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# Local Euler-Maclaurin formula for polytopes

Nicole Berline and Michèle Vergne

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## 1 Introduction

By the name Euler-Maclaurin, one refers to formulas which relate discrete sums to integrals, in particular in the following framework. Let  $\mathbf{p}$  be a rational convex polytope in  $\mathbb{R}^d$  and let  $h(x)$  be a polynomial function on  $\mathbb{R}^d$ . The sum of the values  $h(x)$  at integral points of  $\mathbf{p}$  is written as a sum of terms indexed by the set  $\mathcal{F}(\mathbf{p})$  of faces of  $\mathbf{p}$ ,

$$(1) \quad \sum_{x \in \mathbf{p} \cap \mathbb{Z}^d} h(x) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \int_{\mathbf{f}} D(\mathbf{p}, \mathbf{f}) \cdot h$$

where, for each face  $\mathbf{f}$  of  $\mathbf{p}$ ,  $D(\mathbf{p}, \mathbf{f})$  is a differential operator (of infinite order) with constant coefficients on  $\mathbb{R}^d$ . The basic example is the historical Euler-Maclaurin summation formula in dimension 1: for  $a_1 \leq a_2 \in \mathbb{Z}$ ,

$$\sum_{a_1}^{a_2} h(x) = \int_{a_1}^{a_2} h(t) dt - \sum_{n \geq 1} \frac{b(n)}{n!} h^{(n-1)}(a_1) + \sum_{n \geq 1} (-1)^n \frac{b(n)}{n!} h^{(n-1)}(a_2)$$

where  $b(n)$  are the Bernoulli numbers.

When  $\mathbf{p}$  is an integral polytope, the existence of such operators is the combinatorial counterpart of a homological property of the associated toric variety: the invariant cycles corresponding to the faces of  $\mathbf{p}$  generate the equivariant homology module. An Euler-Maclaurin formula amounts to an explicit Riemann-Roch theorem, as obtained by Khovanskii and Pukhlikov [17] for integral polytopes corresponding to smooth toric varieties, and extended by Cappell and Shaneson [11] to any integral polytope. Furthermore,

by transforming Cappell-Shaneson homological methods into purely combinatorial techniques, valid for any rational (not necessary integral) polytope, Brion and Vergne [9] obtained various expressions for the sum  $\sum_{x \in \mathbf{p} \cap \mathbb{Z}^d} h(x)$ , either as an integral over a deformed polytope followed by differentiation with respect to the deformation parameter, or formulas of type (1).

In this article, we construct differential operators  $D(\mathbf{p}, \mathbf{f})$ , with rational coefficients, which satisfy Formula (1) and which moreover enjoy two essential properties: they are *local* and they are *computable*. The existence of operators with these properties was conjectured in [3].

By local, one means that  $D(\mathbf{p}, \mathbf{f})$  depends only on the equivalence class - modulo integral translations - of the *transverse cone*  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  of  $\mathbf{p}$  along  $\mathbf{f}$  (see Definition 2 and Figure 4). In particular, if  $\mathbf{p}$  is an integral polytope, the operator  $D(\mathbf{p}, \mathbf{f})$  depends only on the cone of feasible directions of  $\mathbf{p}$  along  $\mathbf{f}$ .

By computable, one means that there exists an algorithm which computes the  $m$  lowest order terms of  $D(\mathbf{p}, \mathbf{f})$ , with running time polynomial with respect to the size of the data defining  $\mathbf{p}$ , at least when the dimension  $d$  and the number  $m$  are fixed.

On the contrary, Cappell-Shaneson and Brion-Vergne operators are neither local nor computable.

When applied to the constant polynomial  $h(x) = 1$ , Formula (1) takes the form

$$(2) \quad \text{Card}(\mathbf{p} \cap \mathbb{Z}^d) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \nu_0(\mathbf{p}, \mathbf{f}) \text{vol}(\mathbf{f})$$

where the coefficients  $\nu_0(\mathbf{p}, \mathbf{f})$  are rational numbers. In the context of toric varieties, Danilov [14] asked the question of existence of coefficients  $\nu_0(\mathbf{p}, \mathbf{f})$  with the local property, in the case of an integral polytope. This result was proven by Morelli [18] and McMullen [19]. In [20], Pommersheim and Thomas gave a canonical construction of rational coefficients  $\nu_0(\mathbf{p}, \mathbf{f})$  which satisfy Formula (2), as a consequence of their expression for the Todd class of a toric variety. In a companion article, we will similarly obtain a local formula for the **equivariant** Todd class of any toric variety. The result is stated in Theorem 24.

The computability of our operators  $D(\mathbf{p}, \mathbf{f})$  extends the following remarkable result of Barvinok [2]: when the dimension  $d$  is fixed, the number of integral points  $\text{Card}(\mathbf{p} \cap \mathbb{Z}^d)$  can be computed by a polynomial time algorithm. This result is a consequence of Brion's theorem [7], according to which

the computation can be distributed over the tangent cones at the vertices, and of Barvinok's signed decomposition of a cone into unimodular cones by a polynomial time algorithm. Based on this method, efficient software packages for integer points counting problems and the effective computation of  $\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x)$  have been developed [15], [16], [23], [24].

Let us now explain the construction of the operators  $D(\mathfrak{p}, \mathfrak{f})$ . We define the differential operator  $D(\mathfrak{p}, \mathfrak{f})$  through its symbol, using a scalar product on  $\mathbb{R}^d$ . For this purpose, we first associate an analytic function  $\mu(\mathfrak{a})$  (defined on a neighborhood of 0) on  $\mathbb{R}^d$  to any rational affine cone  $\mathfrak{a} \subset \mathbb{R}^d$ .

For instance, for a half-line  $s + \mathbb{R}_+$ , we have

$$\mu(s + \mathbb{R}_+)(\xi) = \frac{e^{[[s]]\xi}}{1 - e^\xi} + \frac{1}{\xi}.$$

where  $[[s]] = n - s$ , with  $n \in \mathbb{Z}$ ,  $n - 1 < s \leq n$ . Remark that this function is analytic at 0, with value at 0 given by  $\mu(s + \mathbb{R}_+)(0) = \frac{1}{2} - [[s]]$ .

The assignment  $\mathfrak{a} \mapsto \mu(\mathfrak{a})$  has beautiful geometric properties. The most important one is that it is a **valuation** when the vertex of  $\mathfrak{a}$  is fixed (Theorem 20). For instance,

$$(3) \quad \mu(\mathfrak{a}_1 \cup \mathfrak{a}_2) = \mu(\mathfrak{a}_1) + \mu(\mathfrak{a}_2) - \mu(\mathfrak{a}_1 \cap \mathfrak{a}_2).$$

Moreover,  $\mu(\mathfrak{a})$  is unchanged when  $\mathfrak{a}$  is moved by a lattice translation, and the map  $\mathfrak{a} \mapsto \mu(\mathfrak{a})$  is equivariant with respect to lattice-preserving isometries. In particular, with the standard scalar product, we thus get invariants of the group  $O(n, \mathbb{Z})$ .

