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Florent Benaych-Georges

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# RECTANGULAR RANDOM MATRICES, RELATED CONVOLUTION

FLORENT BENAYCH-GEORGES

ABSTRACT. We characterize asymptotic collective behavior of rectangular random matrices, the sizes of which tend to infinity at different rates: when embedded in a space of larger square matrices, independent rectangular random matrices are asymptotically free with amalgamation over a subalgebra. Therefore we can define a “rectangular free convolution”, linearized by cumulants and by an analytic integral transform, called the “rectangular  $R$ -transform”.

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## INTRODUCTION

The first problem we are going to deal with in this paper is the modeling of asymptotic collective behavior of independent rectangular random matrices. In order to explain to the reader the way we will treat this problem, let us recall him the work already done by Wigner, Pastur, Marchenko, Girko, Bai, Voiculescu,... for square random matrices. First of all, in the 50's Wigner considered self-adjoint random matrices with gaussian entries (GUE) and proved that the spectral law (uniform distribution on the set of eigenvalues) of these random matrices converges to the so-called *semicircle law*. This result was improved, and other results giving the asymptotic spectral law of random matrices were proved (see, among many other sources, [PL02]). In the same time, people studied the local structure of the spectrum of random matrices (see, e.g. [M67]), but this is not the kind of problem we are going to study here. A new point of view was adopted in the early 90's by Voiculescu, who proposed a way to compute the asymptotic normalized trace (when the dimension  $n$  of the matrices goes to infinity) of products

$$M(s_1, n)^{\varepsilon_1} \cdots M(s_k, n)^{\varepsilon_k}$$

of random matrices taken among a family  $M(1, n), M(2, n), \dots$  of independent random  $n \times n$  matrices and their adjoints, the only hypothesis being a bound on the norms of the matrices, the fact that the matrices  $M(1, n), M(2, n), \dots$  have all a limit singular law (uniform distribution on the set of singular values, i.e. of eigenvalues of the absolute value of the matrix), and a property of invariance of the distributions under an action of the unitary group (these results can be deduced from the fundamental article [V91], but are presented under this form in [HP00]). The advantage of being able to compute the limit of such normalized traces is that it gives us the asymptotic normalized trace of any noncommutative polynomial in our independent random matrices. Hence, since the normalized trace of the  $k$ -th power of a matrix is the  $k$ -th moment of its spectral law, we are able to give the asymptotic singular law of any polynomial of our

random matrices. This is why this work is said to *model the asymptotic collective behavior of independent square random matrices*. For example, it can be proved (combine results of [HP00] and [HL00]) that the asymptotic singular law of the sum of two independent random matrices whose distributions are invariant under left and right actions of the unitary group and whose singular laws converge weakly to probability measures  $\mu_1, \mu_2$ , only depends on  $\mu_1$  and  $\mu_2$ , and can be expressed easily from  $\mu_1$  and  $\mu_2$ : it is the probability measure on  $[0, \infty)$ , the symmetrization of which is the free convolution of the symmetrizations of  $\mu_1, \mu_2$ . We often present the similar result for hermitian matrices, for which we work with spectral law in the place of singular law, and for which no symmetrization is necessary, but here, we shall work with rectangular matrices, which cannot be hermitian, so the square analogue of our work will be found in non hermitian matrices. In this text, we will propose, similarly, a way to compute asymptotic normalized trace of products of random matrices taken among a family of independent rectangular random matrices, whose sizes tend to infinity, but with different rates. The notion involved, similarly to freeness in Voiculescu's modeling of asymptotics of square matrices, is freeness with amalgamation over a finite dimensional subalgebra. The notion of freeness with amalgamation arises from operator-valued free probability theory, but we chose to use this point of view of operator-valued free probability only when necessary, because this point of view is not satisfying in all cases: if one uses this point of view, then he has to consider the case where the ratio of sizes of our random matrices tends to zero separately. Moreover, the point of view of operator-valued free probability in the asymptotics of rectangular random matrices is developed in a forthcoming paper, where we analyze the related free entropy and Fischer information ([B-G2]).

This modeling of asymptotics of rectangular random matrices will allow us to define, for  $\lambda \in [0, 1]$ , a binary operation  $\boxplus_\lambda$  on the set of symmetric probability measures, called free convolution with ratio  $\lambda$ , and denoted by  $\boxplus_\lambda$ . For  $\mu_1, \mu_2$  symmetric probability measures,  $\mu_1 \boxplus_\lambda \mu_2$  is defined to be the limit of the singular law of a sum of two independent rectangular random matrices, whose dimensions tend to infinity in a ratio  $\lambda$ , one of them being bi-unitarily invariant, and whose singular laws tend to  $\mu_1, \mu_2$ . In the second part of this paper, after having analyzed related cumulants, we construct an analytic integral transform which linearizes  $\boxplus_\lambda$  (like the  $R$ -transform does for free convolution) and give examples. In another paper ([B-G1]), we study the related infinite divisibility. It appears then that the set of  $\boxplus_\lambda$ -infinitely divisible distributions is in a deep correspondence with the set of symmetric classical infinitely divisible distributions.

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## 1. ASYMPTOTIC BEHAVIOR OF RECTANGULAR RANDOM MATRICES

**1.1. Notes on reduction of rectangular random matrices.** In this subsection, we recall some simple facts about polar decomposition of rectangular complex matrices, which can be found in [HJ91]. Consider a  $p \times q$  matrix  $M$ . When  $p \leq q$ , we will denote by  $|M|$  the only  $p \times p$  positive hermitian matrix  $H$  such that we can write  $M = HT$ , with  $T$  a  $p \times q$  matrix such that  $TT^* = I_p$ . In this case, the matrix  $|M|$  is the square root of  $MM^*$ . When  $p > q$ , we will denote by  $|M|$  the only  $q \times q$  positive hermitian matrix  $H$  such that we can write  $M = TH$ , with  $T$  a  $p \times q$  matrix such that  $T^*T = I_q$ . In this case, the matrix  $|M|$  is the square root of  $M^*M$ . In both

cases, the spectrum of  $|M|$  is the only (up to a permutation) family  $(h_1, h_2 \dots)$  of nonnegative real numbers such that one can write

$$M = U[\delta_i^j h_i]_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} V,$$

with  $U, V$  respectively  $p \times p, q \times q$  unitary matrices. These numbers are called *singular values* of  $M$ . The uniform distribution on  $h_1, h_2 \dots$  will be called the *singular law* of  $M$  (whereas the uniform distribution on the spectrum of an hermitian matrix is called its *spectral law*). Note that in both cases, for any  $\alpha > 0$ ,

$$\frac{1}{\min(p, q)} \operatorname{Tr} |M|^\alpha = \frac{1}{\min(p, q)} \operatorname{Tr} (MM^*)^{\alpha/2} = \frac{1}{\min(p, q)} \operatorname{Tr} (M^*M)^{\alpha/2}$$

is the  $\alpha$ -th moment of the singular law of  $M$ .

A random matrix is said to be *bi-unitarily invariant* if its distribution is invariant under the left and right actions of the unitary group. It is easy, transferring the proof of lemma 4.3.10 p. 160 of [HP00], to prove that the distribution of a bi-unitarily invariant  $p \times q$  random matrix can be realized as the distribution of  $UHV$  where  $U, H, V$  are independent,  $U, V$  are respectively  $p \times p, q \times q$  uniform random unitary matrices (*uniform* means distributed according to the Haar measure), and  $H$  is a random positive  $p \times q$  diagonal matrix.

In the same way, an hermitian random matrix is said to be *unitarily invariant* if its distribution is invariant under the action by conjugation of the unitary group. In this case, by the beginning of the proof of Theorem 4.3.5 p. 155 of [HP00], its distribution can be realized as the distribution of  $UHU^*$  where  $U, H$  are independent,  $U$  is a uniform random unitary matrix, and  $H$  is a diagonal random matrix.

**1.2. Definitions.** For definitions of *singular values, singular law, spectral law, (bi-)unitarily invariant and uniform random unitary matrices*, we refer to subsection 1.1. For all positive integer  $d$ , we denote the set  $\{1, \dots, d\}$  by  $[d]$ . We denote the normalized trace of a square matrix  $X$  by  $\operatorname{tr} X$  (whatever the size of  $X$  is).

Consider a positive integer  $d$ , and sequences  $q_1(n), \dots, q_d(n)$  of pairwise distinct positive integers which all tend to infinity as  $n$  tends to infinity, and such that for all  $k \in [d]$ ,

$$(1) \quad \frac{q_k(n)}{n} \xrightarrow{n \rightarrow \infty} \rho_k \geq 0,$$

at most one of the  $\rho_k$ 's being zero. We are going to work with rectangular random matrices of sizes  $q_k(n) \times q_l(n)$ . Free probability's modeling of the asymptotic behavior of square random matrices relies on a comparison between random matrices and elements of an algebra arising from operator algebra theory. To product analogous results for rectangular random matrices, we have embedded our matrices of sizes  $q_k(n) \times q_l(n)$  ( $k, l \in [d]$ ) in an algebra. Let us assume that for all  $n$ ,  $q_1(n) + \dots + q_d(n) = n$  (since what we will prove would obviously also work when  $n$  is replaced by a subsequence, it is not a real restriction). Now,  $n \times n$  matrices will be represented as  $d \times d$  block matrices, such that for all  $k, l \in [d]$ , the  $(k, l)$ -th block is a  $q_k(n) \times q_l(n)$  matrix.

For all  $k, l \in [d]$ , for all  $q_k(n) \times q_l(n)$  matrix  $M$ , let us denote by  $\widetilde{M}$  the " $n \times n$  extension of  $M$ ", that is the  $n \times n$  matrix with  $(i, j)$ -th block  $M$  if  $(i, j) = (k, l)$ , and zero in the other case. Note we have preservation of adjoints ( $\widetilde{M}^* = \widetilde{M}^*$ ), and of products:

$$\forall k, l, k', l' \in [d], \forall M \text{ of size } q_k(n) \times q_l(n), \forall N \text{ of size } q_{k'}(n) \times q_{l'}(n), \widetilde{M}\widetilde{N} = \begin{cases} \widetilde{MN} & \text{if } l = k', \\ 0 & \text{in the other case.} \end{cases}$$

Moreover, the trace of  $\widetilde{M}$  is the one of  $M$  if  $M$  is square, and zero in the other case. Since we are overall interested in singular laws, whose moments are given by normalized traces of square matrices, these embeddings shall not cause any loss of information. Note however that the normalized trace is not exactly preserved by these embeddings (the normalized trace of a matrix  $M$  of size  $q_k(n) \times q_k(n)$  is  $\frac{n}{q_k(n)}$  times the normalized trace of  $\widetilde{M}$ ), and that the well known relation  $\text{Tr } MN = \text{Tr } NM$  is not satisfied anymore when we work with normalized traces of rectangular matrices. Indeed, for all  $k, l \in [d]$ , for all  $M$  of size  $q_k(n) \times q_l(n)$ , for all  $N$  of size  $q_l(n) \times q_k(n)$ , we have

$$(2) \quad q_k(n) \text{tr } MN = q_l(n) \text{tr } NM.$$

At last, let us define, for all  $k \in [d]$ , the projector

$$p_k(n) := \widetilde{I_{q_k(n)}}.$$

Let us formalize the structure we inherit. Consider a  $*$ -algebra  $\mathcal{A}$  endowed with a family  $(p_1, \dots, p_d)$  of non zero self-adjoint projectors (i.e.  $\forall i, p_i^2 = p_i$ ) which are pairwise orthogonal (i.e.  $\forall i \neq j, p_i p_j = 0$ ), and such that  $p_1 + \dots + p_d = 1$ . Any element  $x$  of  $\mathcal{A}$  can then be represented

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix},$$

where  $\forall i, j, x_{ij} = p_i x p_j$ . This notation is compatible with the product and the involution. Assume each subalgebra  $p_k \mathcal{A} p_k$  to be endowed with a tracial state  $\varphi_k$  (i.e.  $\varphi_k(p_k) = 1$  and for all  $x, y \in p_k \mathcal{A} p_k, \varphi_k(xy - yx) = 0$ ) such that for all  $k, l \in [d], x \in p_k \mathcal{A} p_l, y \in p_l \mathcal{A} p_k$ ,

$$(3) \quad \rho_k \varphi_k(xy) = \rho_l \varphi_l(yx),$$

where  $(\rho_1, \dots, \rho_d)$  is still the sequence of nonnegative real numbers defined by (1). Recall that at most one of the  $\rho_k$ 's is zero.

**Definition 1.1.** *Such a family  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$  will be called a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space. Elements of the union of the  $p_k \mathcal{A} p_l$ 's ( $k, l \in [d]$ ) will be called simple elements.*

**Example 1.2.**  *$\mathcal{A}$  is the  $*$ -algebra of  $n \times n$  complex matrices,  $p_1, \dots, p_d$  are  $p_1(n), \dots, p_d(n)$  previously defined, and each  $\varphi_k$  is  $\frac{1}{q_k(n)} \text{Tr}$ . Then by (2), (3) is satisfied when each  $\rho_k$  is replaced by  $q_k(n)/n$ .*

**Example 1.3.** *If  $\mathcal{A}$  is a  $*$ -algebra endowed with an orthogonal family of self-adjoint projectors  $(p_1, \dots, p_d)$  with sum 1, if  $\mathcal{A}$  is endowed with a tracial state  $\varphi$  such that for all  $k \in [d], \rho_k := \varphi(p_k) \neq 0$ , then the family  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$  is a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, where for all  $k, \varphi_k = \frac{1}{\rho_k} \varphi|_{p_k \mathcal{A} p_k}$ .*

Consider a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$ . Denote by  $\mathcal{D}$  the linear span of the  $p_k$ 's, then  $\mathcal{D}$  is an algebra, which can be identified to the set of  $d \times d$  complex diagonal matrices, and then to  $\mathbb{C}^d$  by

$$\sum_{k=1}^d \lambda_k p_k \simeq \text{diag}(\lambda_1, \dots, \lambda_d) \simeq (\lambda_1, \dots, \lambda_d).$$

Let us define  $E : \mathcal{A} \rightarrow \mathbb{C}^d$ , which maps  $x \in \mathcal{A}$  to

$$E(x) = \sum_{k=1}^d \varphi_k(x_{kk})p_k \simeq (\varphi_1(x_{11}), \dots, \varphi_d(x_{dd})).$$

$E$  is then a *conditional expectation* from  $\mathcal{A}$  to  $\mathcal{D}$ , i.e.  $E(1) = 1$  and

$$\forall (d, a, d') \in \mathcal{D} \times \mathcal{A} \times \mathcal{D}, E(dad') = dE(a)d'.$$

The following definition, first appeared in [V95], gives the right notion to describe asymptotics of independent random matrices.

**Definition 1.4.** *In an  $*$ -algebra  $\mathcal{B}$  endowed with a  $*$ -subalgebra  $\mathcal{C}$  and a conditional expectation  $E_{\mathcal{C}}$  from  $\mathcal{B}$  to  $\mathcal{C}$ , a family  $(\mathcal{C} \subset \mathcal{B}_{\alpha})_{\alpha \in A}$  of  $*$ -subalgebras is said to be free with amalgamation over  $\mathcal{C}$  if for all  $m \geq 1$ , for all  $\alpha_1 \neq \dots \neq \alpha_m \in A$ , for all  $x_1, \dots, x_m$  elements of respectively  $\mathcal{B}_{\alpha_1}, \dots, \mathcal{B}_{\alpha_m}$  one has*

$$(4) \quad E_{\mathcal{C}}(x_1) = \dots = E_{\mathcal{C}}(x_m) = 0 \implies E_{\mathcal{C}}(x_1 \cdots x_m) = 0.$$

A family of subsets of  $\mathcal{B}$  is said to be free with amalgamation over  $\mathcal{C}$  if the subsets are contained in  $*$ -subalgebras which are free with amalgamation over  $\mathcal{C}$ .

In the context of a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, it is easy to see that a family  $(\chi_{\alpha})_{\alpha \in A}$  of sets of simple elements are free with amalgamation over  $\mathcal{D}$  if one has: for all  $m \geq 1$ , for all  $\alpha_1 \neq \dots \neq \alpha_m \in A$ , for all  $x_1, \dots, x_m$  elements of the  $*$ -algebras respectively generated by  $\chi_{\alpha_1}, \dots, \chi_{\alpha_m}$ , one has

$$(5) \quad E(x_1) = \dots = E(x_m) = 0 \implies E(x_1 \cdots x_m) = 0.$$

**Definition 1.5.** *In a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, the  $\mathcal{D}$ -distribution of a family  $(a_j)_{j \in J}$  of simple elements is the function which maps any polynomial  $P$  in the noncommutative variables  $(X_j, X_j^*)_{j \in J}$ , with coefficients in  $\mathcal{D}$ , to  $E(P(a_j, a_j^*)_{j \in J})$ .*

The following lemma is well known. For the sake of completeness, we give a proof.

**Lemma 1.6.** *In a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, when  $(\chi_1, \dots, \chi_n)$  is a  $\mathcal{D}$ -free family of sets of simple elements, the  $\mathcal{D}$ -distribution of the union of the  $\chi_i$ 's is completely determined by the  $\mathcal{D}$ -distributions of  $\chi_1, \dots, \chi_n$ .*

**Proof.** The  $*$ -algebra generated by the union is the linear span of the set of elements of the type  $x_1 \cdots x_m$ , where  $m \geq 1$ , and there exists  $\alpha_1 \neq \dots \neq \alpha_m \in [n]$  such that each  $x_i$  is in the  $*$ -algebra generated by  $\chi_{\alpha_i}$ . We prove that  $E(x_1 \cdots x_m)$  is completely determined by the  $\mathcal{D}$ -distributions of  $\chi_1, \dots, \chi_n$  by induction on  $m$ , using the identity

$$x_1 \cdots x_m = \prod_{i=1}^m \left( (x_i - \sum_{k=1}^d \varphi_k(p_k x_i p_k) p_k) + \sum_{k=1}^d \varphi_k(p_k x_i p_k) p_k \right),$$

expanding the right hand side term, and using induction hypothesis and (5).  $\square$

The last notion we have to introduce is the *convergence in  $\mathcal{D}$ -distribution*.

**Definition 1.7.** - *If for all  $n$ ,  $(\mathcal{A}_n, p_{1,n}, \dots, p_{d,n}, \varphi_{1,n}, \dots, \varphi_{d,n})$  is a  $(\rho_{1,n}, \dots, \rho_{d,n})$ -probability space such that*

$$(\rho_{1,n}, \dots, \rho_{d,n}) \xrightarrow{n \rightarrow \infty} (\rho_1, \dots, \rho_d),$$

a family  $(a_j(n))_{j \in J}$  of simple elements of  $\mathcal{A}_n$  is said to converge in  $\mathcal{D}$ -distribution, when  $n$  goes to infinity, to a family  $(a_j)_{j \in J}$  of elements of  $\mathcal{A}$  if the  $\mathcal{D}$ -distributions converge pointwise.

- In the context of example 1.2, if for all  $n$ ,  $a_j(n)$ 's are  $n \times n$  random matrices, convergence in  $\mathcal{D}$ -distribution in probability of the family  $(a_j(n))_{j \in J}$  to  $(a_j)_{j \in J}$  is the convergence in probability (for the definition of convergence in probability, see below), when  $n \rightarrow \infty$ , of  $\mathbb{E}(P(a_j(n), a_j(n)^*)_{j \in J})$  to  $\mathbb{E}(P(a_j, a_j^*)_{j \in J})$  for all polynomial  $P$  in the noncommutative variables  $(X_j, X_j^*)_{j \in J}$ .

*Remarks about convergence in probability.* • Recall that the convergence in probability of a sequence  $X_n$  of random variables in a metric space  $(\mathcal{X}, d)$  to a constant  $l \in \mathcal{X}$  is the convergence of the probability of the event  $\{d(X_n, l) < \varepsilon\}$  to 1 for all positive  $\varepsilon$ .

• For random matrices, instead of using the notion of convergence in  $\mathcal{D}$ -distribution in probability, we could have used *almost sure convergence in  $\mathcal{D}$ -distribution*, or convergence in  $\mathcal{D}$ -distribution with respect to  $\mathbb{E}$  defined by

$$(6) \quad \forall k \in [d], \varphi_k(\cdot) = \mathbb{E} \left( \frac{1}{q_k(n)} \text{Tr}(\cdot) \right), \quad (\mathbb{E} \text{ denotes the expectation}).$$

But we chose to work with convergence in  $\mathcal{D}$ -distribution in probability because we consider it as the right notion for asymptotics of random matrices: since our random matrices have no reason to be defined on the same space when the dimension changes, almost sure convergence has, according to us, no signification, and convergence in  $\mathcal{D}$ -distribution with respect to  $\mathbb{E}$  defined by (6) gives weaker results, because, even though it gives the asymptotic mean value, it does not implies concentration around this value.

• All along this paper, we are going to work with sequences of random probability measures on the real line, and to deal with their convergence in probability. It shall refer to the convergence in probability in the metric space of probability measures on the real line endowed with a distance which defines weak convergence.

**1.3. Statement of the theorems about random matrices.** Let, for  $s \in \mathbb{N}, k, l \in [d], n \geq 1$ ,  $R(s, k, l, n)$  be a  $q_k(n) \times q_l(n)$  random matrix. Consider also, for  $n \geq 1$ , a family  $C(i, n)$  ( $i \in I$ ) of diagonal deterministic matrices, each  $C(i, n)$  having the size  $q_{k_i}(n) \times q_{l_i}(n)$ , for a certain  $(k_i, l_i) \in [d] \times [d]$ . We suppose the family to contain  $p_1(n), \dots, p_d(n)$ , and to be stable under product and taking adjoints. This means that there exists  $i_1, \dots, i_d \in I$  such that for all  $n, k$ ,  $C(i_k, n) = p_k(n)$  and that there exists a map  $\beta$  from  $I \times I$  to  $I$  and an involutive map  $*$  from  $I$  onto itself such that for all  $n \geq 0, i, j \in I$  such that  $l_i = k_j$ ,

$$(7) \quad C(i, n)C(j, n) = C(\beta(i, j), n), \quad C(i, n)^* = C(*i, n).$$

We suppose that for all  $s \in \mathbb{N}, k \neq l \in [d]$ , the moments of the singular law of  $R(s, k, l, n)$  converge in probability. This means that for all integer  $r \geq 0$ ,

$$\frac{1}{\min(q_k(n), q_l(n))} \text{Tr}(R(s, k, l, n)R(s, k, l, n)^*)^r$$

converges in probability to a constant. Note that in the case where one of  $\rho_k, \rho_l$  is zero, it implies that

$$\frac{1}{n} \text{Tr}(R(s, k, l, n)R(s, k, l, n)^*)^r$$

converges in probability to zero.

We suppose also that for all  $s \in \mathbb{N}$  odd, for all  $k \in [d]$ ,  $R(s, k, k, n)$  is almost surely unitary, and that for all  $s \in \mathbb{N}$  even, for all  $k \in [d]$ ,  $R(s, k, k, n)$  is almost surely hermitian and that the moments of its spectral law converge in probability.



We suppose moreover that the family  $C(i, n)$  ( $i \in I$ ) converges in distribution in the following way: for all  $i \in I$  such that  $k_i = l_i$ , the normalized trace

$$\frac{1}{q_{k_i}(n)} \operatorname{Tr} C(i, n)$$

has a finite limit when  $n$  goes to infinity.

Now, consider, in a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$ , a family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ) of elements of  $\mathcal{A}$ , with  $c(i_1) = p_1, \dots, c(i_d) = p_d$ , whose  $\mathcal{D}$ -distribution is defined by the following rules:

- (i) For all  $s \in \mathbb{N}, k, l \in [d]$ ,  $a(s, k, l) \in p_k \mathcal{A} p_l$ , and for all  $i \in I$ ,  $c(i) \in p_{k_i} \mathcal{A} p_{l_i}$ .
  - (ii) For all  $i, j \in I$  such that  $l_i = k_j$ ,
- (8) 
$$c(i)c(j) = c(\beta(i, j)), \quad c(i)^* = c(*i).$$
- (iii) For all  $s \in \mathbb{N}, k \neq l \in [d]$ , for all  $r \geq 0$ ,
- $$\varphi_k[(a(s, k, l)a(s, k, l)^*)^r] = \lim_{n \rightarrow \infty} \operatorname{tr}(R(s, k, l, n)R(s, k, l, n)^*)^r,$$
- $$\varphi_l[(a(s, k, l)^*a(s, k, l))^r] = \lim_{n \rightarrow \infty} \operatorname{tr}(R(s, k, l, n)^*R(s, k, l, n))^r.$$
- (iv) For all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $a(s, k, k)$  is unitary in  $p_k \mathcal{A} p_k$ , and for all  $r \in \mathbb{Z} - \{0\}$ ,  $\varphi_k(a(s, k, k)^r) = 0$ .
  - (v) For all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $a(s, k, k)$  is self-adjoint and for all  $r \geq 0$ ,
- $$\varphi_k[a(s, k, k)^r] = \lim_{n \rightarrow \infty} \operatorname{tr} R(s, k, k, n)^r,$$
- (vi) For all  $i \in I$  such that  $k_i = l_i$ ,  $\varphi_{k_i}(c(i))$  is the limit of the normalized trace of  $C(i, n)$ .
  - (vii) The family  $\{a(s, k, l)\}$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $\{c(i); i \in I\}$  is free with amalgamation over  $\mathcal{D}$ .

