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Florent Benaych-Georges

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**HAL Id: hal-00007541**

**<https://hal.science/hal-00007541v1>**

Preprint submitted on 16 Jul 2005 (v1), last revised 4 Mar 2008 (v6)

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## RECTANGULAR RANDOM MATRICES, RELATED CONVOLUTION

FLORENT BENAYCH-GEORGES

ABSTRACT. We characterize asymptotic collective behaviour of rectangular random matrices, the sizes of which tend to infinity at different rates: when embedded in a space of larger square matrices, independent rectangular random matrices are asymptotically free with amalgamation over a subalgebra. Therefore we can define a “rectangular free convolution”, linearized by cumulants and by an analytic integral transform, the “rectangular  $R$ -transform”.

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*Date:* July 16, 2005.

*MSC 2000 subject classifications.* 15A52, 46L54

*Key words.* random matrices, free probability, free convolution

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## INTRODUCTION

The first problem we are going to deal with in this paper is the modelization of asymptotic collective behaviour of independent rectangular random matrices. In order to explain to the reader the way we will treat this problem, let us recall him the work already done by Wigner, Pastur, Marchenko, Girko, Bai, Voiculescu,... for square random matrices. First of all, in the 50's Wigner considered self-adjoint random matrices with gaussian entries (GUE) and proved that the spectral law (uniform distribution on the set of eigenvalues) of these random matrices converges to the so-called *semicircle law*. This result was improved, and other results giving the asymptotic spectral law of random matrices were proved (see, among many other sources, [PL02]). In the same time, people studied the local structure of the spectrum of random matrices (see, e.g. [M67]), but this is not the kind of problem we are going to study here. A new point of view was adopted in the early 90's by Voiculescu, who proposed a way to compute the asymptotic normalized trace (when the dimension  $n$  of the matrices goes to infinity) of products

$$M(s_1, n)^{\varepsilon_1} \cdots M(s_k, n)^{\varepsilon_k}$$

of random matrices taken among a family  $M(1, n), M(2, n), \dots$  of independent random  $n \times n$  matrices and their adjoints, the only hypothesis being a bound on the norms of the matrices, the fact that the matrices  $M(1, n), M(2, n), \dots$  have all a limit singular law (uniform distribution on the set of singular values, i.e. of eigenvalues of the absolute value of the matrix), and a property of invariance of the distributions under an action of the unitary group (these results can be deduced from the fundamental article [V91], but are presented under this form in [HP00]). The advantage of being able to compute the limit of such normalized traces is that it gives us the asymptotic normalized trace of any noncommutative polynomial in our independent random matrices. Hence, since the normalized trace of the  $k$ -th power of a matrix is the  $k$ -th moment of its spectral law, we are able to give the asymptotic singular law of any polynomial of our random matrices. This is why this work is said to *modelize asymptotic collective behaviour of independent square random matrices*. For example, it can be proved (combine results of [HP00] and [HL00]) that the asymptotic singular law of the sum of two independent random matrices whose distributions are invariant under left and right actions of the unitary group and whose singular laws converge weakly to probability measures  $\mu_1, \mu_2$ , only depends on  $\mu_1$  and  $\mu_2$ , and can be expressed easily from  $\mu_1$  and  $\mu_2$ : it is the probability measure on  $[0, \infty)$ , the symmetrization of which is the free convolution of the symmetrizations of  $\mu_1, \mu_2$ . We often present the similar result for hermitian matrices, for which we work with spectral law in the place of singular law, and for which no symmetrization is necessary, but here, we shall work with rectangular matrices, which cannot be hermitian, so the square analogue of our work will be found in non hermitian matrices. In this text, we will propose, similarly, a way to compute asymptotic normalized trace of products of random matrices taken among a family of independent rectangular random matrices, whose sizes tend to infinity, but with different rates. The notion involved, similarly to freeness in Voiculescu's modelization of asymptotics of square matrices, is freeness with amalgamation over a finite dimensional subalgebra. The notion of freeness with amalgamation arises from operator-valued free probability theory, but we chose to use this point of view of operator-valued free probability only when necessary, because this

point of view is not satisfying in all cases: the case where the ratio of sizes of certain of our random matrices tends to zero would have needed to be treated apart in a work using only operator-valued free probability. Moreover, the point of view of operator-valued free probability in the asymptotics of rectangular random matrices is developed in a forthcoming paper, where we analyze the related free entropy and Fischer information ([B-G2]).

This modelization of asymptotics of rectangular random matrices will allow us to define, for  $\lambda \in [0, 1]$ , a binary operation  $\boxplus_\lambda$  on the set of symmetric probability measures, called free convolution with ratio  $\lambda$ , and denoted by  $\boxplus_\lambda$ . For  $\mu_1, \mu_2$  symmetric probability measures,  $\mu_1 \boxplus_\lambda \mu_2$  is defined to be the limit of the singular law of a sum of two independent rectangular random matrices, whose dimensions tend to infinity in a ratio  $\lambda$ , one of them being bi-unitarily invariant, and whose singular laws tend to  $\mu_1, \mu_2$ . In the second part of this paper, after having analyzed related cumulants, we construct an analytic integral transform which linearizes  $\boxplus_\lambda$  (like the  $R$ -transform does for free convolution). In a forthcoming paper ([B-G1]), we study the related infinite divisibility. It appears then that the set of  $\boxplus_\lambda$ -infinitely divisible distributions is in a deep correspondance with the set of symmetric classical infinitely divisible distributions.

**Acknowledgements.** We would like to thank Philippe Biane, Dan Voiculescu, and Piotr Śniady for usefull discussions, as well as Thierry Cabanal-Duvillard, who organized the workshop “Journée Probabilités Libres” at MAP5 in June 2004, where the author had the opportunity to have some of these discussions. Also, we would like to thank Cécile Martineau for her contribution to the english version of this paper.

## 1. ASYMPTOTIC BEHAVIOUR OF RECTANGULAR RANDOM MATRICES

**1.1. Notes on reduction of rectangular random matrices.** In this subsection, we recall some simple facts about polar decomposition of rectangular complex matrices, which can be found in [HJ91]. Consider a  $p \times q$  matrix  $M$ . When  $p \leq q$ , we will denote by  $|M|$  the only  $p \times p$  positive hermitian matrix  $H$  such that we can write  $M = HT$ , with  $T$  a  $p \times q$  matrix such that  $TT^* = I_p$ . In this case, the matrix  $|M|$  is the square root of  $MM^*$ . When  $p > q$ , we will denote by  $|M|$  the only  $q \times q$  positive hermitian matrix  $H$  such that we can write  $M = TH$ , with  $T$  a  $p \times q$  matrix such that  $T^*T = I_q$ . In this case, the matrix  $|M|$  is the square root of  $M^*M$ . In both cases, the spectrum of  $|M|$  is the only (up to a permutation) family  $(h_1, h_2 \dots)$  of nonnegative real numbers such that one can write

$$M = U[\delta_i^j h_i]_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} V,$$

with  $U, V$  respectively  $p \times p, q \times q$  unitary matrices. These numbers are called *singular values* of  $M$ . The uniform distribution on  $h_1, h_2 \dots$  will be called the *singular law* of  $M$  (whereas the uniform distribution on the spectrum of an hermitian matrix is called its *spectral law*). Note that in both cases, for any  $\alpha > 0$ ,

$$\frac{1}{\min(p, q)} \operatorname{Tr} |M|^\alpha = \frac{1}{\min(p, q)} \operatorname{Tr} (MM^*)^{\alpha/2} = \frac{1}{\min(p, q)} \operatorname{Tr} (M^*M)^{\alpha/2}$$

is the  $\alpha$ -th moment of the singular law of  $M$ .

A random matrix is said to be *bi-unitarily invariant* if its distribution is invariant under the left and right actions of the unitary group. It is easy, transferring the proof of lemma 4.3.10 p. 160 of [HP00], to prove that the distribution of a bi-unitarily invariant  $p \times q$  random matrix can be realized as the distribution of  $UHV$  where  $U, H, V$  are independent,  $U, V$  are respectively

$p \times p$ ,  $q \times q$  uniform random unitary matrices (*uniform* means distributed according to the Haar measure), and  $H$  is a random positive  $p \times q$  diagonal matrix.

In the same way, an hermitian random matrix is said to be *unitarily invariant* if its distribution is invariant under the action by conjugaison of the unitary group. In this case, by the beginning of the proof of Theorem 4.3.5 p. 155 of [HP00], its distribution can be realized as the distribution of  $UHU^*$  where  $U$ ,  $H$  are independent,  $U$  is a uniform random unitary matrix, and  $H$  is a diagonal random matrix.

**1.2. Definitions.** For definitions of *singular values*, *singular law*, *spectral law*, *(bi-)unitarily invariant and uniform random matrix*, we refer to subsection 1.1. For all positive integer  $d$ , we denote the set  $\{1, \dots, d\}$  by  $[d]$ . We denote the normalized trace of a square matrix  $X$  by  $\text{tr } X$  (whatever the size of  $X$  is).

Consider a positive integer  $d$ , and sequences  $q_1(n), \dots, q_d(n)$  of pairwise distinct positive integers which all tend to infinity as  $n$  tends to infinity, and such that for all  $k \in [d]$ ,

$$(1) \quad \frac{q_k(n)}{n} \xrightarrow[n \rightarrow \infty]{} \rho_k \geq 0,$$

at most one of the  $\rho_k$ 's being zero. We are going to work with rectangular random matrices of sizes  $q_k(n) \times q_l(n)$ . Free probability's modelization of asymptotic behavior of square random matrices relies on a comparison between random matrices and elements of an algebra arising from operator algebra theory. To product analogous results for rectangular random matrices, we have embedded our matrices of sizes  $q_k(n) \times q_l(n)$  ( $k, l \in [d]$ ) in an algebra. Let us assume that for all  $n$ ,  $q_1(n) + \dots + q_d(n) = n$  (since what we will prove would obviously also work when  $n$  is replaced by a subsequence, it is not a real restriction). Now,  $n \times n$  matrices will be represented as  $d \times d$  block matrices, such that for all  $k, l \in [d]$ , the  $(k, l)$ -th block is a  $q_k(n) \times q_l(n)$  matrix.

For all  $k, l \in [d]$ , for all  $q_k(n) \times q_l(n)$  matrix  $M$ , let us denote by  $\widetilde{M}$  the " $n \times n$  extension of  $M$ ", that is the  $n \times n$  matrix with  $(i, j)$ -th block  $M$  if  $(i, j) = (k, l)$ , and zero in the other case. Note we have preservation of adjoints ( $\widetilde{M^*} = \widetilde{M}^*$ ), and of products:

$$\forall k, l, k', l' \in [d], \forall M \text{ of size } q_k(n) \times q_l(n), \forall N \text{ of size } q_{k'}(n) \times q_{l'}(n), \widetilde{M}\widetilde{N} = \begin{cases} \widetilde{MN} & \text{if } l = k', \\ 0 & \text{elseif.} \end{cases}$$

Moreover, the trace of  $\widetilde{M}$  is the one of  $M$  if  $M$  is square, and zero in the other case. Since we are overall interested in singular laws, whose moments are given by normalized traces of square matrices, these embeddings shall not cause any loss of information. Note however that the normalized trace is not exactly preserved by these embeddings (the normalized trace of a matrix  $M$  of size  $q_k(n) \times q_k(n)$  is  $\frac{n}{q_k(n)}$  times the normalized trace of  $\widetilde{M}$ ), and that the well known relation  $\text{Tr } MN = \text{Tr } NM$  is not satisfied anymore when we work with normalized traces of rectangular matrices. Indeed, for all  $k, l \in [d]$ , for all  $M$  of size  $q_k(n) \times q_l(n)$ , for all  $N$  of size  $q_l(n) \times q_k(n)$ , we have

$$(2) \quad q_k(n) \text{tr } MN = q_l(n) \text{tr } NM.$$

At last, let us define, for all  $k \in [d]$ , the projector

$$p_k(n) := \widetilde{I_{p_k(n)}}.$$

Let us formalize the structure we inherit. Consider a  $*$ -algebra  $\mathcal{A}$  endowed with a family  $(p_1, \dots, p_d)$  of non zero self-adjoint projectors (i.e.  $\forall i, p_i^2 = p_i$ ) which are pairwise orthogonal

(i.e.  $\forall i \neq j, p_i p_j = 0$ ), and such that  $p_1 + \dots + p_d = 1$ . Any element  $x$  of  $\mathcal{A}$  can then be represented

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \vdots \\ x_{d1} & \cdots & x_{dd} \end{bmatrix},$$

where  $\forall i, j, x_{ij} = p_i x p_j$ . This notation is compatible with the product and the involution. Assume each subalgebra  $p_k \mathcal{A} p_k$  to be endowed with a tracial state  $\varphi_k$  (i.e.  $\varphi_k(p_k) = 1$  and for all  $x, y \in p_k \mathcal{A} p_k, \varphi_k(xy - yx) = 0$ ) such that for all  $k, l \in [d], x \in p_k \mathcal{A} p_l, y \in p_l \mathcal{A} p_k$ ,

$$(3) \quad \rho_k \varphi_k(xy) = \rho_l \varphi_l(yx),$$

where  $(\rho_1, \dots, \rho_d)$  is still the sequence of nonnegative real numbers defined by (1). Recall that at most one of the  $\rho_k$ 's is zero.

**Definition 1.1.** *Such a family  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$  will be called a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space. Elements of the union of the  $p_k \mathcal{A} p_l$ 's ( $k, l \in [d]$ ) will be called simple elements.*

**Example 1.2.**  *$\mathcal{A}$  is the  $*$ -algebra of  $n \times n$  complex matrices,  $p_1, \dots, p_d$  are  $p_1(n), \dots, p_d(n)$  previously defined, and each  $\varphi_k$  is  $\frac{1}{q_k(n)} \text{Tr}$ . Then by (2), (3) is satisfied when each  $\rho_k$  is replaced by  $q_k(n)/n$ .*

**Example 1.3.** *If  $\mathcal{A}$  is a  $*$ -algebra endowed with an orthogonal family of self-adjoint projectors  $(p_1, \dots, p_d)$  with sum 1, if  $\mathcal{A}$  is endowed with a tracial state  $\varphi$  such that for all  $k \in [d], \rho_k := \varphi(p_k) \neq 0$ , then the family  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$  is a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, where for all  $k, \varphi_k = \frac{1}{\rho_k} \varphi|_{p_k \mathcal{A} p_k}$ .*

Consider a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$ . Denote by  $\mathcal{D}$  the linear span of the  $p_k$ 's, then  $\mathcal{D}$  is an algebra, which can be identified to the the set of  $d \times d$  complex diagonal matrices, and then to  $\mathbb{C}^d$  by

$$\sum_{k=1}^d \lambda_k p_k \simeq \text{diag}(\lambda_1, \dots, \lambda_d) \simeq (\lambda_1, \dots, \lambda_d).$$

Let us define  $E : \mathcal{A} \rightarrow \mathbb{C}^n$ , which maps  $x \in \mathcal{A}$  to

$$E(x) = \sum_{k=1}^d \varphi_k(x_{kk}) p_k \simeq (\varphi_1(x_{11}), \dots, \varphi_d(x_{dd})).$$

$E$  is then a conditional expectation from  $\mathcal{A}$  to  $\mathcal{D}$ :

$$\forall (d, a, d') \in \mathcal{D} \times \mathcal{A} \times \mathcal{D}, E(dad') = d E(a) d'.$$

The following definition gives the right notion to describe asymptotics of independent random matrices.

**Definition 1.4.** *In an  $*$ -algebra  $\mathcal{B}$  endowed with a  $*$ -subalgebra  $\mathcal{C}$  and a conditional expectation  $E_{\mathcal{C}}$  from  $\mathcal{B}$  to  $\mathcal{C}$ , a family  $(\mathcal{C} \subset \mathcal{B}_{\alpha})_{\alpha \in A}$  of  $*$ -subalgebras is said to be free with amalgamation over  $\mathcal{C}$  if for all  $m \geq 1$ , for all  $\alpha_1 \neq \dots \neq \alpha_m \in A$ , for all  $x_1, \dots, x_m$  elements of respectively  $\mathcal{B}_{\alpha_1}, \dots, \mathcal{B}_{\alpha_m}$  one has*

$$(4) \quad E(x_1) = \dots = E(x_m) = 0 \implies E(x_1 \cdots x_m) = 0.$$

*A family of subsets of  $\mathcal{B}$  is said to be free with amalgamation over  $\mathcal{C}$  if the subsets are contained in  $*$ -subalgebras which are free with amalgamation over  $\mathcal{C}$ .*

In the context of a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space, it is easy to see that a family  $(\chi_\alpha)_{\alpha \in A}$  of sets of simple elements are free with amalgamation over  $\mathcal{D}$  if one has: for all  $m \geq 1$ , for all  $\alpha_1 \neq \dots \neq \alpha_m \in A$ , for all  $x_1, \dots, x_m$  elements of the  $*$ -algebras respectively generated by  $\chi_{\alpha_1}, \dots, \chi_{\alpha_m}$ , one has

$$(5) \quad \mathbb{E}(x_1) = \dots = \mathbb{E}(x_m) = 0 \implies \mathbb{E}(x_1 \cdots x_m) = 0.$$

**Definition 1.5.** *The  $\mathcal{D}$ -distribution of a family  $(a_j)_{j \in J}$  of simple elements is the function which maps any polynomial  $P$  in the noncommutative variables  $(X_j, X_j^*)_{j \in J}$  to  $\mathbb{E}(P(a_j, a_j^*)_{j \in J})$ .*

**Lemma 1.6.** *When  $(\chi_1, \dots, \chi_n)$  is a  $\mathcal{D}$ -free family of sets of simple elements, the  $\mathcal{D}$ -distribution of the union of the  $\chi_i$ 's is completely determined by the  $\mathcal{D}$ -distributions of  $\chi_1, \dots, \chi_n$ .*

**Proof** The  $*$ -algebra generated by the union is the linear span of the set of elements of the type  $x_1 \cdots x_m$ , where  $m \geq 1$ , and there exists  $\alpha_1 \neq \dots \neq \alpha_m \in [n]$  such that each  $x_i$  is in the  $*$ -algebra generated by  $\chi_{\alpha_i}$ . We prove that  $\mathbb{E}(x_1 \cdots x_m)$  is completely determined by the  $\mathcal{D}$ -distributions of  $\chi_1, \dots, \chi_n$  by induction on  $m$ , using the identity

$$x_1 \cdots x_m = \prod_{i=1}^m \left( (x_i - \sum_{k=1}^d \varphi_k(x_i) p_k) + \sum_{k=1}^d \varphi_k(x_i) p_k \right),$$

expanding the right hand side term, and using induction hypothesis and (5).  $\square$

The last notion we have to introduce is the *convergence in  $\mathcal{D}$ -distribution*.

**Definition 1.7.** - *If for all  $n$ ,  $(\mathcal{A}_n, p_{1,n}, \dots, p_{d,n}, \varphi_{1,n}, \dots, \varphi_{d,n})$  is a  $(\rho_{1,n}, \dots, \rho_{d,n})$ -probability space such that*

$$(\rho_{1,n}, \dots, \rho_{d,n}) \xrightarrow{n \rightarrow \infty} (\rho_1, \dots, \rho_d),$$

*a family  $(a_j(n))_{j \in J}$  of simple elements of  $\mathcal{A}_n$  is said to converge in  $\mathcal{D}$ -distribution, when  $n$  goes to infinity, to a family  $(a_j)_{j \in J}$  of elements of  $\mathcal{A}$  if the  $\mathcal{D}$ -distributions converge pointwise.*

- *If  $a_j(n)$ 's are  $n \times n$  random matrices, convergence in  $\mathcal{D}$ -distribution in probability of the family  $(a_j(n))_{j \in J}$  to  $(a_j)_{j \in J}$  is the convergence in probability, when  $n \rightarrow \infty$ , of  $\mathbb{E}(P(a_j(n), a_j(n)^*)_{j \in J})$  to  $\mathbb{E}(P(a_j, a_j^*)_{j \in J})$  for all polynomial  $P$  in the noncommutative variables  $(X_j, X_j^*)_{j \in J}$ .*

Recall that the *convergence in probability* of a sequence  $X_n$  of random variables in a metric space  $(\mathcal{X}, d)$  to a constant  $l \in \mathcal{X}$  is the convergence of the probability of the event  $\{d(X_n, l) < \varepsilon\}$  to 1 for all positive  $\varepsilon$ . All along this paper, we are going to work with sequences of random probability measures on the real line, and to deal with their convergence in probability. It shall refer to the convergence in probability in the metric space of probability measures on the real line endowed with a distance which defines weak convergence.