If  $\mathfrak{a}$  contains a straight line, we set  $\mu(\mathfrak{a}) = 0$ . If  $\mathfrak{a}$  is pointed with vertex  $s$ , we define  $\mu(\mathfrak{a})$  recursively by the relation:

$$(4) \quad \mu(\mathfrak{a})(\xi) = e^{-\langle \xi, s \rangle} \left( \sum_{x \in \mathfrak{a} \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle} + \sum_{\mathfrak{f}, \dim(\mathfrak{f}) > 0} \mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))(\xi) \int_{\mathfrak{f}} e^{\langle \xi, x \rangle} dm_{\mathfrak{f}}(x) \right)$$

where  $\mathfrak{f}$  denotes a face of  $\mathfrak{a}$  and  $dm_{\mathfrak{f}}(x)$  denotes the canonical Lebesgue measure on  $\mathfrak{f}$  defined by the lattice. In (4), the function  $\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$  is a priori defined only on a subspace of  $\mathbb{R}^d$ , namely the orthogonal to the face  $\mathfrak{f}$ . We extend it to  $\mathbb{R}^d$  by orthogonal projection. Our main point is to show that Formula (4) actually defines an analytic function. Thanks to the valuation property, the proof is reduced to the case of a simplicial unimodular cone.

Then we define  $D(\mathbf{p}, \mathbf{f})$  as the differential operator with symbol  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi)$ . If the scalar product is rational, the Taylor series of  $\mu(\mathbf{a})$  has rational coefficients, in particular the numbers  $\nu_0(\mathbf{p}, \mathbf{f})$  in (2) are rational. Note that  $D(\mathbf{p}, \mathbf{f})$  involves only differentiation in directions perpendicular to  $\mathbf{f}$ .

With this definition, Euler-Maclaurin formula (1) for any rational polytope  $\mathbf{p}$  follows easily from Brion's theorem. Indeed, the defining formula (4) is formally Formula (1) where the polytope  $\mathbf{p}$  is replaced by the cone  $\mathbf{a}$  and the polynomial  $h(x)$  is replaced by the exponential  $e^{\langle \xi, x \rangle}$ .

The computability of the functions  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))$  is also deduced from Barvinok's fast decomposition of cones, thanks to the valuation property.

Moreover, Barvinok proved recently [4] that, given an integer  $m$ , there exists a polynomial time algorithm which computes the  $m$  highest coefficients of the Ehrhart quasipolynomial of any rational simplex in  $\mathbb{R}^d$ , *when the dimension  $d$  is considered as an input*. We hope that our construction of the functions  $\mu(\mathbf{a})$  will lead to another polynomial time algorithm which would compute the  $m$  highest coefficients of the Ehrhart quasipolynomial for any simplex in  $\mathbb{R}^d$  and any polynomial  $h(x)$ , when the dimension  $d$  and the degree of  $h$  are considered as input. We want to point out that our construction involves only cones of dimension less than  $m$  when computing the  $m$  highest order Ehrhart coefficients.

In the forthcoming article [1], we will compare our construction to the mixed valuation method of [4].

Let us illustrate the results in dimension 2. For an affine cone  $\mathbf{a}$  with integral vertex  $s$  and edges generated by two integral vectors  $v_1, v_2$  with  $\det(v_1, v_2) = 1$ , (that is to say,  $\mathbf{a}$  is unimodular), we have:

$$\mu(\mathbf{a})(0) = \frac{1}{4} + \frac{\langle v_1, v_2 \rangle}{12} \left( \frac{1}{\langle v_1, v_1 \rangle} + \frac{1}{\langle v_2, v_2 \rangle} \right).$$

For a general cone, we compute  $\mu(\mathbf{a})(0)$  using the valuation property (3).

According to Pick's theorem, the number of integral points of an integral polygon  $\mathbf{p} \subset \mathbb{R}^2$  is given by

$$\text{Card}(\mathbf{p} \cap \mathbb{Z}^2) = \text{area}(\mathbf{p}) + \frac{1}{2} \text{length}_{\mathbb{Z}^2} \partial(\mathbf{p}) + 1.$$

According to our local formula, as well as Pommersheim-Thomas's, we have:

$$\text{Card}(\mathbf{p} \cap \mathbb{Z}^2) = \text{area}(\mathbf{p}) + \frac{1}{2} \text{length}_{\mathbb{Z}^2} \partial(\mathbf{p}) + \sum_s \nu_0(\mathbf{p}, s)$$

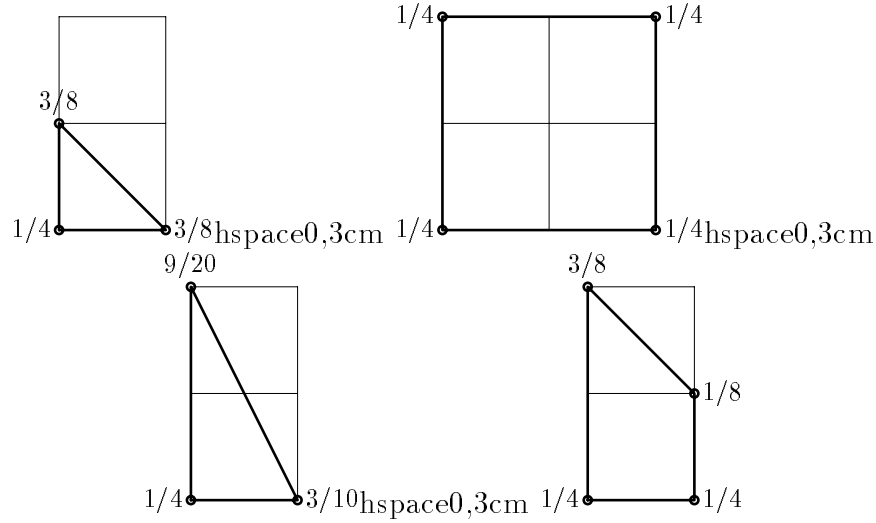


Figure 1: Coefficient  $\nu_0(\mathbf{p}, s) = \mu(\mathbf{t}(\mathbf{p}, s))(0)$  at the vertex  $s$

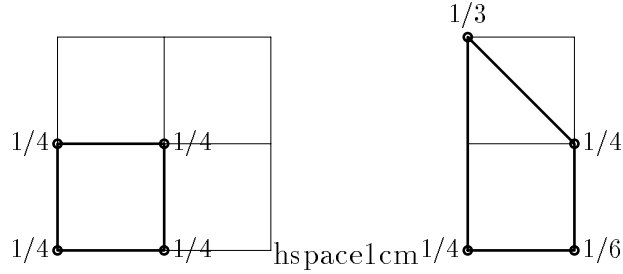


Figure 2: Cappell-Shaneson coefficients

with  $\nu_0(\mathbf{p}, s) = \mu(\mathbf{t}(\mathbf{p}, s))(0)$ , where  $s$  runs over the vertices of  $\mathbf{p}$ . Thus the constant 1 is canonically distributed over the vertices (Figure 1). Cappell-Shaneson coefficients [12] give a different, non local, distribution of the constant 1 over the vertices. For instance, the bottom right coefficients in the square and the trapezoid of Figure 2 are different, although the vertices have the same tangent cone.

