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A separable L-embedded Banach space has property (X) and is therefore the unique predual of its dual.

H. Pfitzner

Abstract

In this note the following is proved. Separable L-embedded spaces - that is separable Banach spaces which are complemented in their biduals such that the norm between the two complementary subspaces is additive - have property (X) which, by a result of Godefroy and Talagrand, entails uniqueness of the space as a predual.

We say that a Banach space X is the unique predual of its dual (more precisely the unique isometric predual of its dual) in case it is isometric to any Banach space whose dual is isometric to the dual of X . (We say that two Banach spaces Y and Z are isomorphic if there is a bounded linear bijective operator $T : Y \rightarrow Z$ with bounded inverse T^{-1} ; if moreover $\|T(y)\| = \|y\|$ for all $y \in Y$ we say that Y and Z are isometric.) In general a Banach space need not be the unique predual of its dual, for example c and c_0 are not isometric Banach spaces although their duals are.

As shown by Grothendieck [10, Rem. 4] in 1955, L^1 -spaces are unique preduals of their duals. Using essentially a result of Dixmier [5] from 1953, Sakai [19, Cor. 1.13.3] observed that more generally preduals of von Neumann algebras are unique, and Barton and Timoney [2] and Horn [13] generalized this to preduals of JBW^* -triples. Ando [1] stated the uniqueness as a predual for the quotient L^1/H_0^1 .

As Banach spaces these examples have in common to be L-summands in their biduals or, for short, to be L-embedded. By definition a Banach space X is L-embedded if there is a projection P on its bidual X^{**} with range X such that $\|Px^{**}\| + \|x^{**} - Px^{**}\| = \|x^{**}\|$ for all $x^{**} \in X^{**}$.

The standard reference for L-embedded spaces is [11], for a survey on unique preduals we refer to [8], for general Banach space theory to [14], [15], or [4].

If not stated otherwise a sequence (z_j) (and similarly a series $\sum z_j$) is indexed by $\mathbb{N} = \{1, 2, \dots\}$; we write $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Recall that a series $\sum z_j$ in a Banach space Z is called weakly unconditionally Cauchy (wuC for short) if $\sum |z^*(z_j)|$ converges for each $z^* \in Z^*$ or, equivalently, if there is a number M such that $\|\sum_{j=1}^n \alpha_j z_j\| \leq M \max_{1 \leq j \leq n} |\alpha_j|$ for all $n \in \mathbb{N}$ and all scalars α_j . It is well known by a result of Bessaga and Pełczyński that a Banach space contains a subspace isomorphic to c_0 if and only if it contains a wuC-series $\sum z_j$ such that $\inf \|z_j\| > 0$. In case $\sum x_j^*$ is a wuC-series in a dual Banach space X^* we denote the w^* -limit (that is the limit in the $\sigma(X^*, X)$ -topology) of the sequence $(\sum_{j=1}^n x_j^*)$ by $\sum^* x_j^*$.

In their study of unique preduals Godefroy and Talagrand [9] defined

Definition 1 (Property (X)) *A Banach space X has property (X) if for each $x^{**} \in X^{**} \setminus X$ there exists a wuC-series $\sum x_j^*$ in X^* such that*

$$\sum x^{**}(x_j^*) \neq x^{**}\left(\sum^* x_j^*\right). \quad (1)$$

They proved

Theorem 2 (Godefroy, Talagrand) *A Banach space X with property (X) is the unique predual of its dual.*

*Moreover, every $x^{**} \in X^{**}$ which is strongly Baire measurable on (X^*, w^*) (for the definition see [8, Th. V.3]) - in particular, every $x^{**} \in X^{**}$ which is Borel on (X^*, w^*) - belongs to X .*

Up to now it has been known that a class of L-embedded spaces, namely the duals of M-embedded spaces (see [11] for the definition), have property (X) [11, p. 148]. Furthermore it has been known ([17], [11, Th. VI.2.7]) that L-embedded spaces have Pełczyński's property (V*); the latter one is similar to and implied by property (X) but is, by an example of Talagrand [20], strictly weaker than (X).