**1.3. Statement of the theorems about random matrices.** Let, for  $s \in \mathbb{N}, k, l \in [d], n \geq 1$ ,  $R(s, k, l, n)$  be a  $q_k(n) \times q_l(n)$  random matrix. Consider also, for  $n \geq 1$ , a family  $C(i, n)$  ( $i \in I$ ) of diagonal deterministic matrices, each  $C(i, n)$  having the size  $q_{k_i}(n) \times q_{l_i}(n)$ , for a certain  $(k_i, l_i) \in [d] \times [d]$ . We suppose the family to contain  $p_1(n), \dots, p_d(n)$ , and to be stable under product and adjonction. This means that there exists  $i_1, \dots, i_d \in I$  such that for all  $n, k$ ,  $C(i_k, n) = p_k(n)$  and that there exists a map  $\beta$  from  $I \times I$  to  $I$  and an involutive map  $*$  from  $I$  onto itself such that for all  $n \geq 0, i, j \in I$  such that  $l_i = k_j$ ,

$$(6) \quad C(i, n)C(j, n) = C(\beta(i, j), n), \quad C(i, n)^* = C(*i, n).$$

We suppose that for all  $s \in \mathbb{N}, k \neq l \in [d]$ , the moments of the singular law of  $R(s, k, l, n)$  converge in probability. This means that for all integer  $r \geq 0$ ,

$$\frac{1}{\min(q_k(n), q_l(n))} \text{Tr}(R(s, k, l, n)R(s, k, l, n)^*)^r$$

converges in probability to a constant. Note that in the case where one of  $\rho_k, \rho_l$  is zero, it implies that

$$\frac{1}{n} \text{Tr}(R(s, k, l, n)R(s, k, l, n)^*)^r$$

converges in probability to zero.

We suppose also that for all  $s \in \mathbb{N}$  odd, for all  $k \in [d]$ ,  $R(s, k, k, n)$  is almost surely unitary, and that for all  $s \in \mathbb{N}$  even, for all  $k \in [d]$ ,  $R(s, k, k, n)$  is almost surely hermitian and that the moments of its spectral law converge in probability.

We suppose moreover that the family  $C(i, n)$  ( $i \in I$ ) converges in distribution in the following way: for all  $i \in I$  such that  $k_i = l_i$ , the normalized trace

$$\frac{1}{q_{k_i}(n)} \text{Tr} C(i, n)$$

has a finite limit when  $n$  goes to infinity.

Now, consider, in a  $(\rho_1, \dots, \rho_d)$ -rectangular probability space  $(\mathcal{A}, p_1, \dots, p_d, \varphi_1, \dots, \varphi_d)$ , a family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ) of elements of  $\mathcal{A}$ , with  $c(i_1) = p_1, \dots, c(i_d) = p_d$ , whose  $\mathcal{D}$ -distribution is defined by the following rules:

- (i) For all  $s \in \mathbb{N}, k, l \in [d]$ ,  $a(s, k, l) \in p_k \mathcal{A} p_l$ , and for all  $i \in I$ ,  $c(i) \in p_{k_i} \mathcal{A} p_{l_i}$ .
  - (ii) For all  $i, j \in I$  such that  $l_i = k_j$ ,
- (7) 
$$c(i)c(j) = c(\beta(i, j)), \quad c(i)^* = c(*i).$$
- (iii) For all  $s \in \mathbb{N}, k \neq l \in [d]$ , for all  $r \geq 0$ ,
- $$\varphi_k[(a(s, k, l)a(s, k, l)^*)^r] = \lim_{n \rightarrow \infty} \text{tr}(R(s, k, l, n)R(s, k, l, n)^*)^r,$$
- $$\varphi_l[(a(s, k, l)^*a(s, k, l))^r] = \lim_{n \rightarrow \infty} \text{tr}(R(s, k, l, n)^*R(s, k, l, n))^r.$$
- (iv) For all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $a(s, k, k)$  is unitary in  $p_k \mathcal{A} p_k$ , and for all  $r \in \mathbb{Z} - \{0\}$ ,  $\varphi_k(a(s, k, k)^r) = 0$ .
  - (v) For all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $a(s, k, k)$  is self-adjoint and for all  $r \geq 0$ ,
- $$\varphi_k[a(s, k, k)^r] = \lim_{n \rightarrow \infty} \text{tr} R(s, k, l, n)^r,$$
- (vi) For all  $i \in I$  such that  $k_i = l_i$ ,  $\varphi_{k_i}(c(i))$  is the limit of the normalized trace of  $C(i, n)$ .
  - (vii) The family  $\{a(s, k, l)\}$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $\{c(i); i \in I\}$  is free with amalgamation over  $\mathcal{D}$ .

Note that such a space and such family exist, they are given by the free product with amalgamation ([VDN91],[S98]).

We shall use the norm  $\|\cdot\|$  on matrix sets: it is the operator norm associated to canonical hermitian norms. The hypothesis of our main theorems are not the same if one supposes all  $\rho_k$ 's to be positive or not. In the following theorem, all  $\rho_k$ 's are supposed to be positive.

**Theorem 1.8** (Case where all  $\rho_k$ 's are positive). *Assume moreover that for all  $s, k \neq l$ ,  $R(s, k, l, n)$  is bi-unitarily invariant, that for all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $R(s, k, k, n)$  is unitarily invariant, and that for all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $R(s, k, k, n)$  is uniform. Assume also the set*

$$\{R(s, k, l, n); s \in \mathbb{N}, k, l \in [d]\}$$



of random matrices to be independent and that the norm  $\|\cdot\|$  of our matrices is bounded uniformly (in  $n, s, k, l, i$ ). Then the family  $(\tilde{R}(s, k, l, n); s \in \mathbb{N}, k, l \in [d]) \cup (\tilde{C}(i, n); i \in I)$  converges in  $\mathcal{D}$ -distribution in probability to the family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ).

**Remark 1.9.** *The hypothesis that  $C(i, n)$ 's are diagonal can be removed. Indeed, we only use it to apply case 1°) of proposition 1.14, and in the case where all  $\rho_k$ 's are positive, this proposition is an immediate consequence of proposition 3.12 of [B-G2], where constant matrices are not supposed to be diagonal. But as the author is writing this paper, [B-G2] has not been refereed yet.*

In the following theorem, we suppose one of the  $\rho_k$ 's, say  $\rho_1$ , to be zero. We supposed that at most one of the  $\rho_k$ 's is zero, so  $\rho_2, \dots, \rho_d$  are positive. The other hypothesis are quite weaker than the one of the previous theorem: we suppose the singular values (or eigenvalues) of our random matrices to be deterministic. In fact, this supposition is done to avoid steps I and IV of the proof, because  $\max_{1 \leq k \leq d} n/q_k(n)$  appears in step IV of the proof of the previous theorem. Note that the singular values (or eigenvalues) of our random matrices could still be supposed to be random, but in this case, we would have to suppose that for all  $s, k, l$  with  $k \neq 1, l \neq k$ , for all  $r$ ,

$$\frac{n}{q_1(n)} (\text{tr}[(R(s, k, l, n)R(s, k, l, n)^*)^r] - \varphi_k[(a(s, k, l)a(s, k, l)^*)^r])$$

converges in probability to zero, and we would have to do the analogous supposition for all  $s$  even,  $1 \neq k = l$ .

**Theorem 1.10** (Case where  $\rho_1 = 0$ , other  $\rho_k$ 's are positive). *Consider the family of random and deterministic matrices introduced above the previous theorem. Assume moreover that for all  $s, k \neq l, n$ ,  $R(s, k, l, n)$  is bi-unitarily invariant with deterministic singular values, that for all  $s \in \mathbb{N}$  even,  $k \in [d]$ ,  $R(s, k, k, n)$  is unitarily invariant with deterministic spectrum, and that for all  $s \in \mathbb{N}$  odd,  $k \in [d]$ ,  $R(s, k, k, n)$  is uniform. Assume also the set*

$$\{R(s, k, l, n); s \in \mathbb{N}, k, l \in [d]\}$$

*of random matrices to be independent and that the norm  $\|\cdot\|$  of our matrices is bounded uniformly (in  $n, s, k, l, i$ ). Then the family  $\tilde{R}(s, k, l, n)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $\tilde{C}(i, n)$  ( $i \in I$ ) converges in  $\mathcal{D}$ -distribution in probability to the family  $a(s, k, l)$  ( $s \in \mathbb{N}, k, l \in [d]$ ),  $c(i)$  ( $i \in I$ ) introduced above.*

*Comparison with a result by Shlyakhtenko.* This section can be skipped by the reader who wants to go straight to the result. Let us compare this results with Theorem 4.1 of [Sh96]. First, in the case where  $\rho_1$  is zero,  $(\rho_1, \dots, \rho_d)$ -probability spaces cannot be seen as particular cases of faithful noncommutative probability spaces, so theorem 1.10 does not really involve the same kind of objects as the ones of many operator algebra articles. But what about the case where all  $\rho_k$ 's are positive? In Theorem 4.1 of [Sh96] the author identifies, for all  $n$ , the algebra of  $n \times n$  diagonal complex matrices with a subalgebra of  $L^\infty([0, 1])$  by

$$\text{diag}(\lambda_1, \dots, \lambda_n) \simeq \sum_{k=1}^n \lambda_k 1_{[\frac{k-1}{n}, \frac{k}{n}[}$$

and defines  $E_{L^\infty}$ , on a set of  $n \times n$  random matrices, by

$$E_{L^\infty}(R) = \text{diag}(\mathbb{E}(R_{1,1}), \dots, \mathbb{E}(R_{n,n})).$$

Then he proves that a family  $(\{Y(s, n)\})_{s \in \mathbb{N}}$  of  $n \times n$  random matrices with complex independent Gaussian entries is asymptotically free with amalgamation over  $L^\infty([0, 1])$ . This means that if we consider  $m \geq 1$ ,  $s_1 \neq \dots \neq s_m$ ,  $P_1, \dots, P_m \in \mathbb{C}\langle Y, X_1, X_2 \dots \rangle$ ,  $D_1(n), D_2(n), \dots$  diagonal  $n \times n$

complex matrices which, according to the previous identification, converge for  $\|\cdot\|_\infty$  to elements  $d_1, d_2, \dots$  of  $L^\infty([0, 1])$ , and if we suppose that for all  $i \in [m]$ ,

$$E_{L^\infty}[P_i(Y(s_i, n), D_1(n), D_2(n), \dots)] \xrightarrow[n \rightarrow \infty]{\|\cdot\|_\infty} 0,$$

then

$$E_{L^\infty}\left[\prod_{i=1}^m P_i(Y(s_i, n), D_1(n), D_2(n), \dots)\right] \xrightarrow[n \rightarrow \infty]{\|\cdot\|_\infty} 0.$$

Note that for all  $k, l, s, n$ , the random matrix  $p_k(n)Y(s, n)p_l(n)$  has the form  $\widetilde{R}$ , with  $R$  a  $q_k(n) \times q_l(n)$  bi-unitarily invariant random matrix, with limit singular law. So, since the  $p_k(n)$ 's are diagonal, and since for all  $P \in \mathcal{C}\langle p_1, \dots, p_d, Y(1), Y(2), \dots \rangle$ , one has

$$E_{L^\infty}[P(p_1(n), \dots, p_d(n), Y(1, n), Y(2, n), \dots)] \xrightarrow[n \rightarrow \infty]{\|\cdot\|_\infty} 0 \quad \iff$$

$$E[P(p_1(n), \dots, p_d(n), Y(1, n), Y(2, n), \dots)] \xrightarrow[n \rightarrow \infty]{\|\cdot\|_\infty} 0,$$

forgetting about the matrices  $C(i, n)$  and replacing convergence in probability by convergence in expectation, part of our result could have been deduced from Theorem 4.1 of [Sh96]. But Shlyakhtenko's result does not imply a quite important part of ours for several reasons: first, it only considers Gaussian random matrices, moreover, it does not allow us to consider a family of constant matrices  $(C(j, n))_{j \in \mathbb{N}}$  such as the one we consider (because convergence in  $\mathcal{D}$ -distribution, for diagonal matrices, is much less restrictive than convergence for  $\|\cdot\|_\infty$ , and at last, convergence in expectation is not as strong as convergence in probability. Indeed, even though it gives the asymptotic mean value, it does not imply concentration around this value. However, it is possible to derive results about bi-unitarily invariant rectangular random matrices from results about Gaussian matrices. This technic will be appear in a forthcoming paper ([B-G2]).

In order to prove the preceding theorems, we are going to prove a preliminary result.

**1.4. Preliminary result.** In this section, we do not make any particular hypothesis about  $\rho_k$ 's: they are nonnegative, and at most one of them is zero. So we can suppose that  $\rho_2, \dots, \rho_d$  are positive.

Recall that, for each integer  $k$ ,  $[k]$  denotes  $\{1, \dots, k\}$ . For all integers  $n, k$ , for each partition  $\mathcal{P}$  of  $[k]$ ,  $[n]^\mathcal{P}$  will design the set of elements  $i = (i_1, \dots, i_k)$  of  $[n]^k$  such that  $h \sim^\mathcal{P} h' \iff i_h = i_{h'}$ . Moreover,  $|\mathcal{P}|$  denotes the number of blocks of  $\mathcal{P}$ .

The following result involves special classes of matrices.

**Definition 1.11.** *An  $n \times n$  matrix is said to be an almost diagonal matrix if it is of the type  $\widetilde{M}$ , with  $M$  diagonal (either square or rectangular) matrix.*

Note that if  $A$  is an  $n \times n$  almost diagonal matrix, then for all  $j \in [n]$ ,  $A$  has at most one nonzero term on its  $j$ -th column.  $A_j$  will denote this term if it exists, and zero in the other case.

**Lemma 1.12.** *Let  $K$  be a positive integer, and let for each  $n \geq 1$ ,  $A(1, n), \dots, A(K, n)$  be a family of almost diagonal  $n \times n$  matrices. Let  $\mathcal{H}$  be a subset of  $[K]$ , and let  $\mathcal{P}$  be a partition of  $[K]$  such that for all  $\{h\}$  singleton class of  $\mathcal{P}$ ,  $A(h, n)$  is a diagonal matrix and  $\text{Tr } A(h, n) = 0$ .*

We suppose too that the family  $(A(i, n))_{i, n}$  is uniformly bounded, and that for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . Then

$$\sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}\right),$$

where  $\mathcal{P}_{\mathcal{H}}$  (resp.  $\overline{\mathcal{P}}_{\mathcal{H}}$ ) is the partition induced by  $\mathcal{P}$  on the union of the classes of  $\mathcal{P}$  that intersect  $\mathcal{H}$  (resp. do not intersect  $\mathcal{H}$ ), and  $p_{\mathcal{H}}$  (resp.  $\overline{p}_{\mathcal{H}}$ ) the number of singletons in  $\mathcal{P}_{\mathcal{H}}$  (resp. of  $\overline{\mathcal{P}}_{\mathcal{H}}$ ).

**Proof** We will prove this result by induction on  $p = p_{\mathcal{H}} + \overline{p}_{\mathcal{H}}$ .

If  $p = 0$ , the result is clear.

Let us suppose the result to be true to the ranks  $0, \dots, p-1$ , with  $p \geq 1$ . Consider  $\mathcal{P}$  such that  $p_{\mathcal{H}} + \overline{p}_{\mathcal{H}} = p$ , and a singleton class  $\{h_0\}$  of  $\mathcal{P}$ . Denote by  $\mathcal{P}'$  the partition  $\mathcal{P} - \{\{h_0\}\}$  of  $\{1, \dots, \hat{h}_0, \dots, K\}$  and let, for each class  $B$  of  $\mathcal{P}$  with  $B \neq \{h_0\}$ ,  $\mathcal{P}(h_0 \rightarrow B)$  be the partition of  $[K]$  obtained from  $\mathcal{P}$  by linking the classes  $\{h_0\}$  and  $B$ . We have

$$\sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) = \sum_{(i_1, \dots, \hat{i}_{h_0}, \dots, i_K) \in [n]^{\mathcal{P}'}} \left[ \left( \sum_{i \in [n] - \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n) \right) \prod_{\substack{h=1 \\ h \neq h_0}}^K A_{i_h}(h, n) \right].$$

But, since  $A(h_0, n)$  is diagonal and has null trace,

$$\sum_{i \in [n] - \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n) = - \sum_{i \in \{i_1, \dots, \hat{i}_{h_0}, \dots, i_K\}} A_i(h_0, n).$$

So

$$\sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) = - \sum_{\substack{B \in \mathcal{P} \\ B \neq \{h_0\}}} \sum_{i \in [n]^{\mathcal{P}(h_0 \rightarrow B)}} \prod_{h=1}^K A_{i_h}(h, n).$$

For example, suppose  $h_0 \in \mathcal{H}$  (the other case is treated in the same way). With the induction hypothesis, and dividing the sum in  $B \in \mathcal{P}_{\mathcal{H}}$  and  $B \in \overline{\mathcal{P}}_{\mathcal{H}}$ , one has (each  $\epsilon_B$  being 1 or 0 according to whether  $B$  is a singleton or not)

$$\begin{aligned} \sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^K A_{i_h}(h, n) &= \sum_{\substack{B \in \mathcal{P}_{\mathcal{H}} \\ B \neq \{h_0\}}} O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - 1 - (p_{\mathcal{H}} - 1 - \epsilon_B)/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}\right) \\ &\quad + \sum_{B \in \overline{\mathcal{P}}_{\mathcal{H}}} O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - (p_{\mathcal{H}} - 1)/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - 1 - (\overline{p}_{\mathcal{H}} - \epsilon_B)/2}\right) \\ &= q_1(n)^{\frac{\epsilon_B - 1}{2}} \sum_{\substack{B \in \mathcal{P}_{\mathcal{H}} \\ B \neq \{h_0\}}} O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}\right) \\ &\quad + q_1(n)^{\frac{1}{2}} n^{\frac{\epsilon_B - 2}{2}} \sum_{B \in \overline{\mathcal{P}}_{\mathcal{H}}} O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}\right). \end{aligned}$$

Since  $q_1(n) = O(n)$ , the result is proved.  $\square$

The following proposition, main result of this subsection, is divided in two cases, which give very similar results with very similar proofs. The proposition involves special classes of matrices.

**Definition 1.13.** For  $k \in [d]$ , we define  $\mathbb{U}_k(n)$  to be the set of  $n \times n$  matrices whose  $(k, k)$ -th block is a  $q_k(n) \times q_k(n)$  unitary matrix, and with other blocks zero.

Note that it is a compact group, isomorphic to the group of  $q_k(n) \times q_k(n)$  unitary matrices, and random matrices of  $\mathbb{U}_k(n)$  distributed according to the Haar measure will be said to be *uniform*. In the following, we are going to use an integer  $N$  and the cyclic order on  $[N]$ . This means that to put the index  $N + 1$  on an element is equivalent to put the index 1.

**Proposition 1.14.** Let, for  $n \geq 1$ ,  $V(s, k, n)$  ( $s \in \mathbb{N}, k \in [d]$ ), be a family of independent random matrices, such that for all  $s, k$ ,  $V(s, k, n)$  is uniform on  $\mathbb{U}_k(n)$ . Fix  $N$  positive integer and  $R > 0$ .

Case 1°) Let, for each  $n \in \mathbb{N}$ ,  $D(1, n), \dots, D(N, n)$  be  $n \times n$  constant matrices such that for all  $r \in [N]$ ,  $\|D(r, n)\| \leq R$ , and there exists  $u(r), v(r) \in [d]$  such that one of the two following conditions is realized:

- (i)  $u(r) = v(r)$  and for all  $n$ ,  $D(r, n) = p_{u(r)}(n)$ ,
- (ii) for all  $n$ ,  $D(r, n)$  is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_{u(h)}(n) \times q_{v(h)}(n)$  diagonal matrix, and  $\mathbb{E}(D(r, n)) \xrightarrow{n \rightarrow \infty} 0$ .

Consider  $(s_l, k_l)_{l \in [N]} \in (\mathbb{N} \times [d])^N$ , and suppose that for all  $r \in [N]$  such that (i) above is satisfied,  $(s_r, k_r) \neq (s_{r+1}, k_{r+1})$ . Consider also  $m_1, \dots, m_N \in \mathbb{Z} - \{0\}$  and  $\eta > 0$ . Then the probability of the event

$$\{ \|\mathbb{E}(V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n))\| \leq \eta \}$$

tends to 1 as  $n$  goes to infinity.