More examples are given at the end of the paper, for polygons  $\mathbf{p}$  with rational (non integral) non unimodular vertices. Based on our Euler-Maclaurin operators, we wrote a Maple program which computes the value of the sum  $\sum_{x \in \mathbf{p} \cap \mathbb{Z}^2} x_1^{m_1} x_2^{m_2}$  and also the (periodic) coefficients of the corresponding

Ehrhart quasipolynomial.

## 2 Definitions and notations

We consider a rational vector space  $V$ , that is to say a finite dimensional real vector space with a lattice denoted by  $\Lambda_V$  or simply  $\Lambda$ . By lattice, we mean a discrete additive subgroup of  $V$  which generates  $V$  as a vector space. Hence, a lattice is generated by a basis of the vector space  $V$ . A basis of  $V$  which is a  $\mathbb{Z}$ -basis of  $\Lambda_V$  is called an integral basis.

We will need to consider subspaces and quotient spaces of  $V$ , this is why we cannot just let  $V = \mathbb{R}^d$  and  $\Lambda = \mathbb{Z}^d$ . The points of  $\Lambda$  are called integral. A point  $x \in V$  is called rational if  $qx \in \Lambda$  for some integer  $q \neq 0$ . The space of rational points in  $V$  is denoted by  $V_{\mathbb{Q}}$ . A subspace  $W$  of  $V$  is called rational if  $W \cap \Lambda$  is a lattice in  $W$ . If  $W$  is a rational subspace, the image of  $\Lambda$  in  $V/W$  is a lattice in  $V/W$ , so that  $V/W$  is a rational vector space.

A rational space  $V$ , with lattice  $\Lambda$ , has a canonical Lebesgue measure, for which  $V/\Lambda$  has measure 1. An affine subspace  $W$  of  $V$  is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will sometimes denote this measure by  $dm_W$ . For example, let  $W$  be a rational line of the form  $W = s + \mathbb{R}v$ . Assume that  $v$  is a generator of the group  $\mathbb{R}v \cap \Lambda$  (we say that  $v$  is a *primitive* vector). Then  $dm_W(s + tv) = dt$ .

If  $v_i \in V_{\mathbb{Q}}$  are linearly independent vectors, we denote by  $\square(v_1, \dots, v_k)$  the semi-open parallelepiped generated by the  $v_i$ 's:

$$\square(v_1, \dots, v_k) = \sum_{i=1}^k [0, 1[v_i.$$

We denote by  $\text{vol}(\square(v_1, \dots, v_k))$  its relative volume, that is to say its volume with respect to the canonical measure on the subspace generated by  $v_1, \dots, v_k$ .

We denote by  $V^*$  the dual space of  $V$ . We will denote elements of  $V$  by latin letters  $x, y, v, \dots$  and elements of  $V^*$  by greek letters  $\xi, \alpha, \dots$ . We denote the duality bracket by  $\langle \xi, x \rangle$ .

$V^*$  is equipped with the dual lattice  $\Lambda^*$  of  $\Lambda$  :

$$\Lambda^* = \{ \xi \in V^* ; \langle \xi, x \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda \}.$$

If  $S$  is a subset of  $V$ , we denote by  $S^\perp$  the subspace of  $V^*$  orthogonal to  $S$ :

$$S^\perp = \{\xi \in V^* ; \langle \xi, x \rangle = 0 \text{ for all } x \in S\}.$$

If  $W$  is a subspace of  $V$ , the dual space  $(V/W)^*$  is canonically identified with the subspace  $W^\perp \subset V^*$ .

If  $S$  is a subset of  $V$ , we denote by  $\langle S \rangle$  the affine subspace generated by  $S$ . If  $S$  consists of rational points, then  $\langle S \rangle$  is rational. Remark that  $\langle S \rangle$  may contain no integral point. We denote by  $\text{lin}(S)$  the vector subspace of  $V$  parallel to  $\langle S \rangle$ .

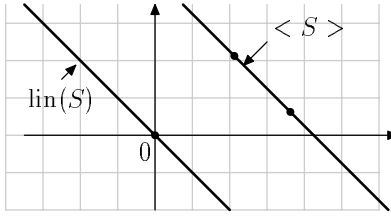


Figure 3: The affine space  $\langle S \rangle$  and the linear space  $\text{lin}(S)$

The set of non negative real numbers is denoted by  $\mathbb{R}_+$ . A convex rational cone  $\mathfrak{c}$  in  $V$  is a closed convex cone  $\sum_{i=1}^k \mathbb{R}_+ v_i$  which is generated by a finite number of elements  $v_i$  of  $V_{\mathbb{Q}}$ . In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone  $\mathfrak{a}$  is, by definition, the translate of a cone in  $V$  by an element  $s \in V_{\mathbb{Q}}$ . This cone is uniquely defined by  $\mathfrak{a}$ ; it is called the cone of directions of  $\mathfrak{a}$  and denoted by  $\text{dir}(\mathfrak{a})$ . Thus  $\mathfrak{a} = s + \text{dir}(\mathfrak{a})$ .

A cone  $\mathfrak{c}$  is called simplicial if it is generated by independent elements of  $V_{\mathbb{Q}}$ . A simplicial cone  $\mathfrak{c}$  is called unimodular if it is generated by independent integral vectors  $v_1, \dots, v_k$  such that  $\text{vol}(\square(v_1, \dots, v_k)) = 1$ . An affine cone  $\mathfrak{a}$  is called simplicial (resp. simplicial unimodular) if  $\text{dir}(\mathfrak{a})$  is simplicial (resp. simplicial unimodular).

An affine cone  $\mathfrak{a}$  is called pointed if it does not contain any straight line.

The set of faces of an affine cone  $\mathfrak{a}$  is denoted by  $\mathcal{F}(\mathfrak{a})$ . If  $\mathfrak{a}$  is pointed, then the vertex of  $\mathfrak{a}$  is the unique face of dimension 0, while  $\mathfrak{a}$  is the unique face of maximal dimension  $\dim \mathfrak{a}$ .