In view of all this it was natural to ask whether L-embedded spaces are unique preduals and have property (X) (see [11, Problem page 123]). At least in the separable case the answer is yes.

Theorem 3 *Separable L-embedded Banach spaces have property (X).*

Proof: Let X be L-embedded and P be the corresponding projection on X^{**} with range X ; we put $Q = \text{id}_{X^{**}} - P$. Denoting the range of Q by X_s we have the decomposition $X^{**} = X \oplus_1 X_s$. Let the sequence (x_n) be dense in X . Let $x^{**} \in X^{**} \setminus X$. Let $\eta = \|x_s\|$ where $x^{**} = x + x_s$, $x \in X$, $x_s \in X_s$. We have that $\eta > 0$ because $x^{**} \notin X$.

Let $1 > \varepsilon > 0$. By the Bishop-Phelps theorem [3], [12] there are $x^* \in X^*$ and $\tilde{x}^{**} \in X^{**}$ such that $\|x^*\| = 1$ and $\|\tilde{x}^{**} - x^{**}\| < \varepsilon\eta/3$ and such that \tilde{x}^{**} attains its norm on x^* that is $\tilde{x}^{**}(x^*) = \|\tilde{x}^{**}\|$. For the decomposition $\tilde{x}^{**} = \tilde{x} + \tilde{x}_s$ we have

$$\tilde{x}_s(x^*) = \|\tilde{x}_s\| \quad \text{and} \quad \|\tilde{x}_s - x_s\| < \varepsilon\eta/3$$

because $\|\tilde{x}_s\| \geq |\tilde{x}_s(x^*)| = |\tilde{x}^{**}(x^*) - x^*(\tilde{x})| \geq \|\tilde{x}^{**}\| - \|\tilde{x}\| = \|\tilde{x}_s\|$ and $\varepsilon\eta/3 > \|\tilde{x}^{**} - x^{**}\| = \|\tilde{x} - x + \tilde{x}_s - x_s\| = \|\tilde{x} - x\| + \|\tilde{x}_s - x_s\|$.

Choose a sequence (ε_j) of strictly positive numbers such that $\prod_{j=1}^{\infty} (1 + \varepsilon_j) < 1 + \varepsilon$ and $\prod_{j=1}^{\infty} (1 - \varepsilon_j) > 1 - \varepsilon$.

By induction over \mathbb{N}_0 we construct two sequences $(x_n^*)_{n \in \mathbb{N}_0}$ and $(y_n^*)_{n \in \mathbb{N}_0}$ in X^* (of which the first members x_0^* and y_0^* are auxiliary values used only for the induction) such that, for all (real or complex) scalars α_j , the following holds:

$$x_0^* = 0, \quad \|y_0^*\| = 1, \tag{2}$$

$$y_n^* = x^* - \sum_{j=0}^n x_j^* \tag{3}$$

$$\begin{aligned} \left(\prod_{j=1}^n (1 - \varepsilon_j) \right) \max_{0 \leq j \leq n} |\alpha_j| &\leq \left\| \alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^* \right\| \\ &\leq \left(\prod_{j=1}^n (1 + \varepsilon_j) \right) \max_{0 \leq j \leq n} |\alpha_j|, \quad \text{if } n \geq 1, \end{aligned} \tag{4}$$

$$\tilde{x}_s(x_j^*) = 0 \quad \text{if } 0 \leq j \leq n, \quad (5)$$

$$x_s(x_j^*) = 0 \quad \text{if } 0 \leq j \leq n, \quad (6)$$

$$y_n^*(x_k) = 0 \quad \text{if } 1 \leq k \leq n. \quad (7)$$

For $n = 0$ we set $x_0^* = 0$ and $y_0^* = x^*$.