Note that such a space and such a family exist, they are given by the free product with amalgamation ([VDN91],[S98]).

We shall use the norm  $\|\cdot\|$  on matrix sets: it is the operator norm associated to canonical hermitian norms. The hypothesis of our main theorems are not the same if one supposes all  $\rho_k$ 's to be positive or not. In the following theorem, all  $\rho_k$ 's are supposed to be positive.

**Theorem 1.8** (Case where all  $\rho_k$ 's are positive). *Assume moreover that for all  $s, k \neq l$ ,  $R(s, k, l, n)$  is bi-unitarily invariant, that for all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $R(s, k, k, n)$  is unitarily invariant, and that for all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $R(s, k, k, n)$  is uniform. Assume also the set*

$$\{R(s, k, l, n); s \in \mathbb{N}, k, l \in [d]\}$$

*of random matrices to be independent and that the norm  $\|\cdot\|$  of our matrices is bounded uniformly (in  $n, s, k, l, i$ ). Then the family  $(\tilde{R}(s, k, l, n); s \in \mathbb{N}, k, l \in [d]) \cup (\tilde{C}(i, n); i \in I)$  converges in  $\mathcal{D}$ -distribution in probability to the family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ).*

**Remark 1.9.** *The hypothesis that  $C(i, n)$ 's are diagonal can be removed. Indeed, we only use it to apply case 1° of proposition 1.14, and in the case where all  $\rho_k$ 's are positive, this proposition is an immediate consequence of proposition 3.12 of [B-G2], where constant matrices are not supposed to be diagonal. But as the author is writing this paper, [B-G2] has not been refereed yet.*

In the following theorem, we suppose one of the  $\rho_k$ 's, say  $\rho_1$ , to be zero. We supposed that at most one of the  $\rho_k$ 's is zero, so  $\rho_2, \dots, \rho_d$  are positive. The other hypothesis are quite stronger than the one of the previous theorem: we suppose the singular values (or eigenvalues) of our random matrices to be deterministic. In fact, this supposition is done to avoid steps I and IV of the proof, because  $\max_{1 \leq k \leq d} n/q_k(n)$  appears in step IV of the proof of the previous theorem. Note that the singular values (or eigenvalues) of our random matrices could still be supposed to be random, but in this case, we would have to suppose that for all  $s, k, l$  with  $k \neq 1, l \neq k$ , for all  $r$ ,

$$\frac{n}{q_1(n)} (\text{tr}[(R(s, k, l, n)R(s, k, l, n)^*)^r] - \varphi_k[(a(s, k, l)a(s, k, l)^*)^r])$$

converges in probability to zero, and we would have to do the analogous supposition for all  $s$  even,  $1 \neq k = l$ .

**Theorem 1.10** (Case where  $\rho_1 = 0$ , other  $\rho_k$ 's are positive). *Consider the family of random and deterministic matrices introduced above the previous theorem. Assume moreover that for all  $s, k \neq l, n$ ,  $R(s, k, l, n)$  is bi-unitarily invariant with deterministic singular values, that for all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $R(s, k, k, n)$  is unitarily invariant with deterministic spectrum, and that for all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $R(s, k, k, n)$  is uniform. Assume also the set*

$$\{R(s, k, l, n); s \in \mathbb{N}, k, l \in [d]\}$$

*of random matrices to be independent and that the norm  $\|\cdot\|$  of our matrices is bounded uniformly (in  $n, s, k, l, i$ ). Then the family  $\tilde{R}(s, k, l, n)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $\tilde{C}(i, n)$  ( $i \in I$ ) converges in  $\mathcal{D}$ -distribution in probability to the family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ) introduced above.*

**1.4. Comparison with already existing results.** There already exists some results which allow to compute asymptotics of normalized traces of products of rectangular random matrices : on one hand the results about Wishart matrices and on the other hand the theorem 4.1 of [Sh96].

Theorems 1.8 and 1.10 allow us to compute the limit (for convergence in probability) of all normalized traces of matrices of the type

$$(9) \quad R(s_1, k_1, l_1, n)^{\varepsilon_1} C(i_1, n) R(s_2, k_2, l_2, n)^{\varepsilon_2} C(i_2, n) \cdots C(i_{p-1}, n) R(s_p, k_p, l_p, n)^{\varepsilon_p} C(i_p, n),$$

with  $p \geq 1$ ,  $s_1, \dots, s_p \in \mathbb{N}$ ,  $k_1, l_1, \dots, k_p, l_p \in [d]$ ,  $\varepsilon_1, \dots, \varepsilon_p \in \{1, *\}$ , and  $i_1, \dots, i_{p-1} \in I$  such that the product is possible and is a square matrix.

The Wishart matrices are matrices of the type  $RR^*$ , where  $R$  is a rectangular matrix. In [HP00], [CC04] or [CDM05], the authors give results of asymptotic freeness for such random matrices. In the case we are interested in, where the random matrix  $R$  is bi-unitarily invariant, this asymptotic freeness is only a consequence of the asymptotic freeness of unitarily invariant hermitian random matrices. With our notations, it only allows to consider the case where  $\rho_1, \dots, \rho_d > 0$ , and in this case, to compute limits of normalized traces of matrices of the type

$$R(s_1, k, l_1, n) R(s_1, k, l_1, n)^* C(i_1, n) R(s_2, k, l_2, n) R(s_2, k, l_2, n)^* C(i_2, n) \cdots \\ \cdots C(i_{p-1}, n) R(s_p, k, l_p, n) R(s_p, k, l_p, n)^*,$$

with  $k \in [d]$ ,  $l_1, \dots, l_p \in [d]$ ,  $s_1, \dots, s_p \in \mathbb{N}$ ,  $i_1, \dots, i_{p-1} \in I$  such that the matrices  $C(i_1, n), \dots, C(i_{p-1}, n)$  are all square. It does not allow to suppose that  $\rho_1 = 0$ , and to consider products **where rectangular matrices are not immediately followed by their adjoint** and where the rectangular matrices are not followed by their adjoints, as the one of (9).

Theorem 4.1 of the paper [Sh96] by Shlyakhtenko allows, embedding rectangular matrices in larger self-adjoint matrices ( $R \mapsto \begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix}$ ), to consider products of the type of the one of (9), but **only when the  $\rho_i$ 's are positive**, the matrices  $R(s, k, l, n)$  are **gaussian**, and the deterministic matrices are **diagonal** and have, in a certain sense, **a limit in  $L^\infty[0, 1]$** . Moreover, this theorem states the convergence of the **expectations** of the normalized traces, but not their convergence in probability (even though this could it is possible to expect to pass, using the concentration results of [GZ00], from convergence in expectation to convergence in probability in Shlyakhtenko's result, but we chose this approach in another paper, [B-G2]).

As an example, no general result had been proved about the convergence, when  $n, p \rightarrow \infty$  such that  $\frac{n}{p} \rightarrow \lambda \geq 0$ , of the singular law of  $M(n, p) + N(n, p)$ , where  $M(n, p), N(n, p)$  are independent bi-unitarily invariant  $n \times p$  random matrices with limit singular laws. We did not even know if this limit existed, even though some computations have been done in some particular cases ([DS04], [HLN04], [HLN05]). Indeed, the computation, by the moments method, of the singular law of  $M(n, p) + N(n, p)$  involves the computation of the normalized trace of

$$[(M(n, p) + N(n, p))(M(n, p) + N(n, p))^*]^k,$$

the expansion of which contains products where some rectangular matrices are not followed by their adjoints.

In order to prove theorems 1.8 and 1.10, we are going to prove a preliminary result.

**1.5. Preliminary result.** In this section, we do not make any particular hypothesis about  $\rho_k$ 's: they are nonnegative, and at most one of them is zero. So we can suppose that  $\rho_2, \dots, \rho_d$  are positive.

Recall that, for each integer  $k$ ,  $[k]$  denotes  $\{1, \dots, k\}$ . For all integers  $n, k$ , for each partition  $\mathcal{P}$  of  $[k]$ ,  $[n]^\mathcal{P}$  will design the set of elements  $i = (i_1, \dots, i_k)$  of  $[n]^k$  such that  $h \stackrel{\mathcal{P}}{\sim} h' \Leftrightarrow i_h = i_{h'}$ . Moreover,  $|\mathcal{P}|$  denotes the number of blocks of  $\mathcal{P}$ .

The following result involves special classes of matrices.

**Definition 1.11.** *An  $n \times n$  matrix is said to be an almost diagonal matrix if it is of the type  $\widetilde{M}$ , with  $M$  diagonal (either square or rectangular) matrix.*

Note that if  $A$  is an  $n \times n$  almost diagonal matrix, then for all  $j \in [n]$ ,  $A$  has at most one nonzero term on its  $j$ -th column.  $A_j$  will denote this term if it exists, and zero in the other case.

**Lemma 1.12.** *Let  $K$  be a positive integer, and let for each  $n \geq 1$ ,  $A(1, n), \dots, A(K, n)$  be a family of almost diagonal  $n \times n$  matrices. Let  $\mathcal{H}$  be a subset of  $[K]$ , and let  $\mathcal{P}$  be a partition of  $[K]$  such that for all  $\{h\}$  singleton class of  $\mathcal{P}$ ,  $A(h, n)$  is a diagonal matrix and  $\text{Tr } A(h, n) = 0$ . We suppose too that the family  $(A(i, n))_{i, n}$  is uniformly bounded, and that for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . Then*

$$\sum_{i \in [n]^\mathcal{P}} \prod_{h=1}^K A_{i_h}(h, n) = O\left(q_1(n)^{|\mathcal{P}_\mathcal{H}| - p_\mathcal{H}/2} n^{|\overline{\mathcal{P}}_\mathcal{H}| - \overline{p}_\mathcal{H}/2}\right),$$

where  $\mathcal{P}_\mathcal{H}$  (resp.  $\overline{\mathcal{P}}_\mathcal{H}$ ) is the partition induced by  $\mathcal{P}$  on the union of the classes of  $\mathcal{P}$  that intersect  $\mathcal{H}$  (resp. do not intersect  $\mathcal{H}$ ), and  $p_\mathcal{H}$  (resp.  $\overline{p}_\mathcal{H}$ ) the number of singletons in  $\mathcal{P}_\mathcal{H}$  (resp. of  $\overline{\mathcal{P}}_\mathcal{H}$ ).

**Proof.** We will prove this result by induction on  $p = p_\mathcal{H} + \overline{p}_\mathcal{H}$ .

If  $p = 0$ , the result is clear.

Let us suppose the result to be true to the ranks  $0, \dots, p-1$ , with  $p \geq 1$ . Consider  $\mathcal{P}$  such that  $p_{\mathcal{H}} + \bar{p}_{\mathcal{H}} = p$ , and a singleton class  $\{h_0\}$  of  $\mathcal{P}$ . Denote by  $\mathcal{P}'$  the partition  $\mathcal{P} - \{\{h_0\}\}$  of  $\{1, \dots, \hat{h}_0, \dots, K\}$  and let, for each class  $B$  of  $\mathcal{P}$  with  $B \neq \{h_0\}$ ,  $\mathcal{P}(h_0 \rightarrow B)$  be the partition of  $[K]$  obtained from  $\mathcal{P}$  by linking the classes  $\{h_0\}$  and  $B$ . We have

$$\sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) = \sum_{(i_1, \dots, \hat{i}_{h_0}, \dots, i_K) \in [n]^{\mathcal{P}'}} \left[ \left( \sum_{i \in [n] - \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n) \right) \prod_{\substack{h=1 \\ h \neq h_0}}^K A_{i_h}(h, n) \right].$$

But, since  $A(h_0, n)$  is diagonal and has null trace,

$$\sum_{i \in [n] - \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n) = - \sum_{i \in \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n).$$

So

$$\sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) = - \sum_{\substack{B \in \mathcal{P} \\ B \neq \{h_0\}}} \sum_{i \in [n]^{\mathcal{P}(h_0 \rightarrow B)}} \prod_{h=1}^K A_{i_h}(h, n).$$

For example, suppose  $h_0 \in \mathcal{H}$  (the other case is treated in the same way). With the induction hypothesis, and dividing the sum in  $B \in \mathcal{P}_{\mathcal{H}}$  and  $B \in \bar{\mathcal{P}}_{\mathcal{H}}$ , one has (each  $\epsilon_B$  being 1 or 0 according to whether  $B$  is a singleton or not)

$$\begin{aligned} \sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) &= \sum_{\substack{B \in \mathcal{P}_{\mathcal{H}} \\ B \neq \{h_0\}}} O \left( q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - 1 - (p_{\mathcal{H}} - 1 - \epsilon_B)/2} n^{|\bar{\mathcal{P}}_{\mathcal{H}}| - \bar{p}_{\mathcal{H}}/2} \right) \\ &\quad + \sum_{B \in \bar{\mathcal{P}}_{\mathcal{H}}} O \left( q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - (p_{\mathcal{H}} - 1)/2} n^{|\bar{\mathcal{P}}_{\mathcal{H}}| - 1 - (\bar{p}_{\mathcal{H}} - \epsilon_B)/2} \right) \\ &= q_1(n)^{\frac{\epsilon_B - 1}{2}} \sum_{\substack{B \in \mathcal{P}_{\mathcal{H}} \\ B \neq \{h_0\}}} O \left( q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\bar{\mathcal{P}}_{\mathcal{H}}| - \bar{p}_{\mathcal{H}}/2} \right) \\ &\quad + q_1(n)^{\frac{1}{2}} n^{\frac{\epsilon_B - 2}{2}} \sum_{B \in \bar{\mathcal{P}}_{\mathcal{H}}} O \left( q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\bar{\mathcal{P}}_{\mathcal{H}}| - \bar{p}_{\mathcal{H}}/2} \right). \end{aligned}$$

Since  $q_1(n) = O(n)$ , the result is proved.  $\square$

The following proposition, main result of this subsection, is divided in two cases, which give very similar results with very similar proofs. The proposition involves special classes of matrices.

**Definition 1.13.** For  $k \in [d]$ , we define  $\mathbb{U}_k(n)$  to be the set of  $n \times n$  matrices whose  $(k, k)$ -th block is a  $q_k(n) \times q_k(n)$  unitary matrix, and with other blocks zero.

Note that it is a compact group, isomorphic to the group of  $q_k(n) \times q_k(n)$  unitary matrices, and random matrices of  $\mathbb{U}_k(n)$  distributed according to the Haar measure will be said to be *uniform*. In the following, we are going to use an integer  $N$  and the cyclic order on  $[N]$ . This means that to put the index  $N+1$  on an element is equivalent to put the index 1.

**Proposition 1.14.** Let, for  $n \geq 1$ ,  $V(s, k, n)$  ( $s \in \mathbb{N}, k \in [d]$ ), be a family of independent random matrices, such that for all  $s, k$ ,  $V(s, k, n)$  is uniform on  $\mathbb{U}_k(n)$ . Fix  $N$  positive integer and  $R > 0$ .

Case 1°) Let, for each  $n \in \mathbb{N}$ ,  $D(1, n), \dots, D(N, n)$  be  $n \times n$  constant matrices such that for all  $r \in [N]$ ,  $\|D(r, n)\| \leq R$ , and there exists  $u(r), v(r) \in [d]$  such that one of the two following conditions is realized:

- (i)  $u(r) = v(r)$  and for all  $n$ ,  $D(r, n) = p_{u(r)}(n)$ ,
- (ii) for all  $n$ ,  $D(r, n)$  is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_{u(r)}(n) \times q_{v(r)}(n)$  diagonal matrix, and  $\mathbb{E}(D(r, n)) \xrightarrow{n \rightarrow \infty} 0$ .

Consider  $(s_l, k_l)_{l \in [N]} \in (\mathbb{N} \times [d])^N$ , and suppose that for all  $r \in [N]$  such that (i) above is satisfied,  $(s_r, k_r) \neq (s_{r+1}, k_{r+1})$ . Consider also  $m_1, \dots, m_N \in \mathbb{Z} - \{0\}$  and  $\eta > 0$ . Then the probability of the event

$$\{\|\mathbb{E}(V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n))\| \leq \eta\}$$

tends to 1 as  $n$  goes to infinity.

Case 2°) Let, for each  $n \in \mathbb{N}$ ,  $D(0, n), \dots, D(N, n)$  be  $n \times n$  constant matrices with norms also  $\leq R$ . We suppose that there exists  $v(0), u(N) \in [d]$  such that  $D(0, n)$  (resp.  $D(N, n)$ ) is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_1(n) \times q_{v(0)}(n)$  (resp.  $q_{u(N)}(n) \times q_1(n)$ ), and that for all  $r \in [N-1]$ , there exists  $u(r), v(r) \in [d]$  such that one of the two following conditions is realized:

- (i)  $u(r) = v(r)$  and for all  $n$ ,  $D(r, n) = p_{u(r)}(n)$ ,
- (ii) for all  $n$ ,  $D(r, n)$  is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_{u(r)}(n) \times q_{v(r)}(n)$  diagonal matrix, and  $\mathbb{E}(D(r, n)) \xrightarrow{n \rightarrow \infty} 0$ .

Consider  $(s_l, k_l)_{l \in [N]} \in (\mathbb{N} \times [d])^N$ , and suppose that for all  $r \in [N-1]$  such that (i) above is satisfied,  $(s_r, k_r) \neq (s_{r+1}, k_{r+1})$ . Consider also  $m_1, \dots, m_N \in \mathbb{Z} - \{0\}$  and  $\eta > 0$ . Then the probability of the event

$$\{\|\mathbb{E}(D(0, n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n))\| \leq \eta\}$$

tends to 1 as  $n$  goes to infinity.

**Remark 1.15.** By linearity of  $\mathbb{E}$ , the result stays true in case 1°), (resp. in case 2°)) if for all  $r$  in  $[N]$  (resp. in  $[N-1]$ ) such that (ii) is satisfied,  $B(r, n)$  is replaced by a sum of constant almost diagonal matrices whose images by  $\mathbb{E}$  tend also to zero.

The proof is inspired from the one of Theorem 4.3.1 p. 147 in [HP00].

**Proof.** Both proofs will be made together, we will only have to separate them sometimes.

Step I. Cases 1°) and 2°) Let us denote, for all  $r$  which satisfies (ii) above and  $u(r) = v(r)$ ,

$$D(r, n) = D'(r, n) + \lambda_r(n) p_{u(r)}(n), \text{ with } \mathbb{E}(D'(r, n)) = 0.$$

Then by hypothesis,  $\lambda_h(n) \xrightarrow{n \rightarrow \infty} 0$ . Hence, by linearity and boundedness hypothesis, it suffices to prove the result for  $D(r, n)$  replaced by  $D'(r, n)$ . Thus we can from now on suppose that for all  $r$  such that (ii) is realized, we have  $\mathbb{E}(D(r, n)) = 0$  for all  $n$ .

Step II. Case 1°) By definition of  $\mathbb{E}$ , it suffices to prove that for all  $k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of the product

$$V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n)$$

converges in probability to zero, i.e. that

$$(10) \quad \frac{1}{q_k(n)} \operatorname{Tr} p_k(n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n) p_k(n)$$

converges in probability to zero. So let us fix  $k \in [d]$ , and let us prove it.

If  $k_1 \neq k$  or  $v(N) \neq k$ , then the  $k$ -th block of the matrix is always zero, so the result is clear. So let us suppose that  $k_1 = v(N) = k$ . We can then remove  $p_k(n)$  in (10). By Markov inequality, it suffices to prove that (10) tends to zero in  $L^2$ . Since for all matrix  $M$ ,  $|\operatorname{Tr} M|^2 = \operatorname{Tr} M \operatorname{Tr} M^*$ , we only have to prove that the expectation of

$$(11) \quad \operatorname{Tr} V(s_1, k_1, n)^{m_1} D_1(n) \cdots D_N(n) \operatorname{Tr} D_N(n)^* V(s_N, k_N, n)^{-m_N} D_{N-1}(n)^* \cdots V(s_1, k_1, n)^{-m_1}$$

is  $o(q_k(n)^2)$ .

Case 2°) We have to prove that

$$\frac{1}{q_1(n)} \operatorname{Tr} D(0, n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n)$$

converges in probability to zero. Since  $\operatorname{Tr} XY = \operatorname{Tr} YX$ , it suffices to prove that

$$(12) \quad \frac{1}{q_1(n)} \operatorname{Tr} V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n) D(0, n)$$

converges in probability to zero. From now on, we will denote  $D(N, n)D(0, n)$  by  $D(N, n)$ , which is now an  $n \times n$  matrix of the type  $\widetilde{M}$  with  $M$   $q_{u(N)}(n) \times q_{v(0)}(n)$ , but which has not more than  $q_1(n)$  nonzero entries. For the same reason as above, we only have to prove that the expectation of

$$(13) \quad \operatorname{Tr} V(s_1, k_1, n)^{m_1} D_1(n) \cdots D_N(n) \operatorname{Tr} B_N(n)^* V(s_N, k_N, n)^{-m_N} D_{N-1}(n)^* \cdots V(s_1, k_1, n)^{-m_1}$$

is  $o(q_1(n)^2)$ .

Step III. Cases 1°) and 2°) Let us expand both traces in the previous product, and re index the sum on partitions.

Let  $M = |m_1| + \cdots + |m_N|$ , let, for  $r \in \{0, \dots, N\}$ ,

$$M(r) = |m_1| + \cdots + |m_r|,$$

let  $M(N+1) = 2M$ , and let, for  $r \in \{N+2, \dots, 2N\}$ ,

$$M(r) = M + |m_N| + |m_{N-1}| + \cdots + |m_{2N+2-r}|.$$

Let also, for  $r \in \{N+1, \dots, 2N\}$ ,  $D(r, n) = D(2N+1-r, n)^*$ . We write, for  $\kappa \in \mathbb{N} \times [d]$ ,

$$\begin{aligned} v_{i,j}(\kappa, 1, n) &= V(\kappa, n)_{i,j}, \\ v_{i,j}(\kappa, -1, n) &= \bar{V}(\kappa, n)_{j,i}. \end{aligned}$$

With those notations, there exists two functions  $\kappa$  and  $\varepsilon$  such that the expectation of (11) is

$$\sum_{i \in [n]^{2M}} \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right],$$

where  $\mathbb{E}$  denotes expectation and for all  $i = (i_1, \dots, i_{2M}) \in [n]^{2M}$ ,

$$j_1 = i_2, j_2 = i_3, \dots, j_M = i_1, j_{M+1} = i_{M+2}, j_{M+2} = i_{M+3}, \dots, j_{2M-1} = i_{2M}, j_{2M} = i_{M+1}.$$

Let us introduce the partition  $\mathcal{Q}$  of  $[2M]$  defined by  $h \overset{\mathcal{Q}}{\sim} h'$  if and only if  $i_h = i_{h'}$ . Then we can rewrite the preceding sum

$$\sum_{\substack{\mathcal{Q} \text{ partition} \\ \text{of } [2M]}} \sum_{i \in [n]^{\mathcal{Q}}} \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right].$$

Thus it suffices to prove that for all partition  $\mathcal{Q}$  of  $[2M]$ , the sum

$$(14) \quad \sum_{i \in [n]^{\mathcal{Q}}} \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right]$$

is  $o(q_k(n)^2)$  in case 1 $^\circ$ ), and  $o(q_1(n)^2)$  in case 2 $^\circ$ ), as  $n$  goes to infinity.

So we fix a partition  $\mathcal{Q}$  of  $[2M]$ .

*Step IV. Cases 1 $^\circ$ ) and 2 $^\circ$ )* Let us denote, for all  $h \in [2M]$ ,  $\kappa(h) = (s(h), k(h))$ . Note that, for  $i \in [n]^{\mathcal{Q}}$ , for

$$(15) \quad \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right]$$

to be nonzero,  $i$  has to satisfy

$$(16) \quad \forall h \in [2M], i_h, j_h \in \{q_1(n) + \dots + q_{k(h)-1}(n) + 1, \dots, q_1(n) + \dots + q_{k(h)-1}(n) + q_{k(h)}(n)\}.$$

Indeed, in the other case, one of the factors in the product is zero, by definition of  $M \mapsto \widetilde{M}$ . Note also that for all elements  $i$  of  $[n]^{\mathcal{Q}}$ , which satisfy (16), the expectation (15) is the same, by invariance of Haar measure on unitary groups under permutation of rows and columns. Let us denote this common expectation by  $\mathbb{E}_{\mathcal{Q}}$ . So, one can write the sum of (14)

$$(17) \quad \mathbb{E}_{\mathcal{Q}} \sum_i \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}}$$

where the sum is taken on elements  $i$  of  $[n]^{\mathcal{Q}}$  which satisfy (16).