The dual cone  $\mathfrak{c}^*$  of a cone  $\mathfrak{c}$  is the set of  $\xi \in V^*$  such that  $\langle \xi, x \rangle \geq 0$  for any  $x \in \mathfrak{c}$ .



A convex rational polyhedron  $\mathfrak{p}$  in  $V$  (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces with boundary a rational affine hyperplane.

**Definition 1** We say that  $\mathfrak{p}$  is solid (in  $V$ ) if  $\langle \mathfrak{p} \rangle = V$ .

The set of faces of  $\mathfrak{p}$  is denoted by  $\mathcal{F}(\mathfrak{p})$  and the set of vertices of  $\mathfrak{p}$  is denoted by  $\mathcal{V}(\mathfrak{p})$ .

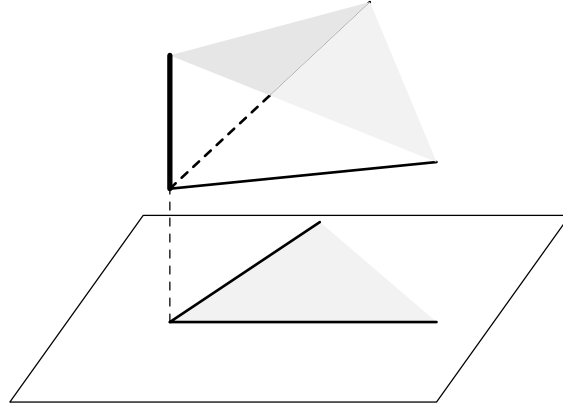


Figure 4: The transverse cone along an edge in dimension 3

We now introduce the main geometrical object in our study, the *transverse cone of a polyhedron*  $\mathfrak{p}$  along one of its faces  $\mathfrak{f}$  (see Figure 4). Let  $x$  be a point in the relative interior of  $\mathfrak{f}$ . Recall that the cone of feasible directions of  $\mathfrak{p}$  at  $x$  is the set  $\mathfrak{c}(\mathfrak{p}, \mathfrak{f}) := \{v \in V ; x + \epsilon v \in \mathfrak{p} \text{ for } \epsilon > 0 \text{ small enough}\}$ . It does not depend on the choice of  $x$  and contains the linear space  $\text{lin}(\mathfrak{f})$ . The supporting cone of  $\mathfrak{p}$  along  $\mathfrak{f}$  is the affine cone  $\langle \mathfrak{f} \rangle + \mathfrak{c}(\mathfrak{p}, \mathfrak{f})$ . We denote the projection  $V \rightarrow V / \text{lin}(\mathfrak{f})$  by  $\pi_{\mathfrak{f}}$ .

**Definition 2** Let  $\mathfrak{p}$  be a polyhedron and  $\mathfrak{f}$  a face of  $\mathfrak{p}$ . The transverse cone  $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$  of  $\mathfrak{p}$  along  $\mathfrak{f}$  is the image  $\pi_{\mathfrak{f}}(\langle \mathfrak{f} \rangle + \mathfrak{c}(\mathfrak{p}, \mathfrak{f}))$  of the supporting cone in  $V / \text{lin}(\mathfrak{f})$ .

We will often write simply *transverse cone along*  $\mathfrak{f}$ , when  $\mathfrak{p}$  is understood.

If  $v$  is a vertex of  $\mathfrak{p}$ , the transverse cone  $\mathfrak{t}(\mathfrak{p}, v)$  coincides with the supporting cone  $v + \mathfrak{c}(\mathfrak{p}, v) \subset V$ .

The transverse cone  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  is a pointed **affine** cone in the quotient space  $V/\text{lin}(\mathbf{f})$ . Its dimension is equal to the codimension of  $\mathbf{f}$  in  $\langle \mathbf{p} \rangle$ . Its vertex is the projection  $\pi_{\mathbf{f}}(x)$  of any point  $x$  of  $\mathbf{f}$  on  $V/\text{lin}(\mathbf{f})$ .

If  $\mathbf{a}$  is an affine cone and  $\mathbf{f}$  is a face of  $\mathbf{a}$ , then the supporting cone is  $\mathbf{a} + \text{lin}(\mathbf{f})$  and the transverse cone  $\mathbf{t}(\mathbf{a}, \mathbf{f})$  along  $\mathbf{f}$  is just the projection  $\pi_{\mathbf{f}}(\mathbf{a})$  of  $\mathbf{a}$  on  $V/\text{lin}(\mathbf{f})$ .

We shall make use of subdivisions of cones.

**Definition 3** *A subdivision of a cone  $\mathbf{c}$  is a finite collection  $\mathcal{C}$  of cones in  $V$  such that:*

- (a) *The faces of any cone in  $\mathcal{C}$  are in  $\mathcal{C}$ .*
  - (b) *If  $\mathfrak{d}_1$  and  $\mathfrak{d}_2$  are two elements of  $\mathcal{C}$ , then the intersection  $\mathfrak{d}_1 \cap \mathfrak{d}_2$  is a face of both  $\mathfrak{d}_1$  and  $\mathfrak{d}_2$ .*
  - (c) *We have  $\mathbf{c} = \cup_{\mathfrak{d} \in \mathcal{C}} \mathfrak{d}$ .*
- If furthermore the elements of  $\mathcal{C}$  are simplicial cones, the subdivision will be called simplicial.*

**Example 4** *The basic example is the subdivision  $\{\mathbb{R}_+, (-\mathbb{R}_+), \{0\}\}$  of the one-dimensional cone  $\mathbb{R}$ .*

It is easy to see that any pointed cone admits a subdivision into simplicial unimodular cones.

If  $S$  is a subset of  $V$ , we denote by  $\chi(S)$  the characteristic function of  $S$  (also called the indicator function of  $S$ ).

Explicit expression of operators  $D(\mathbf{p}, \mathbf{f})$  involves the Bernoulli polynomials  $b(n, t)$ , defined by the generating series

$$(5) \quad \sum_{n=0}^{\infty} \frac{b(n, t)}{n!} X^n = \frac{e^{tX} X}{e^X - 1}.$$

The Bernoulli number is  $b(n, 0)$  and is denoted by  $b(n)$ .

### 3 Meromorphic functions associated to polyhedra

By a meromorphic function on  $V^*$  with rational coefficients, we mean a meromorphic function on the complexification of  $V^*$  which can be written as the

quotient of two holomorphic functions with rational Taylor coefficients with respect to an integral basis of  $V^*$ .

We recall the construction of two meromorphic functions with rational coefficients on  $V^*$ , associated to any polyhedron  $\mathfrak{p}$  in  $V$  (see the survey [3]). The first function  $I(\mathfrak{p})$  is defined via integration over  $\mathfrak{p}$ , the second function  $S(\mathfrak{p})$  via summation over the set of integral points of  $\mathfrak{p}$ .

