)^2}{n^2} \int_0^n \int_0^n g(xy)\mathbb{E}\{g(x\bar{X}y\bar{Y})\} \\ &\quad \left(1 - \frac{x}{n}\right)^{n-2} \left(1 - \frac{y}{n}\right)^{n-2} dy dx, \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left\{g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g(X_1 Y_2)\right\} &= \mathbb{E}\left[g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)\mathbb{E}\left\{g\left(\frac{X_1 \bar{X}}{\bar{X} \bar{Y}} \frac{Y_2 \bar{Y}}{\bar{Y} \bar{Y}}\right) \middle| \frac{X_1}{\bar{X}}, \frac{Y_1}{\bar{Y}}, \frac{Y_2}{\bar{Y}}\right\}\right] \\ &= \frac{(n-1)^2(n-2)}{n^3} \int_0^n \int_0^n \int_0^{n-y} g(xy)\mathbb{E}\{g(x\bar{X}v\bar{Y})\} \\ &\quad \left(1 - \frac{x}{n}\right)^{n-2} \left(1 - \frac{y+v}{n}\right)^{n-3} dv dy dx, \quad (12) \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left\{g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g(X_2 Y_2)\right\} &= \mathbb{E}\left[g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)\mathbb{E}\left\{g\left(\frac{X_2 \bar{X}}{\bar{X} \bar{Y}} \frac{Y_2 \bar{Y}}{\bar{Y} \bar{Y}}\right) \middle| \frac{X_1}{\bar{X}}, \frac{X_2}{\bar{X}}, \frac{Y_1}{\bar{Y}}, \frac{Y_2}{\bar{Y}}\right\}\right] \\ &= \frac{(n-1)^2(n-2)^2}{n^4} \int_0^n \int_0^n \int_0^{n-x} \int_0^{n-y} g(xy)\mathbb{E}\{g(u\bar{X}v\bar{Y})\} \\ &\quad \left(1 - \frac{x+u}{n}\right)^{n-3} \left(1 - \frac{y+v}{n}\right)^{n-3} dv du dy dx. \quad (13) \end{aligned}$$

The following lemma is also needed.

**Lemma 1** *If  $g$  continuous on  $\mathbb{R}^{+*}$  and  $\mathbb{E}g^2(X_1 Y_1) < \infty$ , then  $\forall t \in [0, +\infty[$*

$$\lim_{n \rightarrow \infty} \mathbb{E}g(t\bar{X}\bar{Y}) = g(t).$$



**Proof :**

Denote by  $f_n(u, t) = \frac{n^n}{\Gamma(n)} \frac{u^{n-1}}{t^n} e^{-nu/t}$  the common density of  $t\bar{X}$  and  $t\bar{Y}$ .

$$\begin{aligned} \mathbb{E}g(t\bar{X}\bar{Y}) &= \mathbb{E}g(\sqrt{t}\bar{X}\sqrt{t}\bar{Y}) = \int_0^\infty \int_0^\infty g(uv) f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) du dv \\ &= \int \int_D g(uv) f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) du dv + \int \int_{\bar{D}} g(uv) f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) du dv \end{aligned}$$

where  $D = \{(u, v) \in \mathbb{R}^{+2}; \sqrt{t}/2 < u < 3\sqrt{t}/2; \sqrt{t}/2 < v < 3\sqrt{t}/2\}$ .

It is easy to see that :

$$\sqrt{t}\bar{X} \xrightarrow[n \rightarrow \infty]{P} \sqrt{t} \text{ and } \sqrt{t}\bar{Y} \xrightarrow[n \rightarrow \infty]{P} \sqrt{t}.$$

Introduce the function  $\varphi : D \rightarrow \mathbb{R}$   
 $(x, y) \rightarrow \varphi(x, y) = g(xy).$

From (2) the function  $\varphi$  is continuous and bounded on  $D$ , so by the Helly-Bray theorem one concludes

$$\int \int_D g(uv) f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) du dv = \mathbb{E} \varphi(\sqrt{t}\bar{X}, \sqrt{t}\bar{Y}) \xrightarrow[n \rightarrow \infty]{} \varphi(\sqrt{t}, \sqrt{t}) = g(t).$$

It remains to prove that  $\int \int_{\bar{D}} g(uv) f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) du dv \xrightarrow[n \rightarrow \infty]{} 0.$