It is well known (see [HP00], equation (4.2.11)) that if, for each  $m$ ,  $U(m)$  is a uniform unitary  $m \times m$  random matrix, and  $1 \leq i_m, j_m \leq m$ , then the sequence  $\mathbb{E}(|U(m)_{i_m, j_m}|^{2M})$  does not depend on the choices of  $i_m, j_m$  and

$$(18) \quad \mathbb{E}(|U(m)_{i_m, j_m}|^{2M})^{1/2M} = O(m^{-1/2}).$$

So, with  $i \in [n]^{\mathcal{Q}}$ ,

$$\begin{aligned} |\mathbb{E}_{\mathcal{Q}}| &\leq \prod_{h=1}^{2M} \mathbb{E}(|v_{i_h, j_h}(\kappa(h), \varepsilon(h), n)|^{2M})^{1/2M} \quad (\text{H\"older ineq.}) \\ &= O(q_1(n)^{-M'} n^{-M''}) \end{aligned}$$

where

$$M' = \sum_{\substack{1 \leq r \leq N \\ k_r = 1}} |m_r|, \quad M'' = \sum_{\substack{1 \leq r \leq N \\ k_r \neq 1}} |m_r|.$$

Thus it suffices to prove that

$$(19) \quad \sum_i \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}},$$

is  $o(q_k(n)^2 q_1(n)^{M'} n^{M''})$  in case 1°), and  $o(q_1(n)^{M'+2} n^{M''})$  in case 2°), where the sum is taken on the elements  $i$  of  $[n]^{\mathcal{Q}}$  which satisfy (16).

*Step V. Cases 1°) and 2°)* We are going to prove it as an application of lemma 1.12. The sum of (19) is the sum, on the same  $i$ 's, of

$$\begin{aligned} & D(1, n)_{i_{M(1)+1}} D(2, n)_{i_{M(2)+1}} \cdots D(N-1, n)_{i_{M(N-1)+1}} D(N, n)_{i_1} \\ & \times D(N+1, n)_{i_{M+1}} D(N+2, n)_{i_{M(N+2)}} D(N+3, n)_{i_{M(N+3)}} \cdots D(2N, n)_{i_{M(2N)}}. \end{aligned}$$

i.e. of

$$\begin{aligned} & D(N, n)_{i_1} D(1, n)_{i_{M(1)+1}} D(2, n)_{i_{M(2)+1}} \cdots D(N-1, n)_{i_{M(N-1)+1}} \\ & \times D(N+1, n)_{i_{M+1}} D(N+2, n)_{i_{M(N+2)}} D(N+3, n)_{i_{M(N+3)}} \cdots D(2N, n)_{i_{M(2N)}}. \end{aligned}$$

Adding some matrices of the type  $p_k(n)$  in the product, one sees that this sum is equal to

$$(20) \quad \sum_{i \in [n]^{\mathcal{Q}}} \prod_{h=1}^{2M} A_{i_h}(h, n),$$

where for  $h \in [2M]$ ,

$$A(h, n) = \begin{cases} p_{k(h-1)}(n) D(r, n) p_{k(h)}(n) & \text{if } \exists r \in \{0, \dots, N-1\}, \\ & h = M(r) + 1 \\ p_{k(h)}(n) & \text{if } h \in [M] \text{ and } h \notin \\ & \{M(0) + 1, \dots, M(N-1) + 1\} \\ p_{k(2M)}(n) D(N+1, n) p_{k(M+1)}(n) & \text{if } h = M+1 \\ p_{k(h)}(n) D(r, n) p_{k(h+1)}(n) & \text{if } \exists r \in \{N+2, \dots, 2N\}, \\ & h = M(r) \\ p_{k(h)}(n) & \text{if } h \in [2N] \text{ and } h \notin \\ & [M+1] \cup \{M(N+2), \dots, M(2N)\} \end{cases}$$

Now, note that by lemma 4.2.2 of [HP00] (or because the uniform distribution on the unitary group is left and right invariant), if  $i \in [n]^{\mathcal{Q}}$  corresponds to a nonzero term in equation (14), then for all  $\kappa \in \mathbb{N} \times [d]$ , for all  $1 \leq \alpha, \beta \leq n$ ,

$$\begin{aligned} |\{h; i_h = \alpha, \kappa(h) = \kappa, \varepsilon(h) = 1\}| &= |\{h; j_h = \alpha, \kappa(h) = \kappa, \varepsilon(h) = -1\}|, \\ |\{h; j_h = \beta, \kappa(h) = \kappa, \varepsilon(h) = 1\}| &= |\{h; i_h = \beta, \kappa(h) = \kappa, \varepsilon(h) = -1\}|. \end{aligned}$$

Thus, if  $\{h\}$  is a singleton class in  $\mathcal{Q}$ , and if  $\mathbb{E}_{\mathcal{Q}}$  is non null, one has, using the cyclic orders on  $[M]$  and on  $\{M+1, M+2, \dots, 2M\}$ ,

$$\begin{aligned} \varepsilon(h) = 1 &\Rightarrow \kappa(h) = \kappa(h-1), \varepsilon(h-1) = -1, \\ \text{and } \varepsilon(h) = -1 &\Rightarrow \kappa(h) = \kappa(h-1), \varepsilon(h-1) = 1. \end{aligned}$$

It clearly follows that there exists  $r \in \{0, \dots, 2M\} - \{M+1\}$  such that  $h = M(r) + 1$ . Moreover, by hypothesis, in case 1°),  $A(h, n) = D(r, n)$  is a diagonal matrix with null trace, and, in case 2°),  $A(h, n) = D(r, n)$  is a diagonal matrix with null trace whenever  $h \notin M, M+1$ .



Case 1°) Let us apply the lemma with  $\mathcal{P} = \mathcal{Q}$  and

$$\mathcal{H} = \{h \in [2M]; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . By the lemma, the sum of (19), which is equal to the sum of (20), is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2})$$

But for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| \leq x + (|S| - x)/2$ , so  $|\mathcal{X}| - x/2 \leq |S|/2$ . So,

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M)/2 = M' + M'',$$

Moreover, since for all  $h \in [2M]$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h; k(h) \neq 1\}| = M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M'} n^{M''}\right).$$

The sum of (19) is  $O\left(q_1(n)^{M'} n^{M''}\right)$ , and the proposition is proved.

Case 2°) Let us prove that the sum of (19) is  $o\left(q_1(n)^{2+M'} n^{M''}\right)$  by induction on the number  $|\mathcal{Q}|$  of classes of  $\mathcal{Q}$ . Let us denote, for  $B, B'$  class of  $\mathcal{Q}$ ,  $\mathcal{Q}(M \rightarrow B)$  the partition of  $[2M]$  obtained from  $\mathcal{Q}$  by linking classes  $B$  and  $\{M\}$ , and  $\mathcal{Q}(M \rightarrow B, M+1 \rightarrow B')$  the partition obtained from  $\mathcal{Q}$  by linking classes  $B$  and  $\{M\}$ , and then the classes  $B'$  (which has become  $B \cup \{M\}$  if  $B' = B$ ) and  $\{M+1\}$ . Let  $J(n)$  be the set of elements  $j$  of  $[n]$  such that  $D(N, n)_j \neq 0$ . Note that, as noted to step II,  $|J(n)| \leq q_1(n)$ , and, by definition of  $D(N+1, n)$ ,  $J(n)$  is also the set of elements  $j$  of  $[n]$  such that  $D(N+1, n)_j \neq 0$ .

- If  $\mathcal{Q}$  has only one class, by boundedness hypothesis and since  $|J(n)| \leq q_1(n)$ , the sum of (19) is  $O(q_1(n))$ .

- Assume that the conclusion holds to ranks  $1, \dots, |\mathcal{Q}| - 1$ .

- If neither  $\{M\}$  nor  $\{M+1\}$  are singletons of  $\mathcal{Q}$ , we apply the lemma, as for case 1°), with  $\mathcal{P} = \mathcal{Q}$  and

$$\mathcal{H} = \{h \in [2M]; k(h) = 1\}.$$

It leads to the conclusion in the same way as in case 1°).

- If exactly one of  $\{M\}, \{M+1\}$ , say  $\{M\}$ , is a singleton of  $\mathcal{Q}$ , define  $\mathcal{P} = \mathcal{Q} - \{\{M\}\}$ , partition of the set  $[2M] - \{M\}$ . Note that the sum of (19), which is equal to the sum of (20),

is

$$\begin{aligned}
& \sum_{j \in J(n)} \left[ A(M, n)_j \sum_{i \in ([n] - \{j\})^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right] \\
= & \sum_{j \in J(n)} \left[ A(M, n)_j \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) - \sum_{\substack{i \in [n]^{\mathcal{P}} \\ \exists h, i_h = j}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) \right] \\
= & \left( \sum_{j \in J(n)} A(M, n)_j \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) - \left( \sum_{\substack{B \in \mathcal{Q} \\ B \neq \{M\}}} \sum_{i \in [n]^{\mathcal{Q}(M \rightarrow B)}} \prod_{h=1}^{2N} A_{i_h}(h, n) \right)
\end{aligned}$$

So by the induction hypothesis, for all  $B \in \mathcal{P}, B \neq \{M\}$ , we have

$$\sum_{i \in [n]^{\mathcal{Q}(M \rightarrow B)}} \prod_{h=1}^{2N} A_{i_h}(h, n) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

So it suffices to prove that we have

$$(21) \quad \left( \sum_{j \in J(n)} A(M, n)_j \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

But by boundedness hypothesis,  $\sum_{j \in J(n)} A(M, n)_j = 0(|J(n)|) = 0(q_1(n))$ . So it suffices to prove that the second sum in (21) is  $o\left(q_1(n)^{M'+1} n^{M''}\right)$ . Define

$$\mathcal{H} = \{h \in [2M] - \{M\}; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . So by the lemma, the second sum in (21) is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}).$$

Since, as noted above, for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| - x/2 \leq |S|/2$ , we have

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M - 1)/2 = M' + M'' - 1/2,$$

Moreover, since for all  $h \in [2M] - \{M\}$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h \in [2M] - \{M\}; k(h) \neq 1\}| \leq M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M' - 1/2} n^{M''}\right),$$

thus the conclusion holds.

- If  $\{M\}$  and  $\{M+1\}$  are singletons of  $\mathcal{Q}$ , define  $\mathcal{P} = \mathcal{Q} - \{\{M\}, \{M+1\}\}$ , partition of the set  $[2M] - \{M, M+1\}$ . Note that the sum of (19), which is equal to the sum of (20), is

$$\sum_{\substack{j, j' \in J(n) \\ j \neq j'}} \left[ A(M, n)_j A(M+1, n)_{j'} \sum_{i \in ([n] - \{j, j'\})^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right].$$

Now, let us define  $\mathcal{Q}'$  to be the partition of  $[2M]$  obtained from  $\mathcal{Q}$  by linking classes  $\{M\}$  and  $\{M+1\}$ . The sum of (19) can be written

$$\left( \sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right) - \sum_{i \in [n]^{\mathcal{Q}'}} \prod_{h=1}^{2N} A_{i_h}(h, n) - \sum_{\mathcal{R}} \sum_{i \in [n]^{\mathcal{R}}} \prod_{h=1}^{2N} A_{i_h}(h, n),$$

where in the last sum,  $\mathcal{R}$  is taken in the set

$$\{\mathcal{Q}(M \rightarrow B, M+1 \rightarrow B'); B, B' \in \mathcal{Q}\}.$$

So by the induction hypothesis, it suffices to prove that

$$(22) \quad \left( \sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

But by boundedness hypothesis,  $\sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} = 0(|J(n)|^2) = 0(q_1(n)^2)$ . So it suffices to prove that the second sum in (22) is  $o\left(q_1(n)^{M'} n^{M''}\right)$ . Define

$$\mathcal{H} = \{h \in [2M] - \{M, M+1\}; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . So by the lemma, the second sum in (22) is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}).$$

Since, as noted above, for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| - x/2 \leq |S|/2$ , we have

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M - 2)/2 = M' + M'' - 1,$$

Moreover, since for all  $h \in [2M] - \{M, M+1\}$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h \in [2M] - \{M, M+1\}; k(h) \neq 1\}| \leq M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M'-1} n^{M''}\right),$$

thus the conclusion holds.  $\square$

**1.6. Proof of theorem 1.8 (case where all  $\rho_k$ 's are positive).** *Step I.* Let us introduce, on the space of  $n \times n$  matrices, the Schatten  $p$ -norms  $\|\cdot\|_p$  ( $p \in [1, \infty]$ ), defined, for  $p < \infty$ , by

$$\|X\|_p = (\operatorname{tr} |X|^p)^{\frac{1}{p}},$$

and  $\|\cdot\|_\infty = \|\cdot\|$ . They satisfy classical Hölder inequalities ([N74]).

Fix  $k, l \in [d]$  and consider, for  $n \geq 1$ , a bi-unitarily invariant  $q_k(n) \times q_l(n)$  random matrix  $R(n)$  such that the sequence is uniformly bounded for  $\|\cdot\|$  by  $C > 0$  and such that the moments of the singular law of  $R(n)$  converge in probability to constants. Note first that, by Carleman criterion, it implies that its singular law converges in probability to a probability measure  $\mu$ . Then, as noticed in subsection 1.1, the distribution of  $R(n)$  can be realized as the distribution of  $U(n)\Lambda(n)V(n)$  where  $U(n)$ ,  $\Lambda(n)$ ,  $V(n)$  are independent,  $U(n)$ ,  $V(n)$  are respectively  $q_k(n) \times q_k(n)$ ,  $q_l(n) \times q_l(n)$  uniform random unitary matrices, and

$$\Lambda(n) = [\delta_i^j \lambda(i, n)]_{\substack{1 \leq i \leq q_k(n) \\ 1 \leq j \leq q_l(n)}} \text{ and } 0 \leq \lambda(1, n) \leq \dots \leq \lambda(\min(q_k(n), q_l(n)), n).$$

Choose deterministic

$$0 \leq \xi(1, n) \leq \dots \leq \xi(\min(q_k(n), q_l(n)), n) \leq C$$

such that the uniform distribution on the  $\xi(1, n)$ 's converges weakly to  $\mu$ . Now set

$$\Xi(n) = [\delta_i^j \lambda(i, n)]_{\substack{1 \leq i \leq q_k(n) \\ 1 \leq j \leq q_l(n)}} \text{ and } N(n) = U(n)\Xi(n)V(n).$$

Let us prove that for all  $P$  polynomial in noncommutative random variables  $X, X^*$ , for all  $p \in [1, \infty)$ ,  $\|P(\tilde{R}(n)) - P(\tilde{N}(n))\|_p$  converges in probability to zero.

By lemma 4.3.4 p. 152 of [HP00], for all  $p \geq 1$ ,

$$\|\tilde{R}(n)^* - \tilde{N}(n)^*\|_p = \|\tilde{R}(n) - \tilde{N}(n)\|_p = \|\tilde{\Lambda}(n) - \tilde{\Xi}(n)\|_p$$

converges in probability to zero. So, for all  $m \geq 1$ ,  $\varepsilon_1, \dots, \varepsilon_m \in \{., *\}$ ,

$$\begin{aligned} & \| \tilde{R}(n)^{\varepsilon_1} \dots \tilde{R}(n)^{\varepsilon_m} - \tilde{N}(n)^{\varepsilon_1} \dots \tilde{N}(n)^{\varepsilon_m} \|_p \\ & \leq \sum_{l=1}^m \| \tilde{R}(n)^{\varepsilon_1} \dots \tilde{R}(n)^{\varepsilon_{l-1}} (\tilde{R}(n)^{\varepsilon_l} - \tilde{N}(n)^{\varepsilon_l}) \tilde{N}(n)^{\varepsilon_{l+1}} \dots \tilde{N}(n)^{\varepsilon_m} \|_p \\ & \leq \sum_{l=1}^m C^{m-1} \| \tilde{R}(n)^{\varepsilon_l} - \tilde{N}(n)^{\varepsilon_l} \|_p, \end{aligned}$$

which proves the result.

Note that in the case where  $R$  is a uniformly bounded unitarily invariant  $q_k(n) \times q_k(n)$  hermitian random matrix, the moments of the spectral law of which converge in probability, then the same work can be done, replacing  $V(n)$  by  $U(n)^*$  and singular values by eigenvalues, and the same conclusion holds: for all  $P \in \mathbb{C}[X]$ , for all  $p \in [1, \infty)$ ,  $\|P(\tilde{R}(n)) - P(\tilde{N}(n))\|_p$  converges in probability to zero.

*Step II.* Let us denote by  $\mathcal{C}$  the linear span of the  $c(i)$ 's. By (7) and (8), for all  $n$ , there is an algebra morphism  $\psi_n$  from  $\mathcal{C}$  to the linear span  $\mathcal{C}_n$  of the  $\tilde{C}(i, n)$ 's. By (vi), for all  $c \in \mathcal{C}$ , for all  $k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of  $\psi_n(c)$  tends to  $\varphi_k(c_{kk})$ .

Since  $1 \in \mathcal{C}$ , it suffices to prove that for all  $m \geq 0$ , for all  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m$ , for all  $P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ , for all  $d(0), \dots, d(m) \in \mathcal{C}$ ,

the normalized trace of the  $k$ -th diagonal block of

$$(23) \quad \psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

converges in probability to the  $k$ -th coordinate of

$$(24) \quad \mathbb{E} \left( d(0) \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right),$$

for all  $k \in [d]$ .

*Step III.* Let us prove that we can remove  $d(0)$ , i.e. that in order to prove the convergence in probability, for all  $k \in [d]$  and all  $d(0), \dots, d(m) \in \mathcal{C}$ , of the normalized trace of the  $k$ -th diagonal block of (23) to the  $k$ -th coordinate of (24), it suffices to prove the convergence in probability, for all  $k \in [d]$  and all  $d(1), \dots, d(m) \in \mathcal{C}$ , of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d'(j))$$

to the  $k$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d'(j) \right).$$

The idea is to use the formula  $\text{Tr } XY = \text{Tr } YX$  and to move  $d(0)$  from the left to the right of the product, which comes down to replace  $d(m)$  by  $d(m)d(0)$ . But the formula  $\mathbb{E}(XY) = \mathbb{E}(YX)$  is not always true, so we have to be careful.

First, by linearity of  $\mathbb{E}$ , we can suppose that  $d(0)$  is a simple element. Thus there exists  $l_0 \in [d]$  such that

$$d(0) = d(0)p_{l_0}, \quad \psi_n(d(0)) = \psi_n(d(0))p_{l_0}(n).$$

Now let us fix  $k \in [d]$ . We have to prove that

$$\frac{1}{q_k(n)} \text{Tr } p_k(n) \psi_n(d(0)) p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k(n)$$

converges in probability to

$$\varphi_k \left( p_k d(0) p_{l_0} \left[ \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right] p_k \right),$$

which amounts, since  $\text{Tr } XY = \text{Tr } YX$  and by (3), to prove that

$$\frac{q_{l_0}(n)}{q_k(n)} \frac{1}{q_{l_0}(n)} \text{Tr } p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k(n) \psi_n(d(0)) p_{l_0}(n)$$

converges in probability to

$$\frac{\rho_{l_0}}{\rho_k} \varphi_{l_0} \left( \left[ \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right] p_k d(0) p_{l_0} \right).$$

Since  $\frac{q_{l_0}(n)}{q_k(n)} \xrightarrow{n \rightarrow \infty} \frac{\rho_{l_0}}{\rho_k}$ , it amounts to prove that

$$\frac{1}{q_{l_0}(n)} \text{Tr } p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k \psi_n(d(0)) p_{l_0}(n)$$

converges in probability to

$$\varphi_{l_0} \left( \left[ \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right] p_k d(0) p_{l_0} \right).$$

If one replaces  $d(m)$  by  $d(m)p_k d(0)$ , it appears that it suffices to prove that for all  $l \in [d]$ , the normalized trace of the  $l$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \psi_n(p_k d(0))$$

converges in probability to the  $l$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right).$$

*Step IV.* Then, we claim that it suffices to prove it when the  $R(s, k, l, n)$ 's are replaced by the  $N(s, k, l, n)$ 's, where the  $N(s, k, l, n)$ 's, are defined from the  $R(s, k, l, n)$ 's like  $N(n)$  from  $R(n)$  in the step I. Indeed, the normalized trace of any diagonal block of the difference between

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

and

$$\prod_{j=1}^m P_j(\tilde{N}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

has an absolute value less or equal than the norm  $\|\cdot\|_1$  of their difference times one of the  $n/q_k(n)$ , which is less or equal, by a decomposition like in step I, than

$$\max_{k \in [d]} \frac{n}{q_k(n)} \sum_{j=1}^m C^m D^{m-1} \|P_j(\tilde{R}(s_j, k_j, l_j, n)) - P_j(\tilde{N}(s_j, k_j, l_j, n))\|_1,$$

where  $D = \max_{\substack{|z|, |z'| \leq C \\ j \in [m]}} |P_j(z, z')|$  which converges in probability to zero, by step I and because all  $\rho_k$ 's are positive.

So let us prove that the normalized trace of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{N}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

converges in probability to the  $k$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right),$$

for all  $k \in [d]$ .

*Step V.* We will prove it by induction on  $m$ . If  $m = 0$ , the result is clear. In the other case, let us suppose the result to be proved to the ranks  $0, \dots, m-1$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Since  $\mathcal{C}$  is an algebra containing  $p_1, \dots, p_d$ , if, for a certain  $j \in [m]$ , one would replace  $X(j, n)$  by one of the  $p_k(n)$ 's and  $x(j)$  by the corresponding  $p_k$ , then the result would follow from the induction hypothesis (and from step III if  $j = 1$ ). Thus, by linearity, one can, for all  $j \in [m]$ , add a linear combination of the  $p_k(n)$ 's to  $X(j, n)$  and the same linear combination of the  $p_k$ 's to  $x(j)$ . Therefore we can assume that for all  $j$ ,  $E(x(j)) = 0$ . By linearity, we can moreover suppose that for all  $j$ ,  $x(j)$  is a simple element. Note that these suppositions imply that for all  $n, j$ ,  $X(n, j)$  has only one non zero block, and satisfies

$$E(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

From now on, we use the cyclic order on  $[m]$ . This means that to put the index  $m+1$  on an element amounts to put the index 1. As simple elements, the  $x(j)$ 's have the following property: the product of any of the  $x(j)$ 's by one of the  $p_k$ 's is either zero or  $x(j)$ , and the same holds for  $X(j, n)$ 's with  $p_k(n)$ 's. Thus, if, for a certain  $j \in [m]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , one would replace  $d(j)$  by a linear combination of the  $p_k(n)$ 's, then the result would follow from the induction hypothesis (and from step III if  $j = m$ ). So one can suppose that for all  $j \in [m-1]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ ,  $E(d(j)) = 0$ .

At last, for  $j \in [m]$  such that  $(s_j, k_j, l_j) \neq (s_{j+1}, k_{j+1}, l_{j+1})$ , one can write

$$d(j) = d'(j) + \sum_{k \in [d]} \lambda_k(j) p_k, \quad \text{with } E(d'(j)) = 0.$$

Then, by linearity, one can suppose that for all such  $j$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

To conclude, we have shown that we only have to prove that

$$E \left( \prod_{j=1}^m X(j, n) \psi_n(d(j)) \right)$$

converges in probability to

$$E \left( \prod_{j=1}^m x(j) d(j) \right),$$

under the hypothesis that all  $x(j)$ 's are simple elements and satisfy  $E(x(j)) = 0$ , and for all  $j \in [m]$ ,

$$E(d(j)) = 0 \text{ or } \{d(j) \in \{p_1, \dots, p_d\} \text{ and } (s_j, k_j, l_j) \neq (s_{j+1}, k_{j+1}, l_{j+1})\}.$$

Note that under this hypothesis, by E-freeness,  $E \left( \prod_{j=1}^m x(j) d(j) \right) = 0$ .

*Step VI.* For all  $s, k, l, n$  with  $k \neq l$ , the matrix  $N(s, k, l, n)$  has been defined by the work of step I, so we can write

$$N(s, k, l, n) = U(s, k, l, n) \Xi(s, k, l, n) V(s, k, l, n),$$

with  $U(s, k, l, n)$ ,  $V(s, k, l, n)$  uniform random unitary matrices with respective sizes  $q_k(n)$ ,  $q_l(n)$ , and  $\Xi(s, k, l, n)$  a deterministic diagonal  $q_k(n) \times q_l(n)$  matrix. In the same way, when  $s$  is even,  $N(s, k, k, n)$  can be written

$$N(s, k, l, n) = U(s, k, k, n)\Xi(s, k, k, n)U^*(s, k, k, n),$$

and when  $s$  is odd,  $N(s, k, k, n)$  is a uniform random matrix of  $\mathbb{U}_k(n)$ .