Case 2°) Let, for each  $n \in \mathbb{N}$ ,  $D(0, n), \dots, D(N, n)$  be  $n \times n$  constant matrices with norms also  $\leq R$ . We suppose that there exists  $v(0), u(N) \in [d]$  such that  $D(0, n)$  (resp.  $D(N, n)$ ) is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_1(n) \times q_{v(0)}(n)$  (resp.  $q_{u(N)}(n) \times q_1(n)$ ), and that for all  $r \in [N-1]$ , there exists  $u(r), v(r) \in [d]$  such that one of the two following conditions is realized:

- (i)  $u(r) = v(r)$  and for all  $n$ ,  $D(r, n) = p_{u(r)}(n)$ ,
- (ii) for all  $n$ ,  $D(r, n)$  is an almost diagonal matrix of the type  $\widetilde{M}$ , where  $M$  is a  $q_{u(h)}(n) \times q_{v(h)}(n)$  diagonal matrix, and  $\mathbb{E}(D(r, n)) \xrightarrow{n \rightarrow \infty} 0$ .

Consider  $(s_l, k_l)_{l \in [N]} \in (\mathbb{N} \times [d])^N$ , and suppose that for all  $r \in [N-1]$  such that (i) above is satisfied,  $(s_r, k_r) \neq (s_{r+1}, k_{r+1})$ . Consider also  $m_1, \dots, m_N \in \mathbb{Z} - \{0\}$  and  $\eta > 0$ . Then the probability of the event

$$\{ \|\mathbb{E}(D(0, n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n))\| \leq \eta \}$$

tends to 1 as  $n$  goes to infinity.

**Remark 1.15.** By linearity of  $\mathbb{E}$ , the result stays true in case 1°), (resp. in case 2°)) if for all  $r$  in  $[N]$  (resp. in  $[N-1]$ ) such that (ii) is satisfied,  $B(r, n)$  is replaced by a sum of constant almost diagonal matrices whose images by  $\mathbb{E}$  tend also to zero.

The proof is inspired from the one of Theorem 4.3.1 p. 147 in [HP00].

**Proof** Both proofs will be made together, we will only have to separate them sometimes.

*Step I. Cases 1° and 2°* Let us denote, for all  $r$  which satisfies (ii) above and  $u(r) = v(r)$ ,

$$D(r, n) = D'(r, n) + \lambda_r(n)p_{u(r)}(n), \text{ with } \mathbb{E}(D'(r, n)) = 0.$$

Then by hypothesis,  $\lambda_h(n) \xrightarrow{n \rightarrow \infty} 0$ . Hence, by linearity and boundness hypothesis, it suffices to prove the result for  $D(r, n)$  replaced by  $D'(r, n)$ . Thus we can from now on suppose that for all  $r$  such that (ii) is realized, we have  $\mathbb{E}(D(r, n)) = 0$  for all  $n$ .

*Step II. Case 1°* By definition of  $\mathbb{E}$ , it suffices to prove that for all  $k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of the product

$$V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n)$$

converges in probability to zero, i.e. that

$$(8) \quad \frac{1}{q_k(n)} \text{Tr } p_k(n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n) p_k(n)$$

converges in probability to zero. So let us fix  $k \in [d]$ , and let us prove it.

If  $k_1 \neq k$  or  $v(N) \neq k$ , then the  $k$ -th block of the matrix is always zero, so the result is clear. So let us suppose that  $k_1 = v(N) = k$ . We can then remove  $p_k(n)$  in (8). By Markov inequality, it suffices to prove that (8) tends to zero in  $L^2$ . Since for all matrix  $M$ ,  $|\text{Tr } M|^2 = \text{Tr } M \text{Tr } M^*$ , we only have to prove that the expectation of

$$(9) \quad \text{Tr } V(s_1, k_1, n)^{m_1} B_1(n) \cdots B_N(n) \text{Tr } B_N(n)^* V(s_N, k_N, n)^{-m_N} B_{N-1}(n)^* \cdots V(s_1, k_1, n)^{-m_1}$$

is  $o(q_k(n)^2)$ .

Case 2° We have to prove that

$$\frac{1}{q_1(n)} \text{Tr } D(0, n) V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n)$$

converges in probability to zero. Since  $\text{Tr } XY = \text{Tr } YX$ , it suffices to prove that

$$(10) \quad \frac{1}{q_1(n)} \text{Tr } V(s_1, k_1, n)^{m_1} D(1, n) \cdots D(N-1, n) V(s_N, k_N, n)^{m_N} D(N, n) D(0, n)$$

converges in probability to zero. From now on, we will denote  $D(N, n)D(0, n)$  by  $D(N, n)$ , which is now an  $n \times n$  matrix of the type  $\widetilde{M}$  with  $M$   $q_{u(N)}(n) \times q_{v(0)}(n)$ , but which has not more than  $q_1(n)$  nonzero entries. For the same reason as above, we only have to prove that the expectation of

$$(11) \quad \text{Tr } V(s_1, k_1, n)^{m_1} B_1(n) \cdots B_N(n) \text{Tr } B_N(n)^* V(s_N, k_N, n)^{-m_N} B_{N-1}(n)^* \cdots V(s_1, k_1, n)^{-m_1}$$

is  $o(q_1(n)^2)$ .

*Step III. Cases 1° and 2°* Let us expand both traces in the previous product, and re index the sum on partitions.

Let  $M = |m_1| + \cdots + |m_N|$ , let, for  $r \in \{0, \dots, N\}$ ,

$$M(r) = |m_1| + \cdots + |m_r|,$$

let  $M(N+1) = 2M$ , and let, for  $r \in \{N+2, \dots, 2N\}$ ,

$$M(r) = M + |m_N| + |m_{N-1}| + \cdots + |m_{2N+2-r}|.$$

Let also, for  $r \in \{N + 1, \dots, 2N\}$ ,  $D(r, n) = D(2N + 1 - r, n)^*$ . We write, for  $\kappa \in \mathbb{N} \times [d]$ ,

$$\begin{aligned} v_{i,j}(\kappa, n, 1) &= V(\kappa, n)_{i,j}, \\ v_{i,j}(\kappa, n, -1) &= \bar{V}(\kappa, n)_{j,i}. \end{aligned}$$

With those notations, there exists two functions  $\kappa$  and  $\varepsilon$  such that the expectation of (9) is

$$\sum_{i \in [n]^{2M}} \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right],$$

where  $\mathbb{E}$  denotes expectation and for all  $i = (i_1, \dots, i_{2M}) \in [n]^{2M}$ ,

$$j_1 = i_2, j_2 = i_3, \dots, j_M = i_1, j_{M+1} = i_{M+2}, j_{M+2} = i_{M+3}, \dots, j_{2M-1} = i_{2M}, j_{2M} = i_{M+1}.$$

Let us introduce the partition  $\mathcal{Q}$  of  $[2M]$  defined by  $h \overset{\mathcal{Q}}{\sim} h'$  if and only if  $i_h = i_{h'}$ . Then we can rewrite the preceding sum

$$\sum_{\substack{\mathcal{Q} \text{ partition} \\ \text{of } [2M]}} \sum_{i \in [n]^{\mathcal{Q}}} \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right].$$

Thus it suffices to prove that for all partition  $\mathcal{Q}$  of  $[2M]$ , the sum

$$(12) \quad \sum_{i \in [n]^{\mathcal{Q}}} \prod_{r=0}^{2N} D(r, n)_{j_{M(r)}} \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right]$$

is  $o(q_k(n)^2)$  in case 1 $^\circ$ ), and  $o(q_1(n)^2)$  in case 2 $^\circ$ ), as  $n$  goes to infinity.

So we fix a partition  $\mathcal{Q}$  of  $[2M]$ .

*Step IV. Cases 1 $^\circ$ ) and 2 $^\circ$ )* Let us denote, for all  $h \in [2M]$ ,  $\kappa(h) = (s(h), k(h))$ . Note that, for  $i \in [n]^{\mathcal{Q}}$ , for

$$(13) \quad \mathbb{E} \left[ \prod_{h=1}^{2M} v_{i_h, j_h}(\kappa(h), \varepsilon(h), n) \right]$$

to be nonzero,  $i$  has to satisfy

$$(14) \quad \forall h \in [2M], i_h, j_h \in \{q_1(n) + \dots + q_{k(h)-1}(n) + 1, \dots, q_1(n) + \dots + q_{k(h)-1}(n) + q_{k(h)}(n)\}.$$

Indeed, in the other case, one of the factors in the product is zero, by definition of  $M \mapsto \widetilde{M}$ . Note also that for all elements  $i$  of  $[n]^{\mathcal{Q}}$ , which satisfy (14), the expectation (13) is the same, by invariance of Haar measure on unitary groups under permutation of rows and columns. Let us denote this common expectation by  $\mathbb{E}_{\mathcal{Q}}$ . So, one can write the sum of (12)

$$(15) \quad \mathbb{E}_{\mathcal{Q}} \sum_i \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}}$$

where the sum is taken on elements  $i$  of  $[n]^{\mathcal{Q}}$  which satisfy (14).

It is well known (see [HP00], equation (4.2.11)) that if, for each  $m$ ,  $U(m)$  is a uniform unitary  $m \times m$  random matrix, and  $1 \leq i_m, j_m \leq m$ , then the sequence  $\mathbb{E}(|U(m)_{i_m, j_m}|^{2M})$  does not depend on the choices of  $i_m, j_m$  and

$$(16) \quad \mathbb{E}(|U(m)_{i_m, j_m}|^{2M})^{1/2M} = O(m^{-1/2}).$$

So, with  $i \in [n]^{\mathcal{Q}}$ ,

$$\begin{aligned} |\mathbb{E}_{\mathcal{Q}}| &\leq \prod_{h=1}^{2M} \mathbb{E}(|v_{i_h, j_h}(\kappa(h), \varepsilon(h), n)|^{2M})^{1/2M} \quad (\text{H\"older ineq.}) \\ &= O(q_1(n)^{-M'} n^{-M''}) \end{aligned}$$

where

$$M' = \sum_{\substack{1 \leq r \leq N \\ \bar{k}_r = 1}} |m_r|, \quad M'' = \sum_{\substack{1 \leq r \leq N \\ \bar{k}_r \neq 1}} |m_r|.$$

Thus it suffices to prove that

$$(17) \quad \sum_i \prod_{r=1}^{2N} D(r, n)_{j_{M(r)}},$$

is  $o(q_k(n)^2 q_1(n)^{M'} n^{M''})$  in case 1 $^\circ$ ), and  $o(q_1(n)^{M'+2} n^{M''})$  in case 2 $^\circ$ ), where the sum is taken on the elements  $i$  of  $[n]^{\mathcal{Q}}$  which satisfy (14).

*Step V. Cases 1 $^\circ$ ) and 2 $^\circ$ )* We are going to prove it as an application of lemma 1.12. The sum of (17) is the sum, on the same  $i$ 's, of

$$\begin{aligned} &D(1, n)_{i_{M(1)+1}} D(2, n)_{i_{M(2)+1}} \cdots D(N-1, n)_{i_{M(N-1)+1}} D(N, n)_{i_1} \\ &\times D(N+1, n)_{i_{M+1}} D(N+2, n)_{i_{M(N+2)}} D(N+3, n)_{i_{M(N+3)}} \cdots D(2N, n)_{i_{M(2N)}}. \end{aligned}$$

i.e. of

$$\begin{aligned} &D(N, n)_{i_1} D(1, n)_{i_{M(1)+1}} D(2, n)_{i_{M(2)+1}} \cdots D(N-1, n)_{i_{M(N-1)+1}} \\ &\times (D(2N+1, n) D(N+1, n))_{i_{M+1}} D(N+2, n)_{i_{M(N+2)}} D(N+3, n)_{i_{M(N+3)}} \cdots D(2N, n)_{i_{M(2N)}}. \end{aligned}$$

This sum is equal to

$$(18) \quad \sum_{i \in [n]^{\mathcal{P}}} \prod_{h=1}^{2N} A_{i_h}(h, n),$$

where for  $h \in [2M]$ ,

$$A(h, n) = \begin{cases} p_{k(h-1)}(n) D(r, n) p_{k(h)}(n) & \text{if } \exists r \in \{0, \dots, N-1\}, \\ & h = M(r) + 1 \\ p_{k(h)}(n) & \text{if } h \in [M] \text{ and } h \notin \\ & \{M(0) + 1, \dots, M(N-1) + 1\} \\ p_{k(2M)}(n) D(N+1, n) p_{k(M+1)}(n) & \text{if } h = M+1 \\ p_{k(h)}(n) D(r, n) p_{k(h+1)}(n) & \text{if } \exists r \in \{N+2, \dots, 2N\}, \\ & h = M(r) \\ p_{k(h)}(n) & \text{if } h \in [2M] \text{ and } h \notin \\ & [M+1] \cup \{M(N+2), \dots, M(2N)\} \end{cases}$$

Now, note that by lemma 4.2.2 of [HP00] (or because the uniform distribution on the unitary group is left and right invariant), if  $i \in [n]^{\mathcal{P}}$  corresponds to a nonzero term in equation (12), then for all  $\kappa \in \mathbb{N} \times [d]$ , for all  $1 \leq \alpha, \beta \leq n$ ,

$$\begin{aligned} |\{h; i_h = \alpha, \kappa(h) = \kappa, \varepsilon(h) = 1\}| &= |\{h; j_h = \alpha, \kappa(h) = \kappa, \varepsilon(h) = -1\}|, \\ |\{h; j_h = \beta, \kappa(h) = \kappa, \varepsilon(h) = 1\}| &= |\{h; i_h = \beta, \kappa(h) = \kappa, \varepsilon(h) = -1\}|. \end{aligned}$$

Thus, if  $\{h\}$  is a singleton class in  $\mathcal{P}$ , and if  $\mathbb{E}_{\mathcal{P}}$  is non null, one has, using the cyclic orders on  $[M]$  and on  $\{M+1, M+2, \dots, 2M\}$ ,

$$\begin{aligned} \varepsilon(h) = 1 &\Rightarrow \kappa(h) = \kappa(h-1), \varepsilon(h-1) = -1, \\ \text{and } \varepsilon(h) = -1 &\Rightarrow \kappa(h) = \kappa(h-1), \varepsilon(h-1) = 1. \end{aligned}$$

It clearly follows that there exists  $r \in \{0, \dots, 2M\} - \{M+1\}$  such that  $h = M(r)+1$ . Moreover, by hypothesis, in case 1°),  $A(h, n) = D(r, n)$  is a diagonal matrix with null trace, and, in case 2°),  $A(h, n) = D(r, n)$  is a diagonal matrix with null trace whenever  $h \notin M, M+1$ .

Case 1°) Let us apply the lemma with  $\mathcal{P} = \mathcal{Q}$  and

$$\mathcal{H} = \{h \in [2M]; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . By the lemma, the sum of (17), which is equal to the sum of (18), is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2})$$

But for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| \leq x + (|S| - x)/2$ , so  $|\mathcal{X}| - x/2 \leq |S|/2$ . So,

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M)/2 = M' + M'',$$

Moreover, since for all  $h \in [2M]$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h; k(h) \neq 1\}| = M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M'} n^{M''}\right).$$

The sum of (17) is  $O\left(q_1(n)^{M'} n^{M''}\right)$ , and the proposition is proved.

Case 2°) Let us prove that the sum of (17) is  $o\left(q_1(n)^{2+M'} n^{M''}\right)$  by induction on the number  $|\mathcal{Q}|$  of classes of  $\mathcal{Q}$ . Let us denote, for  $B, B'$  class of  $\mathcal{Q}$ ,  $\mathcal{Q}(M \rightarrow B)$  the partition of  $[2M]$  obtained from  $\mathcal{Q}$  by linking classes  $B$  and  $\{M\}$ , and  $\mathcal{Q}(M \rightarrow B, M+1 \rightarrow B')$  the partition obtained from  $\mathcal{Q}$  by linking classes  $B$  and  $\{M\}$ , and then the classes  $B'$  (which has become  $B \cup \{M\}$  if  $B' = B$ ) and  $\{M+1\}$ . Let  $J(n)$  be the set of elements  $j$  of  $[n]$  such that  $D(N, n)_j \neq 0$ . Note that, as noted to step II,  $|I(n)| \leq q_1(n)$ , and, by definition of  $D(N+1, n)$ ,  $J(n)$  is also the set of elements  $j$  of  $[n]$  such that  $D(N+1, n)_j \neq 0$ .

- If  $\mathcal{Q}$  has only one class, by boundness hypothesis and since  $|I(n)| \leq q_1(n)$ , the sum of (17) is  $O(q_1(n))$ .

- Assume that the conclusion holds to ranks  $1, \dots, |\mathcal{Q}| - 1$ .

- If neither  $\{M\}$  nor  $\{M+1\}$  are singletons of  $\mathcal{Q}$ , we apply the lemma, as for case 1°), with  $\mathcal{P} = \mathcal{Q}$  and

$$\mathcal{H} = \{h \in [2M]; k(h) = 1\}.$$

It leads to the conclusion in the same way as in case 1°).



- If exactly one of  $\{M\}, \{M+1\}$ , say  $\{M\}$ , is a singleton of  $\mathcal{Q}$ , define  $\mathcal{P} = \mathcal{Q} - \{\{M\}\}$ , partition of the set  $[2M] - \{M\}$ . Note that the sum of (17), which is equal to the sum of (18), is

$$\begin{aligned}
& \sum_{j \in J(n)} \left[ A(M, n)_j \sum_{i \in ([n] - \{j\})^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right] \\
= & \sum_{j \in J(n)} \left[ A(M, n)_j \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) - \sum_{\substack{i \in [n]^{\mathcal{P}} \\ \exists h, i_h = j}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) \right] \\
= & \left( \sum_{j \in J(n)} A(M, n)_j \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) - \left( \sum_{\substack{B \in \mathcal{Q} \\ B \neq \{M\}}} \sum_{i \in [n]^{\mathcal{Q}(M \rightarrow B)}} \prod_{h=1}^{2N} A_{i_h}(h, n) \right)
\end{aligned}$$

So by the induction hypothesis, for all  $B \in \mathcal{P}, B \neq \{M\}$ , we have

$$\sum_{i \in [n]^{\mathcal{Q}(M \rightarrow B)}} \prod_{h=1}^{2N} A_{i_h}(h, n) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

So it suffices to prove that we have

$$(19) \quad \left( \sum_{j \in J(n)} A(M, n)_j \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M}}^{2N} A_{i_h}(h, n) \right) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

But by boundness hypothesis,  $\sum_{j \in J(n)} A(M, n)_j = 0(|J(n)|) = 0(q_1(n))$ . So it suffices to prove that the second sum in (19) is  $o\left(q_1(n)^{M'+1} n^{M''}\right)$ . Define

$$\mathcal{H} = \{h \in [2M] - \{M\}; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . So by the lemma, the second sum in (19) is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}).$$

Since, as noted above, for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| - x/2 \leq |S|/2$ , we have

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M - 1)/2 = M' + M'' - 1/2,$$

Moreover, since for all  $h \in [2M] - \{M\}$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h \in [2M] - \{M\}; k(h) \neq 1\}| \leq M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M' - 1/2} n^{M''}\right),$$

thus the conclusion holds.

