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# Local Euler-Maclaurin formula for polytopes

Nicole Berline and Michèle Vergne

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## 1 Introduction

Let  $P$  be a rational convex polytope in  $\mathbb{R}^d$  and  $\mathcal{F}$  be the set of its faces. If  $\phi$  is a function on  $\mathbb{R}^d$ , the comparison between the sum

$$\text{Sum}(\phi, P) = \sum_{n \in \mathbb{Z}^d \cap P} \phi(n)$$

of the value of  $\phi$  at all integral points of  $P$  and the integral  $\int_P \phi(v) dv$  is a classical theme. If  $\phi = 1$ , for example, the number of integral points in  $P$  can be expressed as a sum of linear combination of volumes of the faces  $F$  of  $P$ :

$$\text{Card}(P \cap \mathbb{Z}^d) = \sum_{F \in \mathcal{F}} \mu(F) \text{vol}(F).$$

For a polynomial function  $\phi$ , for each face  $F$  of  $P$ , one seek for (infinite-order) differential operators  $D_F$  with constant coefficients such that

$$\text{Sum}(\phi, P) = \sum_{F \in \mathcal{F}} \int_F D_F \phi$$

(with  $D_F = 1$  for  $F = P$ ).

As explained in [2], essential properties required on the operators  $D_F$  are "locality" and "computability". At each face  $F$  of  $P$ , the operator  $D_F$  should depend only of the translation class modulo  $\mathbb{Z}^d$  of the normal cone  $\text{No}(P, F)$  to  $P$  at a generic point of  $F$  (if  $F$  is of codimension  $k$ , the normal cone is an affine cone of dimension  $k$ ). In particular if  $P$  is an integral polytope, the operator  $D_F$  should depend only of the cone of normal feasible directions at  $F$  to  $P$ .

In this article, using a scalar product, we construct a map  $D$  from rational affine cones in  $\mathbb{R}^k$  (modulo translations by  $\mathbb{Z}^k$ ) to differential operators on  $\mathbb{R}^k$  with rational coefficients such that *For every rational convex polytope  $P$  in  $\mathbb{R}^d$  and any polynomial function  $\phi$  on  $\mathbb{R}^d$ , the formula*

$$\text{Sum}(\phi, P) = \sum_k \sum_{F \in \mathcal{F} | \text{codimension}(F)=k} \int_F D_F(\partial_1, \partial_2, \dots, \partial_k) \phi$$

holds with  $D_F = D(\text{No}(P, F))$ . Furthermore, the formula for  $D_F$  is easily computed from the generating function for the set of integral points *in the normal cone* to  $F$ . Thus computing  $D_F$  can be done by Barvinok's algorithm [1] for the **normal cone**. Here we identify the normal cone to  $P$  at  $F$  (with its integral structure) to the affine tangent cone to  $P$  at a generic point of  $F$ , modulo translations parallel to the face  $F$ . The derivatives  $\partial_1, \partial_2, \dots, \partial_k$  are transversal derivatives. See Theorem 29 for a precise formulation. Our formula for  $\text{Sum}(\phi, P)$ , when  $\phi$  is a polynomial function of degree  $r$ , is thus obtained from computations of the *volumes of faces  $F$*  and from the expansion up to order  $r$  of the generating function of *integral points in the normal cone to  $F$* . This mixed formula is reminiscent of Barvinok's mixed formula for the Ehrhart polynomial [3].

As in Brion's approach [4], our main tool is "localisation" at vertices. Our main theorem is Theorem 17 which expresses, in an explicit recursive way, the generating function for integral points of an affine cone as a linear combination with analytic coefficients of Laplace transforms of its faces.

When  $\phi = 1$ , and  $P$  is an integral polytope, the fact that

$$\text{Card}(P \cap \mathbb{Z}^d) = \sum_{F \in \mathcal{F}} \mu(F) \text{vol}(F)$$

where  $\mu(F)$  depends only of the normal cone to  $F$  was Danilov's conjecture, and this was proven by Pommersheim-Thomas [9] as a corollary of their universal formula for the Todd class of a toric variety. In a companion paper, we will similarly prove a "universal" formula for the equivariant Todd class of any toric variety. We state the formula in Theorem 26.

This problem is already amusing in dimension 2 for an integral polytope in  $\mathbb{R}^2$ . Indeed Pick's formula states:

$$\text{Card}(P \cap \mathbb{Z}^2) = \text{vol}(P) + \frac{1}{2} \text{vol}_{\mathbb{Z}}(\text{boundary}(P)) + 1$$

while Pommersheim-Thomas constructed rational numbers  $\text{pot}(v)$  assigned to each vertex of the polytope  $P$  and depending only of a neighborhood on  $v$  in  $P$  and such that

$$\text{Card}(P \cap \mathbb{Z}^2) = \text{vol}(P) + \frac{1}{2} \text{vol}_{\mathbb{Z}}(\text{boundary}(P)) + \sum_v \text{pot}(v).$$

Thus the constant 1 is distributed over vertices in a canonical way. For example if  $\Delta$  is the simplex with vertices  $[0, 0], [1, 0], [0, 1]$ , the decomposition  $1 = \sum_v \text{pot}(v)$  is  $1 = \frac{1}{4} + \frac{3}{8} + \frac{3}{8}$  while for the simplex  $\Delta'$  with vertices  $[0, 0], [1, 0], [0, 2]$ , the decomposition is  $1 = \frac{1}{4} + \frac{3}{10} + \frac{9}{20}$ .

The name Euler-Maclaurin formula was given to various formulae aimed at relating  $\text{Sum}(\phi, P)$  with integrals (that is finding an explicit Riemann-Roch theorem). For  $P$  an integral polytope, Cappell and Shaneson [7] have obtained a formula for  $\text{Sum}(\phi, P)$  as a sum of integrals of derivatives  $C_F \phi$  of  $\phi$  over faces  $F$  of  $P$ . The fact that such a formula exists is the combinatorial translation of the fact that invariant cycles associated to faces generates the equivariant homology ring of the associated toric variety. However the construction of the operators  $C_F$  depends of the cohomology ring and is not local (see Example 35). Similarly, in Brion-Vergne [6], we reobtained by purely combinatorial methods formulae for  $\text{Sum}(\phi, P)$  either as derivatives of integrals of  $\phi$  over deformed polytopes or as sum of integrals of derivatives over faces, but our formula did not satisfy the locality requirement nor computability.

## 2 Definitions and notations

Let  $V$  be a real vector space of dimension  $d$ . The vector space  $V$  is called rational, if  $V$  is provided with a lattice  $N$  (often left implicit). A point of  $N$  will be called an integral point in  $V$ . A point  $v$  in  $V$  such that there exists  $q \in \mathbb{N}$ ,  $q \neq 0$ , such that  $qv$  is integral is called a rational point and we denote by  $V_{\mathbb{Q}}$  the vector space (over  $\mathbb{Q}$ ) of rational points. A subspace  $W$  of  $V$  is called rational if  $W \cap N$  is a lattice in  $W$ . If  $W$  is a rational subspace, the image of  $N$  in  $V/W$  is a lattice in  $V/W$ , so that  $V/W$  is a rational vector space. A rational space  $V$ , with lattice  $N$ , is equipped with the canonical Lebesgue measure giving volume 1 to  $V/N$ . An affine subspace  $F$  is called rational if it is the translated of a rational subspace by a rational element. It is similarly provided with a canonical measure.

If  $S$  is a subset of  $V$ , we denote by  $\langle S \rangle$  the affine space generated by  $S$ . If  $S$  is a finite subset of  $V_{\mathbb{Q}}$ , the affine space  $\langle S \rangle$  is rational. If  $S$  is a finite subset of  $V_{\mathbb{Q}}$ , we denote by  $\square(S)$  the zonotope generated by  $S$ :

$$\square(S) := \sum_{s \in S} [0, 1]s$$

and by  $\text{rvol}(\square(S))$  the volume of  $\square(S)$  in the rational space  $\langle \{0\}, S \rangle$ .