Consider  $j \in [m]$ . If  $k_j \neq l_j$ , then  $X(j, n)$  is of one of the 4 following types:

- (1)  $\tilde{U}(s_j, k_j, l_j, n)\tilde{\Delta}_j\tilde{V}(s_j, k_j, l_j, n)$ , where  $\Delta_j$  is a  $q_{k_j}(n) \times q_{l_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an odd polynomial.
- (2)  $\tilde{V}(s_j, k_j, l_j, n)^*\tilde{\Delta}_j\tilde{U}(s_j, k_j, l_j, n)^*$ , where  $\Delta_j$  is a  $q_{l_j}(n) \times q_{k_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an odd polynomial.
- (3)  $\tilde{U}(s_j, k_j, l_j, n)\tilde{\Delta}_j\tilde{U}(s_j, k_j, l_j, n)^*$ , where  $\Delta_j$  is a  $q_{k_j}(n) \times q_{k_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an even polynomial.
- (4)  $\tilde{V}(s_j, k_j, l_j, n)^*\tilde{\Delta}_j\tilde{V}(s_j, k_j, l_j, n)$ , where  $\Delta_j$  is a  $q_{l_j}(n) \times q_{l_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an even polynomial.

In the case where  $k_j = l_j$ , when  $s_j$  is even, only the last type is possible, and when  $s_j$  is odd,  $X(j, n)$  is a linear combination of positive or negative powers of  $\tilde{N}(s_j, k_j, l_j, n)$ . By linearity again, this combination can be replaced by a non zero power of  $\tilde{N}(s_j, k_j, l_j, n)$ .

Note that in all types previously presented, the constant diagonal matrix  $\Delta_j$  satisfies

$$E(\tilde{\Delta}_j) \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, for all  $n$ , the family of random matrices from  $\mathbb{U}_k(n)$ 's arising in these decompositions are independent, so the result follows from case 1<sup>o</sup>) of proposition 1.14 and remark 1.15.

**1.7. Proof of theorem 1.10 (case where  $\rho_1 = 0$ , all other  $\rho_k$ 's are positive).** We shall follow closely the proof of theorem 1.8 (case where all  $\rho_k$ 's are positive) given in the previous section, but because of the weaker hypothesis (singular values of non hermitian random matrices and spectrum of hermitian ones are now supposed to be deterministic), we shall skip steps I and IV, in which non hermitian (resp. hermitian) random matrices were approximated by random matrices with deterministic singular values (resp. deterministic spectrum). However, the hypothesis  $\rho_1 = 0$  will make things slightly harder.

As previously, let us denote by  $\mathcal{C}$  the linear span of the  $c(i)$ 's, and define  $\psi_n$  in the same way as before.

As previously again, since  $1 \in \mathcal{C}$ , it suffices to prove that for all  $m \geq 0$ , the following proposition holds:

$\forall (s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, \forall P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ ,

$\forall d(0), \dots, d(m) \in \mathcal{C}, \forall k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of

$$\psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$



$$(25) \quad \text{converges in probability to } \varphi_k \left( d(0) \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right).$$

Note first that by linearity, it suffices to prove it when  $d(0), d(m)$  are supposed to be simple, which implies that there exists  $u, v \in [d]$  such that

$$(26) \quad p_u d(0) = d(0), \quad d(m) p_v = d(m).$$

Let us denote by  $H(m, u, v)$  the proposition (25), where  $d(0), d(m)$  are moreover supposed to satisfy (26).

We have to prove  $H(m, u, v)$  for all  $m \geq 0$  and all  $u, v \in [d]$ . Note first that if  $u \neq v$ ,  $H(m, u, v)$  is immediate. Let us denote by  $H'(m)$  the proposition obtained from (25) removing  $d(0)$ :

$\forall (s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, \forall P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ ,  
 $\forall d(1), \dots, d(m) \in \mathcal{C}, \forall k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

converges in probability to  $\varphi_k \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right)$ .

Following step III of the previous section, we see that for all  $u \in [d] - \{1\}$ ,  $H'(m)$  implies  $H(m, u, u)$ . Thus we only have to prove that for all  $m \geq 0$ ,  $H'(m)$  and  $H(m, 1, 1)$  hold.

Let us prove it by induction on  $m$ . For  $m = 0$ , the result is immediate. Consider  $m \geq 1$  such that  $H'(l)$  and  $H(l, 1, 1)$  hold for all  $l < m$ .

Let us first prove  $H(m)$ . Consider  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*, d(1), \dots, d(m) \in \mathcal{C}$ . Let us prove that

$$\mathbb{E} \left( \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right)$$

converges in probability to  $ds \mathbb{E}(\prod_{j=1}^m \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j))$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Similarly to step V of the previous section, by the induction hypothesis and the fact that  $H'(m-1)$  implies  $H(m-1, u, u)$ , we can suppose that for all  $n, j$ ,  $X(n, j)$  has only one non zero block and satisfies

$$\mathbb{E}(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

We can also suppose that for all  $j \in [m]$  such that with the cyclic order  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

Under these assumptions, we only have to prove that

$$\mathbb{E} \left( \prod_{j=1}^m X(j, n) \psi_n(d(j)) \right)$$

converges in probability to

$$\mathbb{E} \left( \prod_{j=1}^m x(j)d(j) \right) = 0.$$

Following step VI of the previous section, it appears to be an application of case 1°) of proposition 1.14 and remark 1.15.

Let us now prove  $H(m, 1, 1)$ . Consider  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m$ ,  $P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*, d(0), \dots, d(m) \in \mathcal{C}$ . Let us prove that

$$\mathbb{E} \left( \psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right)$$

converges in probability to  $ds \mathbb{E}(d(0) \prod_{j=1}^m \prod_{j=1}^m P_j(a(s_j, k_j, l_j))d(j))$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Again, similarly to step V of the previous section, we can suppose that for all  $n \geq 1, j \in [m-1]$ ,  $X(n, j)$  has only one non zero block and satisfies

$$\mathbb{E}(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

We can also suppose that for all  $j \in [m-1]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

Following again step VI of the previous section, it appears to be an application of case 2°) of proposition 1.14 and remark 1.15.

**1.8. Computational consequences of the asymptotic freeness with amalgamation of the random matrices.** In this section, we are going to explain how Theorems 1.8 and 1.10 can be used to compute the limit of normalized traces of product of rectangular random matrices. In order to make the exposition more simple, we are going to consider only two random matrices, with the same size. So let us consider two sequences  $q_1(n), q_2(n)$  of positive integers which tend to infinity and suppose that the ratio  $q_1(n)/q_2(n)$  tends to a number  $\lambda \in [0, 1]$ . Consider then, for all  $n$ , a pair  $M_1(n), M_2(n)$  of  $q_1(n) \times q_2(n)$  independent random matrices, distributions of which are bi-unitarily invariant, and the singular laws of which are supposed to converge to deterministic probability measures. We make also the technical hypothesis necessary to apply the theorems: we suppose the matrices uniformly bounded (which implies that the limit laws have compact support), and that if  $\lambda = 0$ , the singular values of the matrices are deterministic.

We are going to explain how to compute by induction the limit of the normalized trace of a product of the type

$$(27) \quad A(n) = M_{i_1}(n) M_{i_2}(n)^* \cdots M_{i_{2p-1}}(n) M_{i_{2p}}(n)^*$$

or of the type

$$(28) \quad B(n) = M_{i_1}(n)^* M_{i_2}(n) \cdots M_{i_{2p-1}}(n)^* M_{i_{2p}}(n),$$

with  $p \geq 1$  and  $i_1, \dots, i_{2p} \in \{1, 2\}$ . The first thing to do is to define the sequence  $0 = l_0 < l_1 < l_2 < \dots < l_m = 2p$  ( $m \geq 1$ ) defined by

$$i_1 = i_2 = \dots = i_{l_1} \neq i_{l_1+1} = \dots = i_{l_2} \neq i_{l_2+1} = \dots = i_{l_{m-1}} \neq i_{l_{m-1}+1} = \dots = i_{l_m}.$$

We define, for  $k \in [m]$ ,  $P_k(n)$  (resp.  $Q_k(n)$ ) to be the product of factors with ranks between  $l_{k-1} + 1$  and  $l_k$  in (27) (resp. in (28)). So we have to compute the limit normalized trace of

$P_1(n) \cdots P_m(n)$  (resp. of  $Q_1(n) \cdots Q_m(n)$ ). In order to give an algorithm to do this, we have to consider a more general problem: to compute the limit normalized trace of

$$(P_1(n) - x_1 I) \cdots (P_m(n) - x_m I) \text{ (resp. of } (Q_1(n) - x_1 I) \cdots (Q_m(n) - x_m I)),$$

where in the expression  $P_k(n) - x_k I$  (resp.  $Q_k(n) - x_k I$ ),

- when  $l_k - l_{k-1}$  is even,  $I$  represents the identity matrix with the size of  $P_k(n)$  (resp.  $Q_k(n)$ ) and  $x_k$  is a complex number independent of  $n$ ,
- when  $l_k - l_{k-1}$  is odd,  $x_k = 0$ .

1.8.1. *General strategy for all  $\lambda \in [0, 1]$ . Step I.* If  $m = 1$ , i.e. all  $i'_k$ 's are the same, then the normalized trace of (27) converges in probability to the  $2p$ -th moment of the limit singular law of  $M_{i_1}(n)$  minus  $x_1$ , and the normalized trace of (28) tends to  $\lambda$  times the limit normalized trace of (27) minus  $x_1$ .

*Step II.* If  $m \neq 1$ . Let, for all  $k \in [m]$  such that  $l_k - l_{k-1}$  is even,  $a_k$  denote the limit normalized trace of the matrix  $P_k(n) - x_k I$  (resp.  $Q_k(n) - x_k I$ ) (which can be computed by Step I). Let, for all other  $k$ 's in  $[m]$ ,  $a_k$  be zero.

- (i) If  $a_1, \dots, a_m$  are all zero, then (27) (resp. (28)) has normalized trace converging to zero in probability.
- (ii) If there is  $k \in [m]$  such that  $a_k \neq 0$ , then the normalized trace of (27) tends to  $a_k$  times the limit normalized trace of

$$(P_1(n) - x_1 I) \cdots (P_{k-1}(n) - x_{k-1} I)(P_{k+1}(n) - x_{k+1} I) \cdots (P_m(n) - x_m I)$$

plus the limit normalized trace of

$$(P_1(n) - x_1 I) \cdots (P_{k-1}(n) - x_{k-1} I)(P_k(n) - (x_k + a_k) I)(P_{k+1}(n) - x_{k+1} I) \cdots (P_m(n) - x_m I)$$

It is easy to see that the induction is actually working.

1.8.2. *Case where  $\lambda = 1$ .* In this case, everything happens like if the matrices considered were square: we have random matrix models for  $R$ -diagonal elements.

1.8.3. *Case where  $\lambda = 0$ .* In this case, one can easily see that:

- (28) has always an null limit normalized trace,
- the same holds for (27), except if  $l_1, l_2, \dots, l_m$  are all even numbers, i.e. if for all  $k = 1, \dots, p$ ,  $i_{2k-1} = i_{2k}$ , i.e. if any matrix  $M_1(n)$  or  $M_2(n)$  in the product is followed by respectively  $M_1(n)^*$  or  $M_2(n)^*$ , in which case the limit can be computed using the well known asymptotic freeness of square random matrices. It is the first way to see that if one denotes by  $\mu_1, \mu_2$  the limit singular laws of  $M_1(n), M_2(n)$ , then the singular law of  $M_1(n) + M_2(n)$  converges to the distribution on  $[0, +\infty)$  whose push-forward by the function  $t \rightarrow t^2$  is the free convolution (as defined in Voiculescu's papers) of the push-forwards, by the function  $t \rightarrow t^2$ , of  $\mu_1$  and  $\mu_2$ . This result will be formulated later in terms of rectangular convolution with ratio 0 (see proposition 3.8).

## 2. THE RECTANGULAR FREE ADDITIVE CONVOLUTION

**2.1. Definition for compactly supported probability measures.** For  $\nu$  probability measure on  $\mathbb{R}$ , denote by  $\tilde{\nu}$  the *symmetrization* of  $\nu$ , which is the probability measure defined by  $\tilde{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(-B))$  for all Borel set  $B$ .

One of the interests of modeling with free probabilities the asymptotic behavior of square random matrices is the possibility to compute the asymptotic spectral law of an hermitian matrix which is a function of several independent random matrices. In particular ([V91]), if  $M(1, n)$ ,  $M(2, n)$  are independent hermitian unitarily invariant random matrices whose spectral laws tend respectively to  $\nu_1, \nu_2$  when their dimension  $n$  goes to infinity, then the spectral law of  $M(1, n) + M(2, n)$  tends to  $\nu_1 \boxplus \nu_2$  (free additive convolution of  $\nu_1, \nu_2$ ), and the spectral law of  $\sqrt{M(1, n)}M(2, n)\sqrt{M(1, n)}$  (when the matrices are positive) tends to  $\nu_1 \boxtimes \nu_2$  (free multiplicative convolution of  $\nu_1, \nu_2$ ). In the same way (combine Theorem 4.3.11 of [HP00] and Propositions 3.5, 3.6 of [HL00]), if the matrices  $M(1, n), M(2, n)$  are non hermitian bi-unitarily invariant (and still independent), with asymptotic singular distributions  $\nu_1, \nu_2$ , then the symmetrization of the singular distribution of  $M(1, n) + M(2, n)$  tends to  $\tilde{\nu}_1 \boxplus \tilde{\nu}_2$ , and the push-forward, by the function  $x \rightarrow x^2$ , of the singular distribution of  $M(1, n)M(2, n)$  is the free multiplicative convolution of the push-forwards, by  $x \rightarrow x^2$ , of  $\nu_1, \nu_2$ .

Now, suppose that  $M(1, n), M(2, n)$ , instead of being square, are rectangular (independent and bi-unitarily invariant)  $q_1(n) \times q_2(n)$  random matrices whose singular laws tend to  $\nu_1, \nu_2$ . We keep the notations introduced in section 1.2, but suppose that  $d = 2$ :

$$q_1(n) + q_2(n) = n, \quad q_1(n), q_2(n) \xrightarrow{n \rightarrow \infty} \infty, \quad \frac{q_1(n)}{n} \xrightarrow{n \rightarrow \infty} \rho_1 \geq 0, \quad \frac{q_2(n)}{n} \xrightarrow{n \rightarrow \infty} \rho_2 > 0.$$

Since for  $1 \leq q \leq q'$ , the singular law  $\nu$  of a  $q \times q'$  matrix  $M$  is related to the singular law  $\nu'$  of  $M^*$  by

$$\nu' = \frac{q' - q}{q'} \delta_0 + \frac{q}{q'} \nu,$$

we can, without restriction, suppose that  $q_1(n) \leq q_2(n)$ .

First, the singular distribution of  $M(1, n)M(2, n)^*$  tends to a distribution that can be computed by free probability theory. Indeed, for all  $r \geq 1$  even, the  $r$ -th moment of the singular distribution of  $M(1, n)M(2, n)^*$  is

$$(29) \quad \frac{1}{q_1(n)} \operatorname{Tr}(M(1, n)M(2, n)^* M(2, n)M(1, n)^*)^{r/2},$$

which is equal to

$$\frac{q_2(n)}{q_1(n)} \frac{1}{q_2(n)} \operatorname{Tr}(M(2, n)^* M(2, n)M(1, n)^* M(1, n))^{r/2},$$

which tends to  $\rho_2/\rho_1(n)$  (when  $\rho_1 > 0$ ) times the  $r/2$ -th moment of  $\tau_1 \boxtimes \tau_2$ , where  $\tau_1, \tau_2$  are the limit spectral distributions of  $M(2, n)^* M(2, n), M(1, n)^* M(1, n)$ .  $\tau_1, \tau_2$  can easily be computed from  $\nu_1, \nu_2$ . Thus, if  $\rho_1 > 0$ , the limit of the spectral distribution of  $M(1, n)M(2, n)^*$  can be easily computed using tools of free probability. If  $\rho_1 = 0$ , then  $\rho_1 = 0$ , then the explicit computation of the expectation and of the variance of (29) for  $r = 2$  gives us the convergence of the singular law of  $M(1, n)M(2, n)^*$  to the Dirac measure in zero.

But it does not seem to be possible to compute the singular law of  $M(1, n) + M(2, n)$  with the tools of free probability theory. Indeed, for  $r \geq 1$  even, its  $r$ -th moment is given by

$$\frac{1}{q_1(n)} \operatorname{Tr}((M(1, n) + M(2, n))(M(1, n)^* + M(2, n)^*))^{r/2},$$

which can be expanded into the sum of

$$(30) \quad \frac{1}{q_1(n)} \operatorname{Tr} M(i_1, n) M(i_2, n)^* \cdots M(i_{r-1}, n) M(i_r, n)^*,$$

where the sum is taken over all  $i \in \{1, 2\}^r$ . Asymptotics of normalized traces like in (30) cannot be computed with free probability theory, because our random matrices are not square, and we cannot reduce the problem, as for  $M(1, n)M(2, n)^*$ , to a problem which involves only independent square random matrices. Thus we have to use theorem 1.8 (or theorem 1.10 if  $\rho_1 = 0$ ) to compute the asymptotic singular law of  $M(1, n) + M(2, n)$ .

**Proposition-Definition 2.1.** *Let  $\mu_1, \mu_2$  be two compactly supported symmetric probability measures on the real line.*

(a) *Let, for all  $n \geq 1$ ,  $M(1, n), M(2, n)$  be independent bi-unitarily invariant  $q_1(n) \times q_2(n)$  random matrices, with deterministic singular values if  $\rho_1 = 0$ , uniformly bounded for  $\|\cdot\|$ , and such that for all  $i = 1, 2$ , the symmetrization of the singular law of  $M(d, i)$  converges in probability to  $\mu_i$ . Then the symmetrization of the singular law of  $M(d, 1) + M(d, 2)$  converges in probability to a symmetric probability measure on the real line, denoted  $\mu_{1 \boxplus_\lambda \mu_2}$ , which depends only on  $\mu_1, \mu_2$ , and  $\lambda := q_1(n)/q_2(n)$ .*

(b)  *$\mu_{1 \boxplus_\lambda \mu_2}$  is the unique symmetric measure  $\mu$  such that for all  $r \in \mathbb{N}$  even, the  $r$ -th moment of  $\mu$  is  $\varphi_1(((a_1 + a_2)(a_1 + a_2)^*)^{r/2})$ , where  $a_1, a_2$  are free with amalgamation over  $\mathcal{D}$  elements of  $p_1 \mathcal{A} p_2$ ,  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  is a  $(\rho_1, \rho_2)$  probability space, and for all  $i = 1, 2$ , for all  $r \in \mathbb{N}$  even,  $\varphi_1((a_i a_i^*)^{r/2})$  is the  $r$ -th moment of  $\mu_i$ .*

(c) *The support of  $\mu_{1 \boxplus_\lambda \mu_2}$  is contained in the sum of the convex hulls of the supports of  $\mu_1$  and  $\mu_2$ .*

The binary operation  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures, called *rectangular free convolution with ratio  $\lambda$* , will be extended in section 4.7 to the set of symmetric probability measures, and the same result will stay true without any hypothesis of boundedness.

**Proof.** By theorem 1.8 (or theorem 1.10 if  $\rho_1 = 0$ ), if  $a_1, a_2$  are elements of a  $(\rho_1, \rho_2)$ -probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  as in (b), then  $(\widetilde{M}(1, n), \widetilde{M}(2, n))$  converges in  $\mathcal{D}$ -distribution in probability to  $(a_1, a_2)$ . Thus once the existence of  $\mu_{1 \boxplus_\lambda \mu_2}$  proved, (b) will be immediate.

Let  $[-s_i, s_i]$  be the convex hull of the support of  $\mu_i$  ( $i = 1, 2$ ). By [A61], to prove the convergence in probability of the symmetrization of the singular law of  $M(1, n) + M(2, n)$  to a probability measure with support in  $[-s_1 - s_2, s_1 + s_2]$ , it suffices to prove that for every pair polynomial  $P = \sum_k c_k X^{2k}$  that is non negative on  $[-s_1 - s_2, s_1 + s_2]$ , the normalized trace of  $\sum_k c_k ((M(1, n) + M(2, n))(M(1, n) + M(2, n))^*)^k$  converges to a non negative number. We know that this normalized trace converges to  $\varphi_1[\sum_k c_k ((a_1 + a_2)(a_1 + a_2)^*)^k]$ . To prove that this number is nonnegative, one can suppose that for all  $n$ , for all  $i = 1, 2$ ,

$$M(i, n) = U(i, n) \left[ \delta_k^i \lambda_i(k, n) \right]_{\substack{1 \leq k \leq q_1(n) \\ 1 \leq l \leq q_2(n)}} V(i, n),$$

where  $\lambda_i(1, n), \dots, \lambda_i(q_1(n), n)$  are real numbers of the support of  $\mu_i$  such that the uniform distribution on the  $\lambda_i(k, n)$ 's ( $k = 1, \dots, q_1(n)$ ) converges weakly to  $\mu_i$  and  $U(i, n), V(i, n)$ ,

are independent respectively  $q_1(n) \times q_1(n)$ ,  $q_2(n) \times q_2(n)$  uniform unitary random matrices. In this case, the normalized trace of  $\sum_k c_k ((M(1, n) + M(2, n))(M(1, n) + M(2, n))^*)^k$  is obviously nonnegative.

The fact that the sequence of the  $\varphi_1 \left( ((a_1 + a_2)(a_1 + a_2)^*)^k \right)$  depends only on  $\mu_1$ ,  $\mu_2$  and  $\lambda$  is a direct consequence of the fact that the  $\mathcal{D}$ -distribution of an free with amalgamation over  $\mathcal{D}$  family depends only on the individual  $\mathcal{D}$ -distributions (lemma 1.6).  $\square$

Until the beginning of section 3, we suppose  $\lambda \in (0, 1]$ . Most of the remarks of the present section concern alternative definitions of  $\boxplus_\lambda$ .

**2.2. Another proof of the existence of  $\mu_{1 \boxplus_\lambda \mu_2}$ .** Let  $\mu_1, \mu_2$  be two compactly supported symmetric probability measures on the real line. We consider a  $(\rho_1, \rho_2)$ -probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  with  $\rho_1/\rho_2 = \lambda$ . We also consider  $a_1, a_2 \in p_1 \mathcal{A} p_2$  free with amalgamation over  $\mathcal{D}$ , such that for all  $i = 1, 2$ , for all  $r \in \mathbb{N}$  even,  $\varphi_1((a_i a_i^*)^{r/2})$  is the  $r$ -th moment of  $\mu_i$ . By the GNS representation or by chapter III of [S98], we can suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and that

$$\varphi : \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \rho_1 \varphi_1(x_{11}) + \rho_2 \varphi_2(x_{22}),$$

is a tracial state. So we see that there exists a unique symmetric probability measure  $\mu_{1 \boxplus_\lambda \mu_2}$  on the real line with support contained in the sum of the convex hulls of the supports of  $\mu_1$  and  $\mu_2$ , such that for all  $n$  even, the  $n$ -th moment of  $\mu_{1 \boxplus_\lambda \mu_2}$  is  $\varphi_1((a_i a_i^*)^{n/2})$ . Indeed, the numbers  $\varphi_1((a_i a_i^*)^{r/2})$ 's are the even moments of the distribution of  $((a_1 + a_2)(a_1 + a_2)^*)^{1/2}$ , as a self-adjoint element of  $p_1 \mathcal{A} p_1$ .

### 2.3. Relations between $\boxplus$ and $\boxplus_\lambda$ .

**2.3.1. Introduction.** In all this section, we consider a  $(\rho_1, \rho_2)$ -probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  with  $\rho_1, \rho_2 > 0$ , and we are going to use the following observation: for all  $x \in p_1 \mathcal{A} p_2$ , the symmetrization  $\tilde{\mu}_{(xx^*)^{1/2}}$  of the distribution of  $(xx^*)^{1/2}$  in  $(p_1 \mathcal{A} p_1, \varphi_1)$  and the distribution  $\mu_{x+x^*}$  of  $x + x^*$  in  $(\mathcal{A}, \varphi)$  are linked by the relation

$$(31) \quad \mu_{x+x^*} = 2\rho_1 \tilde{\mu}_{(xx^*)^{1/2}} + (\rho_2 - \rho_1) \delta_0.$$

Indeed, for all  $n \in \mathbb{N}$ ,

$$(x + x^*)^{2n} = \begin{bmatrix} (xx^*)^n & 0 \\ 0 & (x^*x)^n \end{bmatrix} \text{ and } (x + x^*)^{2n+1} = \begin{bmatrix} 0 & (xx^*)^n x \\ (x^*x)^n x^* & 0 \end{bmatrix}.$$

**2.3.2. The case  $\lambda = 1$ .** When  $\lambda = 1$ , the rectangular free convolution with ratio  $\lambda$  is the well known additive free convolution, defined in [VDN91]. There are three ways to prove this. All of them, which are presented just bellow, use the fact that the symmetrization of the distribution of the absolute value of the sum of two free  $R$ -diagonal elements is the free convolution of the symmetrizations of the distributions of their absolute values (Proposition 3.5 of [HL00]).