- If  $\{M\}$  and  $\{M+1\}$  are singletons of  $\mathcal{Q}$ , define  $\mathcal{P} = \mathcal{Q} - \{\{M\}, \{M+1\}\}$ , partition of the set  $[2M] - \{M, M+1\}$ . Note that the sum of (17), which is equal to the sum of (18), is

$$\sum_{\substack{j, j' \in J(n) \\ j \neq j'}} \left[ A(M, n)_j A(M+1, n)_{j'} \sum_{i \in ([n] - \{j, j'\})^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right].$$

Now, let us define  $\mathcal{Q}'$  to be the partition of  $[2M]$  obtained from  $\mathcal{Q}$  by linking classes  $\{M\}$  and  $\{M+1\}$ . The sum of (17) can be written

$$\left( \sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right) - \sum_{i \in [n]^{\mathcal{Q}'}} \prod_{h=1}^{2N} A_{i_h}(h, n) - \sum_{\mathcal{R}} \sum_{i \in [n]^{\mathcal{R}}} \prod_{h=1}^{2N} A_{i_h}(h, n),$$

where in the last sum,  $\mathcal{R}$  is taken in the set

$$\{\mathcal{Q}(M \rightarrow B, M+1 \rightarrow B'); B, B' \in \mathcal{Q}\}.$$

So by the induction hypothesis, it suffices to prove that

$$(20) \quad \left( \sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} \right) \left( \sum_{i \in [n]^{\mathcal{P}}} \prod_{\substack{h=1 \\ h \neq M \\ h \neq M+1}}^{2N} A_{i_h}(h, n) \right) = o\left(q_1(n)^{2+M'} n^{M''}\right).$$

But by boundness hypothesis,  $\sum_{j, j' \in J(n)} A(M, n)_j A(M+1, n)_{j'} = 0(|J(n)|^2) = 0(q_1(n)^2)$ . So it suffices to prove that the second sum in (20) is  $o\left(q_1(n)^{M'} n^{M''}\right)$ . Define

$$\mathcal{H} = \{h \in [2M] - \{M, M+1\}; k(h) = 1\}.$$

By definition of the  $A(h, n)$ 's, for all  $h \in \mathcal{H}$ , for all  $i \in \{q_1(n) + 1, \dots, n\}$ ,  $A_i(h, n) = 0$ . So by the lemma, the second sum in (20) is

$$O(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2}).$$

Since, as noted above, for any partition  $\mathcal{X}$  with  $x$  singletons of a set  $S$ , one has  $|\mathcal{X}| - x/2 \leq |S|/2$ , we have

$$|\mathcal{P}_{\mathcal{H}}| + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} - \frac{\overline{p}_{\mathcal{H}}}{2} \leq (2M - 2)/2 = M' + M'' - 1,$$

Moreover, since for all  $h \in [2M] - \{M, M+1\}$ ,  $k(h) = 1$  implies  $h \in \mathcal{H}$ ,

$$|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 \leq \frac{1}{2} |\{h \in [2M] - \{M, M+1\}; k(h) \neq 1\}| \leq M''.$$

Thus, since  $q_1(n) \leq n$ ,

$$n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''} = O(q_1(n)^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2 - M''} n^{M''}).$$

Hence

$$q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - p_{\mathcal{H}}/2} n^{|\overline{\mathcal{P}}_{\mathcal{H}}| - \overline{p}_{\mathcal{H}}/2} = O\left(q_1(n)^{|\mathcal{P}_{\mathcal{H}}| - \frac{p_{\mathcal{H}}}{2} + |\overline{\mathcal{P}}_{\mathcal{H}}| - \frac{\overline{p}_{\mathcal{H}}}{2} - M''} n^{M''}\right) = O\left(q_1(n)^{M' - 1} n^{M''}\right),$$

thus the conclusion holds.  $\square$

**1.5. Proof of theorem 1.8 (case where all  $\rho_k$ 's are positive).** *Step I.* Let us introduce, on the space of  $n \times n$  matrices, the Schatten  $p$ -norms  $\|\cdot\|_p$  ( $p \in [1, \infty]$ ), defined, for  $p < \infty$ , by

$$\|X\|_p = (\operatorname{tr} |X|^p)^{\frac{1}{p}},$$

and  $\|\cdot\|_\infty = \|\cdot\|$ . They satisfy classical Hölder inequalities ([N74]).

Fix  $k, l \in [d]$  and consider, for  $n \geq 1$ , a bi-unitarily invariant  $q_k(n) \times q_l(n)$  random matrix  $R(n)$  such that the sequence is uniformly bounded for  $\|\cdot\|$  by  $C > 0$  and such that the moments of the singular law of  $R(n)$  converge in probability to constants. Note first that, by Carleman criterion, it implies that its singular law converges in probability to a probability measure  $\mu$ . Then, as noticed in subsection 1.1, the distribution of  $R(n)$  can be realized as the distribution of  $U(n)\Lambda(n)V(n)$  where  $U(n)$ ,  $\Lambda(n)$ ,  $V(n)$  are independent,  $U(n)$ ,  $V(n)$  are respectively  $q_k(n) \times q_k(n)$ ,  $q_l(n) \times q_l(n)$  uniform random unitary matrices, and

$$\Lambda(n) = [\delta_i^j \lambda(i, n)]_{\substack{1 \leq i \leq q_k(n) \\ 1 \leq l \leq q_l(n)}} \text{ and } 0 \leq \lambda(1, n) \leq \dots \leq \lambda(\min(q_k(n), q_l(n)), n).$$

Choose deterministic

$$0 \leq \xi(1, n) \leq \dots \leq \xi(\min(q_k(n), q_l(n)), n) \leq C$$

such that the uniform distribution on the  $\xi(1, n)$ 's converges weakly to  $\mu$ . Now set

$$\Xi(n) = [\delta_i^j \lambda(i, n)]_{\substack{1 \leq i \leq q_k(n) \\ 1 \leq l \leq q_l(n)}} \text{ and } N(n) = U(n)\Xi(n)V(n).$$

Let us prove that for all  $P$  polynomial in noncommutative random variables  $X, X^*$ , for all  $p \in [1, \infty)$ ,  $\|P(\tilde{R}(n)) - P(\tilde{N}(n))\|_p$  converges in probability to zero.

By lemma 4.3.4 p. 152 of [HP00], for all  $m \geq 1$ ,

$$\|\tilde{R}(n)^* - \tilde{N}(n)^*\|_p = \|\tilde{R}(n) - \tilde{N}(n)\|_p = \|\tilde{\Lambda}(n) - \tilde{\Xi}(n)\|_p$$

converges in probability to zero. So, for all  $m \geq 1$ ,  $\varepsilon_1, \dots, \varepsilon_m \in \{., *\}$ ,

$$\begin{aligned} & \|\tilde{R}(n)^{\varepsilon_1} \dots \tilde{R}(n)^{\varepsilon_m} - \tilde{N}(n)^{\varepsilon_1} \dots \tilde{N}(n)^{\varepsilon_m}\|_p \\ & \leq \sum_{l=1}^m \|\tilde{R}(n)^{\varepsilon_1} \dots \tilde{R}(n)^{\varepsilon_{l-1}} (\tilde{R}(n)^{\varepsilon_l} - \tilde{N}(n)^{\varepsilon_l}) \tilde{N}(n)^{\varepsilon_{l+1}} \dots \tilde{N}(n)^{\varepsilon_m}\|_p \\ & \leq \sum_{l=1}^m C^{m-1} \|\tilde{R}(n)^{\varepsilon_l} - \tilde{N}(n)^{\varepsilon_l}\|_p, \end{aligned}$$

which proves the result.

Note that in the case where  $R$  is a uniformly bounded unitarily invariant  $q_k(n) \times q_k(n)$  hermitian random matrix, the moments of the spectral law of which converge in probability, then the same work can be done, replacing  $V(n)$  by  $U(n)^*$  and singular values by eigenvalues, and the same conclusion holds: for all  $P \in \mathbb{C}[X]$ , for all  $p \in [1, \infty)$ ,  $\|P(\tilde{R}(n)) - P(\tilde{N}(n))\|_p$  converges in probability to zero.

*Step II.* Let us denote by  $\mathcal{C}$  the linear span of the  $c(i)$ 's. By (6) and (7), for all  $n$ , there is an algebra morphism  $\psi_n$  from  $\mathcal{C}$  to the linear span  $\mathcal{C}_n$  of the  $\tilde{C}(i, n)$ 's. By (vi), for all  $c \in \mathcal{C}$ , for all  $k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of  $\psi_n(c)$  tends to  $\varphi_k(c_{kk})$ .

Since  $1 \in \mathcal{C}$ , it suffices to prove that for all  $m \geq 0$ , for all  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m$ , for all  $P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ , for all  $d(0), \dots, d(m) \in \mathcal{C}$ ,

the normalized trace of the  $k$ -th diagonal block of

$$(21) \quad \psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

converges in probability to the  $k$ -th coordinate of

$$(22) \quad \mathbb{E} \left( d(0) \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right),$$

for all  $k \in [d]$ .

*Step III.* Let us prove that we can remove  $d(0)$ , i.e. that in order to prove the convergence in probability, for all  $k \in [d]$  and all  $d(0), \dots, d(m) \in \mathcal{C}$ , of the normalized trace of the  $k$ -th diagonal block of (21) to the  $k$ -th coordinate of (22), it suffices to prove the convergence in probability, for all  $k \in [d]$  and all  $d(1), \dots, d(m) \in \mathcal{C}$ , of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d'(j))$$

to the  $k$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d'(j) \right).$$

The idea is to use the formula  $\text{Tr } XY = \text{Tr } YX$  and to move  $d(0)$  from the left to the right of the product, which comes down to replace  $d(m)$  by  $d(m)d(0)$ . But the formula  $\mathbb{E}(XY) = \mathbb{E}(YX)$  is not always true, so we have to be careful.

First, by linearity of  $\mathbb{E}$ , we can suppose that  $d(0)$  is a simple element. Thus there exists  $l_0 \in [d]$  such that

$$d(0) = d(0)p_{l_0}, \quad \psi_n(d(0)) = \psi_n(d(0))p_{l_0}(n).$$

Now let us fix  $k \in [d]$ . We have to prove that

$$\frac{1}{q_k(n)} \text{Tr } p_k(n) \psi_n(d(0)) p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k(n)$$

converges in probability to

$$\varphi_k \left( p_k d(0) p_{l_0} \left[ \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right] p_k \right),$$

which amounts, since  $\text{Tr } XY = \text{Tr } YX$  and by (3), to prove that

$$\frac{q_{l_0}(n)}{q_k(n)} \frac{1}{q_{l_0}(n)} \text{Tr } p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k(n) \psi_n(d(0)) p_{l_0}(n)$$

converges in probability to

$$\frac{\rho_{l_0}}{\rho_k} \varphi_{l_0} \left( \left[ \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right] p_k d(0) p_{l_0} \right).$$

Since  $\frac{q_{l_0}(n)}{q_k(n)} \xrightarrow{n \rightarrow \infty} \frac{\rho_{l_0}}{\rho_k}$ , it amounts to prove that

$$\frac{1}{q_{l_0}(n)} \operatorname{Tr} p_{l_0}(n) \left[ \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right] p_k \psi_n(d(0)) p_{l_0}(n)$$

converges in probability to

$$\varphi_{l_0} \left( \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right) p_k d(0) p_{l_0} \right).$$

If one replaces  $d(m)$  by  $d(m)p_k d(0)$ , it appears that it suffices to prove that for all  $l \in [d]$ , the normalized trace of the  $l$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \psi_n(p_k d(0))$$

converges in probability to the  $l$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right).$$

*Step IV.* Then, we claim that it suffices to prove it when the  $R(s, k, l, n)$ 's are replaced by the  $N(s, k, l, n)$ 's, where the  $N(s, k, l, n)$ 's, are defined from the  $R(s, k, l, n)$ 's like  $N(n)$  from  $R(n)$  in the step I. Indeed, the normalized trace of any diagonal block of the difference between

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

and

$$\prod_{j=1}^m P_j(\tilde{N}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

has an absolute value less or equal than the norm  $\|\cdot\|_1$  of their difference times one of the  $n/q_k(n)$ , which is less or equal, by a decomposition like in step I, than

$$\max_{k \in [d]} \frac{n}{q_k(n)} \sum_{j=1}^m C^m D^{m-1} \|P_j(\tilde{R}(s_j, k_j, l_j, n)) - P_j(\tilde{N}(s_j, k_j, l_j, n))\|_1,$$

where  $D = \max_{\substack{|z|, |z'| \leq C \\ j \in [m]}} |P_j(z, z')|$  which converges in probability to zero, by step I and because all

$\rho_k$ 's are positive.

So let us prove that the normalized trace of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{N}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

converges in probability to the  $k$ -th coordinate of

$$\mathbb{E} \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right),$$

for all  $k \in [d]$ .

*Step V.* We will prove it by induction on  $m$ . If  $m = 0$ , the result is clear. In the other case, let us suppose the result to be proved to the ranks  $0, \dots, m-1$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Since  $\mathcal{C}$  is an algebra containing  $p_1, \dots, p_d$ , if, for a certain  $j \in [m]$ , one would replace  $X(j, n)$  by one of the  $p_k(n)$ 's and  $x(j)$  by the corresponding  $p_k$ , then the result would follow from the induction hypothesis (and from step III if  $j = 1$ ). Thus, by linearity, one can, for all  $j \in [m]$ , add a linear combination of the  $p_k(n)$ 's to  $X(j, n)$  and the same linear combination of the  $p_k$ 's to  $x(j)$ . Therefore we can assume that for all  $j$ ,  $E(x(j)) = 0$ . By linearity, we can moreover suppose that for all  $j$ ,  $x(j)$  is a simple element. Note that these suppositions imply that for all  $n, j$ ,  $X(n, j)$  has only one non zero block, and satisfies

$$E(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

From now on, we use the cyclic order on  $[m]$ . This means that to put the index  $m+1$  on an element amounts to put the index 1. As simple elements, the  $x(j)$ 's have the following property: the product of any of the  $x(j)$ 's by one of the  $p_k$ 's is either zero or  $x(j)$ , and the same holds for  $X(j, n)$ 's with  $p_k(n)$ 's. Thus, if, for a certain  $j \in [m]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , one would replace  $d(j)$  by a linear combination of the  $p_k(n)$ 's, then the result would follow from the induction hypothesis (and from step III if  $j = m$ ). So one can suppose that for all  $j \in [m-1]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ ,  $E(d(j)) = 0$ .

At last, for  $j \in [m]$  such that  $(s_j, k_j, l_j) \neq (s_{j+1}, k_{j+1}, l_{j+1})$ , one can write

$$d(j) = d'(j) + \sum_{k \in [d]} \lambda_k(j) p_k, \quad \text{with } E(d'(j)) = 0.$$

Then, by linearity, one can suppose that for all such  $j$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

To conclude, we have shown that we only have to prove that

$$E \left( \prod_{j=1}^m X(j, n) \psi_n(d(j)) \right)$$

converges in probability to

$$E \left( \prod_{j=1}^m x(j) d(j) \right),$$

under the hypothesis that all  $x(j)$ 's are simple elements and satisfy  $E(x(j)) = 0$ , and for all  $j \in [m]$ ,

$$E(d(j)) = 0 \text{ or } \{d(j) \in \{p_1, \dots, p_d\} \text{ and } (s_j, k_j, l_j) \neq (s_{j+1}, k_{j+1}, l_{j+1})\}.$$

Note that under this hypothesis, by E-freeness,  $E \left( \prod_{j=1}^m x(j) d(j) \right) = 0$ .

*Step VI.* For all  $s, k, l, n$  with  $k \neq l$ , the matrix  $N(s, k, l, n)$  has been defined by the work of step I, so we can write

$$N(s, k, l, n) = U(s, k, l, n) \Xi(s, k, l, n) V(s, k, l, n),$$

with  $U(s, k, l, n)$ ,  $V(s, k, l, n)$  uniform random unitary matrices with respective sizes  $q_k(n)$ ,  $q_l(n)$ , and  $\Xi(s, k, l, n)$  a deterministic diagonal  $q_k(n) \times q_l(n)$  matrix. In the same way, when  $s$  is even,  $N(s, k, k, n)$  can be written

$$N(s, k, l, n) = U(s, k, k, n)\Xi(s, k, k, n)U^*(s, k, k, n),$$

and when  $s$  is odd,  $N(s, k, k, n)$  is a uniform random matrix of  $\mathbb{U}_k(n)$ .

Consider  $j \in [m]$ . If  $k_j \neq l_j$ , then  $X(j, n)$  is of one of the 4 following types:

- (1)  $\tilde{U}(s_j, k_j, l_j, n)\tilde{\Delta}_j\tilde{V}(s_j, k_j, l_j, n)$ , where  $\Delta_j$  is a  $q_{k_j}(n) \times q_{l_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an odd polynomial.
- (2)  $\tilde{V}(s_j, k_j, l_j, n)^*\tilde{\Delta}_j\tilde{U}(s_j, k_j, l_j, n)^*$ , where  $\Delta_j$  is a  $q_{l_j}(n) \times q_{k_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an odd polynomial.
- (3)  $\tilde{U}(s_j, k_j, l_j, n)\tilde{\Delta}_j\tilde{U}(s_j, k_j, l_j, n)^*$ , where  $\Delta_j$  is a  $q_{k_j}(n) \times q_{k_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an even polynomial.
- (4)  $\tilde{V}(s_j, k_j, l_j, n)^*\tilde{\Delta}_j\tilde{V}(s_j, k_j, l_j, n)$ , where  $\Delta_j$  is a  $q_{l_j}(n) \times q_{l_j}(n)$  diagonal matrix whose diagonal terms are the images of the diagonal terms of  $\Xi(s_j, k_j, l_j, n)$  by an even polynomial.

In the case where  $k_j = l_j$ , when  $s_j$  is even, only the last type is possible, and when  $s_j$  is odd,  $X(j, n)$  is a linear combination of positive or negative powers of  $\tilde{N}(s_j, k_j, l_j, n)$ . By linearity again, this combination can be replaced by a non zero power of  $\tilde{N}(s_j, k_j, l_j, n)$ .

Note that in all types previously presented, the constant diagonal matrix  $\Delta_j$  satisfies

$$\mathbb{E}(\tilde{\Delta}_j) \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, for all  $n$ , the family of random matrices from  $\mathbb{U}_k(n)$ 's arising in these decompositions are independent, so the result follows from case 1°) of proposition 1.14 and remark 1.15.

**1.6. Proof of theorem 1.10 (case where  $\rho_1 = 0$ , all other  $\rho_k$ 's are positive).** We shall follow closely the proof of theorem 1.8 (case where all  $\rho_k$ 's are positive) given in the previous section, but because of the weaker hypothesis (singular values of non hermitian random matrices and spectrum of hermitian ones are now supposed to be deterministic), we shall skip steps I and IV, in which non hermitian (resp. hermitian) random matrices were approximated by random matrices with deterministic singular values (resp. deterministic spectrum). However, the hypothesis  $\rho_1 = 0$  will make things slightly harder.

As previously, let us denote by  $\mathcal{C}$  the linear span of the  $c(i)$ 's, and define  $\psi_n$  in the same way as before.

As previously again, since  $1 \in \mathcal{C}$ , it suffices to prove that for all  $m \geq 0$ , the following proposition holds:

$\forall (s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, \forall P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ ,

$\forall d(0), \dots, d(m) \in \mathcal{C}, \forall k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of

$$\psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

$$(23) \quad \text{converges in probability to } \varphi_k \left( d(0) \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right).$$

Note first that by linearity, it suffices to prove it when  $d(0), d(m)$  are supposed to be simple, which implies that there exists  $u, v \in [d]$  such that

$$(24) \quad p_u d(0) = d(0), \quad d(m) p_v = d(m).$$

Let us denote by  $H(m, u, v)$  the proposition (23), where  $d(0), d(m)$  are moreover supposed to satisfy (24).

We have to prove  $H(m, u, v)$  for all  $m \geq 0$  and all  $u, v \in [d]$ . Note first that if  $u \neq v$ ,  $H(m, u, v)$  is immediate. Let us denote by  $H'(m)$  the proposition obtained from (23) removing  $d(0)$ :

$\forall (s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, \forall P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*$ ,

$\forall d(1), \dots, d(m) \in \mathcal{C}, \forall k \in [d]$ , the normalized trace of the  $k$ -th diagonal block of

$$\prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j))$$

$$\text{converges in probability to } \varphi_k \left( \prod_{j=1}^m P_j(a(s_j, k_j, l_j)) d(j) \right).$$

Following step III of the previous section, we see that for all  $u \in [d] - \{1\}$ ,  $H'(m)$  implies  $H(m, u, u)$ . Thus we only have to prove that for all  $m \geq 0$ ,  $H'(m)$  and  $H(m, 1, 1)$  hold.

Let us prove it by induction on  $m$ . For  $m = 0$ , the result is immediate. Consider  $m \geq 1$  such that  $H'(l)$  and  $H(l, 1, 1)$  hold for all  $l < m$ .

Let us first prove  $H(m)$ . Consider  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m, P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*, d(1), \dots, d(m) \in \mathcal{C}$ . Let us prove that

$$\mathbb{E} \left( \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right)$$

converges in probability to  $ds \mathbb{E}(\prod_{j=1}^m \prod_{n=1}^m P_j(a(s_j, k_j, l_j)) d(j))$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Similarly to step V of the previous section, by the induction hypothesis and the fact that  $H'(m-1)$  implies  $H(m-1, u, u)$ , we can suppose that for all  $n, j$ ,  $X(n, j)$  has only one non zero block and satisfies

$$\mathbb{E}(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

We can also suppose that for all  $j \in [m]$  such that with the cyclic order  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

Under these assumptions, we only have to prove that

$$\mathbb{E} \left( \prod_{j=1}^m X(j, n) \psi_n(d(j)) \right)$$



converges in probability to

$$\mathbb{E} \left( \prod_{j=1}^m x(j)d(j) \right) = 0.$$

Following step VI of the previous section, it appears to be an application of case 1°) of proposition 1.14 and remark 1.15.