We denote by  $U$  the dual vector space to  $V$ . It is a rational space, equipped with the dual lattice

$$M := \{u \in U \mid u(N) \subset \mathbb{Z}\}.$$

If  $S$  is a subset of  $V$ , we denote by  $S^{\perp}$  the subspace of  $U$  formed with linear forms vanishing on  $S$ . If  $W$  is a subspace of  $V$ , the dual space to  $V/W$  is the subspace  $W^{\perp}$  of  $U$ .

A rational convex cone  $\mathfrak{c}$  in  $V$  is a convex cone  $\sum_{i=1}^K \mathbb{R}^+ \alpha_i$  generated by a finite number  $\alpha_i$  of elements of  $V_{\mathbb{Q}}$ . A simplicial (rational) cone  $\mathfrak{c}$  is a cone generated by independent elements of  $V_{\mathbb{Q}}$ . In this article, we simply say cone instead of convex rational cone. By definition, the dimension of  $\mathfrak{c}$  is the dimension of the space spanned by  $\mathfrak{c}$ . We say that  $\mathfrak{c}$  is solid if  $\langle \mathfrak{c} \rangle = V$ . The relative interior of  $\mathfrak{c}$  is the interior of  $\mathfrak{c}$  in  $\langle \mathfrak{c} \rangle$ . An affine (rational) cone is the translated of a cone in  $V$  by an element  $r \in V_{\mathbb{Q}}$ .

**Definition 1** *Let  $\mathfrak{c}$  be an affine cone.*

*We denote by  $\text{apex}(\mathfrak{c})$  the greatest affine subspace of  $V$  contained in  $\mathfrak{c}$ .*

*We denote by  $\text{dir}(\mathfrak{c})$  the cone  $\mathfrak{c} - a$ , with  $a \in \text{apex}(\mathfrak{c})$ . We call  $\text{dir}(\mathfrak{c})$  the cone of directions of  $\mathfrak{c}$ .*

An affine cone  $\mathfrak{c}$  is called acute if  $\mathfrak{c}$  does not contain any line. An affine cone  $\mathfrak{c}$  is called simplicial if  $\text{dir}(\mathfrak{c})$  is simplicial.

The set of faces of an affine cone  $\mathfrak{c}$  is denoted by  $\mathcal{F}(\mathfrak{c})$ . If  $\mathfrak{c}$  is acute, then the vertex  $\text{vert}(\mathfrak{c})$  of  $\mathfrak{c}$  is the unique face of dimension 0, while  $\mathfrak{c}$  is the unique face of dimension equal to  $\dim \mathfrak{c}$ .

**Definition 2** *If  $F$  is a face of  $\mathfrak{c}$ , we denote by  $\text{lin}(F)$  the linear space parallel to  $\langle F \rangle$ . We denote by  $\text{cone}(\mathfrak{c}, F) = \mathfrak{c} + \text{lin}(F)$  the affine tangent cone of  $\mathfrak{c}$  at  $F$ . We define the normal cone of  $\mathfrak{c}$  at  $F$  by*

$$\text{No}(\mathfrak{c}, F) = \text{cone}(\mathfrak{c}, F) / \text{lin}(F).$$

The normal cone is an acute affine cone in the vector space  $V/\text{lin}(F)$  and the dual space to  $V/\text{lin}(F)$  is the subspace  $\text{lin}(F)^\perp$  of  $U$ .

If  $F = \mathfrak{c}$ , then  $\text{No}(\mathfrak{c}, F) = \{0\}$ , while if  $F := \{a\}$  is the vertex of an acute affine cone  $\mathfrak{c}$ , the normal cone  $\text{No}(\mathfrak{c}, \{a\})$  is just  $\mathfrak{c}$  itself.

Let  $\mathfrak{c}$  be a cone in  $V$ . We denote by

$$\mathfrak{c}^* := \{u \in U \mid \langle u, \mathfrak{c} \rangle \geq 0\}$$

the dual cone to  $\mathfrak{c}$  (it is a rational cone).

Remark that the dual  $\mathfrak{c}^*$  of an acute cone  $\mathfrak{c}$  is a solid cone, so that it has a non empty interior in  $U$ .

There is a bijective correspondence between faces of a cone  $\mathfrak{c}$  and faces of its dual cone  $\mathfrak{c}^*$ , by taking duals of normal cones: more explicitly, if  $F$  is a face of  $\mathfrak{c}$ , then  $F^\perp \cap \mathfrak{c}^*$  is a face of  $\mathfrak{c}^*$ .

We now recall some results on simplicial decompositions of cones.

**Definition 3** *A subdivision of an acute cone  $\mathfrak{c}$  is a finite collection  $S$  of acute cones in  $V$  such that:*

- 1) *The faces of a cone in  $S$  are in  $S$ .*
- 2) *Intersections of two elements  $\mathfrak{d}_1$  and  $\mathfrak{d}_2$  of  $S$  is a face of  $\mathfrak{d}_1$  and  $\mathfrak{d}_2$ .*
- 3) *We have  $\mathfrak{c} = \cup_{\mathfrak{d} \in S} \mathfrak{d}$ .*

*If furthermore the elements of  $S$  are simplicial cones, the subdivision will be called simplicial.*

Recall the lemma.

**Lemma 4** [10] *Let  $\mathfrak{c}$  be a solid acute cone and let  $\mathbb{R}^+\alpha$  be an edge of  $\mathfrak{c}$ . There exists a simplicial subdivision  $S$  of  $\mathfrak{c}$  such that the edges of a cone in  $S$  are among the edges of  $\mathfrak{c}$  and such that  $\mathbb{R}^+\alpha$  is an edge of all the solid cones in  $S$ .*

Let  $S$  be a subdivision of  $\mathfrak{c}$  by acute cones, and let  $\mathfrak{d}_0 \in S$ .

**Lemma 5** *The set  $S(\mathfrak{d}_0) = \{\mathfrak{d} \mid \mathfrak{d}_0 \subset \mathfrak{d}\}$  induces a subdivision of  $\mathfrak{c} + \langle \mathfrak{d}_0 \rangle / \langle \mathfrak{d}_0 \rangle$ .*

**Proof.** We prove this by induction on  $\dim \mathfrak{d}_0$ . We denote by  $S_0$  the subset of  $S$  formed by cones containing  $\mathfrak{d}_0$ . Assume  $\dim \mathfrak{d}_0 = 1$ , so that  $\mathfrak{d}_0$  is isomorphic to  $\mathbb{R}^+$ . We first prove that cones in  $S(\mathfrak{d}_0)$  cover  $\mathfrak{c} + \langle \mathfrak{d}_0 \rangle / \langle \mathfrak{d}_0 \rangle$ . Let

$x \in \mathfrak{c}$  and consider the half line  $H := x + \mathfrak{d}_0$  contained in  $\mathfrak{c}$ . It is covered by the finite set of intervals  $\mathfrak{d} \cap H$ , with  $\mathfrak{d} \in S$ . Certainly one of these intervals is not bounded. Thus there exists  $\mathfrak{d} \in S$  such that  $\mathfrak{d} \cap H$  is stable by translation by  $\mathfrak{d}_0$ . It follows that the half line  $\mathfrak{d}_0$  is contained in  $\mathfrak{d}$ , so that  $\mathfrak{d}$  is in  $S_0$ . Other properties are easily verified.

Now let  $\mathfrak{d}_0$  be of any dimension, and let  $\mathfrak{d}_1 \subset \mathfrak{d}_0$  a face of  $\mathfrak{d}_0$  of dimension less than  $\mathfrak{d}_0$ . Consider the image  $\mathfrak{c}_0$  of the cone  $\mathfrak{c}$  in  $V / \langle \mathfrak{d}_0 \rangle$  and the image  $\mathfrak{c}_1$  of  $\mathfrak{c}$  in  $V / \langle \mathfrak{d}_1 \rangle$ . The cone  $\mathfrak{c}_1$  is covered by elements of  $S_1 := \{\mathfrak{d}, \mathfrak{d}_1 \subset \mathfrak{d}\}$ . In particular  $\mathfrak{d}_0$  gives an element of  $S_1$ . The cone  $\mathfrak{c}_0 = \mathfrak{c} + \langle \mathfrak{d}_0 \rangle / \langle \mathfrak{d}_0 \rangle$  is the image of the cone  $\mathfrak{c}_1$  modulo the element  $\langle \mathfrak{d}_0 \rangle / \langle \mathfrak{d}_1 \rangle$ . Thus  $\mathfrak{c}_0$  is covered by elements of  $S_1$  containing  $\mathfrak{d}_0$ .