- (1) First, we can apply the previous corollary to sequences of square matrices. And we know (Theorem 4.3.11 of [HP00]) that independent bi-unitarily invariant square matrices are asymptotically free  $R$ -diagonal elements.
- (2) Also, one can prove this constructing  $\mathcal{A}$  to be  $M_2(\mathbb{C}) \otimes \mathcal{B}$ , where  $(\mathcal{B}, \varphi)$  is a  $C^*$ -probability space with two free  $R$ -diagonal elements  $a_1, a_2$ , the absolute values of which have symmetrized distributions  $\mu_1, \mu_2$ . It is then easy to prove (in a similar way as Proposition 3.8

of [NSS99]) that  $A_1 = \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix}$  are free with amalgamation over  $\mathcal{D}$ .  $\mu_{1\boxplus\lambda}\mu_2$

is the symmetrization of the distribution of  $((A_1+A_2)(A_1+A_2)^*)^{1/2}$  in  $(p_1\mathcal{A}p_1, \varphi_1)$ , which is the symmetrization of the distribution of  $((a_1+a_2)(a_1+a_2)^*)^{1/2} = |a_1+a_2|$  in  $(\mathcal{B}, \varphi)$ .

(3) With the same notations, one could also have concluded using the fact that

$$\tilde{\mu}_{((A_1+A_2)(A_1+A_2)^*)^{1/2}}$$

in  $(p_1\mathcal{A}p_1, \varphi_1)$  is, by (31), the distribution of  $\begin{bmatrix} 0 & a_1+a_2 \\ a_1^*+a_2^* & 0 \end{bmatrix}$  in  $(M_2(\mathbb{C}) \otimes \mathcal{B}, \text{tr} \otimes \varphi)$ ,

that is, as noticed in Remark 3.5 of [NSS99], the symmetrization of the distribution of  $|a_1+a_2|$  (in  $(\mathcal{B}, \varphi)$ ), that is  $\mu_{1\boxplus\lambda}\mu_2$ .

**Remark 2.2.** *The trick explained in the third item in the previous list also works in the case that  $\lambda < 1$ . The construction obtained is not faraway from the one of section 2.2. The price one pays is that the resulting state on  $M_2(\mathbb{C})$  is not a trace, but what is known as a KMS state. Using this kind of state would have allowed us to skip the main part of the proof of Proposition-Definition 2.1, but the technical introduction could have been quite heavy for non specialists of operator algebras.*

2.3.3. *Generalization for other values of  $\lambda$  ?* It is natural to ask if for other values of  $\lambda$ , we can still use the relation (31) to give a relation between  $\boxplus_\lambda$  and  $\boxplus$ . When  $\lambda = 1$ , what precedes can be summarized in the following way : for  $a_1, a_2 \in p_1\mathcal{A}p_2$  free with amalgamation over  $\mathcal{D}$ , the distribution of the self adjoint element of  $(\mathcal{A}, \varphi)$

$$\begin{bmatrix} 0 & a_1+a_2 \\ a_1^*+a_2^* & 0 \end{bmatrix}$$

is the free convolution of the distributions of

$$\begin{bmatrix} 0 & a_1 \\ a_1^* & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & a_2 \\ a_2^* & 0 \end{bmatrix}.$$

Does this stay true for other values of  $\lambda$  ? It would mean that for all  $\mu, \nu$  compactly supported symmetric probability measures on the real line, we have

$$(32) \quad f_\lambda(\mu_{\boxplus\lambda}\nu) = f_\lambda(\mu)_{\boxplus} f_\lambda(\nu),$$

where  $f_\lambda$  is the function which maps a probability measure  $P$  on the real line to  $2\rho_1 P + (\rho_2 - \rho_1)\delta_0$ , i.e. to  $2\lambda/(1+\lambda)P + (1-\lambda)/(1+\lambda)\delta_0$ . Let us denote by  $(m_k(P))$  the sequence of the moments of a probability measure  $P$  on the real line. Looking at fourth moments, (32) would imply that

$$\frac{2\lambda}{\lambda+1}(m_4(\mu) + (2\lambda+2)m_2(\mu)m_2(\nu) + m_4(\nu)) = \frac{2\lambda}{\lambda+1}(m_4(\mu) + \frac{8\lambda}{\lambda+1}m_2(\mu)m_2(\nu) + m_4(\nu)),$$

which cannot be true for any  $\mu, \nu$  if  $\lambda \neq 1$ .

However, we will see later that the free convolution with ratio 0 is also related to the free convolution.

### 3. CUMULANTS

In this next section, we are going to study cumulants related to this convolution (and, more generally to the freeness with amalgamation over  $\mathcal{D}$ ), like free cumulants in [S94]. We will use them to define an analytic integral transform of symmetric probability measures which

linearizes  $\boxplus_\lambda$ , and which will, as a consequence, allow us to compute rectangular free convolution of symmetric probability measures.

In this section, the term algebra (resp. subalgebra) will refer to the notion of  $*$ -algebra (resp.  $*$ -subalgebra). The theory of cumulants in a  $\mathcal{D}$ -probability space (i.e. in an algebra endowed with a conditional expectation on a subalgebra  $\mathcal{D}$ ) has been developed in [S98] (for a short abstract, see [NSS02]). In this section, we will begin by giving the main lines of this theory, and then we will investigate the special case of  $(\rho_1, \dots, \rho_d)$ -probability spaces.

**3.1. General theory of cumulants in a  $\mathcal{D}$ -probability space.** In this section, we consider an algebra  $\mathcal{A}$ , a subalgebra  $\mathcal{D}$  of  $\mathcal{A}$ , and a *conditional expectation*  $E$  from  $\mathcal{A}$  to  $\mathcal{D}$ , i.e. a map  $E$  such that

$$\forall(d, a, d') \in \mathcal{D} \times \mathcal{A} \times \mathcal{D}, E(dad') = dE(a)d'.$$

Let us begin with algebraic definitions. A  $\mathcal{D}$ -bimodule is a vector space  $M$  over  $\mathbb{C}$  on which the algebra  $\mathcal{D}$  acts on the right and on the left. The tensor product  $M \otimes_{\mathcal{D}} N$  of two  $\mathcal{D}$ -bimodules  $M, N$  is their tensor product as  $\mathbb{C}$ -vector spaces, where for all  $(m, d, n) \in M \times \mathcal{D} \times N$ ,  $(m.d) \otimes n$  and  $m \otimes (d.n)$  are identified.  $M \otimes_{\mathcal{D}} N$  is endowed with a structure of  $\mathcal{D}$ -bimodule by  $d_1.(m \otimes n).d_2 = (d_1.m) \otimes (n.d_2)$ . Therefore we can define, for  $n$  positive integer,  $\mathcal{A}^{\otimes_{\mathcal{D}} n} = \underbrace{\mathcal{A} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{A}}_{n \text{ times}}$ .

A partition  $\pi$  of a finite totally ordered set  $I$  is said to be non-crossing if there is no  $i < j < k < l \in I$  such that  $i, k$  are connected by  $\pi$ ,  $j, l$  also, but not  $i, j$ . The following lemma can easily be proved by induction over the cardinality of  $I$ .

**Lemma 3.1.** *A partition  $\pi$  of a finite totally ordered set  $I$  is non-crossing if and only if  $\pi$  has only one class or a class  $V$  of  $\pi$  is an interval, and  $\pi - \{V\}$  is a non crossing partition of  $I - V$ .*

Consider a sequence  $(f_n)_{n \geq 1}$  of maps, each  $f_n$  being a  $\mathcal{D}$ -bimodule morphism between  $\mathcal{A}^{\otimes_{\mathcal{D}} n}$  and  $\mathcal{D}$ . For  $n$  positive integer and  $\pi \in \text{NC}(n)$  (non-crossing partition of  $[n]$ ), we define the  $\mathcal{D}$ -bimodule morphism  $f_\pi$  between  $\mathcal{A}^{\otimes_{\mathcal{D}} n}$  and  $\mathcal{D}$  in the following way: if  $\pi = 1_n$  is the one-block partition,  $f_\pi = f_n$ . In the other case, a block  $V$  of  $\pi$  is an interval  $[k, l]$ . If  $k = 1$  (resp.  $l = n$ ), then  $f_\pi(a_1 \otimes \cdots \otimes a_n) = f_{l-k+1}(a_1 \otimes \cdots \otimes a_l) f_{\pi \setminus \{V\}}(a_{l+1} \otimes \cdots \otimes a_n)$  (resp.  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_n)$ ). In the other case, one has  $1 < k \leq l < n$ . Then  $f_\pi(a_1 \otimes \cdots \otimes a_n)$  is defined to be  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1} f_{l-k+1}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n)$  or  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1} \otimes f_{l-k+1}(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n)$ , both are the same by definition of  $\otimes_{\mathcal{D}}$ .

For example, if  $\pi = \{\{1, 6, 8\}, \{2, 5\}, \{3, 4\}, \{7\}, \{9\}\}$ , then

$$f_\pi(a_1 \otimes \cdots \otimes a_9) = f_3(a_1 f_2(a_2 f_2(a_3 \otimes a_4) \otimes a_5) \otimes a_6 f_1(a_7) \otimes a_8) f_1(a_9).$$

Let us define, for all  $n \geq 1$ , the  $\mathcal{D}$ -bimodule morphism  $E_n$  between  $\mathcal{A}^{\otimes_{\mathcal{D}} n}$  and  $\mathcal{D}$  which maps  $a_1 \otimes \cdots \otimes a_n$  to  $E(a_1 \cdots a_n)$ . Then, one can define the sequence  $(c_n)_{n \geq 1}$  of maps, each  $c_n$  being



a  $\mathcal{D}$ -bimodule morphism between  $\mathcal{A}^{\otimes \mathcal{D}^n}$  and  $\mathcal{D}$ , by one of the following equivalent formulae:

$$(33) \quad \forall n, \forall \pi \in \text{NC}(n), \quad E_\pi = \sum_{\sigma \leq \pi} c_\sigma,$$

$$(34) \quad \forall n, \quad E_n = \sum_{\sigma \in \text{NC}(n)} c_\sigma$$

$$(35) \quad \forall n, \forall \pi \in \text{NC}(n), \quad c_\pi = \sum_{\sigma \leq \pi} \mu(\sigma, \pi) E_\sigma,$$

$$(36) \quad \forall n, \quad c_n = \sum_{\sigma \in \text{NC}(n)} \mu(\sigma, 1_n) E_\sigma,$$

where  $\mu$  is the Möbius function ([R64], [S99]) of the lattice  $\text{NC}(n)$  endowed with the refinement order.

We recall Theorem 1 of [SS01].

**Theorem 3.2.** *A family  $(\chi_i)_{i \in I}$  of subsets of  $\mathcal{A}$  is free with amalgamation over  $\mathcal{D}$  if and only if for all  $n \geq 2$ , for all non constant  $i \in I^n$ , for all  $a_1 \in \chi_{i_1}, \dots, a_n \in \chi_{i_n}$ , for all  $d_1, \dots, d_n \in \mathcal{D}$ , one has  $c_n(a_1 d_1 \otimes \dots \otimes a_n d_n) = 0$ .*

For example, if  $a_1, a_2$  are free with amalgamation over  $\mathcal{D}$ , for all  $n \geq 2$  even, one has

$$(37) \quad \begin{aligned} & c_n((a_1 + a_2) \otimes (a_1^* + a_2^*) \otimes (a_1 + a_2) \otimes \dots \otimes (a_1^* + a_2^*)) \\ &= c_n(a_1 \otimes a_1^* \otimes a_1 \otimes \dots \otimes a_1^*) + c_n(a_2 \otimes a_2^* \otimes a_2 \otimes \dots \otimes a_2^*). \end{aligned}$$

**3.2. The special case where  $\mathcal{D} = \text{Span}(p_1, p_2)$ .** In this section, we consider a  $(\rho_1, \rho_2)$ -rectangular probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$ , with  $\rho_1/\rho_2 = \lambda$ . We shall adopt the point of view of operator-valued free probability: the linear span  $\mathcal{D}$  of  $\{p_1, p_2\}$  is an algebra, which can be identified to the set of  $2 \times 2$  complex diagonal matrices, and then to  $\mathbb{C}^2$  by

$$\lambda_1 p_1 + \lambda_2 p_2 \simeq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \simeq (\lambda_1, \lambda_2),$$

and with this identification, the application  $E$  is then a conditional expectation from  $\mathcal{A}$  to  $\mathcal{D}$ .

Let us define, for all  $m$ ,  $c_m^{(1)}$ ,  $c_m^{(2)}$  the linear forms on the complex vector space  $\mathcal{A}^{\otimes \mathcal{D}^m}$  defined by

$$\forall x \in \mathcal{A}, c_m(x) = \begin{bmatrix} c_m^{(1)}(x_{11}) & 0 \\ 0 & c_m^{(2)}(x_{22}) \end{bmatrix}.$$

By (37), an analytic integral transform of a symmetric measure  $\mu$  with coefficients given by  $c_{2n}^{(1)}(a \otimes a^* \otimes \dots \otimes a^*)$ , where  $a \in p_1 \mathcal{A} p_2$  is such that for all  $r$  even,  $\varphi_1((aa^*)^{r/2})$  is the  $r$ -th moment of  $\mu$ , would linearize  $\boxplus_\lambda$ . Thus we have to give an explicit relation between the  $\varphi_1((aa^*)^{r/2})$ 's and the  $c_n^{(1)}(a \otimes a^* \otimes \dots \otimes a^*)$ 's. This relation is given by equation (34), but we shall express  $c_\pi(a \otimes a^* \otimes \dots \otimes a^*)$  as a function of  $c_r^{(1)}(a \otimes a^* \otimes \dots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \dots \otimes a)$ , and give a relation between  $c_r^{(1)}(a \otimes a^* \otimes \dots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \dots \otimes a)$ .

For a more exhaustive study of cumulants in  $(\rho_1, \dots, \rho_d)$ -probability spaces, the reader can refer to [B-G2].

Consider  $a \in p_1 \mathcal{A} p_2$ . First of all, since  $a = p_1 a p_2$  and  $p_1 p_2 = p_2 p_1 = 0$ , by definition of the tensor product of  $\mathcal{D}$ -bimodules, for all  $\varepsilon_1, \dots, \varepsilon_n \in \{., *\}$  such that there exists  $l \in [n-1]$ ,  $\varepsilon_l = \varepsilon_{l+1}$ , we have

$$c_n(a^{\varepsilon_1} \otimes \dots \otimes a^{\varepsilon_n}) = 0.$$

Moreover, since  $c_n$  is a  $\mathcal{D}$ -bimodule morphism with values in  $\mathcal{D}$  and  $p_1 \mathcal{D} p_2 = p_2 \mathcal{D} p_1 = \{0\}$ , if  $\varepsilon_n = \varepsilon_1$ , we also have

$$c_n(a^{\varepsilon_1} \otimes \dots \otimes a^{\varepsilon_n}) = 0.$$

Thus, for  $\pi \in \text{NC}(n)$ , if

$$c_\pi(a \otimes a^* \otimes \dots \otimes a^*) \neq 0,$$

then all blocks of  $\pi$  have even cardinality and consist of an alternating sequence of even and odd numbers. We denote by  $\text{NC}'(n)$  the set of such partitions. For example,  $\{\{1, 4, 5, 6\}, \{2, 3\}\} \in \text{NC}'(6)$ , whereas  $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\} \notin \text{NC}'(n)$ . Using lemma 3.1, we prove easily that in fact, any non-crossing partition in which all blocks have even cardinality is in  $\text{NC}'(n)$ .

At last, because  $c_n$  is a  $\mathcal{D}$ -bimodule morphism with values in  $\mathcal{D}$ ,

$$(38) \quad c_n(a \otimes a^* \otimes \dots \otimes a^*) = c_n^{(1)}(a \otimes a^* \otimes \dots \otimes a^*) p_1,$$

$$(39) \quad c_n(a^* \otimes a \otimes \dots \otimes a) = c_n^{(2)}(a^* \otimes a \otimes \dots \otimes a) p_2.$$

Thus for all  $\pi \in \text{NC}'(n)$ .

$$(40) \quad c_\pi(a \otimes a^* \otimes \dots \otimes a^*) = \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \dots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \dots \otimes a) p_1$$

$$(41) \quad c_\pi(a^* \otimes a \otimes \dots \otimes a) = \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \dots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \dots \otimes a^*) p_2$$

**Remark 3.3.** Note that all we did here stays true if one replaces  $c_\pi, c^{(1)}, \dots$  by  $E_\pi, \varphi_1, \dots$

Now, it remains only to give the relation between  $c_r^{(1)}(a \otimes a^* \otimes \dots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \dots \otimes a)$ .

**Lemma 3.4.** For all  $n$  even, we have  $\rho_1 c_n^{(1)}(a \otimes a^* \otimes \dots \otimes a^*) = \rho_2 c_n^{(2)}(a^* \otimes a \otimes \dots \otimes a)$ .

**Proof.** We are going to prove it by induction over  $n$ . Let us prove it for  $n = 2$ . We have  $c_1 = E$  and  $c_2(x \otimes y) = E(xy) - E(x)E(y)$ , so the result is given by equation (3). Assume now the result to be proved to the ranks  $2, 4, \dots, n-2$ . One has, by formulae (34),(40),(41),

$$\begin{aligned} c_n^{(1)}(a \otimes \dots \otimes a^*) &= \underbrace{\varphi_1(a \dots a^*)}_X \\ &- \underbrace{\sum_{\pi \in \text{NC}'(n)} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \dots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \dots \otimes a)}_Y, \end{aligned}$$

and

$$\begin{aligned} c_n^{(2)}(a^* \otimes \dots \otimes a) &= \underbrace{\varphi_2(a^* \dots a)}_{X'} \\ &- \underbrace{\sum_{\pi \in \text{NC}'(n)} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \dots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \dots \otimes a^*)}_{Y'}. \end{aligned}$$

Since  $\rho_{i_0}X = \rho_{i_1}X'$  (by formula (3)), it suffices to prove that

$$\rho_{i_0}Y = \rho_{i_1}Y'.$$

To do that, it suffices to propose a bijective correspondence  $\pi \mapsto \tilde{\pi}$  from  $\text{NC}'(n) - \{1_n\}$  to  $\text{NC}'(n) - \{1_n\}$  such that for all  $\pi \in \text{NC}'(n) - \{1_n\}$ ,

$$\rho_{i_0} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) = \rho_{i_1} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*).$$

by the induction hypothesis, the correspondence which maps  $\pi \in \text{NC}'(n) - \{1_n\}$  to  $\tilde{\pi}$  defined by  $k \tilde{\pi} l \Leftrightarrow \sigma(k) \tilde{\pi} \sigma(l)$ , where  $\sigma$  is the cycle  $(12 \cdots n)$  of  $[n]$ , is convenient.  $\square$

From (34), (40), and the previous lemma, we deduce:

**Proposition 3.5.** *For  $a \in p_1\mathcal{A}p_2$ ,  $n$  positive integer,*

$$(42) \quad \varphi_1((aa^*)^n) = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{V \in \pi} c_{|V|}(a),$$

where for all  $r$  even,  $c_r(a)$  denotes  $c_r^{(1)}(a \otimes \cdots \otimes a^*)$ , and for all  $\pi$ ,  $e(\pi)$  denotes the number of blocks of  $\pi$  with even minimum.

**3.3. The case  $\lambda = 0$ , characterization of  $\mathbb{E}_0$ .** In all this subsection, we suppose that  $\lambda = 0$ , i.e.  $(\rho_1, \rho_2) = (0, 1)$ . In this case, by equation (3),  $E$  is null on  $p_2\mathcal{A}p_1\mathcal{A}p_2$ , thus on  $p_2\mathcal{A}p_1\mathcal{A}$ . It implies that  $E_n$  is null on  $p_2\mathcal{A}p_1 \otimes_{\mathcal{D}} \mathcal{A}^{\otimes_{\mathcal{D}} n-1}$ . Hence, for any  $\pi \in \text{NC}(n)$ ,  $E_\pi$  is null on any space of the type  $p_{i_0}\mathcal{A}p_{i_1} \otimes_{\mathcal{D}} p_{i_1}\mathcal{A}p_{i_2} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} p_{i_{n-1}}\mathcal{A}p_{i_n}$ , where the minimum  $k$  of a block of  $\pi$  is such that  $(i_{k-1}, i_k) = (2, 1)$ . By the formula (iii) of the definition of the cumulant functions in subsection 3.1, the function  $c_\pi$  is null too on the previous subspace.

Consider  $n$  even and  $\pi \in \text{NC}(n)$ . For  $E_\pi$  or  $c_\pi$  to be nonzero on  $(p_1\mathcal{A}p_2 \otimes_{\mathcal{D}} p_2\mathcal{A}p_1)^{\otimes_{\mathcal{D}} n/2}$ ,  $\pi$  has to belong to  $\text{NC}'(n)$  and no block of  $\pi$  has to have even minimum (i.e.  $e(\pi)$  has to be 0). Let us introduce the set  $\text{NC}_d(n)$  of non-crossing partitions  $\pi$  of  $[n]$  such that for all  $k$  even in  $[n]$ ,  $k-1 \tilde{\pi} k$ . Note that  $\text{NC}_d(n)$  is contained in the set of elements of  $\text{NC}'(n)$  in which no block has even minimum. We claim that the inverse inclusion is true. Indeed, suppose the existence of  $\pi \in \text{NC}'(n) \setminus \text{NC}_d(n)$  in which no block has even minimum. Define  $k = \min\{k \in [n]; k \text{ even and } k-1 \tilde{\pi} k\}$ .  $k$  cannot be the minimum of its block in  $\pi$ , and by minimality of  $k$ , its preceding element  $j$  in this block has to be even. Then, as  $\pi$  is non-crossing, the set  $\{j+1, \dots, k-1\}$  is a union of classes of  $\pi$ , but its cardinality is odd, which is in contradiction with  $\pi \in \text{NC}'(n)$ .

So by (34), for all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$  of  $(p_1\mathcal{A}p_2 \times p_2\mathcal{A}p_1)^{n/2}$ , we have, for all  $\pi \in \text{NC}_d(n)$ ,

$$\varphi_{1,\pi}(a_1, b_1, \dots, a_{n/2}, b_{n/2}) = \sum_{\substack{\sigma \in \text{NC}_d(n) \\ \sigma \leq \pi}} c_\sigma^{(1)}(a_1 \otimes b_1 \otimes \cdots \otimes a_{n/2} \otimes b_{n/2}).$$

But  $\text{NC}_d(n)$  is in correspondence with  $\text{NC}(n/2)$  by the order-preserving bijection  $d$  from  $\text{NC}(n/2)$  onto  $\text{NC}_d(n)$ , that maps any noncrossing partition  $\pi$  to the partition  $d(\pi)$  that links to elements  $i, j$  of  $[n]$  if and only if the upper integer parts of  $i/2$  and  $j/2$  are linked by  $\pi$ . With this definition, the previous equation can be written: for all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$

of  $(p_1\mathcal{A}p_2 \times p_2\mathcal{A}p_1)^{n/2}$ , we have, for all  $\pi \in \text{NC}(n/2)$ ,

$$(43) \quad \varphi_{1,\pi}(a_1b_1, \dots, a_{n/2}b_{n/2}) = \sum_{\substack{\sigma \in \text{NC}(n) \\ \sigma \leq d(\pi)}} c_\sigma^{(1)}(a_1 \otimes b_1 \otimes \dots \otimes a_{n/2} \otimes b_{n/2}).$$

So, from the very definition of free cumulants in a noncommutative probability space (p. 621 of [S94] or definition 4.5 of [S99]), we deduce:

**Proposition 3.6.** *For all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$  of  $(p_1\mathcal{A}p_2 \times p_2\mathcal{A}p_1)^{n/2}$ ,  $c_n^{(1)}(a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes \dots \otimes a_{n/2} \otimes b_{n/2})$  is the free cumulant of the family of the  $n/2$  noncommutative random variables  $(a_1b_1, \dots, a_{n/2}b_{n/2})$  in  $(p_1\mathcal{A}p_1, \varphi_1)$ .*

**Remark 3.7.** *The previous proposition would have been pretty easy to prove using KMS states, as explained in the remark 2.2. Again, the technical introduction could have been quite heavy for non specialists of operator algebras.*

Now we are able to give the link between the free convolution with null ratio  $\boxplus_0$  and the free convolution  $\boxplus$ :

**Proposition 3.8.** *The free convolution with null ratio of two compactly supported symmetric probability measures is the unique symmetric probability whose push-forward by the square function is the free convolution of the push-forwards by the square function of the two probabilities.*

**Proof.** Denote, for  $\nu$  probability measure,  $m_n(\nu)$  its  $n$ -th moment, and  $\mathfrak{K}_n(\nu)$  its  $n$ -th free cumulant. Recall ([S94]) that free cumulants of probability measures are defined by the formula

$$(44) \quad \forall n \geq 1, m_n(\nu) = \sum_{\pi \in \text{NC}(n)} \prod_{B \in \pi} \mathfrak{K}_{|B|}(\nu),$$

and that the free convolution of  $\nu_1, \nu_2$  compactly supported probability measures is the only probability with free cumulants  $\mathfrak{K}_n(\nu_1) + \mathfrak{K}_n(\nu_2)$  ( $n \geq 1$ ).