We notice that the restriction of P^* to X^* is an isometric isomorphism from X^* onto X_s^\perp , that Q is a contractive projection and that $X^{***} = X_s^\perp \oplus_\infty X^\perp$.

For the induction step $n \mapsto n + 1$ suppose now that x_0^*, \dots, x_n^* and y_0^*, \dots, y_n^* have been constructed. Put

$$E = \text{lin}(\{x^*, x_0^*, \dots, x_n^*, y_n^*, P^*x_0^*, \dots, P^*x_n^*, P^*y_n^*\}) \subset X^{***},$$

$$F = \text{lin}(\{x_1, \dots, x_{n+1}, x_s, \tilde{x}_s\}) \subset X^{**}.$$

Note that $Q^*x_j^*, Q^*y_n^* \in E$ for $0 \leq j \leq n$. By the principle of local reflexivity there is an operator $R : E \rightarrow X^*$ such that

$$(1 - \varepsilon_{n+1})\|e^{***}\| \leq \|Re^{***}\| \leq (1 + \varepsilon_{n+1})\|e^{***}\|, \quad (8)$$

$$f^{**}(Re^{***}) = e^{***}(f^{**}), \quad (9)$$

$$R|_{E \cap X^*} = \text{id}_{E \cap X^*} \quad (10)$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define

$$x_{n+1}^* = RP^*y_n^* \quad \text{and} \quad y_{n+1}^* = RQ^*y_n^*.$$

First we notice that (3, $n + 1$) holds because

$$x^* - \sum_{j=0}^{n+1} x_j^* \stackrel{(3)}{=} y_n^* - x_{n+1}^* = R(y_n^* - P^*y_n^*) = RQ^*y_n^* = y_{n+1}^*.$$

In the following we use the convention $\sum_{j=1}^0(\dots) = 0$. Then we have that

$$\alpha_0 y_{n+1}^* + \sum_{j=1}^{n+1} \alpha_j x_j^* = R\left(Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) + P^*(\alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^*)\right).$$

The second inequality of (4, $n + 1$) can be seen as follows:

$$\begin{aligned} \left\| \alpha_0 y_{n+1}^* + \sum_{j=1}^{n+1} \alpha_j x_j^* \right\| &\stackrel{(8)}{\leq} (1 + \varepsilon_{n+1}) \left\| Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) + P^*(\alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right\| \\ &= (1 + \varepsilon_{n+1}) \max \left\{ \left\| Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right\|, \left\| P^*(\alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right\| \right\} \\ &\leq (1 + \varepsilon_{n+1}) \max \left\{ \left\| \alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^* \right\|, \left\| \alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^* \right\| \right\} \\ &\leq \left(\prod_{j=1}^{n+1} (1 + \varepsilon_j) \right) \max \left\{ \max_{0 \leq j \leq n} |\alpha_j|, \max_{1 \leq j \leq n+1} |\alpha_j| \right\} \\ &= \left(\prod_{j=1}^{n+1} (1 + \varepsilon_j) \right) \max_{0 \leq j \leq n+1} |\alpha_j| \end{aligned} \quad (11)$$

where the last inequality comes from (2) if $n = 0$ and from (4) if $n \geq 1$.
For the first inequality of (4, $n + 1$) we estimate

$$\begin{aligned} \left\| \alpha_0 y_{n+1}^* + \sum_{j=1}^{n+1} \alpha_j x_j^* \right\| &\stackrel{(8)}{\geq} (1 - \varepsilon_{n+1}) \left\| Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) + P^*(\alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right\| \\ &= (1 - \varepsilon_{n+1}) \max \left\{ \left\| Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right\|, \left\| \alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^* \right\| \right\}; \end{aligned}$$

in case $|\alpha_0| = \max_{0 \leq j \leq n+1} |\alpha_j|$ we observe that $Q\tilde{x}_s = \tilde{x}_s$, that $\tilde{x}_s(y_n^*) = \tilde{x}_s(x^*) - \sum_{j=1}^n \tilde{x}_s(x_j^*) = \tilde{x}_s(x^*)$ by (5), and we continue the estimate by