### 3 Cones and Rational functions

To a rational affine cone  $\mathfrak{c}$  in  $V$  are associated two meromorphic functions on  $U$ :

$$\theta(\mathfrak{c})(u) = \int_{\mathfrak{c}} e^{-\langle u, v \rangle} dv$$

and

$$\Theta(\mathfrak{c})(u) = \sum_{n \in \mathfrak{c} \cap \mathbb{N}} e^{-\langle u, n \rangle}.$$

In this section, we list some properties of these two functions.

An element  $v$  of  $V$  determines a linear function on  $U$ . Thus the algebra of polynomial functions on  $U$  is identified to the symmetric algebra  $S(V)$  of  $V$  and the field of rational functions on  $U$  is the quotient field of  $S(V)$ .

**Definition 6** *If  $\mathfrak{c}$  is an affine cone in  $V$ , we associate to  $\mathfrak{c}$  the meromorphic function  $\theta(\mathfrak{c})$  on  $U$  such that*

- *If  $\mathfrak{c}$  is not acute, then  $\theta(\mathfrak{c}) = 0$ .*
- *If  $\mathfrak{c}$  is acute, then for any  $u$  in the interior of the dual cone to  $\text{dir}(\mathfrak{c})$ ,*

$$\theta(\mathfrak{c})(-u) = \int_{\mathfrak{c}} e^{-\langle u, v \rangle} dv,$$

*where  $dv$  is the canonical Lebesgue measure on  $\langle \mathfrak{c} \rangle$ .*

For example, if  $\mathbf{c} = \{a\}$ , then  $\theta(\mathbf{c})(u) = e^{\langle a, u \rangle}$ .

The following formula for a simplicial cone follows immediately from the formula  $\int_0^\infty e^{-ux} dx = \frac{1}{u}$ .

**Lemma 7** *If  $\mathbf{c}$  is a simplicial cone generated by independent vectors  $\alpha_1, \dots, \alpha_k$ , then*

$$\theta(\mathbf{c}) = \frac{\text{rvol}(\square(\{\alpha_i\}))}{\prod_{i=1}^k (-\alpha_i)}.$$

We state here some obvious properties of  $\theta(\mathbf{c})$ . These properties enable us in principle to compute  $\theta(\mathbf{c})$  for any affine cone  $\mathbf{c}$ .

**Lemma 8** • *If  $v \in V_{\mathbb{Q}}$ , then*

$$\theta(v + \mathbf{c})(u) = e^{\langle v, u \rangle} \theta(\mathbf{c})(u).$$

• *If  $S$  is a subdivision of a cone  $\mathbf{c}$ , then:*

$$\theta(\mathbf{c}) = \sum_{\mathfrak{d} \in S, \dim(\mathfrak{d}) = \dim(\mathbf{c})} \theta(\mathfrak{d}).$$

The property below although less obvious is well known.

**Lemma 9** *If  $\mathbf{c}$  is an acute cone, and if  $\mathbf{c}^* = \cup_b \gamma_b$  is a subdivision of the dual cone  $\mathbf{c}^*$  of  $\mathbf{c}$ , then*

1) *If  $\mathbf{c}$  is solid:*

$$\theta(\mathbf{c}) = \sum_{b, \dim \gamma_b = d} \theta(\gamma_b^*)$$

2) *If  $\mathbf{c}$  is not solid, then*

$$\sum_{b, \dim \gamma_b = d} \theta(\gamma_b^*) = 0$$

**Example 10** *Consider  $V = \mathbb{R}e_1$  of dimension 1,  $\mathbf{c} = \{0\}$  and the subdivision of  $\mathbf{c}^* = U = \mathbb{R}e^1$  given by  $S := \{\mathbb{R}^+e^1, \mathbb{R}^-e^1, \{0\}\}$ . The dual cones to  $\mathbb{R}^+e^1$  is  $\mathbb{R}^+e_1$ , with  $\theta(\mathbb{R}^+e_1) = \frac{1}{u}$ , while  $\theta(\mathbb{R}^-e_1) = -\frac{1}{u}$ . We have  $0 = \theta(\mathbb{R}^+e_1) + \theta(\mathbb{R}^-e_1)$ .*



Now, we associate to  $\mathfrak{c}$  a meromorphic function  $\Theta(\mathfrak{c})$  on  $U$ , which is the discrete analogue of the function  $\theta(\mathfrak{c})$ .

**Definition 11** *If  $\mathfrak{c}$  is a rational affine cone in  $V$ , define a meromorphic function  $\Theta(\mathfrak{c})$  on  $U$  such that*

- *If  $\mathfrak{c}$  is not acute, then  $\Theta(\mathfrak{c}) = 0$ .*
- *If  $\mathfrak{c}$  is acute, then for  $u$  in the interior of the dual cone to  $\text{dir}(\mathfrak{c})$ , then*

$$\Theta(\mathfrak{c})(-u) = \sum_{n \in \mathfrak{c} \cap N} e^{-\langle u, n \rangle}.$$

If  $\mathfrak{c} := \{a\}$ , then we have two cases. If  $a$  is an integer, then  $\Theta(\mathfrak{c})(u) = e^{\langle a, u \rangle}$  while if  $a$  is not an integer,  $\Theta(\mathfrak{c}) = 0$ .

The following formula for affine simplicial cones is a simple lemma, with proof given for example in [5].

Let  $\mathfrak{c}$  be a solid simplicial affine cone with vertex  $a$ . Choose a basis  $\alpha_i$  of  $V_{\mathbb{Q}}$  such that  $\text{dir}(\mathfrak{c}) = \sum_{i=1}^d \mathbb{R}^+ \alpha_i$ . We denote by  $N(\mathfrak{c})$  the lattice generated by the  $\alpha_i$ . By eventually dividing  $\alpha_i$  by some integers  $q_i$ , we may assume that  $a \in N(\mathfrak{c})$  and that  $N \subset N(\mathfrak{c})$ . Such a lattice  $N(\mathfrak{c})$  will be called adapted to  $\mathfrak{c}$ . The dual lattice  $M(\mathfrak{c})$  to  $N(\mathfrak{c})$  consists of the elements  $m \in M$  such that  $\langle \alpha_i, m \rangle \in \mathbb{Z}$  for all  $i = 1, \dots, d$ . Consider the finite set  $F := M/M(\mathfrak{c})$ . If  $g \in F$ , we denote by  $G$  a representative of  $g$  in  $M$ . Then we have the following formula:

**Lemma 12** *Let  $\mathfrak{c}$  be a solid simplicial affine cone and  $N(\mathfrak{c})$  a lattice adapted to  $\mathfrak{c}$ . Then*

$$\Theta(\mathfrak{c}) = \frac{1}{|F|} \sum_{g \in F} \frac{e^{\langle a, 2\pi\sqrt{-1}G \rangle} e^a}{\prod_{i=1}^d (1 - e^{\alpha_i \langle \alpha_i, 2\pi\sqrt{-1}G \rangle}}.$$

We state some obvious properties of  $\Theta(\mathfrak{c})$ :

**Lemma 13** • *If  $v \in N$  is an integral point of  $V$ , then*

$$\Theta(v + \mathfrak{c})(u) = e^{\langle v, u \rangle} \Theta(\mathfrak{c})(u).$$

- We have the inclusion-exclusion formula: If  $\mathfrak{c} = \cup_{i \in I} \mathfrak{c}_i$ , then  $\Theta(\mathfrak{c}) = \sum_{J \subset I} (-1)^{|J|+1} \Theta(\mathfrak{c}_J)$  with  $\mathfrak{c}_J = \cap_{i \in J} \mathfrak{c}_i$ .