Consider two symmetric compactly supported probability measures  $\mu_1, \mu_2$ . Denote by  $\nu_1, \nu_2$  their respective push-forwards by the square function. It suffices to prove that for all  $n$  even,  $m_n(\mu_1 \boxplus_0 \mu_2) = m_{n/2}(\nu_1 \boxplus \nu_2)$ , i.e. that for all  $n$  even,

$$m_n(\mu_1 \boxplus_0 \mu_2) = \sum_{\pi \in \text{NC}(n/2)} \prod_{B \in \pi} (\mathfrak{K}_{|B|}(\nu_1) + \mathfrak{K}_{|B|}(\nu_2)).$$

Consider free elements  $a_1, a_2 \in p_1\mathcal{A}p_2$  such that

$$\forall i = 1, 2, \forall n \in \mathbb{N} \text{ even}, \varphi_1((a_i a_i^*)^{n/2}) = m_n(\mu_i).$$

Then one has

$$\begin{aligned} m_n(\mu_1 \boxplus_0 \mu_2) &= \varphi_1 \left( ((a_1 + a_2)(a_1^* + a_2^*))^{n/2} \right) \\ &= \sum_{\pi \in \text{NC}_d(n)} \prod_{B \in \pi} \underbrace{c_{|B|}^{(1)}(a_1 + a_2 \otimes \dots \otimes a_1^* + a_2^*)}_{=c_{|B|}^{(1)}(a_1 \otimes \dots \otimes a_1^*) + c_{|B|}^{(1)}(a_2 \otimes \dots)} \quad \text{by eq. (40)} \end{aligned}$$

But for all  $i = 1, 2$ , for all  $m$  even,  $c_m^{(1)}(a_i \otimes \dots \otimes a_i^*)$  is the free cumulant of the family of  $m/2$  variables  $(a_i a_i^*, \dots, a_i a_i^*)$  in the noncommutative probability space  $(p_1\mathcal{A}p_1, \varphi_1)$ , i.e. the  $m/2$ -th

free cumulant of  $\nu_i$ . So

$$m_n(\mu_1 \boxplus_0 \mu_2) = \sum_{\pi \in \text{NC}(n/2)} \prod_{B \in \pi} (\mathfrak{K}_{|B|}(\nu_1) + \mathfrak{K}_{|B|}(\nu_2)) = m_{n/2}(\nu_1 \boxplus \nu_2),$$

which proves the result.  $\square$

**3.4. Generating series of the cumulants of an element of  $p_1 \mathcal{A} p_2$ .** From now on, we suppose that  $\lambda > 0$ . Now, we will derive from (42) a formula which links generating series of the sequence  $\varphi_1((aa^*)^n)$  and  $c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$ . Note that such a formula could be derived from the very general Theorem 2.2.3 of [S98], but the work needed to translate this result to our context is almost as long as this section.

**Lemma 3.9.** *Consider a sequence  $(c_{2n})_{n \in \mathbb{N}^*}$  of complex numbers. Define the sequence  $(m_{2n}^{(e)})_{n \in \mathbb{N}}$  by  $m_0^{(e)} = 1$  and for each  $n \in \mathbb{N}^*$ ,*

$$m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{B \in \pi} c_{|B|}.$$

Define the formal power series

$$C(X) = \sum_{n \geq 1} c_{2n} X^n, \quad M^{(e)}(X) = \sum_{n \geq 1} m_{2n}^{(e)} X^n.$$

Then we have  $C(X(\lambda M^{(e)}(X) + 1)(M^{(e)}(X) + 1)) = M^{(e)}(X)$ .

**Proof.** *Step I.* Define, for  $n$  positive integer and  $\pi \in \text{NC}'(2n)$ ,  $o(\pi)$  to be the number of blocks of  $\pi$  with odd minimum. Then we have

$$\lambda m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}.$$

Indeed, if  $c$  denotes the cycle

$$(2n \rightarrow 2n-1 \rightarrow \cdots \rightarrow 2 \rightarrow 1)$$

of  $[2n]$ , then  $c$  induces a permutation of  $\text{NC}'(2n)$  by

$$C : \pi \in \text{NC}'(2n) \mapsto C(\pi) := \{c(B) ; B \in \pi\}.$$

For example, for  $\pi = \{\{1, 4, 5, 6\}, \{2, 3\}\}$ ,  $C(\pi) = \{\{3, 4, 5, 6\}, \{1, 2\}\}$ . For  $\pi \in \text{NC}'(2n)$ , for  $B$  block of  $\pi$ ,  $\min c(B)$  is  $(\min B) - 1$  if  $\min B > 1$ , and the second element of  $B$  minus 1 in the other case. Note that in the first case,  $\min B$  and  $\min c(B)$  have different parities, and in the second one, they have the same parities, by definition of  $\text{NC}'(2n)$ . Hence we have  $o(C(\pi)) = e(\pi) + 1$ . Thus

$$\lambda m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)+1} \prod_{B \in \pi} c_{|B|} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(C(\pi))} \prod_{B \in \pi} c_{|B|} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}.$$

*Step II.* Thus we have to prove that  $C(X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1)) = M^{(e)}(X)$ , where  $M^{(o)}(X) = \sum_{n \geq 1} m_{2n}^{(o)} X^n$ , and  $m_{2n}^{(o)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}$ .

Define, for each positive  $n$  and each  $j \in [n]$ , the “decomposition map”  $D_j$ , from the set of partitions of  $\text{NC}'(2n)$  in which the class of 1 has cardinality  $2j$  to  $\bigcup_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \cdots + l_{2j} = n-j}} \text{NC}'(2l_1) \times \cdots \times$

$\text{NC}'(2l_{2j})$  (the set  $\text{NC}'(0)$  is considered as a singleton, on which  $e$  and  $o$  have value 0), which maps  $\pi \in \text{NC}'(n)$  to  $(\pi_1, \dots, \pi_{2j})$  defined in the following way: denote by  $\{1 = k_1 < \dots < k_{2j}\}$  the class of 1 in  $\pi$ , and for each  $r$ , let  $\pi_r$  be the restriction of  $\pi$  to the interval  $]k_r, k_{r+1}[$ , with  $k_{2j+1} = 2n + 1$ . It is easy to see that the map  $D_j$  is well defined (recall that for all  $r$ ,  $k_r$  and  $k_{r+1}$  have opposite parity), and that it is a bijection. Moreover, we have

$$\begin{aligned} e(\pi) &= o(\pi_1) + e(\pi_2) + \dots + o(\pi_{2j-1}) + e(\pi_{2j}), \\ o(\pi) &= 1 + e(\pi_1) + o(\pi_2) + \dots + e(\pi_{2j-1}) + o(\pi_{2j}). \end{aligned}$$

We denote by  $\langle X^n \rangle P$  the coefficient of  $X^n$  in a formal power series  $P$  in  $X$ . Let us show that for each  $n \geq 1$ ,

$$\langle X^n \rangle C \left( X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1) \right) = m_{2n}^{(e)}.$$

We have

$$\begin{aligned} & \langle X^n \rangle C \left( X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1) \right) \\ = & \sum_{j=1}^n c_{2j} \langle X^n \rangle \left( X^j (M^{(o)}(X) + 1)^j (M^{(e)}(X) + 1)^j \right) \\ = & \sum_{j=1}^n c_{2j} \langle X^{n-j} \rangle \left( (M^{(o)}(X) + 1)^j (M^{(e)}(X) + 1)^j \right) \\ = & \sum_{j=1}^n c_{2j} \sum_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \dots + l_{2j} = n-j}} m_{2l_1}^{(o)} m_{2l_2}^{(e)} \dots m_{2l_{2j-1}}^{(o)} m_{2l_{2j}}^{(e)} \\ = & \sum_{j=1}^n c_{2j} \sum_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \dots + l_{2j} = n-j}} \sum_{\substack{\pi_1 \in \text{NC}'(2l_1) \\ \vdots \\ \pi_{2j} \in \text{NC}'(2l_{2j})}} \lambda^{o(\pi_1) + e(\pi_2) + \dots + e(\pi_{2j})} \prod_{r=1}^{2j} \prod_{B \in \pi_r} c_{|B|} \end{aligned}$$

The preliminaries about the bijections  $D_j$  tell us that

$$\begin{aligned} \langle X^n \rangle C \left( X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1) \right) &= \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{B \in \pi} c_{|B|} \\ &= m_{2n}^{(e)}. \end{aligned}$$

□

Let us introduce two power series that will play a role in the computation of the generating series of the cumulants:

$$\begin{aligned} T(X) &= (\lambda X + 1)(X + 1), \\ U(X) &= (T - 1)^{\langle -1 \rangle} \\ &= \sum_{n \geq 1} \frac{(-1)^{n-1} \lambda^{n-1} \binom{2n}{n}}{2(\lambda + 1)^{2n-1} (2n - 1)} X^n. \end{aligned}$$

If  $\lambda = 0$  then  $U(z) = z$ , in the other case  $U(z)$  is the power expansion of  $\frac{-\lambda - 1 + [(\lambda + 1)^2 + 4\lambda z]^{1/2}}{2\lambda}$ , where  $z \mapsto z^{1/2}$  is the analytic version of the square root on the complement of the real non positive half line such that  $1^{1/2} = 1$ .

**Theorem 3.10.** Consider  $a \in p_1\mathcal{A}p_2$ , and denote for each  $n \in \mathbb{N}^*$ ,  $m_{2n}(a) = \varphi_1((aa^*)^n)$  and  $c_{2n}(a) = c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$ . Define the formal power series  $M(X) = \sum_{n \geq 1} m_{2n}(a)X^n$  and  $C(X) = \sum_{n \geq 1} c_{2n}(a)X^n$ . Then

$$C = U \circ \left( \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}} - 1 \right).$$

**Proof.** By proposition 3.5 and the previous lemma,

$$C \circ (X \times (T \circ M)) = M.$$

So

$$\begin{aligned} T \circ C \circ (X \times (T \circ M)) &= T \circ M, \\ \frac{T \circ C}{X} \circ (X \times (T \circ M)) &= \frac{1}{X}, \\ \frac{X}{T \circ C} \circ (X \times (T \circ M)) &= X, \\ \frac{X}{T \circ C} &= (X \times (T \circ M))^{\langle -1 \rangle}, \\ T \circ C &= \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}}, \\ C &= U \circ \left( \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}} - 1 \right). \end{aligned}$$

□

#### 4. THE RECTANGULAR $R$ -TRANSFORM

**4.1. The case of compactly supported symmetric measures.** Consider a symmetric compactly supported probability measure  $\mu$ . For  $n \geq 0$ , denote by  $m_n(\mu)$  its  $n$ -th moment and by  $c_{2n}(\mu)$  the number  $c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$  with  $a \in p_1\mathcal{A}p_2$ , where  $\mathcal{A}$  is a  $(\rho_1, \rho_2)$ -probability space, such that for all  $k \geq 1$ ,  $\varphi_1((aa^*)^k) = m_{2k}(a)$ .

Recall that the generating series of the moments  $M_\mu(X) = \sum_{n \geq 1} m_{2n}(\mu)X^n$  and of the cumulants  $C_\mu(X) = \sum_{n \geq 1} c_{2n}(\mu)X^n$  of a symmetric measure  $\mu$  are linked by the relation

$$(45) \quad C_\mu = U \circ \left( \frac{X}{(X \times (T \circ M_\mu))^{\langle -1 \rangle}} - 1 \right).$$

Since the formal power series  $C_\mu$  linearizes the convolution  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures (by equation (37)), we will define an analytic integral transform on the set of symmetric probability measures whose power expansion is  $C_\mu(z)$  when  $\mu$  has compact support.

For any symmetric probability measure  $\mu$  on the real line, we define the *Cauchy transform*  $G_\mu$  of  $\mu$  on the upper half plane by

$$G_\mu(z) = \int_{t \in \mathbb{R}} \frac{d\mu(t)}{z - t}.$$

When  $\mu$  has compact support, the power expansion of the Cauchy transform is

$$G_\mu(z) = \sum_{n \geq 0} m_{2n}(\mu) \frac{1}{z^{2n+1}}.$$

So, when  $z \in \mathbb{C} \setminus [0, +\infty)$  is small enough,

$$M_\mu(z) = \frac{1}{\sqrt{z}} G_\mu\left(\frac{1}{\sqrt{z}}\right) - 1,$$

where we denote by  $\sqrt{\phantom{x}}$  the analytic version of the square root on the complement of the real nonnegative half line in the complex plane such that  $\sqrt{-1} = i$ .

So, for such a  $z$ ,

$$(46) \quad zT \circ M_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1 - \lambda)\sqrt{z}G_\mu\left(\frac{1}{\sqrt{z}}\right).$$

For any symmetric probability measure  $\mu$ , we define the *rectangular Cauchy transform with ratio*  $\lambda$  of  $\mu$  on the complement of the real nonnegative half line to be  $H_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1 - \lambda)\sqrt{z}G_\mu\left(\frac{1}{\sqrt{z}}\right)$ .

When  $\mu$  is compactly supported, from the power expansion of the Cauchy transform  $G_\mu$  we know that  $H_\mu$  is analytic in a neighborhood of zero, that  $H_\mu(0) = 0$ , and that  $H'_\mu(0) = 1$ .

So we can invert the function  $H_\mu$  in a neighborhood of zero, and the inverse function  $H_\mu^{-1}$  has a power series expansion that is, by equation (46), the expansion of  $(X \times (T \circ M_\mu))^{<-1>}$ . Note that  $\frac{z}{H_\mu^{-1}(z)} - 1$  is analytic in a neighborhood of zero, with value zero in zero.

Let us define the analytic function on a neighborhood of zero  $U(z) = \frac{-\lambda - 1 + [(\lambda + 1)^2 + 4\lambda z]^{1/2}}{2\lambda}$  (when  $\lambda = 0$ ,  $U(z) = z$ ), where  $z \mapsto z^{1/2}$  is the analytic version of the square root on the complement of the real non positive half line such that  $1^{1/2} = 1$ . Its power expansion is  $\sum_{n \geq 1} \frac{(-1)^{n-1} \lambda^{n-1} \binom{2n}{n}}{2(\lambda+1)^{2n-1} (2n-1)} z^n$ . So by equation (45), the analytic power expansion of

$$z \mapsto U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right)$$

is  $C_\mu(z)$ .

**4.2. Technical preliminaries.** Note that when  $A$  is a tight set of symmetric probability measures on the real line, for every  $0 < \theta < \pi/2$ , we have

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \frac{\pi}{2}| < \theta}} \frac{1}{z} G_\mu\left(\frac{1}{z}\right) = 1,$$

uniformly in  $\mu \in A$ .

Indeed, if  $\varepsilon > 0$ , if  $M$  is such that  $\frac{1}{\cos \theta} \sup_{\mu \in A} \mu([-M, M]^c) < \varepsilon/2$ , if  $z$  is such that  $|\arg z - \frac{\pi}{2}| < \theta$ ,



and  $\frac{M}{|z|-M} < \varepsilon/2$ , then for each  $\mu \in A$ , we have

$$\begin{aligned} \left| \frac{1}{z} G_\mu\left(\frac{1}{z}\right) - 1 \right| &= \left| \int \frac{u}{\frac{1}{z} - u} d\mu(u) \right| \\ &\leq \frac{M}{|z| - M} \mu([-M, M]) + \frac{1}{\cos \theta} \mu([-M, M]^c) \\ &\leq \varepsilon. \end{aligned}$$

**Proposition 4.1.** *Let  $A$  be a set of symmetric probability measures on the real line. Then the following assertions are equivalent*

- (i)  $A$  is tight,
- (ii) for every  $0 < \theta < \pi$ ,  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \theta}} \frac{1}{z} H_\mu(z) = 1$  uniformly in  $\mu \in A$ ,
- (iii)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} \frac{1}{x} H_\mu(x) = 1$  uniformly in  $\mu \in A$ .

**Proof.** (i)  $\Rightarrow$  (ii) follows from what precedes, (ii)  $\Rightarrow$  (iii) is clear. Suppose (iii). Consider  $\varepsilon > 0$ . Take  $\eta > 0$  such that  $\forall u \in [0, 1], |\lambda u^2 + (1 - \lambda)u - 1| < \eta \Rightarrow |u - 1| < \varepsilon/2$ . Take  $x \in (-\infty, 0)$  such that for all  $\mu \in A$ ,  $|\frac{1}{x} H_\mu(x) - 1| < \eta$ . Put  $\sqrt{x} = iy$ . Then

$$\forall \mu \in A, \underbrace{\left| \frac{1}{iy} G_\mu\left(\frac{1}{iy}\right) - 1 \right|}_{\int_t \frac{t^2 y^2}{t^2 y^2 + 1} d\mu(t)} < \varepsilon/2.$$

Let  $M$  be such that  $\forall t > M, \frac{t^2 y^2}{t^2 y^2 + 1} \geq \frac{1}{2}$ . Then

$$\forall \mu \in A, \mu([-M, M]^c) \leq 2 \int_t \frac{t^2 y^2}{t^2 y^2 + 1} d\mu(t) < \varepsilon.$$

□

Define, for  $\alpha \in (0, \pi)$ ,  $\beta > 0$ ,  $\Delta_{\alpha, \beta}$  to be the set of complex numbers  $z$  such that  $|\arg z - \pi| < \alpha$  and  $|z| < \beta$ .

Let  $\mathcal{H}$  be the set of functions  $f$  which are analytic in a domain  $\mathcal{D}_f$  such that for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that

$$\Delta_{\alpha, \beta} \subset \mathcal{D}_f.$$

A family  $(f_a)_{a \in A}$  of functions of  $\mathcal{H}$  is said to be *uniform* if for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that

$$\forall a \in A, \Delta_{\alpha, \beta} \subset \mathcal{D}_{f_a}.$$

The following theorem has already been used, in other forms, to define Voiculescu's  $R$ - and  $S$ -transforms (see paragraph 5 of [BV93]). Its proof relies on Rouché theorem.

**Theorem 4.2.** *Let  $(H_a)_{a \in A}$  be a uniform family of functions of  $\mathcal{H}$  such that for every  $\alpha \in (0, \pi)$ ,*

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{H_a(z)}{z} = 1 \text{ uniformly in } a \in A.$$

Then there exists a uniform family  $(F_a)_{a \in A}$  of functions of  $\mathcal{H}$  such that for every  $\alpha \in (0, \pi)$ ,

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{F_a(z)}{z} = 1 \text{ uniformly in } a \in A,$$

and there exists  $\beta$  positive such that

$$\forall a \in A, \quad H_a \circ F_a = F_a \circ H_a = I_d \text{ on } \Delta_{\alpha, \beta}.$$

Moreover, the family  $(F_a)_{a \in A}$  is unique in the following sense: if a family  $(\tilde{F}_a)_{a \in A}$  of functions of  $\mathcal{H}$  satisfies the same conditions, then for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that

$$\forall a \in A, \quad F_a = \tilde{F}_a \text{ on } \Delta_{\alpha, \beta}.$$

**4.3. Definition of the rectangular  $R$ -transform.** For every symmetric probability measure on the real line  $\mu$ , let us define the rectangular  $R$ -transform  $R_\mu$  with ratio  $\lambda$  of  $\mu$  by

$$C_\mu(z) = U \left( \frac{z}{H_\mu^{-1}(z)} - 1 \right),$$

where  $H_\mu^{-1}$  is defined by the previous theorem and the function  $U$  is the one defined at the end of section 4.1.

One can summarize the different steps of the construction of the rectangular  $R$ -transform with ratio  $\lambda$  in the following chain

$$\begin{array}{c} \mu \\ \text{sym. prob.} \\ \text{measure} \end{array} \longrightarrow \begin{array}{c} G_\mu \\ \text{Cauchy} \\ \text{transf.} \end{array} \longrightarrow H_\mu(z) = \lambda G_\mu \left( \frac{1}{\sqrt{z}} \right)^2 + (1 - \lambda) \sqrt{z} G_\mu \left( \frac{1}{\sqrt{z}} \right) \longrightarrow \\ C_\mu(z) = U \left( \frac{z}{H_\mu^{-1}(z)} - 1 \right). \\ \text{rect. } R\text{-transf. with ratio } \lambda$$

**4.4. The special cases  $\lambda = 0$  and  $\lambda = 1$ .** Note that the rectangular  $R$ -transform with ratio 1 (resp. 0), for a symmetric distribution  $\mu$ , is linked to the Voiculescu transform  $\varphi_\mu$  of  $\mu$  by the relation  $C_\mu(z) = \sqrt{z} \varphi_\mu(1/\sqrt{z})$  (resp.  $C_\mu(z) = z \varphi_\rho(1/z)$ , where  $\rho$  is the push-forward of  $\mu$  by the function  $t \rightarrow t^2$ ) (see paragraph 5 of [BV93] for the construction of the Voiculescu transform).

**4.5. Remark about the characterization of  $R$ -transforms.** The definition of the rectangular  $R$ -transform of a probability measure  $\mu$  presents many analogies with the definition of its Voiculescu transform  $\varphi_\mu$  (see [BV93]). So it seems natural to state, as the authors of [BV93] did for the Voiculescu transform (Proposition 5.6), a characterization of the functions that are the rectangular  $R$ -transform of a symmetric probability measure. The characterization of the Voiculescu transform is based on the fact that for every probability measure  $\mu$ , the inverse of  $z + \varphi_\mu(z)$  extends to a Pick function (i.e. an analytic function defined on the upper half plane, whose imaginary part does not take negative values), and on the fact that any Pick function equivalent to  $z$  at infinity is of the type  $1/G_\mu$ , so its inverse is  $z + \varphi_\mu(z)$ . But unless  $\lambda = 0$  or 1, Pick functions do not appear in an analogous place in the definition of the rectangular  $R$ -transform, so we cannot proceed similarly to characterize rectangular  $R$ -transforms.

Furthermore, the Lévy Kinchine formula for  $\boxplus_\lambda$  (see [B-G1]), compared with theorem 5.10 of [BV93] will state that when  $\mu$  is  $\boxplus_\lambda$ -infinitely divisible, there exists a unique symmetric  $\boxplus$ -infinitely divisible distribution  $\nu$  such that  $C_\mu(z) = \sqrt{z} \varphi_\nu(1/\sqrt{z})$ . So the question of the

characterization of rectangular  $R$ -transforms joins another question: can we extend the correspondence  $\mu \leftrightarrow \nu$  to a bijective correspondence in the set of symmetric distributions such that  $C_\mu(z) = \sqrt{z}\varphi_\nu(1/\sqrt{z})$ . The analytic functions  $f$  on  $\Delta_{\alpha,\beta}$  of the type  $f(z) = \sqrt{z}\varphi_\nu(1/\sqrt{z})$ , with  $\nu$  symmetric probability measure, are characterized by:

- (i)  $f(\bar{z}) = \overline{f(z)}$ ,
- (ii)  $\lim_{z \rightarrow 0} f(z) = 0$ ,
- (iii) for all  $n$  and all  $z_1, \dots, z_n$ , the matrix  $\left[ \frac{\sqrt{z_j} - \sqrt{z_k}}{\sqrt{z_j}(1+f(z_k)) - \sqrt{z_k}(1+f(z_j))} \right]_{k,j=1}^n$  is positive.

But nothing allows us to claim that the rectangular  $R$ -transform of any symmetric distribution satisfies (iii), and that every function that satisfies (i), (ii), and (iii) is the rectangular  $R$ -transform of a symmetric distribution.

#### 4.6. Properties of the rectangular $R$ -transform.

**Theorem 4.3** (Injectivity of the rectangular  $R$ -transform). *If the rectangular  $R$ -transforms with ratio  $\lambda$  of two symmetric probability measures coincide on a neighborhood of 0 in  $(-\infty, 0)$ , then the measures are equal.*

**Proof.** If the rectangular  $R$ -transforms with ratio  $\lambda$  of two symmetric probability measures  $\mu, \nu$  coincide on a neighborhood of 0 in  $(-\infty, 0)$ , then by uniqueness of analytic continuation, they coincide on a  $\Delta_{\alpha,\beta}$ , and so  $H_\mu = H_\nu$  on this set, and  $zH_\mu(z) = zH_\nu(z)$  on this set. So there exists  $M > 0$  such that for all  $y > M$ ,

$$\lambda(iy2G_\mu(iy))^2 + (1 - \lambda)iyG_\mu(iy) = \lambda(iy2G_\nu(iy))^2 + (1 - \lambda)iyG_\nu(iy).$$

But if  $\rho$  is a symmetric probability measure,  $iyG_\mu(iy) = \int_{t \in \mathbb{R}} \frac{y^2}{y^2 + t^2} d\rho(t) \in (0, 1]$ . So, by injectivity of  $u \mapsto \lambda u^2 + (1 - \lambda)u$  on  $(0, 1]$ ,  $G_\mu(iy) = G_\nu(iy)$  for  $y > 1$ , and then, by analyticity of the Cauchy transform,  $G_\mu = G_\nu$ , and by injectivity of the Cauchy transform (see [A61] or section 3.1 of [HP00])  $\mu = \nu$ .  $\square$

The following remark gives a practical way to derive any symmetric probability measure on the real line  $\mu$  from  $C_\mu$ .