$$\begin{aligned} \dots &\geq (1 - \varepsilon_{n+1}) \left| \left(Q^*(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right) \left(\frac{\tilde{x}_s}{\|\tilde{x}_s\|} \right) \right| \\ &= \frac{(1 - \varepsilon_{n+1})}{\|\tilde{x}_s\|} \left| \tilde{x}_s(\alpha_0 y_n^* + \sum_{j=1}^n \alpha_j x_j^*) \right| \\ &\stackrel{(5)}{=} \frac{(1 - \varepsilon_{n+1})}{\|\tilde{x}_s\|} |\alpha_0| |\tilde{x}_s(y_n^*)| = \frac{(1 - \varepsilon_{n+1})}{\|\tilde{x}_s\|} |\alpha_0| |\tilde{x}_s(x^*)| \\ &= (1 - \varepsilon_{n+1}) |\alpha_0| \end{aligned} \tag{12}$$

whereas in case $|\alpha_0| \neq \max_{0 \leq j \leq n+1} |\alpha_j|$ we get

$$\begin{aligned} \dots &\geq (1 - \varepsilon_{n+1}) \left\| \alpha_{n+1} y_n^* + \sum_{j=1}^n \alpha_j x_j^* \right\| \\ &\geq \left(\prod_{j=1}^{n+1} (1 - \varepsilon_j) \right) \max_{1 \leq j \leq n+1} |\alpha_j| \end{aligned} \tag{13}$$

where the last inequality comes from (2) if $n = 0$ and from (4) if $n \geq 1$. Thus we obtain the first inequality of (4, $n + 1$).

The conditions (5, $n + 1$), (6, $n + 1$) and (7, $n + 1$) are easy to verify because $Px_s = P\tilde{x}_s = Qx_k = 0$ thus

$$\begin{aligned} x_s(x_{n+1}^*) &= x_s(RP^*y_n^*) = (P^*y_n^*)(x_s) = Px_s(y_n^*) = 0, \\ \tilde{x}_s(x_{n+1}^*) &= \tilde{x}_s(RP^*y_n^*) = (P^*y_n^*)(\tilde{x}_s) = P\tilde{x}_s(y_n^*) = 0, \\ y_{n+1}^*(x_k) &= (RQ^*y_n^*)(x_k) = y_n^*(Qx_k) = 0. \end{aligned}$$

This ends the induction.

By (4), $\sum x_j^*$ is wuC (where, as indicated above in the introduction, j runs through \mathbb{N}). We have that $\sum^* x_j^* = x^*$ by (3) (and (2)) because by (7) and the density of the x_k the w^* -limit of (y_n^*) is 0. This easily entails (1) because we have $\sum x_s(x_j^*) = 0$ by (6), we have $\|\tilde{x}_s\| \geq \|x_s\| - \|x_s - \tilde{x}_s\| > (1 - \varepsilon/3)\eta$ and trivially $(\sum^* x_j^*)(x) = \sum x_j^*(x)$ thus

$$x^{**}(\sum^* x_j^*) - \sum x^{**}(x_j^*) = x_s(\sum^* x_j^*) - \sum x_s(x_j^*)$$

$$\begin{aligned}
&= x_s(x^*) = \|\tilde{x}_s\| - (\tilde{x}_s - x_s)(x^*) \\
&> \left(1 - \frac{\varepsilon}{3}\right)\eta - \frac{\varepsilon\eta}{3} \\
&> (1 - \varepsilon)\|Qx^{**}\| > 0.
\end{aligned}$$

This ends the proof. ■

We have already mentioned Godefroy's and Talagrand's result that property (X) implies the uniqueness of a Banach space as a predual; moreover, since (X) is hereditary and stable by equivalent norms we obtain

Corollary 4 *A Banach space that is isomorphic to a subspace of a separable L-embedded space is the unique predual of its dual.*

Remarks:

1. It follows immediately from the first variant of the proof of theorem 3 that if \tilde{x}_s is a non zero norm attaining element of X_s then the two expressions in (1) differ by the greatest possible value, more precisely

$$\tilde{x}_s(\sum^* x_j^*) = \|\tilde{x}_s\| \neq 0 = \sum \tilde{x}_s(x_j^*)$$

with $\|\sum^* x_j^*\| = 1$.