$K$  and  $\tilde{K}$  are two constants. From Beirlant & al (1991), one has  $\forall n \geq 1$

$$f_n(u, \sqrt{t}) \leq K t^{-1/2} n^{1/2} \left( \frac{u}{\sqrt{t}} e^{1-u/\sqrt{t}} \right)^{n-1}.$$

So :  $(u, v) \in \bar{D} \Rightarrow f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) \xrightarrow[n \rightarrow \infty]{} 0.$

Moreover :  $n \geq n_0 \geq 5 \Rightarrow f_n(u, \sqrt{t}) f_n(v, \sqrt{t}) \leq \tilde{K} f_{n_0}(u, \sqrt{t}) f_{n_0}(v, \sqrt{t}).$

And taking  $m \geq 16$ , we get,  $\forall u \in [0, \sqrt{t}/2] \cup [3\sqrt{t}/2, +\infty]$

$$\begin{aligned} \left( \frac{u}{\sqrt{t}} e^{1-u/\sqrt{t}} \right)^m &< e^{-u/\sqrt{t}} \\ \Rightarrow f_{n_0}(u, \sqrt{t}) &< K^{1/2} t^{-1/2} n_0^{1/2} e^{-u} \quad \text{if we take } n_0 > 16\sqrt{t} + 1. \end{aligned}$$

As, from (3),  $(u, v) \rightarrow g(uv) e^{-u} e^{-v} \in \mathcal{L}^1(\mathbb{R}^2)$ , one gets

$$\int \int_{\bar{D}} g(uv) f_{n_0}(u, \sqrt{t}) f_{n_0}(v, \sqrt{t}) du dv < \infty.$$

Lebesgue's dominated-convergence theorem leads to the conclusion. ■

### 3.2.2 Behaviour of $T_{1,n}$

$$\begin{aligned} & \forall (x, y) \in [0, n]^2, \\ & |g^2(xy)(1-x/n)^{n-2}(1-y/n)^{n-2}| \leq g^2(xy)e^4e^{-x}e^{-y} \in \mathcal{L}^1([0, n]^2) \text{ from (3)}. \end{aligned}$$

So applying Lebesgue's dominated-convergence theorem to (9) leads to

$$\lim_{n \rightarrow \infty} \mathbb{E}g^2\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right) = \int_0^\infty \int_0^\infty g^2(xy)e^{-x}e^{-y}dy dx = \mathbb{E}g^2(X_1 Y_1). \quad (14)$$

By Cauchy-Schwarz inequality one gets

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}\left\{\left|g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g(X_1 Y_1)\right|\right\} & \leq \{\mathbb{E}g^2(X_1 Y_1)\}^{1/2} \lim_{n \rightarrow \infty} \left\{\mathbb{E}g^2\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)\right\}^{1/2} \\ & = \mathbb{E}g^2(X_1 Y_1) \text{ from (14)} \\ & < \infty. \end{aligned} \quad (15)$$

(6), (14) and (15) lead to

$$\lim_{n \rightarrow \infty} T_{1,n} = 0.$$

### 3.2.3 Behaviour of $T_{2,n}$

$$\begin{aligned} & \forall (x, y) \in [0, n]^2, \forall v \in [0, n - y], \\ & |g(xy)g(xv)(1-x/n)^{n-2}(1-(y+v)/n)^{n-3}| \leq |g(xy)g(xv)e^5e^{-x}e^{-y}e^{-v}| \in \\ & \mathcal{L}^1([0, n]^2 \times [0, n - y]) \text{ from (3)}. \end{aligned}$$

So applying Lebesgue's dominated-convergence theorem to (10) leads to

$$\lim_{n \rightarrow \infty} \mathbb{E}g\left(\frac{X_1 Y_1}{\bar{X} \bar{Y}}\right)g\left(\frac{X_1 Y_2}{\bar{X} \bar{Y}}\right) = \mathbb{E}g(X_1 Y_1)g(X_1 Y_2). \quad (16)$$

Introduce the function  $h_n : \mathbb{R}^{+*} \rightarrow \mathbb{R}$

$$t \rightarrow h_n(t) = \mathbb{E}g(t\bar{X}\bar{Y}).$$

Lemma 1 gives

$$g(xy)h_n(xv)(1-x/n)^{n-2}(1-(y+v)/n)^{n-3} \xrightarrow[n \rightarrow \infty]{} g(xy)g(xv)e^{-x}e^{-y}e^{-v}.$$