We denote by  $dm_{<\mathfrak{p}>}$  the relative Lebesgue measure on the affine space spanned by  $\mathfrak{p}$ .

**Proposition 5** *There exists a map  $I$  which to every polyhedron  $\mathfrak{p} \subset V$  associates a meromorphic function with rational coefficients  $I(\mathfrak{p})$  on  $V^*$ , so that the following properties hold:*

- (a) *If  $\mathfrak{p}$  contains a straight line, then  $I(\mathfrak{p})=0$ .*
- (b) *If  $\xi \in V^*$  is such that  $|e^{\langle \xi, x \rangle}|$  is integrable over  $\mathfrak{p}$ , then*

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} e^{\langle \xi, x \rangle} dm_{<\mathfrak{p}>}(x).$$

- (c) *For every point  $s \in V_{\mathbb{Q}}$ , we have*

$$I(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} I(\mathfrak{p})(\xi).$$

- (d) *The map  $I$  is a solid valuation: if the characteristic functions  $\chi(\mathfrak{p}_i)$  of a family of polyhedra  $\mathfrak{p}_i$  satisfy a linear relation  $\sum_i r_i \chi(\mathfrak{p}_i) = 0$ , then the functions  $I(\mathfrak{p}_i)$  satisfy the relation*

$$\sum_{\{i, <\mathfrak{p}_i>=V\}} r_i I(\mathfrak{p}_i) = 0.$$

**Example 6** .

- *If  $\mathfrak{p} = \{s\}$  is a point, then  $I(\mathfrak{p})(\xi) = e^{\langle \xi, s \rangle}$ .*
- *In dimension 1, if  $\mathfrak{p} = s + \mathbb{R}^+$ , where  $s \in \mathbb{Q}$ , then  $I(\mathfrak{p})(\xi) = -\frac{e^{\xi s}}{\xi}$ .*
- *If  $\mathfrak{c}$  is a simplicial cone generated by  $k \leq d$  independent vectors  $v_1, \dots, v_k$ , we have*

$$(6) \quad I(\mathfrak{c})(\xi) = (-1)^k \frac{\text{vol}(\square(v_1, \dots, v_k))}{\prod_{i=1}^k \langle \xi, v_i \rangle}.$$

These formulas follow immediately from the computation in dimension 1,  $\int_0^\infty e^{\xi x} dx = \frac{-1}{\xi}$  for  $\xi < 0$ . Thus, for a simplicial cone, the function  $I(\mathfrak{c})$  can indeed be extended as a rational function on the whole space  $V^*$ .

In a similar way, one defines the second meromorphic function, which is the discrete analogue of  $I(\mathfrak{p})$ .

**Proposition 7** *There exists a map  $S$  which to every polyhedron  $\mathfrak{p} \subset V$  associates a meromorphic function with rational coefficients  $S(\mathfrak{p})$  on  $V^*$  so that the following properties hold:*

- (a) *If  $\mathfrak{p}$  contains a straight line, then  $S(\mathfrak{p})=0$ .*
- (b) *If  $\xi \in V^*$  is such that  $\sum_{x \in \mathfrak{p} \cap \Lambda} |e^{\langle \xi, x \rangle}| < \infty$ , then*

$$(7) \quad S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle}.$$

- (c) *For every integral point  $s \in \Lambda$ , we have*

$$S(s + \mathfrak{p})(\xi) = e^{\langle \xi, s \rangle} S(\mathfrak{p})(\xi).$$

(d) *The map  $S$  is a valuation: if the characteristic functions  $\chi(\mathfrak{p}_i)$  of a family of polyhedra  $\mathfrak{p}_i$  satisfy a linear relation  $\sum_i r_i \chi(\mathfrak{p}_i) = 0$ , then the functions  $S(\mathfrak{p}_i)$  satisfy the same relation*

$$\sum_i r_i S(\mathfrak{p}_i) = 0.$$

**Example 8 .**

- *If  $\mathfrak{p} = \{s\}$  is a point, then we have two cases. If  $s$  is an integral point, then  $S(\{s\})(\xi) = e^{\langle \xi, s \rangle}$ , otherwise  $S(\{s\}) = 0$ .*
- *In dimension 1, if  $\mathfrak{p} = s + \mathbb{R}^+$ , where  $s \in \mathbb{Q}$ , then*

$$S(s + \mathbb{R}_+)(\xi) = \frac{e^{k\xi}}{1 - e^\xi}$$

*where  $k$  is the smallest integer greater or equal than  $s$ .*

- *Let  $\mathfrak{a}$  be a solid simplicial affine cone with vertex  $s \in V_{\mathbb{Q}}$ . Let  $v_1, \dots, v_d$  be integral generators of the edges of  $\mathfrak{a}$ . Then*

$$(8) \quad S(\mathfrak{a})(\xi) = \left( \sum_{x \in (s + \square(v_1, \dots, v_d)) \cap \Lambda} e^{\langle \xi, x \rangle} \right) \prod_{i=1}^d \frac{1}{1 - e^{\langle \xi, v_i \rangle}}.$$

We obtain this formula by observing that any element  $x$  of the affine cone  $\mathfrak{a}$  can be written in a unique way as a sum  $y + \sum_{i=1}^d n_i v_i$  where  $y$  lies in the semi-open parallelepiped  $s + \square(v_1, \dots, v_d)$  and the coefficients  $n_i$  are non negative integers, and that the point  $x$  is integral if and only if  $y$  is.

Thus indeed  $S(s + \mathfrak{c})$  can be extended to a meromorphic function on the whole of  $V^*$ .

Let us check the valuation property in dimension one:

**Example 9** In dimension one, let  $s \in \mathbb{Q}$ . The relation  $\chi(s + \mathbb{R}_+) + \chi(s - \mathbb{R}_+) - \chi(\{s\}) = \chi(\mathbb{R})$  must imply  $S(s + \mathbb{R}_+) + S(s - \mathbb{R}_+) = S(\{s\})$ . If  $s = 0$ , we have indeed

$$\frac{1}{1 - e^\xi} + \frac{1}{1 - e^{-\xi}} = 1 = S(\{s\})(\xi),$$

while, if  $0 < s < 1$ , then

$$S(s + \mathbb{R}_+)(\xi) = \frac{e^\xi}{1 - e^\xi},$$

$$S(s - \mathbb{R}_+)(\xi) = \frac{1}{1 - e^{-\xi}},$$

thus  $S(s + \mathbb{R}_+) + S(s - \mathbb{R}_+) = 0 = S(\{s\})$ .