Let us now prove  $H(m, 1, 1)$ . Consider  $(s_j, k_j, l_j)_{j \in [m]} \in (\mathbb{N} \times [d] \times [d])^m$ ,  $P_1, \dots, P_m$  polynomials in the noncommutative variables  $X, X^*, d(0), \dots, d(m) \in \mathcal{C}$ . Let us prove that

$$\mathbb{E} \left( \psi_n(d(0)) \prod_{j=1}^m P_j(\tilde{R}(s_j, k_j, l_j, n)) \psi_n(d(j)) \right)$$

converges in probability to  $ds \mathbb{E}(d(0) \prod_{j=1}^m \prod_{j=1}^m P_j(a(s_j, k_j, l_j))d(j))$ . Let us denote  $x(j) = P_j(a(s_j, k_j, l_j))$  and  $X(j, n) = P_j(\tilde{N}(s_j, k_j, l_j, n))$ . Again, similarly to step V of the previous section, we can suppose that for all  $n \geq 1, j \in [m-1]$ ,  $X(n, j)$  has only one non zero block and satisfies

$$\mathbb{E}(X(j, n)) \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{convergence in probability}).$$

We can also suppose that for all  $j \in [m-1]$  such that  $(s_j, k_j, l_j) = (s_{j+1}, k_{j+1}, l_{j+1})$ , we have

$$E(d(j)) = 0 \text{ or } \exists k \in [d], d(j) = p_k.$$

Following again step VI of the previous section, it appears to be an application of case 2°) of proposition 1.14 and remark 1.15.

## 2. DEFINITION OF RECTANGULAR FREE ADDITIVE CONVOLUTION

For  $\nu$  probability measure on  $\mathbb{R}$ , denote by  $\tilde{\nu}$  the *symmetrization* of  $\nu$ , which is the probability measure defined by  $\tilde{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(-B))$  for all Borel set  $B$ .

One of the interests of modelization with free probabilities of the asymptotic behavior of square random matrices is the possibility to compute the asymptotic spectral law of an hermitian matrix which is a function of several independent random matrices. In particular ([V91]), if  $M(1, n), M(2, n)$  are independent hermitian unitarily invariant random matrices whose spectral laws tend respectively to  $\nu_1, \nu_2$  when their dimension  $n$  goes to infinity, then the spectral law of  $M(1, n) + M(2, n)$  tends to  $\nu_1 \boxplus \nu_2$  (free additive convolution of  $\nu_1, \nu_2$ ), and the spectral law of  $\sqrt{M(1, n)}M(2, n)\sqrt{M(1, n)}$  (when the matrices are positive) tends to  $\nu_1 \boxtimes \nu_2$  (free multiplicative convolution of  $\nu_1, \nu_2$ ). In the same way (combine Theorem 4.3.11 of [HP00] and Propositions 3.5, 3.6 of [HL00]), if the matrices  $M(1, n), M(2, n)$  are non hermitian bi-unitarily invariant (and still independent), with asymptotic singular distributions  $\nu_1, \nu_2$ , then the symmetrization of the singular distribution of  $M(1, n) + M(2, n)$  tends to  $\tilde{\nu}_1 \boxplus \tilde{\nu}_2$ , and the push-forward, by the function  $x \rightarrow x^2$ , of the singular distribution of  $M(1, n)M(2, n)$  is the free multiplicative convolution of the push-forwards, by  $x \rightarrow x^2$ , of  $\nu_1, \nu_2$ .

Now, suppose that  $M(1, n), M(2, n)$ , instead of being square, are rectangular (independent and bi-unitarily invariant)  $q_1(n) \times q_2(n)$  random matrices whose singular laws tend to  $\nu_1, \nu_2$ . We keep the notations introduced in section 1.2, but suppose that  $d = 2$ :

$$q_1(n) + q_2(n) = n, \quad q_1(n), q_2(n) \xrightarrow[n \rightarrow \infty]{} \infty, \quad \frac{q_1(n)}{n} \xrightarrow[n \rightarrow \infty]{} \rho_1 \geq 0, \quad \frac{q_2(n)}{n} \xrightarrow[n \rightarrow \infty]{} \rho_2 > 0.$$

Since for  $1 \leq q \leq q'$ , the singular law  $\nu$  of a  $q \times q'$  matrix  $M$  is related to the singular law  $\nu'$  of  $M^*$  by

$$\nu' = \frac{q' - q}{q'} \delta_0 + \frac{q}{q'} \nu,$$

we can, without restriction, suppose that  $q_1(n) \leq q_2(n)$ .

First, the singular distribution of  $M(1, n)M(2, n)^*$  tends to a distribution that can be computed by free probability theory. Indeed, for all  $r \geq 1$  even, the  $r$ -th moment of the singular distribution of  $M(1, n)M(2, n)^*$  is

$$(25) \quad \frac{1}{q_1(n)} \operatorname{Tr}(M(1, n)M(2, n)^* M(2, n)M(1, n)^*)^{r/2},$$

which is equal to

$$\frac{q_2(n)}{q_1(n)} \frac{1}{q_2(n)} \operatorname{Tr}(M(2, n)^* M(2, n)M(1, n)^* M(1, n))^{r/2},$$

which tends to  $\rho_2/\rho_1(n)$  (when  $\rho_1 > 0$ ) times the  $r/2$ -th moment of  $\tau_1 \boxtimes \tau_2$ , where  $\tau_1, \tau_2$  are the limit spectral distributions of  $M(2, n)^* M(2, n), M(1, n)^* M(1, n)$ .  $\tau_1, \tau_2$  can easily be computed from  $\nu_1, \nu_2$ . Thus, if  $\rho_1 > 0$ , the limit of the spectral distribution of  $M(1, n)M(2, n)^*$  can be easily computed using tools of free probability. If  $\rho_1 = 0$ , then  $\rho_1 = 0$ , then the explicit computation of the expectation and of the variance of (25) for  $r = 2$  gives us the convergence of the singular law of  $M(1, n)M(2, n)^*$  to the Dirac measure in zero.

But it does not seem to be possible to compute the singular law of  $M(1, n) + M(2, n)$  with the tools of free probability theory. Indeed, for  $r \geq 1$  even, its  $r$ -th moment is given by

$$\frac{1}{q_1(n)} \operatorname{Tr}((M(1, n) + M(2, n))(M(1, n)^* + M(2, n)^*))^{r/2},$$

which can be expanded into the sum of

$$(26) \quad \frac{1}{q_1(n)} \operatorname{Tr} M(i_1, n)M(i_2, n)^* \cdots M(i_{r-1}, n)M(i_r, n)^*,$$

where the sum is taken over all  $i \in \{1, 2\}^r$ . Asymptotics of normalized traces like in (26) cannot be computed with free probability theory, because our random matrices are not square, and we cannot reduce the problem, as for  $M(1, n)M(2, n)^*$ , to a problem which involves only independent square random matrices. Thus we have to use theorem 1.8 (or theorem 1.10 if  $\rho_1 = 0$ ) to compute the asymptotic singular law of  $M(1, n) + M(2, n)$ .

**Proposition-Definition 2.1.** *Let  $\mu_1, \mu_2$  be two compactly supported symmetric probability measures on the real line.*

(a) *Let, for all  $n \geq 1$ ,  $M(1, n), M(2, n)$  be independent bi-unitarily invariant  $q_1(n) \times q_2(n)$  random matrices, with deterministic singular values if  $\rho_1 = 0$ , uniformly bounded for  $\|\cdot\|$ , and such that for all  $i = 1, 2$ , the symmetrization of the singular law of  $M(d, i)$  converges in probability to  $\mu_i$ . Then the symmetrization of the singular law of  $M(d, 1) + M(d, 2)$  converges in probability to a symmetric probability measure on the real line, denoted  $\mu_{1 \boxplus_\lambda \mu_2}$ , which depends only on  $\mu_1, \mu_2$ , and  $\lambda := q_1(n)/q_2(n)$ .*

(b)  *$\mu_{1 \boxplus_\lambda \mu_2}$  is the unique symmetric measure  $\mu$  such that for all  $r \in \mathbb{N}$  even, the  $r$ -th moment of  $\mu$  is  $\varphi_1(((a_1 + a_2)(a_1 + a_2)^*)^{r/2})$ , where  $a_1, a_2$  are free with amalgamation over  $\mathcal{D}$  elements of  $p_1 \mathcal{A} p_2$ ,  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  is a  $(\rho_1, \rho_2)$  probability space, and for all  $i = 1, 2$ , for all  $r \in \mathbb{N}$  even,  $\varphi_1((a_i a_i^*)^{r/2})$  is the  $r$ -th moment of  $\mu_i$ .*

(c) *The support of  $\mu_{1 \boxplus_\lambda \mu_2}$  is contained in the sum of the convex hulls of the supports of  $\mu_1$  and  $\mu_2$ .*

The binary operation  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures, called *rectangular free convolution with ratio  $\lambda$* , will be extended to the set of symmetric probability measures, and the same result will stay true without any hypothesis of boundness.

**Proof** By theorem 1.8 (or theorem 1.10 if  $\rho_1 = 0$ ), if  $a_1, a_2$  are elements of a  $(\rho_1, \rho_2)$ -probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  as in (b), then  $(\widetilde{M}(1, n), \widetilde{M}(2, n))$  converges in  $\mathcal{D}$ -distribution in probability to  $(a_1, a_2)$ . Thus once the existence of  $\mu_{1\boxplus_\lambda\mu_2}$  proved, (b) will be immediate.

Let  $[-s_i, s_i]$  be the convex hull of the support of  $\mu_i$  ( $i = 1, 2$ ). By [A61], to prove the convergence in probability of the symmetrization of the singular law of  $M(1, n) + M(2, n)$  to a probability measure with support in  $[-s_1 - s_2, s_1 + s_2]$ , it suffices to prove that for every pair polynomial  $P = \sum_k c_k X^{2k}$  that is non negative on  $[-s_1 - s_2, s_1 + s_2]$ , the normalized trace of  $\sum_k c_k ((M(1, n) + M(2, n))(M(1, n) + M(2, n))^*)^k$  converges to a non negative number. We know that this normalized trace converges to  $\varphi_1[\sum_k c_k ((a_1 + a_2)(a_1 + a_2)^*)^k]$ . To prove that this number is nonnegative, one can suppose that for all  $n$ , for all  $i = 1, 2$ ,

$$M(i, n) = U(i, n) \left[ \delta_k^l \lambda_i(k, n) \right]_{\substack{1 \leq k \leq q_1(n) \\ 1 \leq l \leq q_2(n)}} V(i, n),$$

where  $\lambda_i(1, n), \dots, \lambda_i(q_1(n), n)$  are real numbers of the support of  $\mu_i$  such that the uniform distribution on the  $\lambda_i(k, n)$ 's ( $k = 1, \dots, q_1(n)$ ) converges weakly to  $\mu_i$  and  $U(i, n), V(i, n)$ , are independent respectively  $q_1(n) \times q_1(n), q_2(n) \times q_2(n)$  uniform unitary random matrices. In this case, the normalized trace of  $\sum_k c_k ((M(1, n) + M(2, n))(M(1, n) + M(2, n))^*)^k$  is obviously nonnegative.

The fact that the sequence of the  $\varphi_1 \left( ((a_1 + a_2)(a_1 + a_2)^*)^k \right)$  depends only on  $\mu_1, \mu_2$  and  $\lambda$  is a direct consequence of the fact that the  $\mathcal{D}$ -distribution of an free with amalgamation over  $\mathcal{D}$  family depends only on the individual  $\mathcal{D}$ -distributions (lemma 1.6).  $\square$

Now, let us make a couple of remarks about this convolution.

(1) When  $\rho_1 > 0$ , the proof of the existence of a symmetric probability measure with support contained in the sum of the convex hulls of the supports of  $\mu_1$  and  $\mu_2$  and which satisfies (b) can be done using the theory of  $C^*$ -algebras (the reader who has no knowledge of operator algebra can skip (1) without it being problem for the rest of the article): consider  $a_1, a_2$  are elements of a  $(\rho_1, \rho_2)$ -probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$  as in (b). By the GNS representation or by chapter III of [S98], we can suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and that

$$\varphi : \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \rho_1 \varphi_1(x_{11}) + \rho_2 \varphi_2(x_{22}),$$

is a tracial state. In this case,  $p_1 \mathcal{A} p_1$  is also a  $C^*$ -algebra, and  $\varphi_1$  is a tracial state on  $p_1 \mathcal{A} p_1$ . Then  $\varphi_1([((a_1 + a_2)(a_1 + a_2)^*)^n])$ 's are the even moments of the distribution of  $((a_1 + a_2)(a_1 + a_2)^*)^{1/2}$ , as a self-adjoint element of  $p_1 \mathcal{A} p_1$ . Moreover, it is easy to see that for all  $x \in p_1 \mathcal{A} p_2$ , the symmetrization  $\tilde{\mu}_{(xx^*)^{1/2}}$  of the distribution of  $(xx^*)^{1/2}$  in  $(p_1 \mathcal{A} p_1, \varphi_1)$  and the distribution  $\mu_{x+x^*}$  of  $x + x^*$  in  $(\mathcal{A}, \varphi)$  are linked by the relation

$$(27) \quad \mu_{x+x^*} = 2\rho_1 \tilde{\mu}_{(xx^*)^{1/2}} + (\rho_2 - \rho_1) \delta_0.$$

(2) When  $\lambda = 1$ , the rectangular free convolution is the well known additive free convolution, defined in [VDN91]. There are three ways to prove this. All use the fact that the symmetrization of the distribution of the absolute value of the sum of two free  $R$ -diagonal elements is the free convolution of the symmetrizations of the distributions of their absolute values (Proposition 3.5 of [HL00]).

- First, we can apply the previous corollary to sequences of square matrices. And we know (Theorem 4.3.11 of [HP00]) that independent bi-unitarily invariant square matrices are asymptotically free  $R$ -diagonal elements.
- Also, one can prove this constructing  $\mathcal{A}$  to be  $M_2(\mathbb{C}) \otimes \mathcal{B}$ , where  $(\mathcal{B}, \varphi)$  is a  $C^*$ -probability space with two free  $R$ -diagonal elements  $a_1, a_2$ , whose absolute values have symmetrized distributions  $\mu_1, \mu_2$ . It is then easy to prove (in a similar way as Proposition 3.8 of [NSS99]) that  $A_1 = \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix}$  are free with amalgamation over  $\mathcal{D}$ .  $\mu_{1 \boxplus_\lambda \mu_2}$  is the symmetrization of the distribution of  $((A_1 + A_2)(A_1 + A_2)^*)^{1/2}$  in  $(p_1 \mathcal{A} p_1, \varphi_1)$ , which is the symmetrization of the distribution of  $((a_1 + a_2)(a_1 + a_2)^*)^{1/2} = |a_1 + a_2|$  in  $(\mathcal{B}, \varphi)$ .
- With the same notations, one could also have concluded using the fact that  $\tilde{\mu}_{((A_1 + A_2)(A_1 + A_2)^*)^{1/2}}$  in  $(p_1 \mathcal{A} p_1, \varphi_1)$  is, by (27), the distribution of  $\begin{bmatrix} 0 & a_1 + a_2 \\ a_1^* + a_2^* & 0 \end{bmatrix}$  in  $(M_2(\mathbb{C}) \otimes \mathcal{B}, \text{tr} \otimes \varphi)$ , that is, as noticed in Remark 3.5 of [NSS99], the symmetrization of the distribution of  $|a_1 + a_2|$  (in  $(\mathcal{B}, \varphi)$ ), that is  $\mu_{1 \boxplus \mu_2}$ .

(3) We will see later that the free convolution with ratio 0 is also related to the free convolution.

In the next section, we are going to study cumulants related to this convolution (and, more generally to the freeness with amalgamation over  $\mathcal{D}$ ), like free cumulants in [S94]. We will use them to define an analytic integral transform of symmetric probability measures which linearizes  $\boxplus_\lambda$ , and which will, as a consequence, allow us to compute rectangular free convolution of symmetric probability measures.

### 3. CUMULANTS

In this section, the term algebra (resp. subalgebra) will refer to the notion of  $*$ -algebra (resp.  $*$ -subalgebra). The theory of cumulants in a  $\mathcal{D}$ -probability space (i.e. in an algebra endowed with a conditional expectation on a subalgebra  $\mathcal{D}$ ) has been developed in [S98] (for a short abstract, see [NSS02]). In this section, we will begin by giving the main lines of this theory, and then we will investigate the special case of  $(\rho_1, \dots, \rho_d)$ -probability spaces.

**3.1. General theory of cumulants in a  $\mathcal{D}$ -probability space.** In this section, we consider an algebra  $\mathcal{A}$ , a subalgebra  $\mathcal{D}$  of  $\mathcal{A}$ , and a *conditional expectation*  $E$  from  $\mathcal{A}$  to  $\mathcal{D}$ , i.e. a map  $E$  such that

$$\forall (d, a, d') \in \mathcal{D} \times \mathcal{A} \times \mathcal{D}, E(dad') = dE(a)d'.$$

Let us begin with algebraic definitions. A  $\mathcal{D}$ -bimodule is a vector space  $M$  over  $\mathbb{C}$  on which the algebra  $\mathcal{D}$  acts on the right and on the left. The tensor product  $M \otimes_{\mathcal{D}} N$  of two  $\mathcal{D}$ -bimodules  $M, N$  is their tensor product as  $\mathbb{C}$ -vector spaces, where for all  $(m, d, n) \in M \times \mathcal{D} \times N$ ,  $(m.d) \otimes n$  and  $m \otimes (d.n)$  are identified.  $M \otimes_{\mathcal{D}} N$  is endowed with a structure of  $\mathcal{D}$ -bimodule by  $d_1.(m \otimes n).d_2 = (d_1.m) \otimes (n.d_2)$ . Therefore we can define, for  $n$  positive integer,  $\mathcal{A}^{\otimes_{\mathcal{D}} n} = \underbrace{\mathcal{A} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{A}}_{n \text{ times}}$ .

A partition  $\pi$  of a finite totally ordered set  $I$  is said to be noncrossing if there is no  $i < j < k < l \in I$  such that  $i, k$  are connected by  $\pi$ ,  $j, l$  also, but not  $i, j$ . The following lemma can easily be proved by induction over the cardinality of  $I$ .

**Lemma 3.1.** *A partition  $\pi$  of a finite totally ordered set  $I$  is noncrossing if and only if  $\pi$  has only one class or a class  $V$  of  $\pi$  is an interval, and  $\pi - \{V\}$  is a non crossing partition of  $I - V$ .*

Consider a sequence  $(f_n)_{n \geq 1}$  of maps, each  $f_n$  being a  $\mathcal{D}$ -bimodule morphism between  $\mathcal{A}^{\otimes \mathcal{D}^n}$  and  $\mathcal{D}$ . For  $n$  positive integer and  $\pi \in \text{NC}(n)$  (noncrossing partition of  $[n]$ ), we define the  $\mathcal{D}$ -bimodule morphism  $f_\pi$  between  $\mathcal{A}^{\otimes \mathcal{D}^n}$  and  $\mathcal{D}$  in the following way: if  $\pi = 1_n$  is the one-block partition,  $f_\pi = f_n$ . In the other case, a block  $V$  of  $\pi$  is an interval  $[k, l]$ . If  $k = 1$  (resp.  $l = n$ ), then  $f_\pi(a_1 \otimes \cdots \otimes a_n) = f_{l-k+1}(a_1 \otimes \cdots \otimes a_l) f_{\pi \setminus \{V\}}(a_{l+1} \otimes \cdots \otimes a_n)$  (resp.  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_n)$ ). In the other case, one has  $1 < k \leq l < n$ . Then  $f_\pi(a_1 \otimes \cdots \otimes a_n)$  is defined to be  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1}) f_{l-k+1}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n$  or  $f_{\pi \setminus \{V\}}(a_1 \otimes \cdots \otimes a_{k-1} \otimes f_{l-k+1}(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n)$ , both are the same by definition of  $\otimes_{\mathcal{D}}$ .

For example, if  $\pi = \{\{1, 6, 8\}, \{2, 5\}, \{3, 4\}, \{7\}, \{9\}\}$ , then

$$f_\pi(a_1 \otimes \cdots \otimes a_9) = f_3(a_1 f_2(a_2 f_2(a_3 \otimes a_4) \otimes a_5) \otimes a_6 f_1(a_7) \otimes a_8) f_1(a_9).$$

Let us define, for all  $n \geq 1$ , the  $\mathcal{D}$ -bimodule morphism  $E_n$  between  $\mathcal{A}^{\otimes \mathcal{D}^n}$  and  $\mathcal{D}$  which maps  $a_1 \otimes \cdots \otimes a_n$  to  $E(a_1 \cdots a_n)$ . Then, one can define the sequence  $(c_n)_{n \geq 1}$  of maps, each  $c_n$  being a  $\mathcal{D}$ -bimodule morphism between  $\mathcal{A}^{\otimes \mathcal{D}^n}$  and  $\mathcal{D}$ , by one of the following equivalent formulae:

$$(28) \quad \forall n, \forall \pi \in \text{NC}(n), \quad E_\pi = \sum_{\sigma \leq \pi} c_\sigma,$$

$$(29) \quad \forall n, \quad E_n = \sum_{\sigma \in \text{NC}(n)} c_\sigma$$

$$(30) \quad \forall n, \forall \pi \in \text{NC}(n), \quad c_\pi = \sum_{\sigma \leq \pi} \mu(\sigma, \pi) E_\sigma,$$

$$(31) \quad \forall n, \quad c_n = \sum_{\sigma \in \text{NC}(n)} \mu(\sigma, 1_n) E_\sigma,$$

where  $\mu$  is the Möbius function ([R64], [S99]) of the lattice  $\text{NC}(n)$  endowed with the refinement order.