Denote  $t_1 \in \mathbb{R}^{+*}$ . From (5), one gets

$$\exists \psi : \mathbb{R} \rightarrow \mathbb{R}, \forall t > t_1, \forall x \in \mathbb{R}^+, \left|\frac{g(txy)}{g(t)}\right| < \psi(x)$$

$$\begin{aligned} \Rightarrow \left|\frac{h_n(t)}{g(t)}\right| & = \left|\int_0^\infty \int_0^\infty \frac{g(txy)}{g(t)} f_n(x, 1) f_n(y, 1) dy dx\right| \\ & < \int_0^\infty \int_0^\infty \psi(xy) f_n(x, 1) f_n(y, 1) dy dx = \mathbb{E}\psi(\bar{X}\bar{Y}) = \Psi(n). \end{aligned}$$

So,  $\forall (x, v) \in \mathbb{R}^{+2}$ ,  $xv > t_1 \Rightarrow |g(xy)h_n(xv)(1-x/n)^{n-2}(1-(y+v)/n)^{n-3}| < |g(xy)g(xv)\Psi(n)e^5e^{-x}e^{-y}e^{-v}| \in \mathcal{L}^1([0, n]^2 \times [0, n-y])$ .

From (4), one gets

$$\begin{aligned} & \exists \varphi : \mathbb{R} \rightarrow \mathbb{R}, \forall t < t_1, \forall x \in \mathbb{R}^+, |g(tx)| < \varphi(x) \\ \Rightarrow & \forall t < t_1, |h_n(t)| < \mathbb{E}\varphi(\bar{X}\bar{Y}) = \Phi(n). \end{aligned}$$

So,  $\forall (x, v) \in \mathbb{R}^{+2}$ ,  $xv < t_1 \Rightarrow |g(xy)h_n(xv)(1-x/n)^{n-2}(1-(y+v)/n)^{n-3}| < |g(xy)\Phi(n)e^5e^{-x}e^{-y}e^{-v}| \in \mathcal{L}^1([0, n]^2 \times [0, n-y])$ .

So applying Lebesgue's dominated-convergence theorem to (12) leads to

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g(X_1 Y_2) \right\} = \mathbb{E} g(X_1 Y_1) g(X_1 Y_2). \quad (17)$$

(7), (16) and (17) lead to

$$\lim_{n \rightarrow \infty} T_{2,n} = 0.$$

### 3.2.4 Behaviour of $T_{3,n}$

From (11), one gets

$$\begin{aligned} & \frac{(n-1)^2}{n} \mathbb{E} \left\{ g \left( \frac{X_1 Y_1}{\bar{X} \bar{Y}} \right) g \left( \frac{X_2 Y_2}{\bar{X} \bar{Y}} \right) \right\} \\ &= \frac{(n-1)^4(n-2)^2}{n^5} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g(xy)g(uv) \left( 1 + \frac{3(x+u) - (x+u)^2/2}{n} \right) \\ & \quad \left( 1 + \frac{3(y+v) - (y+v)^2/2}{n} \right) e^{-x}e^{-y}e^{-u}e^{-v} dv du dy dx + I_n \\ &= \frac{(n-1)^4(n-2)^2}{n^5} \mu^2 + 12\mu \frac{(n-1)^4(n-2)^2}{n^6} \mathbb{E}[X_1 g(X_1 Y_1)] + \mathcal{O}(n^{-1}) \quad (18) \\ & - 2\mu \frac{(n-1)^4(n-2)^2}{n^6} \mathbb{E}[X_1^2 g(X_1 Y_1)] - 2 \frac{(n-1)^4(n-2)^2}{n^6} \mathbb{E}^2[X_1 g(X_1 Y_1)] + I_n \end{aligned}$$

where  $I_n =$

$$\begin{aligned} & \frac{(n-1)^4(n-2)^2}{n^5} \left\{ \int_0^n \int_0^n \int_0^{n-x} \int_0^{n-y} g(xy)g(uv) \left( 1 - \frac{x+u}{n} \right)^{n-3} \left( 1 - \frac{y+v}{n} \right)^{n-3} \right. \\ & dv du dy dx - \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty g(xy)g(uv) \left( 1 + \frac{3(x+u) - (x+u)^2/2}{n} \right) \\ & \left. \left( 1 + \frac{3(y+v) - (y+v)^2/2}{n} \right) e^{-x}e^{-y}e^{-u}e^{-v} dv du dy dx \right\}. \end{aligned}$$