The valuation property of the maps  $\mathfrak{p} \rightarrow S(\mathfrak{p})$  and  $\mathfrak{p} \rightarrow I(\mathfrak{p})$  have the following important corollary. This was first obtained by Brion [7] using toric varieties:

**Theorem 10** (Brion) Let  $\mathfrak{p}$  be a polyhedron in  $V$ . Then

$$I(\mathfrak{p}) = \sum_{v \in \mathcal{V}(\mathfrak{p})} I(\mathfrak{t}(\mathfrak{p}, v)),$$

$$S(\mathfrak{p}) = \sum_{v \in \mathcal{V}(\mathfrak{p})} S(\mathfrak{t}(\mathfrak{p}, v)).$$

The singularities of the functions  $I(\mathfrak{a})$  and  $S(\mathfrak{a})$  are easy to compute for a pointed affine cone:

**Lemma 11** Let  $\mathfrak{a} = s + \mathfrak{c}$  be a pointed affine cone with vertex  $s$  and let  $v_1, \dots, v_k$  be rational generators of the edges of the cone  $\mathfrak{c}$ . The products

$$\left( \prod_{i=1}^k \langle \xi, v_i \rangle \right) I(s + \mathfrak{c})(\xi) \text{ and } \left( \prod_{i=1}^k \langle \xi, v_i \rangle \right) S(s + \mathfrak{c})(\xi)$$

are analytic near 0.

**Proof.** It is easy to see that the cone  $\mathfrak{c}$  admits a subdivision into simplicial cones whose edges are already edges of  $\mathfrak{c}$ . Thus, thanks to the valuation properties of Proposition 7, it is enough to prove the lemma when  $\mathfrak{c}$  itself is a simplicial cone. In this case it follows immediately from Formulas (6) and (8).  $\square$

## 4 The main construction

In this section, we will perform the main construction of this article: to any affine cone  $\mathfrak{a}$  in  $V$  or in a rational quotient  $V/L$  of  $V$ , we will associate an **analytic** function  $\mu(\mathfrak{a})$  defined in a neighborhood of 0 in  $V^*$ . In the next section, if  $\mathfrak{p}$  is a convex rational polytope in  $V$ , we will obtain a local Euler-Maclaurin formula for  $\mathfrak{p}$  in terms of the functions  $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))$  associated to the transverse cones  $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$  of  $\mathfrak{p}$  along its various faces  $\mathfrak{f}$ .

We will denote the ring of analytic functions with rational coefficients, defined in a neighborhood of 0 in  $V^*$ , by  $\mathcal{H}(V^*)$  and the ring of meromorphic functions with rational coefficients, defined in a neighborhood of 0 in  $V^*$ , by  $\mathcal{M}(V^*)$ .

We will need to extend to the space  $V^*$  some meromorphic functions which are a priori defined only on a subspace of the form  $(V/L)^* = L^\perp$  of  $V^*$ . For that purpose, we fix a scalar product  $Q(x, y)$  on  $V$ . We assume that  $Q$  is rational, meaning that  $Q(x, y)$  is rational for  $x, y \in V_{\mathbb{Q}}$ . We denote also by  $Q(\xi, \eta)$  the dual scalar product on  $V^*$  and we use the orthogonal projection  $\text{proj}_{L^\perp} : V^* \rightarrow L^\perp$ . If  $\phi$  is a meromorphic function (with rational coefficients) on  $L^\perp$ , we still write  $\phi$  for the function on  $V^*$  defined by  $\xi \mapsto \phi(\text{proj}_{L^\perp}(\xi))$ . It is meromorphic with rational coefficients.

Actually, we will do this not only for the space  $V$  itself, but also when  $V$  is replaced by a rational quotient space  $W$ . The dual  $W^*$  is a subspace of  $V^*$ , thus it inherits the scalar product of  $V^*$ .

**A word of caution.** Let  $L$  be a rational subspace of  $V$ . By means of the scalar product  $Q$ , we can identify  $V/L$  with  $L^{\perp_Q} \subset V$ , the orthogonal of  $L$  with respect to  $Q$ . However these two spaces are not isomorphic as rational spaces. The lattice of  $V/L$  corresponds to the orthogonal projection of  $\Lambda$  on  $L^{\perp_Q}$ ; it contains the lattice  $L^{\perp_Q} \cap \Lambda$ , and the inclusion is strict in general, see Figure 5.

Let  $\mathfrak{a}$  be an affine cone in  $V$  and let  $\mathfrak{f}$  be a face of  $\mathfrak{a}$ . Recall that the transverse cone  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$  is the projection  $\pi_{\mathfrak{f}}(\mathfrak{a})$  of  $\mathfrak{a}$  in  $V/\text{lin}(\mathfrak{f})$ . When we

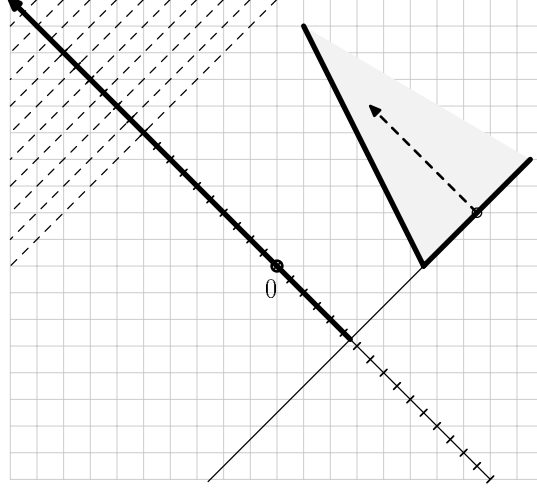


Figure 5: In dimension 2, the transverse cone along an edge with its lattice

identify  $V/\text{lin}(\mathfrak{f})$  with the orthogonal  $\text{lin}(\mathfrak{f})^{\perp_Q}$ , the transverse cone  $\mathbf{t}(\mathfrak{a}, \mathfrak{f})$  is a pointed affine cone in  $\text{lin}(\mathfrak{f})^{\perp_Q}$ .

We denote by  $\mathcal{C}_{\text{pointed}}(V)$  the set of pointed affine cones in  $V$ .

**Proposition 12** *Let  $V$  be a rational space and let  $Q$  be a rational scalar product on  $V$ . There exists a unique family of maps  $\mu_W$ , indexed by the rational quotient spaces  $W$  of  $V$ , such that the family enjoys the following properties.*

- (a)  $\mu_W$  maps  $\mathcal{C}_{\text{pointed}}(W)$  to  $\mathcal{M}(W^*)$ .
- (b) If  $W = \{0\}$ , then  $\mu_W(\{0\}) = 1$ .
- (c) For any pointed affine cone  $\mathfrak{a}$  in  $W$ , one has

$$(9) \quad S(\mathfrak{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu_{W/\text{lin}(\mathfrak{f})}(\mathbf{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f})$$

where the sum is over the set of faces of  $\mathfrak{a}$ .