**Remark 4.4** (How to compute  $\mu$  when we know  $C_\mu$ ?). *First, we have  $z/H_\mu^{-1}(z) = T(C_\mu(z))$ , for  $z \in \mathbb{C} \setminus \mathbb{R}^+$  small enough. From this, we can compute  $H_\mu(z)$  for  $z \in \mathbb{C} \setminus \mathbb{R}^+$  small enough. Then we can use the equation, for  $z \in \mathbb{C} \setminus \mathbb{R}^+$ ,*

$$\frac{1}{z}H_\mu(z) = \lambda \left( \frac{1}{\sqrt{z}}G_\mu \left( \frac{1}{\sqrt{z}} \right) \right)^2 + (1 - \lambda) \frac{1}{\sqrt{z}}G_\mu \left( \frac{1}{\sqrt{z}} \right).$$

Moreover, when  $z \in \mathbb{C} \setminus \mathbb{R}^+$  is small enough,  $1/\sqrt{z}$  is large and in  $\mathbb{C}^-$ , so  $\frac{1}{\sqrt{z}}G_\mu \left( \frac{1}{\sqrt{z}} \right)$  is closed to 1.  $\frac{1}{z}H_\mu(z)$  is also closed to 1, and for  $h, g$  complex numbers closed to 1,

$$h = \lambda g^2 + (1 - \lambda)g \Leftrightarrow g = V(h), \text{ with } V(z) = \frac{\lambda - 1 + ((\lambda - 1)^2 + 4\lambda z)^{\frac{1}{2}}}{2\lambda} = U(z - 1) + 1.$$

So one has, for  $z \in \mathbb{C} \setminus \mathbb{R}^+$  small enough,

$$\frac{1}{\sqrt{z}}G_\mu \left( \frac{1}{\sqrt{z}} \right) = V \left( \frac{H_\mu(z)}{z} \right).$$

The following lemma is an easy consequence of proposition 4.1 and theorem 4.2.

**Lemma 4.5** (Tightness and rectangular  $R$ -transform). *Let  $A$  be a set of symmetric probability measures. Then we have equivalence between*

- (i)  $A$  is tight,
- (ii) for any  $0 < \alpha < \pi$ ,  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_\mu(z) = 0$  uniformly in  $\mu \in A$ ,
- (iii)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} R_\mu(x) = 0$  uniformly in  $\mu \in A$ .

**Theorem 4.6** (Paul Lévy's theorem for rectangular  $R$ -transform). *Let  $(\mu_n)$  be a sequence of symmetric probability measures. Then we have equivalence between:*

- (i)  $(\mu_n)$  converges weakly to a symmetric probability measure;
- (ii) there exists  $\alpha, \beta$  such that
  - (a)  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_{\mu_n}(z) = 0$  uniformly in  $n$ ,
  - (b) the sequence  $(C_{\mu_n})$  converges uniformly on every compact set of  $\Delta_{\alpha, \beta}$ ;
- (iii) (a)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} C_{\mu_n}(x) = 0$  uniformly in  $n$ ,
- (b) there exists  $\beta > 0$  such that the sequence  $(C_{\mu_n})$  converges pointwise on  $[-\beta, 0)$ .

Moreover, in this case, denoting by  $\mu$  the weak limit of  $(\mu_n)$ , for every  $\alpha$ , there exists  $\beta$  such that the sequence  $(C_{\mu_n})$  converges uniformly to  $C_\mu$  on every compact set of  $\Delta_{\alpha, \beta}$ .

**Proof.** (i)  $\Rightarrow$  (ii): suppose that  $(\mu_n)$  converges weakly to  $\mu$ . Then by the previous lemma, we have (a) of (ii). So there exists  $\beta > 0$  such that on  $\Delta_{\alpha, \beta}$ , for all  $n$ ,  $|C_{\mu_n}| \leq 1$ . So, by Montel's theorem, it suffices to show that the only possible limit, for uniform convergence on every compact, of any subsequence of  $(C_{\mu_n})$  is  $C_\mu$ . Let  $C$  be an analytic function on  $\Delta_{\alpha, \beta}$  such that a subsequence  $(C_{\mu_{k_n}})$  of  $(C_{\mu_n})$  converges uniformly to  $C$  on every compact of  $\Delta_{\alpha, \beta}$ . We know (see [A61] or section 3.1 of [HP00]) that the sequence  $(G_{\mu_n})$  converges uniformly on every compact of the upper half plane to  $G_\mu$ . So the sequence  $(H_{\mu_n})$  converges uniformly on every compact of the complement of  $[0, +\infty)$  to  $H_\mu$ . Since  $\lim_{z \rightarrow 0} C(z) = 0$ , to prove  $C = C_\mu$ , it suffices to prove that

$$\begin{aligned} (\lambda C + 1)(C + 1) &= (\lambda C_\mu + 1)(C_\mu + 1). \\ \text{But } (\lambda C_\mu(z) + 1)(C_\mu(z) + 1) &= \frac{z}{H_\mu^{-1}(z)} \end{aligned}$$

So it suffices to prove that

$$\frac{z}{(\lambda C(z) + 1)(C(z) + 1)} = H_\mu^{-1}(z),$$

so by theorem 4.2, it suffices to prove that

$$H_\mu \left( \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right) = z.$$

We have

$$\begin{aligned}
& \left| H_\mu \left[ \frac{z}{(\lambda C(z)+1)(C(z)+1)} \right] - z \right| \\
= & \left| H_\mu \left[ \frac{z}{(\lambda C(z)+1)(C(z)+1)} \right] - H_{\mu_{k_n}} \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z)+1)(C_{\mu_{k_n}}(z)+1)} \right] \right| \\
\leq & \underbrace{\left| H_\mu \left[ \frac{z}{(\lambda C(z)+1)(C(z)+1)} \right] - H_\mu \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z)+1)(C_{\mu_{k_n}}(z)+1)} \right] \right|}_{(1)} \\
& + \underbrace{\left| H_\mu \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z)+1)(C_{\mu_{k_n}}(z)+1)} \right] - H_{\mu_{k_n}} \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z)+1)(C_{\mu_{k_n}}(z)+1)} \right] \right|}_{(2)}
\end{aligned}$$

By continuity of  $H_\mu$ , (1) tends to zero when  $n$  tends to infinity, and, since  $H_{\mu_{k_n}}$  converges uniformly to  $H_\mu$  on every compact, (2) tends to zero. So  $H_\mu \left( \frac{z}{(\lambda C(z)+1)(C(z)+1)} \right) = z$ .

(ii)  $\Rightarrow$  (iii) is clear.

(iii)  $\Rightarrow$  (i): suppose (iii). Then by (i)  $\Rightarrow$  (ii), every limit of a subsequence of  $(\mu_n)$  has a rectangular  $R$ -transform equal to the pointwise limit of  $(C_{\mu_n})$  on  $[\beta, 0)$ . By uniqueness of analytic continuation, all the limits of subsequences of  $(\mu_n)$  have the same rectangular  $R$ -transform, so, by injectivity of this transform (previous theorem), there cannot be more than one limit of subsequence of  $(\mu_n)$ . As by (a) of (iii), the set  $\{\mu_n; n \in \mathbb{N}\}$  is tight,  $(\mu_n)$  converges weakly to a symmetric probability measure.  $\square$

**4.7. Rectangular  $R$ -transform and rectangular free convolution.** Recall the definition of the binary operation  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures : let  $\mu, \nu$  be two compactly supported symmetric probability measures. Let  $a, b$  be two free elements in  $p_1 \mathcal{A} p_2$  with  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$   $(\rho_1, \rho_2)$ -probability space such that  $\lambda = \rho_1/\rho_2$  and such that for all  $n \in \mathbb{N}$ ,  $m_{2n}(\mu) = \varphi((aa^*)^n)$  and  $m_{2n}(\nu) = \varphi((bb^*)^n)$ . Then there exists a unique compactly supported probability measure  $\mu_{\boxplus_\lambda \nu}$ , that depends only on  $\mu$  and  $\nu$  such that for all  $n \in \mathbb{N}$ ,  $m_{2n}(\mu_{\boxplus_\lambda \nu}) = \varphi(((a+b)(a+b)^*)^n)$ . By equation (37), we know that

$$(47) \quad C_{\mu_{\boxplus_\lambda \nu}} = C_\mu + C_\nu,$$

as a formal power series. But, by section 4.1, we know that this formal series are the expansion of the analytic functions with the same name. So (47) holds also for analytic functions.

Let us extend the binary operation  $\boxplus_\lambda$  to a binary operation on the set of symmetric probability measures on the real line, that will be commutative, associative, and continuous with respect to the weak convergence. Let  $\mu, \nu$  be symmetric probability measures. If  $(\mu_n)$  (resp.  $(\nu_n)$ ) is a sequence of compactly supported symmetric probability measures that converges weakly to  $\mu$  (resp.  $\nu$ ), then by theorem 4.6 and equation (47) the sequence  $(\mu_n \boxplus_\lambda \nu_n)$  converges to a measure whose rectangular  $R$ -transform is  $C_\mu + C_\nu$  (so this measure does not depend on the choice of the sequences  $(\mu_n)$  and  $(\nu_n)$  and is equal to  $\mu_{\boxplus_\lambda \nu}$  when  $\mu$  and  $\nu$  are compactly supported). This measure will be called the *rectangular  $R$ -transform with ratio  $\lambda$*  of  $\mu$  and  $\nu$ , and denoted by  $\mu_{\boxplus_\lambda \nu}$ . Note that the equation (47) stays true for all symmetric probability measures  $\mu$  and  $\nu$ . Moreover, this equation shows that  $\boxplus_\lambda$  is a commutative, associative, and continuous binary operation on the set of symmetric probability measures. All this facts are collected in the following theorem.

**Theorem 4.7.** *The binary operation  $\boxplus_\lambda$  defined on the set of compactly supported symmetric probability measures in section 2, extends in a unique way to a commutative, associative, and continuous binary operation on the set of symmetric probability measures on the real line. This operation is called the rectangular free convolution with ratio  $\lambda$ . Moreover, for all  $\mu, \nu$  symmetric probability measures one has*

$$(48) \quad C_{\mu \boxplus_\lambda \nu} = C_\mu + C_\nu.$$

Note that the continuity of  $\boxplus_\lambda$  is with respect to the weak convergence: for any  $\mu, \nu$  symmetric probability measures and any sequences  $(\mu_n), (\nu_n)$  of symmetric probability measures which converge weakly respectively to  $\mu, \nu$ , the sequence  $(\mu_n \boxplus_\lambda \nu_n)$  converges weakly to  $\mu \boxplus_\lambda \nu$ .

Now we can extend the proposition 2.1 to the case where  $\mu_1, \mu_2$  are not compactly supported:

**Theorem 4.8.** *Consider sequences  $q_1(n), q_2(n)$  such that*

$$q_1(n) \leq q_2(n), \quad q_1(n) \xrightarrow{n \rightarrow \infty} +\infty, \quad q_1(n)/q_2(n) \xrightarrow{n \rightarrow \infty} \lambda \in [0, 1].$$

*Let, for all  $n$ ,  $A_n, B_n$  be independent  $q_1(n) \times q_2(n)$  random matrices, one of them being bi-unitarily invariant, such that the symmetrizations of the singular laws of  $A_n, B_n$  converge in probability to probability measures  $\mu, \nu$ . Then the symmetrization of the singular law of  $A_n + B_n$  converges in probability to  $\mu \boxplus_\lambda \nu$ .*

To prove the theorem, we will need the following lemma, for which we need to introduce an extended functional calculus. When  $F$  is a real Borel function on the real line, for any  $q_1(n) \times q_2(n)$  matrix  $C$ , we define  $F(C)$  to be the matrix

$$U[\delta_i^j F(h_i)]_{\substack{1 \leq i \leq q_1(n) \\ 1 \leq j \leq q_2(n)}} V,$$

where  $h_1, \dots, h_{q_1(n)} \geq 0$ , and  $U, V$  are respectively  $q_1(n) \times q_1(n)$ ,  $q_2(n) \times q_2(n)$  unitary matrices such that

$$c = U[\delta_i^j h_i]_{\substack{1 \leq i \leq q_1(n) \\ 1 \leq j \leq q_2(n)}} V.$$

Note that  $(F + G)(C) = F(C) + G(C)$  and  $F(C)G(C)^* = 0$  when  $FG = 0$ . For any probability measure  $\sigma$ , we will denote by  $F(\sigma)$  (for example  $\sigma^2, \sigma^{1/2}, |\sigma| \dots$ ) the push-forward of  $\sigma$  by  $F$ , that is the distribution of the random variable  $F(X)$ , when  $X$  is a random variable with distribution  $\sigma$ . We denote by  $\mu_H$  the spectral measure of an hermitian matrix  $H$ . Recall that we denote by  $\tilde{\sigma}$  the symmetrization of any probability measure  $\sigma$ . For example, the symmetrization of the singular law of a rectangular matrix  $A$  will be denoted by  $\tilde{\mu}_{|A|}$ .

**Lemma 4.9.** *Let  $(C_n)$  be a sequence of  $q_1(n) \times q_2(n)$  random matrices such that  $\tilde{\mu}_{|C_n|}$  converges in probability to a probability measure  $\sigma$ . Then, for any odd real function  $F$  continuous at  $\sigma$ -almost every point of  $\mathbb{R}$  and such that  $F(\mathbb{R}^+) \subset \mathbb{R}^+$ ,  $\tilde{\mu}_{|F(C_n)|}$  converges in probability to  $F(\sigma)$ .*

**Proof.**  $F(\sigma)$  is symmetric, the symmetrization is a continuous operation, so it suffices to prove that  $\mu_{|F(C_n)|}$  converges in probability to  $|F(\sigma)| = F(|\sigma|)$ . But  $\mu_{|F(C_n)|} = F(\mu_{|C_n|})$ ,  $\mu_{|C_n|}$  converges in probability to  $|\sigma|$ , and  $\sigma \mapsto F(\sigma)$  on the set of probability measures on the real line is weakly continuous at  $|\sigma|$  because  $F$  is continuous at  $|\sigma|$ -almost every point of the real line ([B68]).  $\square$

**Proof of the theorem.** First of all, for  $U_n, V_n$  independent  $q_1(n) \times q_1(n)$ ,  $q_2(n) \times q_2(n)$  uniform random unitary matrices which are independent from  $\{A_n, B_n\}$ , the random matrices  $U_n A_n V_n, U_n B_n V_n$  are independent (it is a property of the Haar measure on compact groups),

have the same singular values as respectively  $A_n, B_n$ , both of them are bi-unitarily invariant and the singular values of  $U_n A_n V_n + U_n B_n V_n = U_n (A_n + B_n) V_n$  are the same ones as the ones of  $A_n + B_n$ . Hence one can suppose that both  $A_n, B_n$  are bi-unitarily invariant.

The function that maps a probability measure  $\sigma$  on  $\mathbb{R}^+$  to the symmetrization  $\widetilde{\sigma^{1/2}}$  of its push-forward by  $x \rightarrow x^{1/2}$  is continuous. So it suffices to prove that the push-forward of the spectral law of  $|A_n + B_n|$  by  $x \rightarrow x^2$ , i.e. the spectral law of  $M_n := (A_n + B_n)(A_n + B_n)^*$  converges in probability to  $(\mu_{\boxplus \lambda \nu})^2$ .

We can define a distance on the set of probability measures on the real line with the Cauchy transform by

$$(\sigma_1, \sigma_2) \mapsto \sup_{\Im z \geq 1} |G_{\sigma_1}(z) - G_{\sigma_2}(z)|.$$

This distance defines the topology of weak convergence ([A61], [PL02]). The Cauchy transform of the spectral distribution of an hermitian matrix  $M$  is the normalized trace of its resolvent  $\mathfrak{R}_z(M) = (z - M)^{-1}$ . So the spectral distribution of a sequence  $(X_n)$  of  $q_1(n) \times q_1(n)$  hermitian random matrices converges in probability to a probability measure  $\sigma$  on the real line if and only if for each  $\varepsilon > 0$ , the probability of the event

$$\left\{ \sup_{\Im z \geq 1} |\operatorname{tr} \mathfrak{R}_z(X_n) - G_\sigma(z)| > \varepsilon \right\}$$

tends to zero as  $n$  tends to infinity.

Choose  $\varepsilon > 0$ . We will show that

$$\mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu_{\boxplus \lambda \nu})^2}(z) \right| > \varepsilon \right\} \xrightarrow{n \rightarrow \infty} 0,$$

where  $\mathbb{P}$  designs the probability measure of the probability space where the random matrices are defined. Let us define, for  $t > 0$ , the function  $F_t$  on the real line by

$$F_t : x \mapsto \begin{cases} x & \text{if } |x| \leq t, \\ 0 & \text{elseif.} \end{cases}$$

For every probability measure  $\sigma$ ,  $F_t(\sigma)$  converges weakly to  $\sigma$  when  $t$  tends to infinity, so, by continuity of  $\boxplus_\lambda$ , there exists  $t \in (0, +\infty)$  such that  $t$  and  $-t$  are not atoms of the measures  $\mu$  and  $\nu$ , and such that

$$(49) \quad \sup_{\Im z \geq 1} \left| G_{(F_t(\mu) \boxplus_\lambda F_t(\nu))^2}(z) - G_{(\mu_{\boxplus \lambda \nu})^2}(z) \right| < \frac{\varepsilon}{3},$$

$$(50) \quad \mu(\mathbb{R} - [-t - 1, t + 1]) + \nu(\mathbb{R} - [-t - 1, t + 1]) < \frac{\varepsilon}{18}.$$

We will now use the notations  $F_t = F$ ,  $G(x) = x - F(x)$  and  $M_{n,t} = (F(A_n) + F(B_n))(F(A_n) + F(B_n))^*$ .

Using triangular inequality and (49), we have

$$(51) \quad \begin{aligned} & \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu_{\boxplus \lambda \nu})^2}(z) \right| > \varepsilon \right\} \\ & \leq \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left[ \mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t}) \right] \right| > \frac{\varepsilon}{3} \right\} \\ & + \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \mathfrak{R}_z(M_{n,t}) - G_{(F(\mu) \boxplus_\lambda F(\nu))^2}(z) \right| > \frac{\varepsilon}{3} \right\} \end{aligned}$$

To treat the first term of the right hand side of (51), recall that for any  $q_1(n) \times q_1(n)$  matrix  $T$ ,  $|\operatorname{tr} T| \leq \frac{1}{q_1(n)} \|T\| \operatorname{rg} T$ . The operator norm  $\|\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t})\|$  is not less than 2, because  $\Im z \geq 1$ . Moreover, due to the equation

$$\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t}) = -\mathfrak{R}_z(M_n)(M_n - M_{n,t})\mathfrak{R}_z(M_{n,t}),$$

the rank is not more than the one of  $M_n - M_{n,t}$ . One has (omitting the indices  $n$  in  $A_n, B_n$ )

$$M_n - M_{n,t} = G(A)G(A)^* + G(B)G(B)^* + G(A)B^* + F(A)G(B)^* + G(B)A^* + F(B)G(A)^*,$$

so its rank is not more than  $3 \operatorname{rg} G(A_n) + 3 \operatorname{rg} G(B_n)$ . Hence

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{\Im z \geq 1} |\operatorname{tr} [\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t})]| > \frac{\varepsilon}{3} \right\} \\ & \leq \mathbb{P} \left\{ \frac{6}{q_1(n)} (\operatorname{rg} G(A_n) + \operatorname{rg} G(B_n)) > \frac{\varepsilon}{3} \right\} \\ & = \mathbb{P} \left\{ \mu_{A_n}(\mathbb{R} - [-t, t]) + \mu_{B_n}(\mathbb{R} - [-t, t]) > \frac{\varepsilon}{18} \right\}, \end{aligned}$$

which tends to zero when  $n$  goes to infinity, by (50). On the other side, the second term of right hand side of (51) goes to zero when  $n$  goes to infinity by definition (Proposition-Definition 2.1) of the rectangular convolution of compactly supported symmetric probability measures (we didn't suppose that  $q_1(n) + q_2(n) = n$ , but as we said in the second paragraph of section 1.2, this hypothesis is made to simplify the notations, but is not really necessary in results like Proposition-Definition 2.1) and by the previous lemma. So

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu \boxplus_{\lambda} \nu)^2}(z) \right| > \varepsilon \right\} = 0,$$

and the result is proved.  $\square$

## 5. EXAMPLES

In this section, we give examples of symmetric probability measures rectangular  $R$ -transforms of which can be computed, and examples of computations of rectangular free convolutions.

The case where  $\lambda = 1$  (see section 4.4), where the rectangular  $R$ -transform and the rectangular free convolution are essentially the Voiculescu transform and the free convolution defined by Voiculescu and Bercovici, doesn't present any new interest (one can find the little number of examples known in, e.g., [HP00]), so we will suppose that  $\lambda \in [0, 1)$ . Unfortunately, for  $\lambda \in (0, 1)$ , few computations can be done, still less than for the well known case  $\lambda = 1$ .

Before giving the examples, let us explain the way we found most of them and the way we will present them. It is proved, in [B-G1], that for every  $\lambda \in [0, 1]$ , there exists a bijection  $\Lambda_{\lambda}$  between the set of classical symmetric infinitely divisible distributions and the set of probability measures which are infinitely divisible with respect to  $\boxplus_{\lambda}$ , and that this bijection is a pretty deep correspondence, since it is an homeomorphism with respect to the weak convergence, a morphism (i.e.  $\Lambda_{\lambda}(\mu * \nu) = \Lambda_{\lambda}(\mu) \boxplus_{\lambda} \Lambda_{\lambda}(\nu)$ ) and it preserves limit theorems (i.e. for all  $\nu_n$  sequence of symmetric probability measures, for all  $\mu$  symmetric classical infinitely divisible distribution,  $\nu_n^{*n}$  converges weakly to  $\mu$  if and only if  $\nu_n^{\boxplus_{\lambda} n}$  converges weakly to  $\Lambda_{\lambda}(\mu)$ ). Note that the last property is not a consequence of the two first ones, since one doesn't apply  $\Lambda_{\lambda}$  to  $\nu_n$ , which is by the way not supposed to be infinitely divisible. In the following, we shall mostly consider  $\boxplus_{\lambda}$ -infinitely divisible laws, and present them as the analogues of their pre-images by the bijection  $\Lambda_{\lambda}$ , called *the Bercovici-Pata bijection with ratio  $\lambda$* .