From (13), one gets

$$\begin{aligned}
U_n &= \frac{(n-1)^2}{n} \mathbb{E} \left\{ g \left( \frac{X_1}{\bar{X}} \frac{Y_1}{\bar{Y}} \right) g(X_2 Y_2) \right\} \\
&= \frac{(n-1)^4 (n-2)^2}{n^5} \int_0^n \int_0^n \int_0^{n-x} \int_0^{n-y} \left( \int_0^\infty \int_0^\infty g(ab) f_n(a, u) f_n(b, v) da db \right) \\
&\quad g(xy) \left(1 - \frac{x+u}{n}\right)^{n-3} \left(1 - \frac{y+v}{n}\right)^{n-3} dv du dy dx \\
&= \frac{(n-1)^4 (n-2)^2}{n^5} \int_0^n \int_0^n \int_0^\infty \int_0^\infty g(xy) \left\{ \int_0^{n-x} f_n(a, u) \left(1 - \frac{x+u}{n}\right)^{n-3} du \right\} \\
&\quad g(ab) \left\{ \int_0^{n-y} f_n(b, v) \left(1 - \frac{y+v}{n}\right)^{n-3} dv \right\} db da dy dx.
\end{aligned}$$

Now the first integral in braces is equal to

$$\begin{aligned}
&\frac{n}{n-1} e^{-a} \left(1 - \frac{x}{n}\right)^{n-3} e^{-ax/(n-x)} \left(1 + \frac{na}{(n-x)(n-2)}\right) \\
\Rightarrow U_n &= \frac{(n-1)^2 (n-2)^2}{n^3} \int_0^n \int_0^n \int_0^\infty \int_0^\infty g(xy) g(ab) e^{-a} \left(1 - \frac{x}{n}\right)^{n-3} \\
&\quad e^{-ax/(n-x)} \left(1 + \frac{na}{(n-x)(n-2)}\right) e^{-b} \left(1 - \frac{y}{n}\right)^{n-3} e^{-ay/(n-y)} \\
&\quad \left(1 + \frac{nb}{(n-y)(n-2)}\right) db da dy dx \\
&= \frac{(n-1)^2 (n-2)^2}{n^3} \int_0^n \int_0^n \int_0^\infty \int_0^\infty g(xy) g(ab) e^{-a} \left(1 - \frac{x}{n}\right)^{n-3} \\
&\quad e^{-b} \left(1 - \frac{y}{n}\right)^{n-3} \left(1 - \frac{ax}{n-x} + \frac{na}{(n-x)(n-2)}\right) \\
&\quad \left(1 - \frac{by}{n-y} + \frac{nb}{(n-y)(n-2)}\right) db da dy dx + J_n
\end{aligned}$$

where  $J_n =$

$$\begin{aligned}
&\frac{(n-1)^2 (n-2)^2}{n^3} \int_0^n \int_0^n \int_0^\infty \int_0^\infty g(xy) g(ab) e^{-a} \left(1 - \frac{x}{n}\right)^{n-3} e^{-b} \left(1 - \frac{y}{n}\right)^{n-3} \\
&\quad \left\{ e^{-ax/(n-x)} e^{-by/(n-y)} \left(1 + \frac{na}{(n-x)(n-2)}\right) \left(1 + \frac{nb}{(n-y)(n-2)}\right) \right. \\
&\quad \left. - \left(1 - \frac{ax}{n-x} + \frac{na}{(n-x)(n-2)}\right) \left(1 - \frac{by}{n-y} + \frac{nb}{(n-y)(n-2)}\right) \right\} db da dy dx.
\end{aligned}$$

Substituting  $\frac{ax}{n} \left(1 - \frac{x}{n}\right)^{-1}$  to  $\frac{ax}{n-x}$ , one gets that  $U_n$  equals