Similarly to Lemma 9, we have the following property for dual cones decompositions:

**Lemma 14** *If  $\mathfrak{c}$  is an acute cone, and if  $\mathfrak{c}^* = \cup_b \gamma_b$  is a subdivision of the dual cone to  $\mathfrak{c}$ , then for any  $v \in V_{\mathbb{Q}}$ ,*

$$\Theta(v + \mathfrak{c}) = \sum_{b, \dim \gamma_b = d} \Theta(v + \gamma_b^*).$$

**Example 15** *Return to the Example 10. Let  $a \in \mathbb{Q}$  and  $v = ae_1$ . If  $a = 0$ , we need to prove  $\Theta(\{0\}) = \Theta(\mathbb{R}^+e_1) + \Theta(\mathbb{R}^-e_1)$ , which is the identity:*

$$1 = \frac{1}{1 - e^u} + \frac{1}{1 - e^{-u}}.$$

*For  $0 < a < 1$ , then  $\Theta(\{ae_1\}) = 0$  and we need to prove that  $\Theta(ae_1 + \mathbb{R}^+e_1) + \Theta(ae_1 + \mathbb{R}^-e_1) = 0$ . We have  $\Theta(ae_1 + \mathbb{R}^+e_1) = \frac{e^u}{1 - e^u}$  while  $\Theta(ae_1 + \mathbb{R}^-e_1) = \frac{1}{1 - e^{-u}}$  and the sum of the two is equal to 0.*

For a cone  $\mathfrak{c}$  with  $K$  edges, let us choose  $\Delta := \{\alpha_i | i = 1, \dots, K\}$  a set of non zero vectors of  $V_{\mathbb{Q}}$ , one on each edge of  $\mathfrak{c}$ . We denote by  $R_{\Delta}$  the set of rational functions on  $U$  of the form  $\Phi = \frac{P}{\prod_{i=1}^K \alpha_i^{n_i}}$  where  $P \in S(V)$  is a polynomial function on  $U$  and  $n_i$  are non negative integers. If  $n_i \leq 1$ , we say that  $\Phi$  has at most a simple pole on  $\alpha_i = 0$ .

We denote by  $\hat{S}(V)$  the ring of formal power series on  $U$  and by  $\mathcal{A}(U)$  the ring of germs of analytic functions on  $U$  at 0. Such a germ  $\Phi$  gives rise to an element of  $\hat{S}(V)$  by its Taylor expansion at 0. We say that  $\Phi$  has rational coefficients, if  $\Phi$  belongs to  $\hat{S}(V_{\mathbb{Q}})$ . We often identify a germ of analytic function on  $U$  and the element of  $\hat{S}(V)$  given by its (convergent) Taylor expansion. Denote by  $\hat{R}_{\Delta}$  the ring of functions  $\Phi = \frac{P}{\prod_i \alpha_i^{n_i}}$  where  $P \in \hat{S}(V)$ . If  $n_i \leq 1$ , we say that  $\Phi$  has at most a simple pole on  $\alpha_i = 0$ .

**Lemma 16** *Let  $\mathfrak{c}$  be an affine acute cone and let  $\Delta := \{\mathbb{R}^+\alpha_i | i = 1, \dots, K\}$  be the set of edges of the cone  $\text{dir}(\mathfrak{c})$  (with  $\alpha_i \in V_{\mathbb{Q}}$ ). Let  $a$  be the vertex of  $\mathfrak{c}$ . Then*

- The function  $e^{-\langle a, u \rangle} \theta(\mathbf{c})$  is in  $R_\Delta$ , and this function has at most a simple pole on  $\alpha = 0$  for each edge  $\mathbb{R}^+ \alpha$  of  $\text{dir}(\mathbf{c})$ .
- The function  $e^{-\langle a, u \rangle} \Theta(\mathbf{c})$  is in  $\hat{R}_\Delta$ , and this function has at most a simple pole on  $\alpha = 0$  for each edge  $\mathbb{R}^+ \alpha$  of  $\text{dir}(\mathbf{c})$ .

Furthermore functions  $(\prod_{i=1}^K \alpha_i) \theta(\mathbf{c})$  and  $(\prod_{i=1}^K \alpha_i) \Theta(\mathbf{c})$  are analytic functions with rational coefficients.

**Proof.** We just prove the formula stated in the lemma for the function  $\Theta(\mathbf{c})$ . Using a simplicial decomposition of  $\text{dir}(\mathbf{c})$  without adding new edges (as the one given in Lemma 4), and the inclusion-exclusion formula, it is sufficient to prove the lemma for a simplicial affine cone. We use expansions

$$\frac{1}{1 - e^{-x}} = \frac{1}{x} \left( \frac{x}{1 - e^{-x}} \right) = \frac{1}{x} + \frac{1}{2} + \dots$$

and if  $g \neq 1$  the expansion

$$\frac{1}{1 - ge^{-x}} = \frac{1}{(1-g)} \frac{1}{1 + \frac{g(1-e^{-x})}{(1-g)}} = \frac{1}{(1-g)} \sum_{k=0}^{\infty} \left( -\frac{g(1-e^{-x})}{(1-g)} \right)^k.$$

We use Lemma 12. If  $\mathbf{c}$  is simplicial, the edges of  $\text{dir}(\mathbf{c})$  are the half-lines  $\mathbb{R}^+ \alpha_i$ . Then if  $g \in F$  is such that  $e^{\langle \alpha_i, 2\pi\sqrt{-1}G \rangle} = 1$ , the term

$$\frac{e^{\langle a, 2\pi\sqrt{-1}G \rangle} e^a}{\prod_{j=1}^d (1 - e^{\alpha_j} e^{\langle \alpha_j, 2\pi\sqrt{-1}G \rangle})}$$

has a simple pole at  $\alpha_i = 0$ . If  $g \in F$  is such that  $e^{2\pi\sqrt{-1}\langle \alpha_i, G \rangle} \neq 1$ , the term

$$\frac{e^{\langle a, 2\pi\sqrt{-1}G \rangle} e^a}{\prod_{j=1}^d (1 - e^{\alpha_j} e^{\langle \alpha_j, 2\pi\sqrt{-1}G \rangle})}$$

has no pole at  $\alpha_i = 0$ . Thus the full sum over  $g \in F$  has at most a simple pole at  $\alpha_i = 0$ .

## 4 The main theorem

Our aim is to associate in an "universal" way to any affine cone  $\mathfrak{c}$  in  $V$  an analytic function  $\mu(\mathfrak{c})$  on  $U$  (defined in a neighborhood of 0) such that for any affine acute cone  $\mathfrak{c}$  in  $V$

$$\Theta(\mathfrak{c}) = \sum_{F \in \mathcal{F}(\mathfrak{c})} \mu(\text{cone}(\mathfrak{c}, F))\theta(F).$$

Let  $a$  be the vertex of  $\mathfrak{c}$ . For  $F := \{a\}$ , the tangent cone at  $a$  is just equal to  $\mathfrak{c}$ . Thus the formula above reads

$$\Theta(\mathfrak{c})(u) = e^{\langle a, u \rangle} \mu(\mathfrak{c})(u) + \sum_{F \in \mathcal{F}(\mathfrak{c}), \dim(F) > 0} \mu(\text{cone}(\mathfrak{c}, F))(u)\theta(F)(u),$$

and we will find a formula for  $\mu$  so that  $\mu(\text{cone}(\mathfrak{c}, F))$  depends only on the normal cone to  $\mathfrak{c}$  at  $F$ , which is of dimension equal to the codimension of  $F$  in  $\mathfrak{c}$ .

We need an additional data: let us choose a scalar product  $s$  on  $V$ . We still denote by  $s$  the scalar product induced by  $s$  on  $U$ , on subspaces and quotient spaces of  $V, U$ , etc. The scalar product  $s$  allows us to identify polynomial functions on a subspace  $Y$  of  $U$  with a subspace of analytic functions on  $U$  via the orthogonal projection  $\text{proj}^* \phi(u) = \phi(\text{proj}_Y u)$ , where  $\text{proj}_Y$  is the orthogonal (with respect to  $s$ ) projection from  $U$  to  $Y$ .