In Formula (9), the transverse cone  $\mathbf{t}(\mathfrak{a}, \mathfrak{f})$  is a pointed affine cone in the quotient space  $W/\text{lin}(\mathfrak{f})$ . The function  $\mu_{W/\text{lin}(\mathfrak{f})}(\mathbf{t}(\mathfrak{a}, \mathfrak{f}))$  is a meromorphic function on a neighborhood of 0 in the dual  $(W/\text{lin}(\mathfrak{f}))^*$ . We give a meaning to the formula by extending this function to a neighborhood of 0 in the whole

space  $W^*$  by means of orthogonal projection. The function  $I(\mathfrak{f})$  is defined as a meromorphic function on  $W^*$  as in Section 3.

**Proof.** The result is easily obtained by induction on the dimension of  $W$ . If  $W = \{0\}$ , the only cone is  $\mathfrak{a} = \{0\}$  and Formula (9) is true. Let  $\mathfrak{a}$  be a pointed affine cone in  $W$ . Let  $s$  be the vertex of  $\mathfrak{a}$ . The transverse cone at the zero-dimensional face  $s$  is  $\mathfrak{a}$  itself. Formula (9) gives

$$(10) \quad S(\mathfrak{a})(\xi) = e^{\langle \xi, s \rangle} \mu_W(\mathfrak{a})(\xi) + \sum_{\mathfrak{f}, \dim \mathfrak{f} > 0} \mu_{W/\text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))(\xi) I(\mathfrak{f})(\xi).$$

For a face  $\mathfrak{f}$  of positive dimension, the transverse cone  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$  is a pointed affine cone in the vector space  $W/\text{lin}(\mathfrak{f})$ . Therefore,  $\mu_{W/\text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$  being defined by the induction hypothesis, Formula (10) defines  $\mu_W(\mathfrak{a})$  in a unique way, as a meromorphic function on  $W^*$ .  $\square$

The following property follows immediately from the definition:

**Proposition 13** *If  $V_1 \subset V_2$  and  $\mathfrak{a}$  is an affine cone contained in  $V_1$ , then the function  $\mu_{V_2}(\mathfrak{a})$  is the lift to  $V_2^*$  of  $\mu_{V_1}(\mathfrak{a})$ , by the natural restriction map  $V_2^* \rightarrow V_1^*$ .*

In the rest of this article, we will omit the subscript  $W$  in the notation  $\mu_W(\mathfrak{a})$ .

**Proposition 14** *The functions defined in Proposition 12 have the following properties:*

- (a) *For any  $s \in \Lambda$ , one has  $\mu(s + \mathfrak{a}) = \mu(\mathfrak{a})$ .*
- (b) *The map  $\mathfrak{a} \mapsto \mu(\mathfrak{a})$  is equivariant with respect to lattice-preserving linear isometries. In other words, let  $g$  be a linear isometry of  $W$  which preserves the lattice  $\Lambda$  and denote its transpose by  ${}^t g$ , then  $\mu(g(\mathfrak{a}))({}^t g^{-1} \xi) = \mu(\mathfrak{a})(\xi)$ .*
- (c) *The map  $\mathfrak{a} \mapsto \mu(\mathfrak{a})$  is multiplicative with respect to orthogonal sums of cones. More precisely, if  $W$  is an orthogonal sum  $W = W_1 \oplus W_2$  and  $\mathfrak{a}_i$  is an affine cone in  $W_i$  for  $i = 1, 2$ , then*

$$\mu(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu(\mathfrak{a}_1) \mu(\mathfrak{a}_2).$$

- (d) *If  $\mathfrak{a} \in \mathcal{C}_{\text{pointed}}(W)$  is such that  $\langle \mathfrak{a} \rangle \cap \Lambda_W = \emptyset$ , then  $\mu_W(\mathfrak{a}) = 0$ .*

**Proof.** The invariance in (a) and (b) and the multiplication rule in (c) follow immediately from the definition, by induction. To prove (d), assume



that  $\langle \mathfrak{a} \rangle \cap \Lambda_V = \emptyset$ . Then  $S(\mathfrak{a}) = 0$  and, for any face  $\mathfrak{f}$  of  $\mathfrak{a}$ , the transverse cone  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$  does not contain any integral point of  $V/\text{lin}(\mathfrak{f})$ , therefore, by induction,  $\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) = 0$  for  $\dim \mathfrak{f} > 0$ , hence  $\mu(\mathfrak{a}) = 0$ .  $\square$

If  $\mathfrak{a}$  is an affine cone in  $V$  which contains a straight line, we define  $\mu(\mathfrak{a}) = 0$ . Since all faces  $\mathfrak{f}$  of  $\mathfrak{a}$  contain a straight line, Formula (9) still holds in this case. Thus we have defined  $\mu(\mathfrak{a})$  for any rational affine cone in any quotient space of  $V$ .

Our objective is to show that  $\mu(\mathfrak{a})$  is indeed analytic near 0, but we will first observe some further properties of this family of functions.

It is easy to compute  $\mu$  in dimension 1. Let  $t \in \mathbb{Q}$  such that  $0 \leq t < 1$ . Then  $\mu(-t + \mathbb{R}_+)(\xi)$  is defined by

$$S(-t + \mathbb{R}_+)(\xi) = \frac{1}{1 - e^\xi} = e^{-t\xi} \mu(-t + \mathbb{R}_+)(\xi) + \int_{-t}^{\infty} e^{x\xi} dx$$

hence

$$(11) \quad \mu(-t + \mathbb{R}_+)(\xi) = \frac{e^{t\xi}}{1 - e^\xi} + \frac{1}{\xi}.$$

We may write this in terms of the Bernoulli polynomials  $b(n, t)$ , defined by the generating series (5). We obtain

$$(12) \quad \mu(-t + \mathbb{R}_+)(\xi) = - \sum_{n=0}^{\infty} \frac{b((n+1), t)}{(n+1)!} \xi^n.$$

Let  $\mathfrak{a}$  be a 1-dimensional pointed cone in  $V$ , that is to say a half-line. If  $\mathfrak{a}$  does not contain any integral point, then  $\mu(\mathfrak{a}) = 0$ . If  $\mathfrak{a}$  contains integral points, then there exists an integral point  $a \in \mathfrak{a}$  such that the translated half-line  $\mathfrak{a} - a$  is of the form

$$(-t + \mathbb{R}_+)v$$

where  $v$  is a primitive integral vector and  $t \in [0, 1[$ .