We recall Theorem 1 of [SS01].

**Theorem 3.2.** *A family  $(\chi_i)_{i \in I}$  of subsets of  $\mathcal{A}$  is free with amalgamation over  $\mathcal{D}$  if and only if for all  $n \geq 2$ , for all non constant  $i \in I^n$ , for all  $a_1 \in \chi_{i_1}, \dots, a_n \in \chi_{i_n}$ , for all  $d_1, \dots, d_n \in \mathcal{D}$ , one has  $c_n(a_1 d_1 \otimes \cdots \otimes a_n d_n) = 0$ .*

For example, if  $a_1, a_2$  are free with amalgamation over  $\mathcal{D}$ , for all  $n \geq 2$  even, one has

$$(32) \quad \begin{aligned} & c_n((a_1 + a_2) \otimes (a_1^* + a_2^*) \otimes (a_1 + a_2) \otimes \cdots \otimes (a_1^* + a_2^*)) \\ &= c_n(a_1 \otimes a_1^* \otimes a_1 \otimes \cdots \otimes a_1^*) + c_n(a_2 \otimes a_2^* \otimes a_2 \otimes \cdots \otimes a_2^*). \end{aligned}$$

**3.2. The special case where  $\mathcal{D} = \text{Span}(p_1, p_2)$ .** In this section, we consider a  $(\rho_1, \rho_2)$ -rectangular probability space  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$ , with  $\rho_1/\rho_2 = \lambda$ . We shall adopt the point of view of operator-valued free probability: the linear span  $\mathcal{D}$  of  $\{p_1, p_2\}$  is an algebra, which can be identified to the set of  $2 \times 2$  complex diagonal matrices, and then to  $\mathbb{C}^2$  by

$$\lambda_1 p_1 + \lambda_2 p_2 \simeq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \simeq (\lambda_1, \lambda_2),$$

and with this identification, the application  $E$  is then a conditional expectation from  $\mathcal{A}$  to  $\mathcal{D}$ .

Let us define, for all  $m$ ,  $c_m^{(1)}$ ,  $c_m^{(2)}$  the linear forms on the complex vector space  $\mathcal{A}^{\otimes \mathcal{D}^m}$  defined by

$$\forall x \in \mathcal{A}, c_m(x) = \begin{bmatrix} c_m^{(1)}(x_{11}) & 0 \\ 0 & c_m^{(2)}(x_{22}) \end{bmatrix}.$$

By (32), an analytic integral transform of a symmetric measure  $\mu$  with coefficients given by  $c_{2n}^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*)$ , where  $a \in p_1 \mathcal{A} p_2$  is such that for all  $r$  even,  $\varphi_1((aa^*)^{r/2})$  is the  $r$ -th moment of  $\mu$ , would linearize  $\boxplus_\lambda$ . Thus we have to give an explicit relation between the  $\varphi_1((aa^*)^{r/2})$ 's and the  $c_n^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*)$ 's. This relation is given by equation (29), but we shall express  $c_\pi(a \otimes a^* \otimes \cdots \otimes a^*)$  as a function of  $c_r^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \cdots \otimes a)$ , and give a relation between  $c_r^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \cdots \otimes a)$ .

For a more exhaustive study of cumulants in  $(\rho_1, \dots, \rho_d)$ -probability spaces, the reader can refer to [B-G2].

Consider  $a \in p_1 \mathcal{A} p_2$ . First of all, since  $a = p_1 a p_2$  and  $p_1 p_2 = p_2 p_1 = 0$ , by definition of the tensor product of  $\mathcal{D}$ -bimodules, for all  $\varepsilon_1, \dots, \varepsilon_n \in \{., *\}$  such that there exists  $l \in [n-1]$ ,  $\varepsilon_l = \varepsilon_{l+1}$ , we have

$$c_n(a^{\varepsilon_1} \otimes \cdots \otimes a^{\varepsilon_n}) = 0.$$

Moreover, since  $c_n$  is a  $\mathcal{D}$ -bimodule morphism with values in  $\mathcal{D}$  and  $p_1 \mathcal{D} p_2 = p_2 \mathcal{D} p_1 = \{0\}$ , if  $\varepsilon_n = \varepsilon_1$ , we also have

$$c_n(a^{\varepsilon_1} \otimes \cdots \otimes a^{\varepsilon_n}) = 0.$$

Thus, for  $\pi \in \text{NC}(n)$ , if

$$c_\pi(a \otimes a^* \otimes \cdots \otimes a^*) \neq 0,$$

then all blocks of  $\pi$  have even cardinality and consist of an alternating sequence of even and odd numbers. We denote by  $\text{NC}'(n)$  the set of such partitions. For example,  $\{\{1, 4, 5, 6\}, \{2, 3\}\} \in \text{NC}'(6)$ , whereas  $\{\{1\}, \{2, 3\}, \{4, 5, 6\}\} \notin \text{NC}'(n)$ . Using lemma 3.1, we prove easily that in fact, any noncrossing partition in which all blocks have even cardinality is in  $\text{NC}'(n)$ .

At last, because  $c_n$  is a  $\mathcal{D}$ -bimodule morphism with values in  $\mathcal{D}$ ,

$$(33) \quad c_n(a \otimes a^* \otimes \cdots \otimes a^*) = c_n^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*) p_1,$$

$$(34) \quad c_n(a^* \otimes a \otimes \cdots \otimes a) = c_n^{(2)}(a^* \otimes a \otimes \cdots \otimes a) p_2.$$

Thus for all  $\pi \in \text{NC}'(n)$ .

$$(35) \quad c_\pi(a \otimes a^* \otimes \cdots \otimes a^*) = \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a). p_1$$

$$(36) \quad c_\pi(a^* \otimes a \otimes \cdots \otimes a) = \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*). p_2$$

**Remark 3.3.** Note that all we did here stays true if one replaces  $c_\pi, c^{(1)}, \dots$  by  $E_\pi, \varphi_1, \dots$

Now, it remains only to give the relation between  $c_r^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*)$  and  $c_r^{(2)}(a^* \otimes a \otimes \cdots \otimes a)$ .

**Lemma 3.4.** For all  $n$  even, we have  $\rho_1 c_n^{(1)}(a \otimes a^* \otimes \cdots \otimes a^*) = \rho_2 c_n^{(2)}(a^* \otimes a \otimes \cdots \otimes a)$ .

**Proof** We are going to prove it by induction over  $n$ . Let us prove it for  $n = 2$ . We have  $c_1 = E$  and  $c_2(x \otimes y) = E(xy) - E(x)E(y)$ , so the result is given by equation (3). Assume now the result to be proved to the ranks  $2, 4, \dots, n - 2$ . One has, by formulae (29),(35),(36),

$$c_n^{(1)}(a \otimes \cdots \otimes a^*) = \underbrace{\varphi_1(a \cdots a^*)}_X - \underbrace{\sum_{\pi \in \text{NC}'(n)} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a)}_Y,$$

and

$$c_n^{(2)}(a^* \otimes \cdots \otimes a) = \underbrace{\varphi_2(a^* \cdots a)}_{X'} - \underbrace{\sum_{\pi \in \text{NC}'(n)} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*)}_{Y'}.$$

Since  $\rho_{i_0} X = \rho_{i_1} X'$  (by formula (3)), it suffices to prove that

$$\rho_{i_0} Y = \rho_{i_1} Y'.$$

To do that, it suffices to propose a bijective correspondance  $\pi \mapsto \tilde{\pi}$  from  $\text{NC}'(n) - \{1_n\}$  to  $\text{NC}'(n) - \{1_n\}$  such that for all  $\pi \in \text{NC}'(n) - \{1_n\}$ ,

$$\rho_{i_0} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) = \rho_{i_1} \prod_{\substack{V \in \pi \\ \min V \text{ odd}}} c_{|V|}^{(2)}(a^* \otimes \cdots \otimes a) \prod_{\substack{V \in \pi \\ \min V \text{ even}}} c_{|V|}^{(1)}(a \otimes \cdots \otimes a^*).$$

by the induction hypothesis, the correspondance which maps  $\pi \in \text{NC}'(n) - \{1_n\}$  to  $\tilde{\pi}$  defined by  $k \tilde{\pi} l \Leftrightarrow \sigma(k) \tilde{\pi} \sigma(l)$ , where  $\sigma$  is the cycle  $(12 \cdots n)$  of  $[n]$ , is convenient.  $\square$

From (29), (35), and the previous lemma, we deduce:

**Proposition 3.5.** *For  $a \in p_1 \mathcal{A} p_2$ ,  $n$  positive integer,*

$$(37) \quad \varphi_1((aa^*)^n) = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{V \in \pi} c_{|V|}(a),$$

where for all  $r$  even,  $c_r(a)$  denotes  $c_r^{(1)}(a \otimes \cdots \otimes a^*)$ , and for all  $\pi$ ,  $e(\pi)$  denotes the number of blocks of  $\pi$  with even minimum.

**3.3. The case  $\lambda = 0$ , characterization of  $\mathbb{M}_0$ .** In all this subsection, we suppose that  $\lambda = 0$ , i.e.  $(\rho_1, \rho_2) = (0, 1)$ . In this case, by equation (3),  $E$  is null on  $p_2 \mathcal{A} p_1 \mathcal{A} p_2$ , thus on  $p_2 \mathcal{A} p_1 \mathcal{A}$ . It implies that  $E_n$  is null on  $p_2 \mathcal{A} p_1 \otimes_{\mathcal{D}} \mathcal{A}^{\otimes \mathcal{D} n - 1}$ . Hence, for any  $\pi \in \text{NC}(n)$ ,  $E_\pi$  is null on any space of the type  $p_{i_0} \mathcal{A} p_{i_1} \otimes_{\mathcal{D}} p_{i_1} \mathcal{A} p_{i_2} \otimes_{\mathcal{D}} \dots \otimes_{\mathcal{D}} p_{i_{n-1}} \mathcal{A} p_{i_n}$ , where the minimum  $k$  of a block of  $\pi$  is such that  $(i_{k-1}, i_k) = (2, 1)$ . By the formula (iii) of the definition of the cumulant functions in subsection 3.1, the function  $c_\pi$  is null too on the previous subspace.

Consider  $n$  even and  $\pi \in \text{NC}(n)$ . For  $E_\pi$  or  $c_\pi$  to be nonzero on  $(p_1 \mathcal{A} p_2 \otimes_{\mathcal{D}} p_2 \mathcal{A} p_1)^{\otimes_{\mathcal{D}} n/2}$ ,  $\pi$  has to belong to  $\text{NC}'(n)$  and no block of  $\pi$  has to have even minimum (i.e.  $e(\pi)$  has to be 0). Let us introduce the set  $\text{NC}_d(n)$  of noncrossing partitions  $\pi$  of  $[n]$  such that for all  $k$  even in  $[n]$ ,  $k-1 \tilde{\pi} k$ . Note that  $\text{NC}_d(n)$  is contained in the set of elements of  $\text{NC}'(n)$  in which no block has

even minimum. We claim that the inverse inclusion is true. Indeed, suppose the existence of  $\pi \in \text{NC}'(n) \setminus \text{NC}_d(n)$  in which no block has even minimum. Define  $k = \min\{k \in [n]; k \text{ even and } k-1 \not\approx k\}$ .  $k$  cannot be the minimum of its block in  $\pi$ , and by minimality of  $k$ , its preceding element  $j$  in this block has to be even. Then, as  $\pi$  is noncrossing, the set  $\{j+1, \dots, k-1\}$  is a union of classes of  $\pi$ , but its cardinality is odd, which is in contradiction with  $\pi \in \text{NC}'(n)$ .

So by (29), for all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$  of  $(p_1 \mathcal{A} p_2 \times p_2 \mathcal{A} p_1)^{n/2}$ , we have, for all  $\pi \in \text{NC}_d(n)$ ,

$$\varphi_{1, \pi}(a_1, b_1, \dots, a_{n/2}, b_{n/2}) = \sum_{\substack{\sigma \in \text{NC}_d(n) \\ \sigma \leq \pi}} c_{\sigma}^{(1)}(a_1 \otimes b_1 \otimes \dots \otimes a_{n/2} \otimes b_{n/2}).$$

But  $\text{NC}_d(n)$  is in correspondance with  $\text{NC}(n/2)$  by the order-preserving bijection  $d$  from  $\text{NC}(n/2)$  onto  $\text{NC}_d(n)$ , that maps any noncrossing partition  $\pi$  to the partition  $d(\pi)$  that links to elements  $i, j$  of  $[n]$  if and only if the upper integer parts of  $i/2$  and  $j/2$  are linked by  $\pi$ . With this definition, the previous equation can be written: for all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$  of  $(p_1 \mathcal{A} p_2 \times p_2 \mathcal{A} p_1)^{n/2}$ , we have, for all  $\pi \in \text{NC}(n/2)$ ,

$$(38) \quad \varphi_{1, \pi}(a_1 b_1, \dots, a_{n/2} b_{n/2}) = \sum_{\substack{\sigma \in \text{NC}(n/2) \\ \sigma \leq d(\pi)}} c_{\sigma}^{(1)}(a_1 \otimes b_1 \otimes \dots \otimes a_{n/2} \otimes b_{n/2}).$$

So, from the very definition of free cumulants in a noncommutative probability space (p. 621 of [S94] or definition 4.5 of [S99]), we deduce:

**Proposition 3.6.** *For all  $n$  even, for all family  $(a_i, b_i)_{i=1, \dots, n/2}$  of  $(p_1 \mathcal{A} p_2 \times p_2 \mathcal{A} p_1)^{n/2}$ ,  $c_n^{(1)}(a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes \dots \otimes a_{n/2} \otimes b_{n/2})$  is the free cumulant of the family of the  $n/2$  noncommutative random variables  $(a_1 b_1, \dots, a_{n/2} b_{n/2})$  in  $(p_1 \mathcal{A} p_1, \varphi_1)$ .*

Now we are able to give the link between the free convolution with null ratio  $\boxplus_0$  and the free convolution  $\boxplus$ :

**Proposition 3.7.** *The free convolution with null ratio of two compactly supported symmetric probability measures is the unique symmetric probability whose push-forward by the square function is the free convolution of the push-forwards by the square function of the two probabilities.*

**Proof** Denote, for  $\nu$  probability measure,  $m_n(\nu)$  its  $n$ -th moment, and  $\mathfrak{K}_n(\nu)$  its  $n$ -th free cumulant. Recall ([S94]) that free cumulants of probability measures are defined by the formula

$$(39) \quad \forall n \geq 1, m_n(\nu) = \sum_{\pi \in \text{NC}(n)} \prod_{B \in \pi} \mathfrak{K}_{|B|}(\nu),$$

and that the free convolution of  $\nu_1, \nu_2$  compactly supported probability measures is the only probability with free cumulants  $\mathfrak{K}_n(\nu_1) + \mathfrak{K}_n(\nu_2)$  ( $n \geq 1$ ).

Consider two symmetric compactly supported probability measures  $\mu_1, \mu_2$ . Denote by  $\nu_1, \nu_2$  their respective push-forwards by the square function. It suffices to prove that for all  $n$  even,  $m_n(\mu_1 \boxplus_0 \mu_2) = m_{n/2}(\nu_1 \boxplus \nu_2)$ , i.e. that for all  $n$  even,

$$m_n(\mu_1 \boxplus_0 \mu_2) = \sum_{\pi \in \text{NC}(n/2)} \prod_{B \in \pi} (\mathfrak{K}_{|B|}(\nu_1) + \mathfrak{K}_{|B|}(\nu_2)).$$



Consider free elements  $a_1, a_2 \in p_1\mathcal{A}p_2$  such that

$$\forall i = 1, 2, \forall n \in \mathbb{N} \text{ even}, \varphi_1((a_i a_i^*)^{n/2}) = m_n(\mu_i).$$

Then one has

$$\begin{aligned} m_n(\mu_1 \boxplus_0 \mu_2) &= \varphi_1 \left( ((a_1 + a_2)(a_1^* + a_2^*))^{n/2} \right) \\ &= \sum_{\pi \in \text{NC}_d(n)} \prod_{B \in \pi} \underbrace{c_{|B|}^{(1)}(a_1 + a_2 \otimes \cdots \otimes a_1^* + a_2^*)}_{=c_{|B|}^{(1)}(a_1 \otimes \cdots \otimes a_1^*) + c_{|B|}^{(1)}(a_2 \otimes \cdots)} \quad \text{by eq. (35)} \end{aligned}$$

But for all  $i = 1, 2$ , for all  $m$  even,  $c_m^{(1)}(a_i \otimes \cdots \otimes a_i^*)$  is the free cumulant of the family of  $m/2$  variables  $(a_i a_i^*, \dots, a_i a_i^*)$  in the noncommutative probability space  $(p_1\mathcal{A}p_1, \varphi_1)$ , i.e. the  $m/2$ -th free cumulant of  $\nu_i$ . So

$$m_n(\mu_1 \boxplus_0 \mu_2) = \sum_{\pi \in \text{NC}(n/2)} \prod_{B \in \pi} (\mathfrak{K}_{|B|}(\nu_1) + \mathfrak{K}_{|B|}(\nu_2)) = m_{n/2}(\nu_1 \boxplus \nu_2),$$

which proves the result.  $\square$

**3.4. Generating series of the cumulants of an element of  $p_1\mathcal{A}p_2$ .** From now on, we suppose that  $\lambda > 0$ . Now, we will derive from (37) a formula which links generating series of the sequence  $\varphi_1((aa^*)^n)$  and  $c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$ . Note that such a formula could be derived from the very general Theorem 2.2.3 of [S98], but the work needed to translate this result to our context is almost as long as this section.

**Lemma 3.8.** *Consider a sequence  $(c_{2n})_{n \in \mathbb{N}^*}$  of complex numbers. Define the sequence  $(m_{2n}^{(e)})_{n \in \mathbb{N}}$  by  $m_0^{(e)} = 1$  and for each  $n \in \mathbb{N}^*$ ,*

$$m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{B \in \pi} c_{|B|}.$$

Define the formal power series

$$C(X) = \sum_{n \geq 1} c_{2n} X^n, \quad M^{(e)}(X) = \sum_{n \geq 1} m_{2n}^{(e)} X^n.$$

Then we have  $C(X(\lambda M^{(e)}(X) + 1)(M^{(e)}(X) + 1)) = M^{(e)}(X)$ .

**Proof Step I** Define, for  $n$  positive integer and  $\pi \in \text{NC}'(2n)$ ,  $o(\pi)$  to be the number of blocks of  $\pi$  with odd minimum. Then we have

$$\lambda m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}.$$

Indeed, if  $c$  denotes the cycle

$$(2n \rightarrow 2n - 1 \rightarrow \cdots \rightarrow 2 \rightarrow 1)$$

of  $[2n]$ , then  $c$  induces a permutation of  $\text{NC}'(2n)$  by

$$C : \pi \in \text{NC}'(2n) \mapsto C(\pi) := \{c(B); B \in \pi\}.$$

For example, for  $\pi = \{\{1, 4, 5, 6\}, \{2, 3\}\}$ ,  $C(\pi) = \{\{3, 4, 5, 6\}, \{1, 2\}\}$ . For  $\pi \in \text{NC}'(2n)$ , for  $B$  block of  $\pi$ ,  $\min c(B)$  is  $(\min B) - 1$  if  $\min B > 1$ , and the second element of  $B$  minus 1 in the other case. Note that in the first case,  $\min B$  and  $\min c(B)$  have different parities, and in the second

one, they have the same parities, by definition of  $\text{NC}'(2n)$ . Hence we have  $o(C(\pi)) = e(\pi) + 1$ . Thus

$$\lambda m_{2n}^{(e)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)+1} \prod_{B \in \pi} c_{|B|} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(C(\pi))} \prod_{B \in \pi} c_{|B|} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}.$$

*Step II* Thus we have to prove that  $C\left(X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1)\right) = M^{(e)}(X)$ , where  $M^{(o)}(X) = \sum_{n \geq 1} m_{2n}^{(o)} X^n$ , and  $m_{2n}^{(o)} = \sum_{\pi \in \text{NC}'(2n)} \lambda^{o(\pi)} \prod_{B \in \pi} c_{|B|}$ .