**Theorem 17** *Let  $V$  be a rational space of dimension  $d$  and  $U$  its dual vector space. Let  $\mathcal{CS}(V)$  be the set of acute solid rational affine cones on  $V$  and let  $s$  be a scalar product on  $V$ . There exists a unique function*

$$\mu := \mu(V, s) : \mathcal{CS}(V) \mapsto \mathcal{A}(U)$$

*such that: For every acute rational solid affine cone  $\mathfrak{c}$  in  $U$ , we have:*

$$\Theta(\mathfrak{c}) = \sum_{F \in \mathcal{F}(\mathfrak{c})} \mu(\text{No}(\mathfrak{c}, F))\theta(F).$$

**Remark 18** *The normal cone  $\text{No}(\mathfrak{c}, F)$  is a rational solid acute cone in  $V/\text{lin}(F)$ , a rational vector space with the scalar product deduced from  $s$ . Thus the function  $\mu(\text{No}(\mathfrak{c}, F))$  is an analytic function on  $\text{lin}(F)^\perp \subset U$ . In the formula above, this function is identified to an analytic function on  $U$  by orthogonal projection.*

Clearly, if the formula exists, it is unique. So, before going to the proof of the existence of  $\mu$ , we note some properties of  $\mu$  easily deduced from the uniqueness of  $\mu$ .

**Proposition 19** *For any  $v \in N$ , we have  $\mu(v + \mathfrak{c}) = \mu(\mathfrak{c})$ . In particular if the vertex of  $\mathfrak{c}$  is integral, the function  $\mu(\mathfrak{c})$  is equal to  $\mu(\text{dir}(\mathfrak{c}))$ .*

*If  $s$  is a rational scalar product, then  $\mu(\mathfrak{c})$  has rational coefficients. In particular  $\mu(\mathfrak{c})(0)$  is a rational number.*

*The expansion at a fixed order  $o$  of the function  $\mu(\mathfrak{c})$  can be computed in polynomial time, when  $\dim V$  is fixed.*

Indeed, from Barvinok's fundamental signed decomposition formula [1], the meromorphic functions  $\Theta(\mathfrak{c})$  and  $\theta(F)$  can be computed in polynomial time.

**Definition 20** *The constant  $\mu(\mathfrak{c})(0)$  will be called the normalized constant term of  $\Theta(\mathfrak{c})$ . This normalization depends of the choice of a scalar product.*

Let us give the formula for  $\mu$  in dimension 1. Let  $a \in \mathbb{Q}$ , and assume  $-1 < a \leq 0$ . Then

$$\mu(a + \mathbb{R}^+)(u) = \frac{e^{-au}}{1 - e^u} + \frac{1}{u}.$$

Let us start the proof of Theorem 17. **Proof.** Let  $a$  be the vertex of  $\mathfrak{c}$ . For  $F := \{a\}$ , the normal cone at  $a$  is just equal to  $\mathfrak{c}$ . Thus the formula of the theorem reads

$$\Theta(\mathfrak{c})(u) = e^{\langle a, u \rangle} \mu(\mathfrak{c})(u) + \sum_{F \in \mathcal{F}(\mathfrak{c}), \dim(F) > 0} \mu(\text{No}(\mathfrak{c}, F))(u) \theta(F)(u).$$

If  $\dim F > 0$ , the cone  $\text{No}(\mathfrak{c}, F)$  is an affine cone in the space  $V/\text{lin}(F)$ , which is of dimension strictly less than  $V$ . Thus the formula above determines uniquely  $\mu(\mathfrak{c})$  by induction. It remains to see that  $\mu(\mathfrak{c})(u)$  is analytic.

Let  $\Delta$  be the set of edges of  $\text{dir}(\mathfrak{c})$ . The formula above determines by induction a function  $\mu(\mathfrak{c})$  in  $\hat{R}_\Delta$ . Furthermore, for each edge  $\mathbb{R}^+\alpha$  of  $\text{dir}(\mathfrak{c})$ , the function  $\mu(\mathfrak{c})$  has at most a simple pole at  $\alpha = 0$ . The partial residue map  $\text{Res}_{\alpha=0} \Phi$  associates to a function  $\Phi$  with at most a simple pole at  $\alpha = 0$  the rational function  $(\text{Simplify}(\alpha\Phi))|_{\alpha=0}$ . A function has no pole at  $\alpha = 0$  if and only if  $\text{Res}_\alpha \Phi = 0$ . Thus we will prove our theorem by proving that  $\text{Res}_{\alpha=0} \mu(\mathfrak{c}) = 0$  for all edges  $\mathbb{R}^+\alpha$  of  $\text{dir}(\mathfrak{c})$ .

Let us fix an edge  $\mathbb{R}^+\alpha$ . Let  $U_0 = \{\alpha = 0\}$  and  $M_0 = M \cap U_0$ . We need to prove that for  $u_0 \in U_0$ , then

$$(\text{Res}_{\alpha=0} \Theta(\mathbf{c}))(u_0) = \sum_{F \in \mathcal{F}(\mathbf{c}), \dim F > 0} \mu(\text{No}(\mathbf{c}, F))(\text{proj}(u_0))(\text{Res}_{\alpha=0} \theta(F))(u_0).$$

Thanks to Lemma 16, in the above sum, we can restrict ourselves to the set of elements  $F \in \mathcal{F}(\mathbf{c})$  having  $a + \mathbb{R}^+\alpha$  as edge. For such a face, the space  $\text{lin}(F)^\perp$  is a subset of  $U_0$ . Consider the projection  $h(\mathbf{c})$  of  $\mathbf{c}$  on the rational vector space  $V/\mathbb{R}\alpha$ . The set of faces of  $h(\mathbf{c})$  is in bijective correspondence with the set of faces of  $\mathbf{c}$  having  $a + \mathbb{R}^+\alpha$  as edge. Our formula thus follows by induction on the dimension of  $V$  from the following result:

**Lemma 21** *Let  $\mathbf{c}$  be a solid acute affine cone in  $V$ . Let  $\alpha$  be an edge of  $\text{dir}(\mathbf{c})$  and let  $h : V \rightarrow V/\mathbb{R}\alpha$  be the projection on the rational space  $V/\mathbb{R}\alpha$ . Let  $m \in U$  be the generator of the group  $M/U_0 \cap M$ , such that  $\langle \alpha, m \rangle > 0$ . Then*

$$\text{Res}_{\alpha=0} \theta(\mathbf{c}) = \langle \alpha, m \rangle \theta(h(\mathbf{c})),$$

$$\text{Res}_{\alpha=0} \Theta(\mathbf{c}) = \langle \alpha, m \rangle \Theta(h(\mathbf{c})).$$

**Proof.** We prove only the second formula. We first prove it for an affine simplicial cone  $\mathbf{c}$ . We choose a basis  $m_i$  of  $U$  consisting of elements of  $M$  such that the dual cone  $\text{dir}(\mathbf{c})^*$  to  $\text{dir}(\mathbf{c})$  is generated by the elements  $m_i$ . Let  $\alpha_i$  be the dual basis of  $m_i$ . Then  $\text{dir}(\mathbf{c}) = \sum_{i=1}^d \mathbb{R}^+\alpha_i$ , and we number the vectors  $\alpha_i$  so that  $\mathbb{R}^+\alpha = \mathbb{R}^+\alpha_d$ . We denote by  $a_i$  the image of the elements  $\alpha_i$  ( $i = 1, \dots, d-1$ ) in  $V/\mathbb{R}\alpha$ . The dual lattice to  $N(\mathbf{c}) = \sum_{i=1}^d \mathbb{Z}\alpha_i$  is the lattice  $M(\mathbf{c}) = \sum_{i=1}^d \mathbb{Z}m_i \subset M$ . Choosing  $m_i$  sufficiently large such that  $\langle a, m_i \rangle$  are integers, we may assume that  $a \in N(\mathbf{c})$ , so that  $N(\mathbf{c})$  is adapted to  $\mathbf{c}$ . The image of the lattice  $N(\mathbf{c})$  in  $V/\mathbb{R}\alpha$  is an adapted lattice for  $h(\mathbf{c})$  with dual lattice  $M(\mathbf{c}) \cap M_0$ . In the formula for  $\Theta(\mathbf{c})$  only elements  $G \in F$  such that  $\langle \alpha, G \rangle \in \mathbb{Z}$  contribute to the residue. For such a  $G$ ,

$$\text{Res}_{\alpha=0} \left( \frac{e^{\langle a, 2\pi\sqrt{-1}G \rangle} e^a}{\prod_{i=1}^d (1 - e^{\alpha_i} e^{\langle \alpha_i, 2\pi\sqrt{-1}G \rangle})} \right) = \frac{e^{\langle a, 2\pi\sqrt{-1}G \rangle} e^{h(a)}}{\prod_{i=1}^{d-1} (1 - e^{a_i} e^{\langle \alpha_i, 2\pi\sqrt{-1}G \rangle})}.$$