### 5.1. The case where $\lambda \in (0, 1)$ .

5.1.1. *Symmetrized Dirac mass.* a) *Dirac mass at zero.* The Dirac mass at zero has 0 for rectangular  $R$ -transform and its rectangular free convolution with any law is the law itself.

b) *Computation of the rectangular  $R$ -transform of  $(\delta_{-c} + \delta_c)/2$ .* Now, let us fix  $a > 0$  to compute the rectangular  $R$ -transform of

$$\mu := (\delta_{-c} + \delta_c)/2 \quad \text{for } c = a^{\frac{1}{2}}.$$

We have

$$G_\mu(z) = \frac{1}{2} \left( \frac{1}{z + a^{\frac{1}{2}}} + \frac{1}{z - a^{\frac{1}{2}}} \right) = \frac{z}{z^2 - a}.$$

$$G_\mu(1/\sqrt{z}) = \frac{\sqrt{z}}{1 - az}.$$

$$H_\mu(z) = \frac{\lambda z}{(1 - az)^2} + (1 - \lambda) \frac{z(1 - az)}{(1 - az)^2} = \frac{(\lambda - 1)az^2 + z}{(1 - az)^2}.$$

Hence for  $x, y \in \mathbb{C} \setminus \mathbb{R}^+$  in a neighborhood of zero, the equation  $H_\mu(y) = x$  is equivalent to

$$(\lambda - 1)ay^2 + y = x - 2axy + a^2xy^2$$

$$y^2(a^2x + (1 - \lambda)a) - y(1 + 2ax) + x = 0.$$

Since  $y = H_\mu^{-1}(x)$  must tend to zero as  $x$  tends to zero, one has

$$H_\mu^{-1}(x) = y = \frac{1 + 2ax - (1 + 4\lambda ax)^{\frac{1}{2}}}{2(a^2x + (1 - \lambda)a)},$$

$$\frac{x}{H_\mu^{-1}(x)} - 1 = \frac{2a^2x^2 + 2(1 - \lambda)ax - 1 - 2ax + (1 + 4\lambda ax)^{\frac{1}{2}}}{1 + 2ax - (1 + 4\lambda ax)^{\frac{1}{2}}}$$

$$= \frac{2a^2x^2 - 2\lambda ax - 1 + (1 + 4\lambda ax)^{\frac{1}{2}}}{1 + 2ax - (1 + 4\lambda ax)^{\frac{1}{2}}}$$

$$= \frac{(2a^2x^2 - 2\lambda ax - 1 + (1 + 4\lambda ax)^{\frac{1}{2}})(1 + 2ax + (1 + 4\lambda ax)^{\frac{1}{2}})}{1 + 4ax + 4a^2x^2 - 1 - 4\lambda ax}$$

$$= \frac{(2a^2x^2 - 2\lambda ax)2ax + 4\lambda ax + (2a^2x^2 - 2\lambda ax)((1 + 4\lambda ax)^{\frac{1}{2}} + 1) + 2ax((1 + 4\lambda ax)^{\frac{1}{2}} - 1)}{4ax(ax + 1 - \lambda)}$$

$$= \frac{2a^2x^2 - 2\lambda ax + 2\lambda + (ax - \lambda)((1 + 4\lambda ax)^{\frac{1}{2}} + 1) + (1 + 4\lambda ax)^{\frac{1}{2}} - 1}{2(ax + 1 - \lambda)}$$

$$= \frac{(ax + 1 - \lambda)(1 + 4\lambda ax)^{\frac{1}{2}} + ax - \lambda - 1 + 2\lambda + 2a^2x^2 - 2\lambda ax}{2(ax + 1 - \lambda)}$$

$$= \frac{(ax + 1 - \lambda)((1 + 4\lambda ax)^{\frac{1}{2}} - 1 + 2ax)}{2(ax + 1 - \lambda)}$$

$$= \frac{(1 + 4\lambda ax)^{\frac{1}{2}} - 1 + 2ax}{2}$$

Note that

$$(\lambda + 1)^2 + 4\lambda \left( \frac{x}{H_\mu^{-1}(x)} - 1 \right) = \lambda^2 + 2\lambda + 1 + 2\lambda(1 + 4\lambda ax)^{\frac{1}{2}} - 2\lambda + 4\lambda ax = \left( \lambda + (1 + 4\lambda ax)^{\frac{1}{2}} \right)^2,$$

hence

$$C_\mu(x) = \frac{-\lambda - 1 + \left[ (\lambda + 1)^2 + 4\lambda \left( \frac{x}{H_\mu^{-1}(x)} - 1 \right) \right]^{\frac{1}{2}}}{2\lambda} = \frac{1}{2\lambda} \left[ (1 + 4\lambda ax)^{\frac{1}{2}} - 1 \right]$$

c) *Computation of*  $[(\delta_{-1} + \delta_1)/2]_{\boxplus_\lambda} [(\delta_{-1} + \delta_1)/2]$ . It could be interesting, for  $c, d > 0$  to compute  $\tau := [(\delta_{-c} + \delta_c)/2]_{\boxplus_\lambda} [(\delta_{-d} + \delta_d)/2]$ . By (48), we have, for  $z \in \mathbb{C} \setminus \mathbb{R}^+$  in the neighborhood of zero,

$$C_\tau(z) = C_{[(\delta_{-c} + \delta_c)/2]}(z) + C_{[(\delta_{-d} + \delta_d)/2]}(z).$$

By remark 4.4, in order to obtain the explicit computation of the probability measure  $\tau$  from its rectangular  $R$ -transform  $C_\tau$ , one needs to invert the function  $H_\tau^{-1}$ , which one can compute by the formula  $z/H_\tau^{-1}(z) = T(C_\tau(z))$ . But if one denotes  $c^2 = a, d^2 = b$ , one has

$$H_\tau^{-1}(z) = \frac{2\lambda z}{1 + 2\lambda(a+b)z + (1 + 4\lambda az)^{\frac{1}{2}}(1 + 4\lambda bz)^{\frac{1}{2}} + (\lambda - 1)((1 + 4\lambda az)^{\frac{1}{2}} + (1 + 4\lambda bz)^{\frac{1}{2}})},$$

and unless  $a = b$ , we didn't manage to invert this function.

However, for  $a = b$ , the computation can be done. The following remark allows us to restrict our computations to the case  $a = b = 1$ .

**Remark 5.1.** *Let us denote, For  $\nu$  probability measure on the real line, let us denote by  $D_c(\nu)$  the push-forward of  $\nu$  by  $x \mapsto cx$ . Clearly, by theorem 4.8, for any pair  $\nu_1, \nu_2$  of probability measures,*

$$D_c(\nu_1 \boxplus_\lambda \nu_2) = D_c(\nu_1) \boxplus_\lambda D_c(\nu_2).$$

**Proposition 5.2.** *The rectangular free convolution with ratio  $\lambda$*

$$\tau := \frac{\delta_{-1} + \delta_1}{2} \boxplus_\lambda \frac{\delta_{-1} + \delta_1}{2}$$

has support

$$[-(2 + \kappa)^{1/2}, -(2 - \kappa)^{1/2}] \cup [(2 - \kappa)^{1/2}, (2 + \kappa)^{1/2}],$$

with  $\kappa = 2(\lambda(2 - \lambda))^{1/2} \in (0, 2)$ , and it admits a density with formula

$$\frac{[\kappa^2 - (x^2 - 2)^2]^{1/2}}{\pi\lambda|x|(4 - x^2)}$$

on its support.

**Proof.** By what precedes, for  $y \in \mathbb{C} \setminus \mathbb{R}^+$  in the neighborhood of zero, if one denotes  $(1 + 4\lambda y)^{\frac{1}{2}}$  by  $w$ , one has

$$H_\tau^{-1}(y) = \frac{\lambda y}{1 + 4\lambda y + (\lambda - 1)(1 + 4\lambda y)^{\frac{1}{2}}} = \frac{(w^2 - 1)/4}{w^2 + (\lambda - 1)w}.$$

Hence for  $x \in \mathbb{C} \setminus \mathbb{R}^+$  in the neighborhood of zero, if one denotes  $H_\tau(x)$  by  $y$  and  $(1 + 4\lambda y)^{\frac{1}{2}}$  by  $w$ , one has  $H_\tau^{-1}(y) = x$ , i.e.  $w^2 - 1 = 4xw^2 + 4x(\lambda - 1)w$ , i.e., since  $w$  tends to 1 as  $x$  tends to zero,

$$w = \frac{1}{1 - 4x} [2x(\lambda - 1) + [1 - 4x + 4(\lambda - 1)^2 x^2]^{\frac{1}{2}}],$$

hence

$$\begin{aligned} 1 + 4\lambda y &= w^2 = \frac{[2x(\lambda - 1) + [1 - 4x + 4(\lambda - 1)^2 x^2]^{\frac{1}{2}}]^2}{(1 - 4x)^2} \\ &= \frac{8x^2(\lambda - 1)^2 + 1 - 4x + 4x(\lambda - 1)[1 - 4x + 4(\lambda - 1)^2 x^2]^{\frac{1}{2}}}{(1 - 4x)^2}, \end{aligned}$$

$$\frac{4\lambda y}{x} = \frac{8x(\lambda-1)^2 + 4 - 16x + 4x(\lambda-1)[1-4x+4(\lambda-1)^2x^2]^{\frac{1}{2}}}{(1-4x)^2}.$$

Hence an easy computation leads to

$$(\lambda-1)^2 + \frac{4\lambda y}{x} = \left( \frac{\lambda-1 + 2[1-4x+4(\lambda-1)^2x^2]^{\frac{1}{2}}}{1-4x} \right)^2.$$

Hence by remark 4.4, for  $z \in \mathbb{C} \setminus \mathbb{R}^+$  in the neighborhood of zero,

$$\frac{1}{\sqrt{z}} G_\tau \left( \frac{1}{\sqrt{z}} \right) = V \left( \frac{H_\tau(z)}{z} \right) = \frac{1}{2\lambda} \left( \lambda - 1 + \frac{\lambda - 1 + 2[1 - 4z + 4(\lambda - 1)^2 z^2]^{\frac{1}{2}}}{1 - 4z} \right),$$

hence

$$G_\tau \left( \frac{1}{\sqrt{z}} \right) = \frac{1}{2\lambda \frac{1}{\sqrt{z}}} \left( \lambda - 1 + \frac{\lambda - 1 + 2[1 - 4/(1/\sqrt{z})^2 + 4(\lambda - 1)^2/(1/\sqrt{z})^4]^{\frac{1}{2}}}{1 - 4 \frac{1}{(1/\sqrt{z})^2}} \right),$$

so for  $z \in i\mathbb{R}^-$  large enough,

$$\begin{aligned} G_\tau(z) &= \frac{1}{2\lambda z} \left( \lambda - 1 + \frac{\lambda - 1 + 2[1 - 4/z^2 + 4(\lambda - 1)^2/z^4]^{\frac{1}{2}}}{1 - 4/z^2} \right) \\ &= \frac{(4 - z^2)(\lambda - 1) - z^2(\lambda - 1) - 2z^2[1 - 4/z^2 + 4(\lambda - 1)^2/z^4]^{\frac{1}{2}}}{2\lambda z(4 - z^2)} \\ &= \frac{(2z^2 - 4)(\lambda - 1) + 2[z^4 - 4z^2 + 4(\lambda - 1)^2]^{\frac{1}{2}}}{2\lambda z(z^2 - 4)} = \frac{(2z^2 - 4)(\lambda - 1) + 2[(z^2 - 2)^2 - \kappa^2]^{\frac{1}{2}}}{2\lambda z(z^2 - 4)} \end{aligned}$$

so by analytic continuation, for any  $z \in \mathbb{C}^-$ ,

$$G_\tau(z) = \frac{(4 - 2z^2)(\lambda - 1) + 2s(z)}{2\lambda z(4 - z^2)}$$

where  $s(z)$  is analytic on  $\mathbb{C}^-$  and satisfies  $s(z)^2 = (z^2 - 2)^2 - \kappa^2$ . In order to compute  $\tau$ , we shall use the following result (for reference, see lemma 2.17 of [B06]): for Lebesgue-almost all real number  $x$ , the limit, as  $z = u + iv \in \mathbb{C}^-$  tends to  $x$  in such a way that  $(u - x)/v$  stays bounded, of  $\Im G_\tau(z)/\pi$  is the density of the absolutely continuous part of  $\tau$  at  $x$ .

Let us define

$$N := (-(2 + \kappa)^{1/2}, -(2 - \kappa)^{1/2}) \cup ((2 - \kappa)^{1/2}, (2 + \kappa)^{1/2}).$$

By what precedes, for Lebesgue-almost all  $x \in \mathbb{R}$ , there is no sequence  $w_n > 0$  tending to 0 such that  $\lim_{n \rightarrow \infty} \Im G_\tau(x - iw_n) < 0$ . Hence since for all  $x \in N$ ,  $(x^2 - 2)^2 - \kappa^2$  is negative, for Lebesgue-almost all  $x \in N$ , there is  $\varepsilon > 0$  such that for all  $w \in (0, \varepsilon)$ , for  $z = x - iw$ ,

$$G_\tau(z) = \begin{cases} \frac{(4-2z^2)(\lambda-1) + 2\sqrt{(z^2-2)^2 - \kappa^2}}{2\lambda z(4-z^2)} & \text{if } x > 0, \\ \frac{(4-2z^2)(\lambda-1) - 2\sqrt{(z^2-2)^2 - \kappa^2}}{2\lambda z(4-z^2)} & \text{if } x < 0. \end{cases}$$

It implies that for Lebesgue-almost all  $x \in N$ ,

$$\lim_{y \rightarrow 0^-} \Im G_\tau(x + iy)/\pi = \frac{\Im(\sqrt{(x^2 - 2)^2 - \kappa^2})}{\pi \lambda |x|(4 - x^2)} = \frac{[\kappa^2 - (x^2 - 2)^2]^{1/2}}{\pi \lambda |x|(4 - x^2)}.$$

Now, to complete the proof, it suffices to prove that  $\tau$  is equal the absolutely continuous part of its restriction to  $N$ , i.e. that

$$\int_N \frac{[\kappa^2 - (x^2 - 2)^2]^{1/2}}{\pi\lambda|x|(4 - x^2)} dx = 1.$$

Let us denote this integral by  $I$ .

One has

$$\begin{aligned} I &= \frac{2}{\pi\lambda} \int_{(2-\kappa)^{1/2}}^{(2+\kappa)^{1/2}} \frac{[\kappa^2 - (x^2 - 2)^2]^{1/2}}{x(4 - x^2)} dx = \frac{1}{\pi\lambda} \int_{(2-\kappa)^{1/2}}^{(2+\kappa)^{1/2}} \frac{[\kappa^2 - (x^2 - 2)^2]^{1/2}}{(2 + x^2 - 2)(2 - (x^2 - 2))} d(x^2 - 2) \\ &= \frac{1}{\pi\lambda} \int_{-\kappa}^{\kappa} \frac{[\kappa^2 - u^2]^{1/2}}{4 - u^2} du = \frac{1}{\pi\lambda} \int_{-\kappa}^{\kappa} \frac{[1 - (u/\kappa)^2]^{1/2}}{4/\kappa^2 - (u/\kappa)^2} du/\kappa = \frac{1}{\pi\lambda} \int_{-1}^1 \frac{(1 - v^2)^{1/2}}{4/\kappa^2 - v^2} dv \end{aligned}$$

Let us define  $y = \frac{v}{(1-v^2)^{1/2}}$ . When  $v$  goes from  $-1$  to  $1$ ,  $y$  goes increasingly from  $-\infty$  to  $+\infty$ .

One has  $dy = \frac{dv}{(1-v^2)^{3/2}}$ , and  $v^2 = \frac{y^2}{1+y^2}$ , hence

$$(1 - v^2)^{1/2} dv = (1 - v^2)^2 dy = \frac{dy}{(1 + y^2)^2}$$

and since  $\kappa^2 = 4\lambda(2 - \lambda)$ ,

$$\frac{1}{4/\kappa^2 - v^2} = \frac{1}{\frac{1}{\lambda(2-\lambda)} - \frac{y^2}{1+y^2}} = \frac{\lambda(2-\lambda)(1+y^2)}{(1-2\lambda+\lambda^2)y^2+1} = \frac{\lambda(2-\lambda)}{(1-\lambda)^2} \frac{1+y^2}{y^2+1/(1-\lambda)^2}.$$

So

$$\begin{aligned} I &= \frac{2-\lambda}{(1-\lambda)^2} \int_{\mathbb{R}} \frac{1}{y^2+1/(1-\lambda)^2} \frac{dy}{\pi(1+y^2)} \\ &= \frac{2-\lambda}{(1-\lambda)^2} \int_{\mathbb{R}} \frac{i(\lambda-1)}{2} \left( \frac{1}{y-i/(1-\lambda)} - \frac{1}{y+i/(1-\lambda)} \right) \frac{dy}{\pi(1+y^2)}, \end{aligned}$$

so the well known formula of the Cauchy transform of the Cauchy distribution with parameter 1 gives us

$$I = \frac{2-\lambda}{(1-\lambda)^2} \frac{i(\lambda-1)}{2} \left( \frac{1}{-i-i/(1-\lambda)} - \frac{1}{i+i/(1-\lambda)} \right) = \frac{2-\lambda}{1-\lambda} \frac{1}{1+1/(1-\lambda)} = 1,$$

which closes the proof of the proposition.  $\square$

5.1.2. *The rectangular analogues of the Gaussian distributions.* For all  $\sigma > 0$ , the rectangular analogue of the symmetric Gaussian distribution with variance  $\sigma^2$  is the law  $N_{\sigma^2}$  with density

$$\frac{[4\lambda - (\sigma^2 x^2 - 1 - \lambda)^2]^{1/2}}{2\pi\lambda\sigma^2|x|} \chi(\sigma^2 x^2),$$

where  $\chi$  stands for the characteristic function of the interval  $[(1 - \lambda^{1/2})^2, (1 + \lambda^{1/2})^2]$ . Its support is  $[-\sigma(1 + \lambda^{1/2}), -\sigma(1 - \lambda^{1/2})] \cup [\sigma(1 - \lambda^{1/2}), \sigma(1 + \lambda^{1/2})]$ .

It is proved in [B-G1] that its rectangular  $R$ -transform is  $\sigma^2 z$ .

Hence one has, for all  $\sigma, \alpha > 0$ ,  $N_{\sigma^2 \boxplus \lambda} N_{\alpha^2} = N_{\sigma^2 + \alpha^2}$ .

**Remark 5.3** (Growth of the support in the related semigroup). *Note that in the additive semigroup  $(\{N_c; c > 0\}, \boxplus_\lambda)$  the size of the support of  $N_c$  is not linear in  $c$  but in  $c^{1/2}$ . This kind of phenomenon had already been observed in free probability.*

5.1.3. *The rectangular analogues of the Cauchy distributions.* For all  $t > 0$ , the image of the symmetric Cauchy law with parameter  $t$  by the Bercovici-Pata bijection with ratio  $\lambda$  is

$$\nu_t := 1_{\mathbb{R} \setminus [-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2}]}(x) \frac{t}{\pi(\lambda t^2 + x^2)} \left[ 1 - \frac{t^2(\lambda - 1)^2}{4x^2} \right]^{\frac{1}{2}} dx.$$

Its support is  $\mathbb{R} \setminus \left( -\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2} \right)$ .

Again, it is proved in [B-G1] that the  $R$ -transform with ratio  $\lambda$  of  $\nu_t$  admits an analytic extension to  $\mathbb{C} \setminus \mathbb{R}^+$  given by the formula  $it\sqrt{z}$ . Hence one has, for all  $t_1, t_2 > 0$ ,  $\nu_{t_1} \boxplus_{\lambda} \nu_{t_2} = \nu_{t_1+t_2}$ .

5.1.4. *The rectangular analogues of the symmetric Poisson distributions.* It is proved, in [B-G1], that for all  $c > 0$ , the sequence

$$\left( \left( 1 - \frac{c}{n} \right) \delta_0 + \frac{c}{2n} (\delta_{-1} + \delta_1) \right)^{\boxplus_{\lambda} n}$$

has a weak limit, that we shall denote by  $P_c$ , and that the rectangular  $R$ -transform of  $P_c$  is given by the formula

$$\frac{cz}{1-z}.$$

Hence

$$H_{P_c}^{-1}(z) = \frac{z}{T(C_{P_c}(z))} = \frac{z(1-z)^2}{[(\lambda c - 1)z + 1][(c - 1)z + 1]},$$

the inversion of which would be very heavy because of a third degree equation. So we know the rectangular  $R$ -transform of this law, but we don't give the law.

However, we know that for all  $c, c' > 0$ ,  $P_c \boxplus_{\lambda} P_{c'} = P_{c+c'}$ .

## 5.2. The case where $\lambda = 0$ .

5.2.1. *Symmetric Bernoulli distributions.* Let, for  $a \geq 0$ ,  $B_{a^2}$  be  $\frac{1}{2}(\delta_{-a} + \delta_a)$ . By the section 4.4, the rectangular  $R$ -transform with ratio 0 of  $B_{a^2}$  is  $za^2$ . Hence one has, for all  $a, b \geq 0$ ,  $B_{a^2} \boxplus_0 B_{b^2} = B_{a^2+b^2}$ .  $B_{a^2}$  is in fact the image, by the rectangular Bercovici-Pata bijection with ratio 0, of the Gaussian distribution with variance  $a^2$ . Note again that in the additive semigroup  $(\{B_c; c > 0\}, \boxplus_0)$  the bounds of the support of  $B_c$  are not linear in  $c$  but in  $c^{\frac{1}{2}}$ .

5.2.2. *The analogues of the Cauchy distributions.* For all  $t > 0$ , the image of the symmetric Cauchy law with parameter  $t$  by the Bercovici-Pata bijection with ratio 0 is

$$\nu_t := 1_{\mathbb{R} \setminus [-\frac{t}{2}, \frac{t}{2}]}(x) \frac{t}{\pi x^2} \left[ 1 - \frac{t^2}{4x^2} \right]^{\frac{1}{2}} dx.$$

Its support is  $\mathbb{R} \setminus \left( -\frac{t}{2}, \frac{t}{2} \right)$ . Note that its push-forward by the function  $t \rightarrow 1/t$  is the symmetric semicircle law with radius  $2/t$ .

Again, it is proved in [B-G1] that the  $R$ -transform with ratio 0 of  $\nu_t$  admits an analytic extension to  $\mathbb{C} \setminus \mathbb{R}^+$  given by the formula  $it\sqrt{z}$ .

Hence one has, for all  $t_1, t_2 > 0$ ,  $\nu_{t_1} \boxplus_0 \nu_{t_2} = \nu_{t_1+t_2}$ .

5.2.3. *Rectangular symmetric Poisson distributions.* Let us recall the definition, for  $c > 0$ , of the *Marchenko-Pastur law* with parameter  $c$ , also called *free Poisson law* with parameter  $c$ . It is the probability measure on  $[0, +\infty)$

$$M_c = \begin{cases} \frac{(4c-(x-1-c)^2)^{\frac{1}{2}}}{2\pi x} \chi_c(x) dx & \text{if } c \geq 1, \\ (1-c)\delta_0 + \frac{(4c-(x-1-c)^2)^{\frac{1}{2}}}{2\pi x} \chi_c(x) dx & \text{if } 0 < c < 1, \end{cases}$$

where  $\chi_c$  is the characteristic function of  $[(1-c^{\frac{1}{2}})^2, (1+c^{\frac{1}{2}})^2]$ .

It is well known (see [HP00]) that its Voiculescu transform is  $cz/(z-1)$ . Hence by the section 4.4, the rectangular  $R$ -transform with ratio 0 of the symmetric law

$$m_c := \begin{cases} \frac{(4c-(x^2-1-c)^2)^{\frac{1}{2}}}{2\pi} \chi_c(x^2) dx & \text{if } c \geq 1, \\ (1-c)\delta_0 + \frac{(4c-(x^2-1-c)^2)^{\frac{1}{2}}}{2\pi} \chi_c(x^2) dx & \text{if } 0 < c < 1. \end{cases}$$

whose push-forward by  $t \rightarrow t^2$  is  $M_c$  is  $cz/(1-z)$ , and that for all  $c, c' > 0$ ,  $m_c \boxplus_0 m_{c'} = m_{c+c'}$ .

**Remark 5.4** (Growth of the support in the semigroup). *Note that we observe the same kind of phenomenon as in the remark 5.3: in the additive semigroup  $(\{m_c; c > 0\}, \boxplus_0)$ , the size of the support of  $m_c$  is not linear in  $c$ , but in  $c^{\frac{1}{2}}$ .*

5.2.4. *Dirac measures and arcsinus law.* The computation of the Voiculescu transform of  $\rho := \frac{1}{2}(\delta_0 + \delta_1)$  is easy and one gets

$$\varphi_\rho(1/z) = \frac{z-1+(1+z^2)^{\frac{1}{2}}}{2z},$$

which allows to claim that  $\rho \boxplus \rho$  is the arcsinus law

$$1_{(0,2)}(x) \frac{dx}{\pi[x(2-x)]^{\frac{1}{2}}}.$$

Thus we get, for  $\mu = \frac{1}{4}(\delta_{-1} + 2\delta_0 + \delta_1)$ , that  $C_\mu(z) = \frac{z-1+(1+z^2)^{\frac{1}{2}}}{2}$ , and that

$$\mu \boxplus_0 \mu = 1_{\{x^2 < 2\}}(x) \frac{dx}{(2-x^2)^{\frac{1}{2}}}.$$

Note that even though the support of  $\mu$  has convex hull  $[-1, 1]$ , the support of  $\mu \boxplus_0 \mu$  is  $[-2^{\frac{1}{2}}, 2^{\frac{1}{2}}]$ .

5.2.5. *An example derived from the semicircle law.* It is well known (see [HP00]) that for  $m \in \mathbb{R}$  and  $r > 0$ , the Voiculescu transform of the semicircle law  $sc(m, r)$  with center  $m$  and radius  $r$  is given by the formula  $m + \frac{r^2}{4}z$ . Hence if for  $0 < r \leq m$ , one denotes by  $srsc(m, r)$  the symmetric distribution whose push-forward by the function  $t \rightarrow t^2$  is  $sc(m, r)$ , then the rectangular  $R$ -transform of  $srsc(m, r)$  is given by the formula  $mz + \frac{r^2}{4}z^2$  and one has, for all  $0 < r \leq m$ ,  $0 < r' \leq m'$ ,

$$srsc(m, r) \boxplus_0 srsc(m', r') = srsc(m + m', r''),$$

with  $r''^2 = r^2 + r'^2$ .

Note that the distributions  $srsc(m, r)$  do not allow us to define any continuous semigroup of convolution for  $\boxplus_0$  of the type  $\{srsc(m, f(m)); m > 0\}$  for  $f$  function on  $(0, +\infty)$  such that for

all  $m > 0$ ,  $0 < f(m) \leq m$ . Indeed, the existence of such a semi-goup would imply that the function  $f$  satisfies

$$\forall m, m' > 0, f(m + m')^2 = f(m)^2 + f(m')^2,$$

i.e. that there exists  $a > 0$  such that for all  $m > 0$ ,  $f(m) = am^{\frac{1}{2}}$ , which is in contradiction with  $f(m) \leq m$ .

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