$$\begin{aligned}
& \frac{(n-1)^2(n-2)^2}{n^3} \mu^2 + \frac{6(n-1)^2(n-2)^2}{n^4} \mu \mathbb{E}[X_1 g(X_1 Y_1)] \\
& - \frac{2(n-1)^2(n-2)^2}{n^4} \mathbb{E}[X_1 g(X_1 Y_1)] \int_0^n \int_0^n xg(xy) \left(1 - \frac{x}{n}\right)^{n-4} \left(1 - \frac{y}{n}\right)^{n-3} dy dx \\
& + \frac{2(n-1)^2(n-2)^2}{n^3} \mathbb{E}[X_1 g(X_1 Y_1)] \int_0^n \int_0^n g(xy) \left(1 - \frac{x}{n}\right)^{n-4} \left(1 - \frac{y}{n}\right)^{n-3} dy dx \\
& - \frac{(n-1)^2(n-2)^2}{n^4} \mu \mathbb{E}[X_1^2 g(X_1 Y_1)] + J_n + K_n + \mathcal{O}(n^{-1}) \tag{19}
\end{aligned}$$

where  $K_n =$

$$\begin{aligned}
& \frac{(n-1)^2(n-2)^2}{n^3} \mu \left[ \int_0^n \int_0^n g(xy) \left(1 - \frac{x}{n}\right)^{n-3} \left(1 - \frac{y}{n}\right)^{n-3} dy dx \right. \\
& \left. - \int_0^\infty \int_0^\infty g(xy) e^{-x} e^{-y} \left(1 + \frac{3x - x^2/2}{n}\right) \left(1 + \frac{3y - y^2/2}{n}\right) dy dx \right].
\end{aligned}$$

So, from (8), (18) et (19) we get

$$T_{3,n} = A_n + \mathcal{O}(n^{-1}) + I_n + J_n + K_n - 2c^2 \tag{20}$$

where  $A_n = (n-8)\mu^2 + 12\mu \mathbb{E}[X_1 g(X_1 Y_1)] - 2\mu \mathbb{E}[X_1^2 g(X_1 Y_1)] - 2\mathbb{E}^2[X_1 g(X_1 Y_1)]$   
 $- 2(n-6)\mu^2 - 12\mu \mathbb{E}[X_1 g(X_1 Y_1)] + 2\mu \mathbb{E}[X_1^2 g(X_1 Y_1)] + (n-2)\mu^2$   
 $+ 4\mathbb{E}[X_1 g(X_1 Y_1)] \int_0^n \int_0^n xg(xy) \left(1 - \frac{x}{n}\right)^{n-4} \left(1 - \frac{y}{n}\right)^{n-3} dy dx$   
 $- 4\mathbb{E}[X_1 g(X_1 Y_1)] \int_0^n \int_0^n g(xy) \left(1 - \frac{x}{n}\right)^{n-4} \left(1 - \frac{y}{n}\right)^{n-3} dy dx + \mathcal{O}(n^{-1})$   
 $\Rightarrow A_n \xrightarrow[n \rightarrow \infty]{} 2\mu^2 - 4\mu \mathbb{E}[X_1 g(X_1 Y_1)] + 2\mathbb{E}^2[X_1 g(X_1 Y_1)]$   
 $\Rightarrow A_n \xrightarrow[n \rightarrow \infty]{} 2c^2.$  \tag{21}

It now suffices to show that  $I_n + J_n + K_n = o(1)$ .

A Taylor-expansion leads to

$$\forall x \in [1, \sqrt{n}] : \left(1 - \frac{x}{n}\right)^{n-3} = e^{-x} \left[1 + \frac{3x - x^2/2}{n} + \mathcal{O}(n^{-2}(x^2 + x^4))\right]. \tag{22}$$

Denote  $b_n = 4 \log n$ . Choosing  $n$  large enough gives

$$\forall x \in [b_n, n], \left(1 - \frac{x}{n}\right)^{n-3} < e^{-x}. \tag{23}$$

By Cauchy-Schwarz inequality one gets

$$\begin{aligned}
\frac{n^3}{(n-1)^2(n-2)^2\mu} |K_n| &= \left| \int_0^\infty \int_0^\infty g(xy) \left[ \left\{ \left(1 - \frac{x}{n}\right)^+ \left(1 - \frac{y}{n}\right)^+ \right\}^{n-3} \right. \right. \\
&\quad \left. \left. - e^{-x}e^{-y} \left(1 + \frac{3x - x^2/2}{n}\right) \left(1 + \frac{3y - y^2/2}{n}\right) \right] dy dx \right| \\
&\leq [\mathbb{E}g^2(X_1Y_1)]^{1/2} \left[ \int_0^\infty \int_0^\infty \left[ \left\{ \left(1 - \frac{x}{n}\right)^+ \left(1 - \frac{y}{n}\right)^+ \right\}^{n-3} \right. \right. \\
&\quad \left. \left. - e^{-x}e^{-y} \left(1 + \frac{3x - x^2/2}{n}\right) \left(1 + \frac{3y - y^2/2}{n}\right) \right]^2 e^x e^y dy dx \right]^{1/2}
\end{aligned}$$

where  $x^+ = x$  if  $x > 0$ , 0 elsewhere.