Write  $G = \sum_{i=1}^d g_i m_i$  with  $g_i \in \mathbb{Q}$ . Then  $\langle \alpha, G \rangle = g_d$  is in  $\mathbb{Z}$ . Let  $G_0 = G - g_d m_d$ . The element  $G_0$  is in  $M_0 = M \cap U_0$ . As  $a, \alpha_i$  are in the

dual lattice to  $M(\mathbf{c})$ , we may replace  $G$  by  $G_0$  in the formula just above. Furthermore, the map  $G \mapsto G_0$  provides an isomorphism between the set of elements  $g \in F$  with  $\langle \alpha, G \rangle \in \mathbb{Z}$  and the set  $F_0 = M_0/M_0 \cap M(\mathbf{c})$ . To prove the formula of the Lemma, it remains to see that  $\frac{1}{|F|} = \frac{1}{|F_0|} \langle \alpha, m \rangle$ .

We can compute  $|F_0|, |F|$  as follows. The lattice  $M(\mathbf{c})$  is generated by  $m_1, m_2, \dots, m_d$ . Thus,  $|F| = \text{vol}(\square(\{m_i\}_{i=1}^d))$ . The lattice  $M_0(\mathbf{c})$  is generated by  $m_1, m_2, \dots, m_{d-1}$  so that  $|F_0| = \text{rvol}(\square(\{m_i\}_{i=1}^{d-1}))$ . Consider  $m \in M$  such that  $M = M_0 \oplus \mathbb{Z}m$  and  $\langle \alpha, m \rangle > 0$ . We write  $m_d = m_0 + xm$  with  $x \in \mathbb{Z}$  and  $m_0 \in M_0$ . Then  $|F| = |x||F_0|$ . As  $\langle \alpha, m_d \rangle = 1$ , we have  $x \langle \alpha, m \rangle = 1$ . This proves the lemma for simplicial affine cones.

Now, let  $\mathbf{c}$  be any cone and  $\mathbb{R}^+\alpha$  an edge of  $\text{dir}(\mathbf{c})$ . We choose a simplicial decomposition of  $\text{dir}(\mathbf{c})$ , where all solid cones contain  $\mathbb{R}^+\alpha$  as edge as in Lemma 4. Let  $S_0$  be the set of cones in the simplicial decomposition  $S$  containing  $\mathbb{R}^+\alpha$  as an edge, and  $S_1$  the complement. Each cone  $\mathfrak{d}$  in  $S_0$  is sent by the projection  $h : V \rightarrow V/\mathbb{R}\alpha$  to a cone  $h(\mathfrak{d})$  of dimension one less. The set of cones  $Q_0 = \{h(\mathfrak{d}) | \mathfrak{d} \in S_0\}$  form a simplicial decomposition of  $h(\text{dir}(\mathbf{c}))$ . We use the inclusion-exclusion formula to compute  $\Theta(\mathbf{c}) = \Theta(a + \text{dir}(\mathbf{c}))$  in functions of  $\Theta(a + \mathfrak{d})$  for  $\mathfrak{d}$  in  $S$ . When taking the residue in  $\alpha = 0$ , only cones in  $S_0$  contribute. Using the inclusion exclusion formula for the simplicial decomposition  $h(\mathbf{c})$ , indexed by  $S_0$ , we obtain our theorem.

We extend the definition of the function  $\mu$  to all affine cones of  $V$ .

**Definition 22** a) If  $\mathbf{c}$  is not acute, we define  $\mu(\mathbf{c}) = 0$ .

b) If  $\mathbf{c}$  is acute, we define  $\mu(\mathbf{c}) = 0$ , if  $\langle \mathbf{c} \rangle \cap N = \emptyset$ . Otherwise, we choose  $n$  such that  $\mathbf{c} - n$  is a solid acute affine cone in  $\text{lin}(\mathbf{c})$  and define  $\mu(\mathbf{c}) = \mu(\mathbf{c} - n)$ .

Clearly if  $\mathbf{c}$  is such that  $\langle \mathbf{c} \rangle \cap N = 0$ , the function  $\Theta(\mathbf{c})$  is equal to 0. Thus with this definition, for every affine cone  $\mathbf{c}$  in  $V$ , we still have:

$$\Theta(\mathbf{c}) = \sum_{F \in \mathcal{F}(\mathbf{c})} \mu(\text{No}(\mathbf{c}, F)) \theta(F).$$

There is a duality between rational cones in  $V$  and in  $U$  by taking dual cones. Consider the dual cone  $\sigma^*$  of a cone  $\sigma$  in  $U$ . Its projection on  $V/\sigma^\perp$  is always an acute cone. The function  $\mu(\sigma^*/\sigma^\perp)$  is thus an analytic function on the vector space  $\langle \sigma \rangle$ . If  $a \in V_\mathbb{Q}$ , we denote sometimes by  $a + \sigma^*/\sigma^\perp$  the affine cone  $(a + \sigma^*)/\sigma^\perp$  obtained from  $\sigma^*/\sigma^\perp$  by translation by the image of  $a$  in  $V/\sigma^\perp$ .

**Definition 23** For  $\sigma$  a cone in  $U$ , and  $a \in V_{\mathbb{Q}}$ , we denote by  $\mu_a^*(\sigma)$  the analytic function on  $\langle \sigma \rangle$  defined by

$$\mu_a^*(\sigma) = \mu(a + \sigma^*/\sigma^\perp).$$

If  $a = 0$ , we write simply  $\mu^*(\sigma)$  for  $\mu(\sigma^*/\sigma^\perp)$ .

A fundamental property of  $\mu_a^*(\sigma)$  is the following.

**Theorem 24** Let  $\sigma = \cup_{s \in S} \sigma_s$  be a subdivision of  $\sigma$ . Introduce  $S' := \{g \in S \mid \dim(\sigma_g) = \dim(\sigma)\}$ . Then, we have:

$$\mu_a^*(\sigma) = \sum_{g \in S'} \mu_a^*(\sigma_g).$$

**Proof.** Obviously, we can assume  $\sigma$  solid. Using Example 15, this is true for  $\sigma = U$  subdivided in quadrants. Then using eventually a refinement of the subdivision, we may assume  $\sigma$  solid and acute.