Define, for each positive  $n$  and each  $j \in [n]$ , the “decomposition map”  $D_j$ , from the set of partitions of  $\text{NC}'(2n)$  in which the class of 1 has cardinality  $2j$  to  $\bigcup_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \dots + l_{2j} = n-j}} \text{NC}'(2l_1) \times \dots \times$

$\text{NC}'(2l_{2j})$  (the set  $\text{NC}'(0)$  is considered as a singleton, on which  $e$  and  $o$  have value 0), which maps  $\pi \in \text{NC}'(n)$  to  $(\pi_1, \dots, \pi_{2j})$  defined in the following way: denote by  $\{k_1 < \dots < k_{2j}\}$  the class of 1 in  $\pi$ , and for each  $r$ , let  $\pi_r$  be the restriction of  $\pi$  to the interval  $]k_r, k_{r+1}[$ , with  $k_{2j+1} = 2n + 1$ . It is easy to see that the map  $D_j$  is well defined (recall that for all  $r$ ,  $k_r$  and  $k_{r+1}$  have opposite parity), and that it is a bijection. Moreover, we have

$$\begin{aligned} e(\pi) &= o(\pi_1) + e(\pi_2) + \dots + o(\pi_{2j-1}) + e(\pi_{2j}), \\ o(\pi) &= 1 + e(\pi_1) + o(\pi_2) + \dots + e(\pi_{2j-1}) + o(\pi_{2j}). \end{aligned}$$

We denote by  $\langle X^n \rangle P$  the coefficient of  $X^n$  in a formal power series  $P$  in  $X$ . Let us show that for each  $n \geq 1$ ,

$$\langle X^n \rangle C\left(X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1)\right) = m_{2n}^{(e)}.$$

We have

$$\begin{aligned} & \langle X^n \rangle C\left(X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1)\right) \\ &= \sum_{j=1}^n c_{2j} \langle X^n \rangle \left(X^j (M^{(o)}(X) + 1)^j (M^{(e)}(X) + 1)^j\right) \\ &= \sum_{j=1}^n c_{2j} \langle X^{n-j} \rangle \left((M^{(o)}(X) + 1)^j (M^{(e)}(X) + 1)^j\right) \\ &= \sum_{j=1}^n c_{2j} \sum_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \dots + l_{2j} = n-j}} m_{2l_1}^{(o)} m_{2l_2}^{(e)} \dots m_{2l_{2j-1}}^{(o)} m_{2l_{2j}}^{(e)} \\ &= \sum_{j=1}^n c_{2j} \sum_{\substack{l \in \mathbb{N}^{2j} \\ l_1 + \dots + l_{2j} = n-j}} \sum_{\substack{\pi_1 \in \text{NC}'(2l_1) \\ \vdots \\ \pi_{2j} \in \text{NC}'(2l_{2j})}} \lambda^{o(\pi_1) + e(\pi_2) + \dots + e(\pi_{2j})} \prod_{r=1}^{2j} \prod_{B \in \pi_r} c_{|B|} \end{aligned}$$

The preliminaries about the bijections  $D_j$  tell us that

$$\begin{aligned} \langle X^n \rangle C\left(X(M^{(o)}(X) + 1)(M^{(e)}(X) + 1)\right) &= \sum_{\pi \in \text{NC}'(2n)} \lambda^{e(\pi)} \prod_{B \in \pi} c_{|B|} \\ &= m_{2n}^{(e)}. \end{aligned}$$

□

Let us introduce two power series that will play a role in the computation of the generating series of the cumulants:

$$\begin{aligned} T(X) &= (\lambda X + 1)(X + 1), \\ U(X) &= (T - 1)^{\langle -1 \rangle} \\ &= \sum_{n \geq 1} \frac{(-1)^{n-1} \lambda^{n-1} \binom{2n}{n}}{2(\lambda + 1)^{2n-1} (2n - 1)} X^n. \end{aligned}$$

If  $\lambda = 0$  then  $U(z) = z$ , in the other case  $U(z)$  is the power expansion of  $\frac{-\lambda - 1 + [(\lambda + 1)^2 + 4\lambda z]^{1/2}}{2\lambda}$ , where  $z \mapsto z^{1/2}$  is the analytic version of the square root on the complement of the real non positive half line such that  $1^{1/2} = 1$ .

**Theorem 3.9.** *Consider  $a \in p_1 \mathcal{A} p_2$ , and denote for each  $n \in \mathbb{N}^*$ ,  $m_{2n}(a) = \varphi_1((aa^*)^n)$  and  $c_{2n}(a) = c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$ . Define the formal power series  $M(X) = \sum_{n \geq 1} m_{2n}(a) X^n$  and  $C(X) = \sum_{n \geq 1} c_{2n}(a) X^n$ . Then*

$$C = U \circ \left( \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}} - 1 \right).$$

**Proof** By proposition 3.5 and the previous lemma,

$$C \circ (X \times (T \circ M)) = M.$$

So

$$\begin{aligned} T \circ C \circ (X \times (T \circ M)) &= T \circ M, \\ \frac{T \circ C}{X} \circ (X \times (T \circ M)) &= \frac{1}{X}, \\ \frac{X}{T \circ C} \circ (X \times (T \circ M)) &= X, \\ \frac{X}{T \circ C} &= (X \times (T \circ M))^{\langle -1 \rangle}, \\ T \circ C &= \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}}, \\ C &= U \circ \left( \frac{X}{(X \times (T \circ M))^{\langle -1 \rangle}} - 1 \right). \end{aligned}$$

□

#### 4. DEFINITION AND FIRST PROPERTIES OF THE RECTANGULAR $R$ -TRANSFORM

**4.1. The case of compactly supported symmetric measures.** Consider a symmetric compactly supported probability measure  $\mu$ . For  $n \geq 0$ , denote by  $m_n(\mu)$  its  $n$ -th moment and by  $c_{2n}(\mu)$  the number  $c_{2n}^{(1)}(a \otimes \cdots \otimes a^*)$  with  $a \in p_1 \mathcal{A} p_2$ , where  $\mathcal{A}$  is a  $(\rho_1, \rho_2)$ -probability space, such that for all  $k \geq 1$ ,  $\varphi_1((aa^*)^k) = m_{2k}(a)$ .

Recall that the generating series of the moments  $M_\mu(X) = \sum_{n \geq 1} m_{2n}(\mu) X^n$  and of the cumulants  $C_\mu(X) = \sum_{n \geq 1} c_{2n}(\mu) X^n$  of a symmetric measure  $\mu$  are linked by the relation

$$(40) \quad C_\mu = U \circ \left( \frac{X}{(X \times (T \circ M_\mu))^{\langle -1 \rangle}} - 1 \right).$$

Since the formal power series  $C_\mu$  linearizes the convolution  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures (by equation (32)), we will define an analytic integral transform on the set of symmetric probability measures whose power expansion is  $C_\mu(z)$  when  $\mu$  has compact support.

For any symmetric probability measure  $\mu$  on the real line, we define the *Cauchy transform*  $G_\mu$  of  $\mu$  on the upper half plane by

$$G_\mu(z) = \int_{t \in \mathbb{R}} \frac{d\mu(t)}{z - t}.$$

When  $\mu$  has compact support, the power expansion of the Cauchy transform is

$$G_\mu(z) = \sum_{n \geq 0} m_{2n}(\mu) \frac{1}{z^{2n+1}}.$$

So, when  $z \in \mathbb{C} \setminus [0, +\infty)$  is small enough,

$$M_\mu(z) = \frac{1}{\sqrt{z}} G_\mu\left(\frac{1}{\sqrt{z}}\right) - 1,$$

where we denote by  $\sqrt{\phantom{z}}$  the analytic version of the square root on the complement of the real nonnegative half line in the complex plane such that  $\sqrt{-1} = i$ .

So, for such a  $z$ ,

$$(41) \quad zT \circ M_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1 - \lambda)\sqrt{z}G_\mu\left(\frac{1}{\sqrt{z}}\right).$$

For any symmetric probability measure  $\mu$ , we define the *rectangular Cauchy transform with ratio*  $\lambda$  of  $\mu$  on the complement of the real nonnegative half line to be  $H_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1 - \lambda)\sqrt{z}G_\mu\left(\frac{1}{\sqrt{z}}\right)$ .

When  $\mu$  is compactly supported, from the power expansion of the Cauchy transform  $G_\mu$  we know that  $H_\mu$  is analytic in a neighborhood of zero, that  $H_\mu(0) = 0$ , and that  $H'_\mu(0) = 1$ .

So we can invert the function  $H_\mu$  in a neighborhood of zero, and the inverse function  $H_\mu^{-1}$  has a power series expansion that is, by equation (41), the expansion of  $(X \times (T \circ M_\mu))^{<-1>}$ . Note that  $\frac{z}{H_\mu^{-1}(z)} - 1$  is analytic in a neighborhood of zero, with value zero in zero.

Let us define the analytic function on a neighborhood of zero  $U(z) = \frac{-\lambda - 1 + [(\lambda + 1)^2 + 4\lambda z]^{1/2}}{2\lambda}$  (when  $\lambda = 0$ ,  $U(z) = z$ ), where  $z \mapsto z^{1/2}$  is the analytic version of the square root on the complement of the real non positive half line such that  $1^{1/2} = 1$ . Its power expansion is  $\sum_{n \geq 1} \frac{(-1)^{n-1} \lambda^{n-1} \binom{2n}{n}}{2(\lambda+1)^{2n-1} (2n-1)} z^n$ . So by equation (40), the analytic power expansion of

$$z \mapsto U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right)$$

is  $C_\mu(z)$ .

**4.2. Definition of the rectangular  $R$ -transform.** Note that when  $A$  is a tight set of symmetric probability measures on the real line, for every  $0 < \theta < \pi/2$ , we have

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \frac{\pi}{2}| < \theta}} \frac{1}{z} G_{\mu}\left(\frac{1}{z}\right) = 1,$$

uniformly in  $\mu \in A$ .

Indeed, if  $\varepsilon > 0$ , if  $M$  is such that  $\frac{1}{\cos \theta} \sup_{\mu \in A} \mu([-M, M]^c) < \varepsilon/2$ , if  $z$  is such that  $|\arg z - \frac{\pi}{2}| < \theta$ ,

and  $\frac{M}{\frac{1}{|z|} - M} < \varepsilon/2$ , then for each  $\mu \in A$ , we have

$$\begin{aligned} \left| \frac{1}{z} G_{\mu}\left(\frac{1}{z}\right) - 1 \right| &= \left| \int \frac{u}{\frac{1}{z} - u} d\mu(u) \right| \\ &\leq \frac{M}{\frac{1}{|z|} - M} \mu([-M, M]) + \frac{1}{\cos \theta} \mu([-M, M]^c) \\ &\leq \varepsilon. \end{aligned}$$

**Proposition 4.1.** *Let  $A$  be a set of symmetric probability measures on the real line. Then the following assertions are equivalent*

- (i)  $A$  is tight,
- (ii) for every  $0 < \theta < \pi$ ,  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \theta}} \frac{1}{z} H_{\mu}(z) = 1$  uniformly in  $\mu \in A$ ,
- (iii)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} \frac{1}{x} H_{\mu}(x) = 1$  uniformly in  $\mu \in A$ .

**Proof** (i)  $\Rightarrow$  (ii) follows from what precedes, (ii)  $\Rightarrow$  (iii) is clear. Suppose (iii). Consider  $\varepsilon > 0$ . Take  $\eta > 0$  such that  $\forall u \in [0, 1], |\lambda u^2 + (1 - \lambda)u - 1| < \eta \Rightarrow |u - 1| < \varepsilon/2$ . Take  $x \in (-\infty, 0)$  such that for all  $\mu \in A$ ,  $|\frac{1}{x} H_{\mu}(x) - 1| < \eta$ . Put  $\sqrt{x} = iy$ . Then

$$\forall \mu \in A, \underbrace{\left| \frac{1}{iy} G_{\mu}\left(\frac{1}{iy}\right) - 1 \right|}_{\int_t \frac{t^2 y^2}{t^2 y^2 + 1} d\mu(t)} < \varepsilon/2.$$

Let  $M$  be such that  $\forall t > M, \frac{t^2 y^2}{t^2 y^2 + 1} \geq \frac{1}{2}$ . Then

$$\forall \mu \in A, \mu([-M, M]^c) \leq 2 \int_t \frac{t^2 y^2}{t^2 y^2 + 1} d\mu(t) < \varepsilon.$$

□

Define, for  $\alpha \in (0, \pi)$ ,  $\beta > 0$ ,  $\Delta_{\alpha, \beta}$  to be the set of complex numbers  $z$  such that  $|\arg z - \pi| < \alpha$  and  $|z| < \beta$ .

Let  $\mathcal{H}$  be the set of functions  $f$  which are analytic in a domain  $\mathcal{D}_f$  such that for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that

$$\Delta_{\alpha, \beta} \subset \mathcal{D}_f.$$

A family  $(f_a)_{a \in A}$  of functions of  $\mathcal{H}$  is said to be *uniform* if for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that

$$\forall a \in A, \quad \Delta_{\alpha, \beta} \subset \mathcal{D}_{f_a}.$$

The following theorem has already been used, in other forms, to define Voiculescu's  $R$ - and  $S$ -transforms (see paragraph 5 of [BV93]). Its proof relies on Rouché theorem.

**Theorem 4.2.** *Let  $(H_a)_{a \in A}$  be a uniform family of functions of  $\mathcal{H}$  such that for every  $\alpha \in (0, \pi)$ ,*

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{H_a(z)}{z} = 1 \text{ uniformly in } a \in A.$$

*Then there exists a uniform family  $(F_a)_{a \in A}$  of functions of  $\mathcal{H}$  such that for every  $\alpha \in (0, \pi)$ ,*

$$\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} \frac{F_a(z)}{z} = 1 \text{ uniformly in } a \in A,$$

*and there exists  $\beta$  positive such that*

$$\forall a \in A, \quad H_a \circ F_a = F_a \circ H_a = I_d \text{ on } \Delta_{\alpha, \beta}.$$

*Moreover, the family  $(F_a)_{a \in A}$  is unique in the following sense: if a family  $(\tilde{F}_a)_{a \in A}$  of functions of  $\mathcal{H}$  satisfies the same conditions, then for all  $\alpha \in (0, \pi)$ , there exists  $\beta$  positive such that*

$$\forall a \in A, \quad F_a = \tilde{F}_a \text{ on } \Delta_{\alpha, \beta}.$$

### Application: definition of the rectangular $R$ -transform

For every symmetric probability measure on the real line  $\mu$ , let us define the *rectangular  $R$ -transform*  $R_\mu$  with ratio  $\lambda$  of  $\mu$  by

$$C_\mu(z) = U \left( \frac{z}{H_\mu^{-1}(z)} - 1 \right),$$

where  $H_\mu^{-1}$  is defined by the previous theorem and the function  $U$  is the one defined at the end of section 4.1.

One can summarize the different steps of the construction of the rectangular  $R$ -transform with ratio  $\lambda$  in the following chain

$$\begin{array}{ccccccc} \mu & \longrightarrow & G_\mu & \longrightarrow & H_\mu(z) = \lambda G_\mu \left( \frac{1}{\sqrt{z}} \right)^2 + (1 - \lambda) \sqrt{z} G_\mu \left( \frac{1}{\sqrt{z}} \right) & \longrightarrow & \\ \text{sym. prob.} & & \text{Cauchy} & & & & \\ \text{measure} & & \text{transf.} & & & & \\ C_\mu(z) = U \left( \frac{z}{H_\mu^{-1}(z)} - 1 \right) & & & & & & \\ \text{rect. } R\text{-transf. with ratio } \lambda & & & & & & \end{array}$$

Note that the rectangular  $R$ -transform with ratio 1 (resp. 0), for symmetric distributions, is linked to the Voiculescu transform by the relation  $C_\mu(z) = \sqrt{z} \varphi_\mu(1/\sqrt{z})$  (resp.  $C_\mu(z) = z \varphi_\rho(z)$ , where  $\rho$  is the push-forward of  $\mu$  by the function  $t \rightarrow t^2$ ) (see paragraph 5 of [BV93] for the construction of the Voiculescu transform  $\varphi_\nu$  of a probability measure  $\nu$ ).

*Remark about the characterization of  $R$ -transforms.* The definition of the rectangular  $R$ -transform of a probability measure  $\mu$  presents many analogies with the definition of its Voiculescu transform  $\varphi_\mu$  (see [BV93]). So it seems natural to state, as the authors of [BV93] did for the Voiculescu transform (Proposition 5.6), a characterization of the functions that are the rectangular  $R$ -transform of a symmetric probability measure. The characterization of the Voiculescu transform is based on the fact that for every probability measure  $\mu$ , the inverse of  $z + \varphi_\mu(z)$  extends to a Pick function, and on the fact that any Pick function equivalent to  $z$  at infinity is

of the type  $1/G_\mu$ , so its inverse is  $z + \varphi_\mu(z)$ . But unless  $\lambda = 0$  or  $1$ , Pick functions do not appear in an analogous place in the definition of the rectangular  $R$ -transform, so we cannot proceed similarly to characterize rectangular  $R$ -transforms.

Furthermore, the Lévy Kinchine formula for  $\boxplus_\lambda$  (see [B-G1]), compared with theorem 5.10 of [BV93]) will state that when  $\mu$  is  $\boxplus_\lambda$ -infinitely divisible, there exists a unique symmetric  $\boxplus$ -infinitely divisible distribution  $\nu$  such that  $C_\mu(z) = \sqrt{z}\varphi_\nu(1/\sqrt{z})$ . So the question of the characterization of rectangular  $R$ -transforms joins another question: can we extend the correspondence  $\mu \leftrightarrow \nu$  to a bijective correspondence in the set of symmetric distributions such that  $C_\mu(z) = \sqrt{z}\varphi_\nu(1/\sqrt{z})$ . The analytic functions  $f$  on  $\Delta_{\alpha,\beta}$  of the type  $f(z) = \sqrt{z}\varphi_\nu(1/\sqrt{z})$ , with  $\nu$  symmetric probability measure, are characterized by:

- (i)  $f(\bar{z}) = \overline{f(z)}$ ,
- (ii)  $\lim_{z \rightarrow 0} f(z) = 0$ ,
- (iii) for all  $n$  and all  $z_1, \dots, z_n$ , the matrix  $\left[ \frac{\sqrt{z_j} - \sqrt{z_k}}{\sqrt{z_j}(1+f(z_k)) - \sqrt{z_k}(1+f(z_j))} \right]_{k,j=1}^n$  is positive.

But nothing allows us to claim that the rectangular  $R$ -transform of any symmetric distribution satisfies (iii), and that every function that satisfies (i), (ii), and (iii) is the rectangular  $R$ -transform of a symmetric distribution.

### 4.3. Properties of the rectangular $R$ -transform.

**Theorem 4.3** (Injectivity of the rectangular  $R$ -transform). *If the rectangular  $R$ -transforms with ratio  $\lambda$  of two symmetric probability measures coincide on a neighborhood of 0 in  $(-\infty, 0)$ , then the measures are equal.*

**Proof** If the rectangular  $R$ -transforms with ratio  $\lambda$  of two symmetric probability measures  $\mu, \nu$  coincide on a neighborhood of 0 in  $(-\infty, 0)$ , then by uniqueness of analytic continuation, they coincide on a  $\Delta_{\alpha,\beta}$ , and so  $H_\mu = H_\nu$  on this set, and  $zH_\mu(z) = zH_\nu(z)$  on this set. So there exists  $M > 0$  such that for all  $y > M$ ,

$$\lambda(iy2G_\mu(iy))^2 + (1 - \lambda)iyG_\mu(iy) = \lambda(iy2G_\nu(iy))^2 + (1 - \lambda)iyG_\nu(iy).$$

But if  $\rho$  is a symmetric probability measure,  $iyG_\mu(iy) = \int_{t \in \mathbb{R}} \frac{y^2}{y^2 + t^2} d\rho(t) \in (0, 1]$ . So, by injectivity of  $u \mapsto \lambda u^2 + (1 - \lambda)u$  on  $(0, 1]$ ,  $G_\mu(iy) = G_\nu(iy)$  for  $y > 1$ , and then, by analyticity of the Cauchy transform,  $G_\mu = G_\nu$ , and by injectivity of the Cauchy transform (see [A61] or section 3.1 of [HP00])  $\mu = \nu$ .  $\square$

The following lemma is an easy consequence of proposition 4.1 and theorem 4.2.