2. Is it possible to refine the proof of theorem 3 so to produce a sequence spanning c_0 almost or asymptotically isometrically? We say that a sequence (z_j) in a Banach space Z spans c_0 almost isometrically if there exists a sequence (δ_m) satisfying $[0, 1[\ni \delta_m \rightarrow 0$ such that $(1 - \delta_m) \sup_{m \leq j \leq n} |\alpha_j| \leq \left\| \sum_{j=m}^n \alpha_j z_j \right\| \leq (1 + \delta_m) \sup_{m \leq j \leq n} |\alpha_j|$ for all $m \leq n$. If we have even $\sup_{j \leq n} (1 - \delta_j) |\alpha_j| \leq \left\| \sum_{j=1}^n \alpha_j z_j \right\| \leq \sup_{j \leq n} (1 + \delta_j) |\alpha_j|$ for all $n \in \mathbb{N}$ then (z_j) is said to span c_0 asymptotically (or asymptotically isometrically). While, by James' distortion theorem, a Banach space isomorphic to c_0 always contains an almost isomorphic copy of c_0 , Dowling, Johnson, Lennard and Turett [6] proved the existence of a c_0 -copy which does not contain asymptotic copies of c_0 . Note that the L-structure of an L-embedded Banach space and, respectively, the M-structure of its dual have an influence on the existence of asymptotic copies of ℓ^1 and, respectively, c_0 . For example, it has been proved in [18] that each almost isometric copy of ℓ^1 inside an L-embedded space contains an asymptotic copy of ℓ^1 (see [18] also for the definitions) and it has been proved there that if an L-embedded space is the dual of an M-embedded space then its dual contains asymptotic copies of c_0 .

3. There is an interesting difference of the construction of a c_0 -copy in the present proof and in the proof of property (V*). The latter one works for both separable and non-separable L-embedded spaces whereas the present proof of property (X) runs into an obstacle in the non-separable case: Edgar [7, Prop. 12] showed that $\ell^1(\Gamma)$ has property (X) if and only if $\text{card}(\Gamma)$ is not a real measurable cardinality (that is if and only if there is no non-zero measure on Γ vanishing on singletons). For a discussion of (X) and measurable cardinals we refer to [16]. It seems reasonable to conjecture that an L-embedded Banach space may have property (X) if it does not

contain a subspace isomorphic to $\ell^1(\Gamma)$ with $\text{card}(\Gamma)$ measurable, or, perhaps, if it has a dense subset of non-measurable cardinality.

4. Given a Banach space Z it might occur that its bidual contains an element z_0^{**} which is L-direct to Z that is

$$\|z + z_0^{**}\| = \|z\| + \|z_0^{**}\| \text{ for all } z \in Z. \quad (14)$$

Godefroy has shown ([11, IV.2] or [14, I.18.5.6]) that viewed as a function on the unit ball of the dual (B_{Z^*}, w^*) such an element z_0^{**} is "very" discontinuous, for example it is nowhere continuous on (B_{Z^*}, w^*) . The space $Z = C([0, 1])$ and the function $z_0^{**} = 1_{\mathbb{Q} \cap [0, 1]} - 1_{(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]}$ serve as an example. This function z_0^{**} is of second Baire class but does not belong to Z . In other words, the "local" property (14) is definitely weaker than the "global" one of being L-embedded because if Z were l-embedded then the second Baire class function z_0^{**} would belong to Z (cf. the second part of theorem 2).

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