We denote by  $\mathbf{c} = \sigma^*$  the dual cone to  $\sigma$  and by  $\mathbf{c}_g$  the dual to  $\sigma_g$ , with  $g \in S'$ . Thus we obtain

$$\Theta(a + \mathbf{c}) = \sum_g \Theta(a + \mathbf{c}_g)$$

and

$$e^{-a} \Theta(a + \mathbf{c}_g) = \sum_{F \in \mathcal{F}(\mathbf{c}_g)} \mu(a + \text{No}(\mathbf{c}, F)) \theta(F).$$

Let us use the duality between faces of  $\mathbf{c}_g$  and faces of  $\sigma_g$  (which are elements of  $S$ ). Let  $\mathcal{L}$  be the collection of linear spaces spanned by elements of  $S$ . For given  $L \in \mathcal{L}$ , we denote by  $S'_L$  the subset of elements  $\delta$  of  $S$ , such that  $\langle \delta \rangle = L$ . Thus we may rewrite the right hand side (RHS) of the formula above as

$$RHS = \sum_{L \in \mathcal{L}} \sum_{\delta \in S'_L} \mu_a^*(\delta) \sum_{\{g \mid \delta \in \mathcal{F}(\sigma_g)\}} \theta(L^\perp \cap \mathbf{c}_g).$$

Fix  $L \in \mathcal{L}$ ,  $\delta \in S'_L$ . Consider the projection  $p_L : U \rightarrow U/L$ . Then  $p_L(\sigma) = \cup_{\{s \mid \delta \subset \sigma_s\}} p_L(\sigma_s)$  is a subdivision of  $p_L(\sigma)$ . The solid cones of this subdivision are the cones  $p_L(\sigma_g)$ , with  $\delta \in \mathcal{F}(\sigma_g)$ . If  $L$  is not spanned by a face of  $\sigma$ , the cone  $p_L(\sigma)$  is not acute in  $U/L$ . Equivalently,  $L^\perp \cap \mathbf{c}$  is not a



solid cone in  $L^\perp$ . It follows from Lemma 9 that  $\sum_{\{g|\delta\subset\sigma_g\}}\theta(L^\perp\cap\mathfrak{c}_g)=0$ . It remains to sum over the subspaces  $L$  attached to faces of  $\sigma$ . Then we obtain by the same lemma that  $\sum_{\{g,\delta\subset\sigma_g\}}\theta(L^\perp\cap\mathfrak{c}_g)=\theta(L^\perp\cap\mathfrak{c})$ .

Reindexing by faces  $F=L^\perp\cap\mathfrak{c}$  of  $\mathfrak{c}$ , the formula becomes:

$$RHS = \sum_{F\in\mathcal{F}(\mathfrak{c})} \left( \sum_{\{\delta\in S, \langle\delta\rangle=F^\perp\}} \mu_a^*(\delta) \right) \theta(F).$$

For a face  $F$  of  $\mathfrak{c}$ , the set of elements  $\delta\in S$ , with  $\delta\subset F^\perp$  form a subdivision of the face  $\sigma\cap F^\perp$ . Using induction argument on the dimension of  $F$ , we then have that

$$\sum_{\{\delta\in S, \langle\delta\rangle=F^\perp\}} \mu_a^*(\delta) = \mu_a^*(\sigma\cap F^\perp) = \mu(\text{No}(a+\mathfrak{c}, a+F))$$

for any  $F$  of positive dimension.

Finally we obtain

$$\Theta(a+\mathfrak{c}) = \left( \sum_{g\in S'} \mu_a^*(\sigma_g) \right) \theta(a+\mathfrak{c}) + \sum_{\{F|\dim F>0\}} \mu(\text{No}(a+\mathfrak{c}, a+F)) \theta(a+F).$$

Comparing with the definition of  $\mu(a+\mathfrak{c}) = \mu_a^*(\sigma)$ , we obtain the desired equality.

**Corollary 25** *Let  $\text{Ev} := \{\sigma_s\}$  be a complete fan in  $U$ . Then for any  $a\in V_{\mathbb{Q}}$ , we have:*

*If  $a\in N$ , then*

$$\sum_{s\in\text{Ev}|\sigma_s\text{solid}} \mu_a^*(\sigma_s) = 1$$

*If  $a\notin N$ , then*

$$\sum_{s\in\text{Ev}|\sigma_s\text{solid}} \mu_a^*(\sigma_s) = 0.$$

In a companion paper, we will give the proof of the following theorem.

**Theorem 26** *Let  $X$  be a toric variety associated to a fan  $\text{Ev}$ . Then the equivariant Todd class of  $X$  is equal to*

$$\sum_{\sigma\in\text{Ev}} \mu^*(\sigma)[O_\sigma]$$

*in the equivariant homology ring of  $X$ .*

## 5 Euler-Maclaurin formula

As before, we consider our vector space  $V$  with lattice  $N$  and scalar product  $s$ . We may identify elements of  $S(V)$  to polynomial functions on  $U$  or to constant coefficients differential operators on  $V$ . Using the scalar product, if  $W$  is a subspace of  $V$ , we identify  $W^\perp$  to a subspace of  $V$  and we identify  $S(V/W)$  to  $S(W^\perp) \subset S(V)$ , and thus to constant coefficients differential operators on  $V$  involving only transversal derivatives to  $W$ .

Recall that  $\mu$  associates to an affine cone  $\mathfrak{c}$  in a rational vector space  $V$  (with scalar product) an element of  $\hat{S}(V)$ . Thus the following definition makes sense.

**Definition 27** *Let  $\mathfrak{d}$  be an acute affine cone in  $V/W$ . We denote by  $D(\mathfrak{d})$  the constant coefficient operator on  $V$  associated to  $\mu(\mathfrak{d})$ .*

By definition of the correspondence between differential operators on  $V$  and polynomial functions on  $U$ , we have for any  $v \in V$ ,

$$(1) \quad (D(\mathfrak{d})e^u)(v) = \mu(\mathfrak{d})(u)e^{\langle v, u \rangle}.$$

Let  $P$  be a convex rational polytope in  $V$  and denote by  $\mathcal{F}(P)$  the set of faces of  $P$ . If  $F$  is a face of  $P$ , we denote by  $\text{lin}(F)$  the subspace of  $V$  parallel to  $\langle F \rangle$ . The cone  $\text{No}(P, F)$  is an affine acute cone in  $V/\text{lin}(F)$ . To see it "concretely", we take a generic point  $p$  in  $F$ , the affine space  $G$  perpendicular to the face  $F$  through  $p$ , and consider the cut  $P \cap G$  of  $P$  by  $G$ . The neighborhood of  $p$  in  $P \cap G$  is isomorphic to the normal cone at  $F$  near its vertex. Its integral structure is obtained by projection of  $N$  on  $G$  and not by intersection.

**Definition 28** *If  $F$  is a face of  $P$ , we denote by  $D(F)$  the differential operator  $D(\text{No}(P, F))$ . We denote by  $\text{pot}(F)$  the constant  $D(\text{No}(P, F)) \cdot 1$ .*

By construction, the operator  $D(F)$  depends only of the class modulo translations by  $N$  of the cone  $\text{No}(P, F)$ . In this sense, the operator  $D(F)$  is **local** as well as the constant  $\text{pot}(F)$ . In particular, if  $P$  is a polytope with integral vertices, the operator  $D(F)$  depends only of the cone of normal feasible directions at a generic point of  $F$ .

We are now ready to state the local Euler-Maclaurin formula for any rational polytope.

**Theorem 29** (*Local Euler-Maclaurin formula*). *Let  $P$  be a convex rational polytope in  $V$ . Then for any polynomial function  $\phi$ , we have*

$$\sum_{n \in P \cap N} \phi(n) = \sum_{F \in \mathcal{F}(P)} \int_F (D(F) \cdot \phi).$$

Remark: the integrals over  $F$  are taken with respect to the canonical Lebesgue measure on  $F$  determined by the lattice  $N \cap \text{lin}(F)$ . The term corresponding to  $F$  is local: we use only a neighborhood of the face  $F$  in  $P$ . More precisely, the operator  $D(F)$  is only depending of the local cut of  $P$  by a normal space to  $F$  at a generic point of  $F$ . Then we need to integrate over  $F$  a polynomial function depending of the restriction of  $\phi$  on  $F$  and of a certain number of normal derivatives.