**Lemma 4.4** (Tightness and rectangular  $R$ -transform). *Let  $A$  be a set of symmetric probability measures. Then we have equivalence between*

- (i)  $A$  is tight,
- (ii) for any  $0 < \alpha < \pi$ ,  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_\mu(z) = 0$  uniformly in  $\mu \in A$ ,
- (iii)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} R_\mu(x) = 0$  uniformly in  $\mu \in A$ .

**Theorem 4.5** (Paul Lévy's theorem for rectangular  $R$ -transform). *Let  $(\mu_n)$  be a sequence of symmetric probability measures. Then we have equivalence between:*

- (i)  $(\mu_n)$  converges weakly to a symmetric probability measure;
- (ii) there exists  $\alpha, \beta$  such that
  - (a)  $\lim_{\substack{z \rightarrow 0 \\ |\arg z - \pi| < \alpha}} C_{\mu_n}(z) = 0$  uniformly in  $n$ ,
  - (b) the sequence  $(C_{\mu_n})$  converges uniformly on every compact set of  $\Delta_{\alpha, \beta}$ ;
- (iii) (a)  $\lim_{\substack{x \rightarrow 0 \\ x \in (-\infty, 0)}} C_{\mu_n}(x) = 0$  uniformly in  $n$ ,
- (b) there exists  $\beta > 0$  such that the sequence  $(C_{\mu_n})$  converges pointwise on  $[-\beta, 0)$ .

Moreover, in this case, denoting by  $\mu$  the weak limit of  $(\mu_n)$ , for every  $\alpha$ , there exists  $\beta$  such that the sequence  $(C_{\mu_n})$  converges uniformly to  $C_\mu$  on every compact set of  $\Delta_{\alpha, \beta}$ .

**Proof** (i)  $\Rightarrow$  (ii): suppose that  $(\mu_n)$  converges weakly to  $\mu$ . Then by the previous lemma, we have (a) of (ii). So there exists  $\beta > 0$  such that on  $\Delta_{\alpha, \beta}$ , for all  $n$ ,  $|C_{\mu_n}| \leq 1$ . So, by Montel's theorem, it suffices to show that the only possible limit, for uniform convergence on every compact, of any subsequence of  $(C_{\mu_n})$  is  $C_\mu$ . Let  $C$  be an analytic function on  $\Delta_{\alpha, \beta}$  such that a subsequence  $(C_{\mu_{k_n}})$  of  $(C_{\mu_n})$  converges uniformly to  $C$  on every compact of  $\Delta_{\alpha, \beta}$ . We know (see [A61] or section 3.1 of [HP00]) that the sequence  $(G_{\mu_n})$  converges uniformly on every compact of the upper half plane to  $G_\mu$ . So the sequence  $(H_{\mu_n})$  converges uniformly on every compact of the complement of  $[0, +\infty)$  to  $H_\mu$ . Since  $\lim_{z \rightarrow 0} C(z) = 0$ , to prove  $C = C_\mu$ , it suffices to prove that

$$\begin{aligned} (\lambda C + 1)(C + 1) &= (\lambda C_\mu + 1)(C_\mu + 1). \\ \text{But } (\lambda C_\mu(z) + 1)(C_\mu(z) + 1) &= \frac{z}{H_\mu^{-1}(z)} \end{aligned}$$

So it suffices to prove that

$$\frac{z}{(\lambda C(z) + 1)(C(z) + 1)} = H_\mu^{-1}(z),$$

so by theorem 4.2, it suffices to prove that

$$H_\mu \left( \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right) = z.$$

We have

$$\begin{aligned} & \left| H_\mu \left[ \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right] - z \right| \\ = & \left| H_\mu \left[ \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right] - H_{\mu_{k_n}} \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z) + 1)(C_{\mu_{k_n}}(z) + 1)} \right] \right| \\ \leq & \underbrace{\left| H_\mu \left[ \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right] - H_\mu \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z) + 1)(C_{\mu_{k_n}}(z) + 1)} \right] \right|}_{(1)} \\ & + \underbrace{\left| H_\mu \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z) + 1)(C_{\mu_{k_n}}(z) + 1)} \right] - H_{\mu_{k_n}} \left[ \frac{z}{(\lambda C_{\mu_{k_n}}(z) + 1)(C_{\mu_{k_n}}(z) + 1)} \right] \right|}_{(2)} \end{aligned}$$

By continuity of  $H_\mu$ , (1) tends to zero when  $n$  tends to infinity, and, since  $H_{\mu_{k_n}}$  converges uniformly to  $H_\mu$  on every compact, (2) tends to zero. So  $H_\mu \left( \frac{z}{(\lambda C(z) + 1)(C(z) + 1)} \right) = z$ .



(ii)  $\Rightarrow$  (iii) is clear.

(iii)  $\Rightarrow$  (i): suppose (iii). Then by (i)  $\Rightarrow$  (ii), every limit of a subsequence of  $(\mu_n)$  has a rectangular  $R$ -transform equal to the pointwise limit of  $(C_{\mu_n})$  on  $[\beta, 0)$ . By uniqueness of analytic continuation, all the limits of subsequences of  $(\mu_n)$  have the same rectangular  $R$ -transform, so, by injectivity of this transform (previous theorem), there cannot be more than one limit of subsequence of  $(\mu_n)$ . As by (a) of (iii), the set  $\{\mu_n; n \in \mathbb{N}\}$  is tight,  $(\mu_n)$  converges weakly to a symmetric probability measure.  $\square$

**4.4. Rectangular  $R$ -transform and rectangular free convolution.** Recall the definition of the binary operation  $\boxplus_\lambda$  on the set of compactly supported symmetric probability measures : let  $\mu, \nu$  be two compactly supported symmetric probability measures. Let  $a, b$  be two free elements in  $p_1\mathcal{A}p_2$  with  $(\mathcal{A}, p_1, p_2, \varphi_1, \varphi_2)$   $(\rho_1, \rho_2)$ -probability space such that  $\lambda = \rho_1/\rho_2$  and such that for all  $n \in \mathbb{N}$ ,  $m_{2n}(\mu) = \varphi((aa^*)^n)$  and  $m_{2n}(\nu) = \varphi((bb^*)^n)$ . Then there exists a unique compactly supported probability measure  $\mu_{\boxplus_\lambda\nu}$ , that depends only on  $\mu$  and  $\nu$  such that for all  $n \in \mathbb{N}$ ,  $m_{2n}(\mu_{\boxplus_\lambda\nu}) = \varphi(((a+b)(a+b)^*)^n)$ . By equation (32), we know that

$$(42) \quad C_{\mu_{\boxplus_\lambda\nu}} = C_\mu + C_\nu,$$

as a formal power series. But, by section 4.1, we know that this formal series are the expansion of the analytic functions with the same name. So (42) holds also for analytic functions.

Let us extend the binary operation  $\boxplus_\lambda$  to a binary operation on the set of symmetric probability measures on the real line, that will be commutative, associative, and continuous with respect to the weak convergence. Let  $\mu, \nu$  be symmetric probability measures. If  $(\mu_n)$  (resp.  $(\nu_n)$ ) is a sequence of compactly supported symmetric probability measures that converges weakly to  $\mu$  (resp.  $\nu$ ), then by theorem 4.5 and equation (42) the sequence  $(\mu_n \boxplus_\lambda \nu_n)$  converges to a measure whose rectangular  $R$ -transform is  $C_\mu + C_\nu$  (so this measure does not depend on the choice of the sequences  $(\mu_n)$  and  $(\nu_n)$  and is equal to  $\mu_{\boxplus_\lambda\nu}$  when  $\mu$  and  $\nu$  are compactly supported). This measure will be called the *rectangular  $R$ -transform with ratio  $\lambda$*  of  $\mu$  and  $\nu$ , and denoted by  $\mu_{\boxplus_\lambda\nu}$ . Note that the equation (42) stays true for all symmetric probability measures  $\mu$  and  $\nu$ . Moreover, this equation shows that  $\boxplus_\lambda$  is a commutative, associative, and continuous binary operation on the set of symmetric probability measures.

Now we can extend the proposition 2.1 to the case where  $\mu_1, \mu_2$  are not compactly supported:

**Theorem 4.6.** *Consider sequences  $q_1(n), q_2(n)$  such that*

$$q_1(n) + q_2(n) = n, \quad q_1(n) \leq q_2(n), \quad q_1(n) \xrightarrow[n \rightarrow \infty]{} \infty, \quad q_1(n)/q_2(n) \xrightarrow[n \rightarrow \infty]{} \lambda \in [0, 1].$$

*Let, for all  $n$ ,  $A_n, B_n$  be independent  $q_1(n) \times q_2(n)$  bi-unitarily invariant random matrices such that the symmetrizations of the singular laws of  $A_n, B_n$  converge in probability to probability measures  $\mu, \nu$ . Then the symmetrization of the singular law of  $A_n + B_n$  converges in probability to  $\mu_{\boxplus_\lambda\nu}$ .*

To prove the theorem, we will need the following lemma, for which we need to introduce an extended functional calculus. When  $F$  is a real Borel function on the real line, for any  $q_1(n) \times q_2(n)$  matrix  $C$ , we define  $F(C)$  to be the matrix

$$U[\delta_i^j F(h_i)]_{\substack{1 \leq i \leq q_1(n) \\ 1 \leq j \leq q_2(n)}} V,$$

where  $h_1, \dots, h_{q_1(n)} \geq 0$ , and  $U, V$  are respectively  $q_1(n) \times q_1(n)$ ,  $q_2(n) \times q_2(n)$  unitary matrices such that

$$c = U \left[ \delta_i^j h_i \right]_{\substack{1 \leq i \leq q_1(n) \\ 1 \leq j \leq q_2(n)}} V.$$

Note that  $(F + G)(C) = F(C) + G(C)$  and  $F(C)G(C)^* = 0$  when  $FG = 0$ . For any probability measure  $\sigma$ , we will denote by  $F(\sigma)$  (for example  $\sigma^2, \sigma^{1/2}, |\sigma| \dots$ ) the push-forward of  $\sigma$  by  $F$ , that is the distribution of the random variable  $F(X)$ , when  $X$  is a random variable with distribution  $\sigma$ . We denote by  $\mu_H$  the spectral measure of an hermitian matrix  $H$ . Recall that we denote by  $\tilde{\sigma}$  the symmetrization of any probability measure  $\sigma$ . For example, the symmetrization of the singular law of a rectangular matrix  $A$  will be denoted by  $\tilde{\mu}_{|A|}$ .

**Lemma 4.7.** *Let  $(C_n)$  be a sequence of  $q_1(n) \times q_2(n)$  random matrices such that  $\tilde{\mu}_{|C_n|}$  converges in probability to a probability measure  $\sigma$ . Then, for any odd real function  $F$  continuous at  $\sigma$ -almost every point of  $\mathbb{R}$  and such that  $F(\mathbb{R}^+) \subset \mathbb{R}^+$ ,  $\tilde{\mu}_{|F(C_n)|}$  converges in probability to  $F(\sigma)$ .*

**Proof**  $F(\sigma)$  is symmetric, the symmetrization is a continuous operation, so it suffices to prove that  $\mu_{|F(C_n)|}$  converges in probability to  $|F(\sigma)| = F(|\sigma|)$ . But  $\mu_{|F(C_n)|} = F(\mu_{|C_n|})$ ,  $\mu_{|C_n|}$  converges in probability to  $|\sigma|$ , and  $\sigma \mapsto F(\sigma)$  on the set of probability measures on the real line is weakly continuous at  $|\sigma|$  because  $F$  is continuous at  $|\sigma|$ -almost every point of the real line ([B68]).  $\square$

**Proof of the theorem** The function that maps a probability measure  $\sigma$  on  $\mathbb{R}^+$  to the symmetrization  $\widetilde{\sigma^{1/2}}$  of its push-forward by  $x \rightarrow x^{1/2}$  is continuous. So it suffices to prove that the push-forward of the spectral law of  $|A_n + B_n|$  by  $x \rightarrow x^2$ , i.e. the spectral law of  $M_n := (A_n + B_n)(A_n + B_n)^*$  converges in probability to  $(\mu_{\boxplus \lambda \nu})^2$ .

We can define a distance on the set of probability measures on the real line with the Cauchy transform by

$$(\sigma_1, \sigma_2) \mapsto \sup_{\Im z \geq 1} |G_{\sigma_1}(z) - G_{\sigma_2}(z)|.$$

This distance defines the topology of weak convergence ([A61], [PL02]). The Cauchy transform of the spectral distribution of an hermitian matrix  $M$  is the normalized trace of its resolvent  $\mathfrak{R}_z(M) = (z - M)^{-1}$ . So the spectral distribution of a sequence  $(X_n)$  of  $q_1(n) \times q_1(n)$  hermitian random matrices converges in probability to a probability measure  $\sigma$  on the real line if and only if for each  $\varepsilon > 0$ , the probability of the event

$$\left\{ \sup_{\Im z \geq 1} |\operatorname{tr} \mathfrak{R}_z(X_n) - G_\sigma(z)| > \varepsilon \right\}$$

tends to zero as  $n$  tends to infinity.

Choose  $\varepsilon > 0$ . We will show that

$$\mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu_{\boxplus \lambda \nu})^2}(z) \right| > \varepsilon \right\} \xrightarrow{n \rightarrow \infty} 0,$$

where  $\mathbb{P}$  designs the probability measure of the probability space where the random matrices are defined. Let us define, for  $t > 0$ , the function  $F_t$  on the real line by

$$F_t : x \mapsto \begin{cases} x & \text{if } |x| \leq t, \\ 0 & \text{elseif.} \end{cases}$$

For every probability measure  $\sigma$ ,  $F_t(\sigma)$  converges weakly to  $\sigma$  when  $t$  tends to infinity, so, by continuity of  $\boxplus$ , there exists  $t \in (0, +\infty)$  such that  $t$  and  $-t$  are not atoms of the measures  $\mu$

and  $\nu$ , and such that

$$(43) \quad \sup_{\Im z \geq 1} \left| G_{(F_t(\mu) \boxplus_\lambda F_t(\nu))^2}(z) - G_{(\mu \boxplus_\lambda \nu)^2}(z) \right| < \frac{\varepsilon}{3},$$

$$(44) \quad \mu(\mathbb{R} - [-t-1, t+1]) + \nu(\mathbb{R} - [-t-1, t+1]) < \frac{\varepsilon}{18}.$$

We will now use the notations  $F_t = F$ ,  $G(x) = x - F(x)$  and  $M_{n,t} = (F(A_n) + F(B_n))(F(A_n) + F(B_n))^*$ .

Using triangular inequality and (43), we have

$$(45) \quad \begin{aligned} & \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu \boxplus_\lambda \nu)^2}(z) \right| > \varepsilon \right\} \\ & \leq \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left[ \mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t}) \right] \right| > \frac{\varepsilon}{3} \right\} \\ & + \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \mathfrak{R}_z(M_{n,t}) - G_{(F(\mu) \boxplus_\lambda F(\nu))^2}(z) \right| > \frac{\varepsilon}{3} \right\} \end{aligned}$$

To treat the first term of right hand side of (45), recall that for any  $q_1(n) \times q_1(n)$  matrix  $T$ ,  $|\operatorname{tr} T| \leq \frac{1}{q_1(n)} \|T\| \operatorname{rg} T$ . The operator norm  $\|\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t})\|$  is not less than 2, because  $\Im z \geq 1$ . Moreover, due to the equation

$$\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t}) = -\mathfrak{R}_z(M_n)(M_n - M_{n,t})\mathfrak{R}_z(M_{n,t}),$$

the rank is not more than the one of  $M_n - M_{n,t}$ . One has (omitting the indices  $n$  in  $A_n, B_n$ )

$$M_n - M_{n,t} = G(A)G(A)^* + G(B)G(B)^* + G(A)B^* + F(A)G(B)^* + G(B)A^* + F(B)G(A)^*,$$

so its rank is not more than  $3 \operatorname{rg} G(A_n) + 3 \operatorname{rg} G(B_n)$ . Hence

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{\Im z \geq 1} |\operatorname{tr} [\mathfrak{R}_z(M_n) - \mathfrak{R}_z(M_{n,t})]| > \frac{\varepsilon}{3} \right\} \\ & \leq \mathbb{P} \left\{ \frac{6}{q_1(n)} (\operatorname{rg} G(A_n) + \operatorname{rg} G(B_n)) > \frac{\varepsilon}{3} \right\} \\ & = \mathbb{P} \left\{ \mu_{A_n}(\mathbb{R} - [-t, t]) + \mu_{B_n}(\mathbb{R} - [-t, t]) > \frac{\varepsilon}{18} \right\}, \end{aligned}$$

which tends to zero when  $n$  goes to infinity, by (44). On the other side, the second term of right hand side of (45) goes to zero when  $n$  goes to infinity by definition of rectangular convolution of compactly supported symmetric probability measures and by the previous lemma. So

$$\lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{\Im z \geq 1} \left| \operatorname{tr} \left( \mathfrak{R}_z(M_n) \right) - G_{(\mu \boxplus_\lambda \nu)^2}(z) \right| > \varepsilon \right\} = 0,$$

and the result is proved.  $\square$

## REFERENCES

- [A61] Akhiezer, N.I. *The classical moment problem*, Moscou, 1961
- [B-G1] Benaych-Georges, F. *Infinitely divisible distributions for rectangular free convolution: classification and matricial interpretation* available on <http://www.dma.ens.fr/~benaych/>
- [B-G2] Benaych-Georges, F. *Rectangular random matrices, related free entropy and free Fisher's information* available on <http://www.dma.ens.fr/~benaych/>
- [BV93] Bercovici, H., Voiculescu, D. *Free convolution of measures with unbounded supports* Indiana Univ. Math. J. **42** (1993) 733-773
- [B68] Billingsley, P. *Convergence of probability measures* Wiley, 1968

- [D74] Donoghue, W. *Monotone matrix functions and analytic continuation*, Springer, New-York, 1974 Adisson-Wesley Publ. Co., Cambridge, Mass., 1954
- [HL00] Haagerup, U., Larsen, F. *Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras* Journ. Functional Analysis 176, 331-367 (2000).
- [HP00] Hiai, F., Petz, D. *The semicircle law, free random variables, and entropy* Amer. Math. Soc., Mathematical Surveys and Monographs Volume 77, 2000
- [HJ91] Horn, R.A., Johnson, C.R. *Topics in matrix analysis* Cambridge University Press, Cambridge, 1991
- [M67] Mehta, M.L. *Random matrices and the statistical theory of energy levels* New York Academic Press, 1967
- [N74] Nelson, E. *Notes on non-commutative integration* J. Functional Analysis 15 (1974), 103–116
- [NSS99] Nica, Alexandru; Shlyakhtenko, Dimitri; Speicher, Roland *Some minimization problems for the free analogue of the Fisher information* Adv. Math. 141 (1999), no. 2, 282–321.
- [NSS02] Nica, Alexandru; Shlyakhtenko, Dimitri; Speicher, Roland *Operator-valued distributions. I. Characterizations of freeness* Int. Math. Res. Not. 2002, no. 29, 1509–1538.
- [PL02] Pastur, L., Lejay, A. *Matrices alatoires : statistique asymptotique des valeurs propres* Seminaire de Probabilités XXXVI, Lecutre notes in M. 1801, Springer 2002
- [R64] Rota, Gian-Carlo *On the foundations of combinatorial theory. I. Theory of Möbius functions* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 340–368 (1964)
- [Sh96] Shlyakhtenko, D. *Random Gaussian band matrices and freeness with amalgamation* Internat. Math. Res. Notices 1996, no. 20, 1013–1025.
- [S94] Speicher, R. *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Annalen 298 (1994) 611-628
- [S98] Speicher, R. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc. 132 (1998), no. 627
- [S99] Speicher, R. (1999). Notes of my lectures on Combinatorics of Free Probability (IHP, Paris, 1999). Available on <http://www.mast.queensu.ca/~speicher>
- [ŚS01] Śniady, Piotr, Speicher, Roland, *Continuous family of invariant subspaces for R-diagonal operators* Invent. Math. 146, 329-363 (2001)
- [V91] Voiculescu, Dan *Limit laws for random matrices and free products* Invent. Math. 104 (1991), no. 1, 201–220
- [V95] Voiculescu, Dan *Operations on certain non-commutative operator-valued random variables* Recent advances in operator algebras (Orlans, 1992). Astrisque No. 232 (1995), 243–275.
- [V98] Voiculescu, Dan *A strengthened asymptotic freeness result for random matrices with applications to free entropy* Internat. Math. Res. Notices 1998, no. 1, 41–63.
- [VDN91] Voiculescu, D.V., Dykema, K., Nica, A. *Free random variables* CRM Monographs Series No.1, Amer. Math. Soc., Providence, RI, 1992
- [W58] Wigner, E. *On the distribution of the roots of certain symmetric matrices* Ann. Math. 67 325-327 (1958)

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