Denote  $E_n$  as the double integral

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left[ \left\{ \left(1 - \frac{x}{n}\right)^+ \left(1 - \frac{y}{n}\right)^+ \right\}^{n-3} - e^{-x}e^{-y} \right. \\
&\quad \left. \left(1 + \frac{3x - x^2/2}{n}\right) \left(1 + \frac{3y - y^2/2}{n}\right) \right]^2 e^x e^y dy dx.
\end{aligned}$$

Using (22) and (23) and following the technique of Does & Klaassen (1984), one can prove :  $E_n = \mathcal{O}(n^{-4})$

$$\Rightarrow K_n = \mathcal{O}(n^{-1}). \quad (24)$$

The same arguments are used for  $I_n$

$$\Rightarrow I_n = \mathcal{O}(n^{-1}). \quad (25)$$

Using the inequalities  $1 - z \geq e^{-z} \geq 1 - z + z^2/2$ ,  $z \geq 0$ , the expression in braces in the definition of  $J_n$  is bounded below by

$$\begin{aligned}
&-\frac{a^2nx}{(n-x)^2(n-2)} - \frac{n^2a^2bx}{(n-2)^2(n-x)^2(n-y)} \\
&- \frac{b^2ny}{(n-y)^2(n-2)} - \frac{n^2b^2ay}{(n-2)^2(n-y)^2(n-x)}
\end{aligned}$$

and bounded above by

$$\begin{aligned}
& \frac{a^2x^2}{2(n-x)^2} + \frac{na^2bx^2}{2(n-2)(n-x)^2(n-y)} \\
+ & \frac{a^2b^2x^2y^2}{4(n-x)^2(n-y)^2} + \frac{na^2b^3x^2y^2}{4(n-2)(n-x)^2(n-y)^3} \\
+ & \frac{na^3x^2}{2(n-2)(n-x)^3} + \frac{n^2a^3bx^2}{2(n-2)^2(n-x)^3(n-y)} \\
+ & \frac{na^3b^2x^2y^2}{4(n-2)(n-x)^3(n-y)^2} + \frac{n^2a^3b^3x^2y^2}{4(n-2)^2(n-x)^3(n-y)^3} \\
+ & \frac{b^2y^2}{2(n-y)^2} + \frac{nb^2ay^2}{2(n-2)(n-y)^2(n-x)} \\
+ & \frac{b^2a^2y^2x^2}{4(n-y)^2(n-x)^2} + \frac{nb^2a^3y^2x^2}{4(n-2)(n-y)^2(n-x)^3} \\
+ & \frac{nb^3y^2}{2(n-2)(n-y)^3} + \frac{n^2b^3ay^2}{2(n-2)^2(n-y)^3(n-x)} \\
+ & \frac{nb^3a^2y^2x^2}{4(n-2)(n-y)^3(n-x)^2} + \frac{n^2b^3a^3y^2x^2}{4(n-2)^2(n-y)^3(n-x)^3}.
\end{aligned}$$

Hence one gets

$$\begin{aligned}
\frac{n^3}{(n-1)^2(n-2)^2}|J_n| & \leq \frac{1}{2n^2} \int_0^n \int_0^n \int_0^\infty \int_0^\infty |g(xy)g(ab)| a^2x^2e^{-a}e^{-b} \\
& \quad \left(1 - \frac{x}{n}\right)^{n-5} \left(1 - \frac{y}{n}\right)^{n-3} db da dy dx \\
& + \dots \\
& + \frac{1}{n(n-2)^2} \int_0^n \int_0^n \int_0^\infty \int_0^\infty |g(xy)g(ab)| ab^2ye^{-a}e^{-b} \\
& \quad \left(1 - \frac{x}{n}\right)^{n-4} \left(1 - \frac{y}{n}\right)^{n-5} db da dy dx \\
& \Rightarrow J_n = \mathcal{O}(n^{-1}). \tag{26}
\end{aligned}$$

(20), (21), (24), (25) and (26) lead to

$$\lim_{n \rightarrow \infty} T_{3,n} = 0.$$

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