**Proof.** It is enough to prove this formula for an exponential function  $\phi = e^u$  with a small  $u$ . Consider the set Vertex of vertices  $v$  of  $P$ . Recall Brion's formulae ([4], see also [5]):

$$\sum_{n \in P \cap N} e^{\langle n, u \rangle} = \sum_{v \in \text{Vertex}} \Theta(\text{No}(P, \{v\}))(u),$$

$$\int_F e^{\langle u, v \rangle} = \sum_{v \in \text{Vertex} \cap F} \theta(\text{No}(F, \{v\}))(u).$$

Thus for  $\phi = e^u$ , computing both sides of the formula of the theorem in function of vertices, we need to prove the equality:

$$\sum_{v \in \text{Vertex}} \Theta(\text{No}(P, \{v\}))(u)$$

$$= \sum_F \sum_{v \in F} \mu(\text{No}(P, F))(u) \theta(\text{No}(F, \{v\}))(u).$$

Fix a vertex  $v$ . The set of faces of  $P$  containing  $v$  is in bijection with the set of faces of  $\text{No}(P, \{v\})$ . We associate to a face  $F$  of  $P$  the affine cone  $\text{No}(F, \{v\})$ . The formula

$$\Theta(\text{No}(P, \{v\}))(u) = \sum_{F, v \in F} \mu(\text{No}(P, F))(u) \theta(\text{No}(F, \{v\}))(u)$$

is the formula of Theorem 17

## 6 Ehrhart polynomial

Let  $P$  be a rational polytope and let  $q$  be an integer such that  $qP$  is with integral vertices.

Consider the dilation  $kP$  of a rational convex polytope by any integer  $k$ . If  $\phi$  is a polynomial function on  $V$  of degree  $r$ , then the function

$$\text{Sum}(\phi, P)(k) = \sum_{n \in kP \cap N} \phi(n)$$

is given by a quasi-polynomial formula for all  $k \geq 0$  (and even in a slightly further range including negative values). Thus there exists periodic functions  $k \mapsto e_i(k)$  of period  $q$  such that

$$\text{Sum}(\phi, P)(k) = \sum_{i=0}^{d+r} e_i(k) k^i.$$

Clearly Euler-Maclaurin formula above give  $\text{Sum}(\phi, P)(k)$  as such an expression. Let us be more specific.

Recall that  $\mu$  is a function on acute affine cones invariant by translation by  $N$ . We need a small definition.

Let  $\mathbf{c}$  be an acute cone with vertex  $a$ . If  $k$  is a non zero integer, we define  $\mu(\mathbf{c}, k) = \mu(k\mathbf{c})$ . If  $k = 0$ , we define  $\mu(\mathbf{c}, 0) = \mu(\text{dir}(\mathbf{c}))$ . Then if  $q$  is an integer such that  $qa$  is integral, the function  $k \mapsto \mu(\mathbf{c}, k)$  is periodic of period  $q$  for all  $k \geq 0$ .

**Definition 30** *If  $P$  is a rational polytope and  $F$  a face, we denote by  $D(F, k)$  the operator  $D(\text{No}(P, F), k)$ . For  $k = 0$ , the operator  $D(\text{No}(P, F), k)$  is the operator associated to the cone of normal feasible directions at  $F$ .*

If  $F$  is of codimension  $h$ , the space perpendicular to  $\text{lin}(F)$  has a rational basis  $w_1, w_2, \dots, w_h$ . The operator  $D(F, k)$  has the following expression

$$D(F, k) = d_0(F, k) + \sum_{A, |A|=1}^{\infty} d_A(F, k) \partial^A$$

where  $A = (a_1, a_2, \dots, a_z)$  is a tuple of non negative integers and  $\partial_A = \partial(w_1)^{a_1} \dots \partial(w_h)^{a_h}$ . The constants  $d_A(F, k)$  are rational and periodic in  $k$  with period the smallest integer  $q_F$  such that  $q < F >$  contains an integral point.

**Definition 31** We denote by  $\text{pot}(F, k) \in \mathbb{Q}$  the constant  $d_0(F, k)$ .

The function  $k \mapsto \text{pot}(F, k)$  is periodic modulo  $q_F$ .

**Proposition 32** Let  $P$  be a rational cone and  $\phi$  be a polynomial function of degree  $r$ . Then for any  $k \geq 0$ , we have

$$\text{Sum}(\phi, P)(k) = \sum_{F \in \mathcal{F}(P)} \int_{kF} D(F, k) \phi.$$

Furthermore, we have

$$\int_{kF} D(F, k) \phi = \sum_{u=\dim F}^{\dim F+r} j_u(k) k^u$$

where the coefficients  $j_u(k)$  are periodic of period  $q_F$ .

**Proof.** For  $k > 0$ , this is just Theorem 29, and obvious estimates on polynomial behavior of integrals.

For  $k = 0$ , as both sides of the equality are quasi-polynomials, they coincide. We may also deduce this property for  $k = 0$  from Corollary 25. Indeed, for  $k = 0$ , the only contribution is from the operators  $D(v, 0)$  associated to vertices and the result of the left hand side is  $\phi(0)$ . We need to prove

$$\sum_{v \in \text{Vertex}} (D(\text{dir}(\text{No}(P, \{v\})))\phi)(0) = \phi(0).$$

Now, the dual cones  $\text{dir}(\text{No}(P, \{v\}))^*$  to the cones  $\text{dir}(\text{No}(P, \{v\}))$  are the solid cones of the complete fan associated to the polytope. By Corollary 25, we obtain that the sum of the operators  $D(\text{dir}(\text{No}(P, \{v\})))$  over vertices  $v$  is identically equal to 1. Thus the formula is indeed true for  $k = 0$ .

If we apply this theorem to the function  $\phi = 1$ , we obtain a formula for the number of integral points in  $P$  as follows:

**Proposition 33** The Ehrhart polynomial of  $P$  is given by:

$$\text{Card}(kP \cap N) = \sum_{F \in \mathcal{F}(P)} \text{pot}(F, k) \text{vol}(F) k^{\dim F}.$$

The rational numbers  $\text{pot}(F, k)$  depends only of the class of the normal cone to  $P$  at  $F$  modulo lattice translations.

Our description of the constant  $\text{pot}(F, k)$  depends of the expansion at order 0 of the function  $\Theta(\text{No}(\mathbb{P}, F))$  (which is meromorphic with a denominator of degree equal to the codimension of  $F$ ).

**Example 34** *If  $v$  is an integral vertex of a polygone  $\mathbb{P}$  in dimension 2,  $\alpha, \beta$  are the two integral edges starting from  $v$ , and if we assume that  $\alpha$  and  $\beta$  form a basis of  $\mathbb{Z}^2$ , then*

$$\text{pot}(v) = \frac{1}{4} + \frac{\langle \alpha, \beta \rangle}{12} \left( \frac{1}{\langle \alpha, \alpha \rangle} + \frac{1}{\langle \beta, \beta \rangle} \right).$$

**Example 35** *In Cappell-Shaneson formula [8] for a polygon*

$$\text{Sum}(\phi, \mathbb{P}) = \sum_F C_F \phi$$

*the differential operators associated to faces are not local operators. The operator associated to a face involves next face, and the operator associated to a vertex involves 3 faces. For example, for the square  $S$  with vertices  $v_0 = [0, 0], v_1 = [1, 0], v_2 = [1, 1], v_3 = [0, 1]$ , Cappell-Shaneson operators are  $C_{v_1, S} = \frac{1}{4} + \dots$ , while for the simplex  $B$  with vertices  $v_0, v_1, v_2, v'_3 = [0, 2]$ ,  $C_{v_1, B} = \frac{1}{6} + \dots$  although in both polytopes  $S$  and  $B$  the normal cones at  $v_1$  are the same. On the other hand Cappell-Shaneson operators differentiates along edges, while we introduce via an arbitrary scalar product normal directions to faces.*

It is very likely that our explicit formula for  $\mu(F)$  gives an alternate proof of the following theorem of Barvinok [3].

**Theorem 36** *(Barvinok) Fix an integer  $i$ . There exists a polynomial time algorithm which, given an integer  $d \geq i$ , a rational simplex  $\mathbb{P} \subset \mathbb{R}^d$  and an integer  $k$ , computes the value of the coefficient  $eh_{d-i}(\mathbb{P})(k)$ .*

Indeed  $eh_{d-i}(\mathbb{P})$  involves only computations of generating functions of cones of dimensions  $i$  and of volumes of the faces of  $\mathbb{P}$ . We will come back to this theme in the near future.

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