By a similar computation, we get, for  $\xi \in V^*$ ,

$$(13) \quad \mu(\mathfrak{a})(\xi) = \frac{e^{t\langle \xi, v \rangle}}{1 - e^{\langle \xi, v \rangle}} + \frac{1}{\langle \xi, v \rangle}.$$

The next step is crucial to our construction; we will prove that the map  $\mathfrak{c} \rightarrow \mu(s + \mathfrak{c})$  enjoys the valuation property:

**Proposition 15** *Let  $\mathfrak{c}_i$  be a finite family of cones in  $V$ . Assume that there exists a linear relation between their characteristic functions  $\sum_i r_i \chi(\mathfrak{c}_i) = 0$ . Then, for any  $s \in V_{\mathbb{Q}}$ , we have the corresponding relation  $\sum_i r_i \mu(s + \mathfrak{c}_i) = 0$ .*

**Example 16** *Consider the subdivision of Example 4 in dimension one. Let  $s \in \mathbb{Q}$ . Then*

$$0 = \mu(\mathbb{R}) = \mu(s + \mathbb{R}_+) + \mu(s - \mathbb{R}_+) - \mu(\{s\}).$$

*Indeed, if  $s$  is an integer, we have*

$$\mu(s + \mathbb{R}_+)(\xi) + \mu(s - \mathbb{R}_+)(\xi) = \left(\frac{1}{1 - e^{\xi}} + \frac{1}{\xi}\right) + \left(\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}\right) = 1 = \mu(\{s\})(\xi)$$

*while, if  $s \in ]-1, 0[$ , we have*

$$\mu(s + \mathbb{R}_+)(\xi) + \mu(s - \mathbb{R}_+)(\xi) = \left(\frac{e^{-s\xi}}{1 - e^{\xi}} + \frac{1}{\xi}\right) + \left(\frac{e^{-(s+1)\xi}}{1 - e^{-\xi}} - \frac{1}{\xi}\right) = 0 = \mu(\{s\})(\xi).$$

**Proof.**

We will prove the proposition by induction on  $\dim V$ . By a standard argument ([21], see also [8]), it suffices to prove the result in the following particular case (see Figure 6). Let  $\mathfrak{a}$  be a pointed solid affine cone in  $V$  with vertex  $s$ , let  $H$  be an affine hyperplane through  $s$ . Denote by  $H^{\pm}$  the closed half-spaces separated by  $H$ . Then we have

$$\chi(\mathfrak{a}) = \chi(\mathfrak{a} \cap H^+) + \chi(\mathfrak{a} \cap H^-) - \chi(\mathfrak{a} \cap H)$$

and we must prove:

$$(14) \quad \mu(\mathfrak{a}) = \mu(\mathfrak{a} \cap H^+) + \mu(\mathfrak{a} \cap H^-) - \mu(\mathfrak{a} \cap H).$$

We proceed to prove (14).

The functions  $S(\mathfrak{a})$  have the valuation property

$$S(\mathfrak{a}) - S(\mathfrak{a} \cap H^+) - S(\mathfrak{a} \cap H^-) + S(\mathfrak{a} \cap H) = 0.$$

By applying Formula (9), we obtain the following expansion of the left hand side

$$(15) \quad \begin{aligned} & \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f}) - \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H^+)} \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f})) I(\mathfrak{f}) \\ & - \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H^-)} \mu(\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f})) I(\mathfrak{f}) + \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H)} \mu(\mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f})) I(\mathfrak{f}). \end{aligned}$$

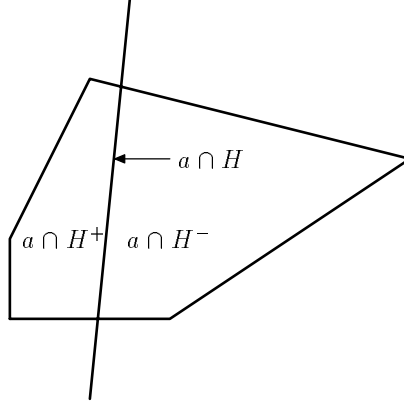


Figure 6: The cut of a 3-dimensional cone with 5 edges by a 2-dimensional plane  $H$ . The cone is represented by its slice in the figure plane. Thus  $H$  is represented by a line.

Let  $L$  be a affine subspace of  $V$  of dimension  $> 0$ . We will show

$$\begin{aligned}
 (16) \quad & \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a}), \langle \mathfrak{f} \rangle = L} \mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f}) - \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H^+), \langle \mathfrak{f} \rangle = L} \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f})) I(\mathfrak{f}) \\
 & - \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H^-), \langle \mathfrak{f} \rangle = L} \mu(\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f})) I(\mathfrak{f}) + \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a} \cap H), \langle \mathfrak{f} \rangle = L} \mu(\mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f})) I(\mathfrak{f}) \\
 & = 0.
 \end{aligned}$$

From the relation (16), it follows that the terms in (15) corresponding to the faces  $\mathfrak{f}$  of positive dimension add up to 0. Therefore the contribution of the 0-dimensional vertex  $\{s\}$  to (15) is also equal to 0, which proves the relation (14).

We fix  $L$  and we proceed to prove (16).

Remark that all the transverse cones which appear in (16) are affine cones in  $V/\text{lin}(L)$ . We will apply the induction hypothesis to  $V/\text{lin}(L)$ .

I) First we consider the case where there is a face  $\mathfrak{f}$  of  $\mathfrak{a}$  such that  $\langle \mathfrak{f} \rangle = L$ . There are three cases, according to whether the relative interior of  $\mathfrak{f}$  meets both the interiors of  $H^\pm$ , or only one, or none of them (in the third case,  $\mathfrak{f}$  is contained in  $H$ ).

- Case I.1: The relative interior of  $\mathfrak{f}$  meets both the interiors of  $H^\pm$ . Then

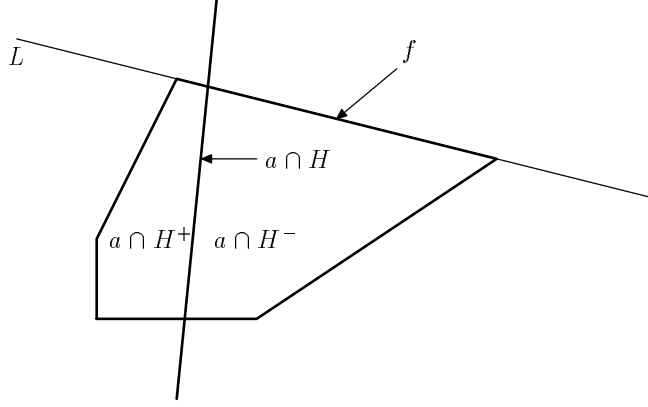


Figure 7: Case I.1

$\mathfrak{f} \cap H^\pm$  is a face of  $\mathfrak{a} \cap H^\pm$  and  $\langle \mathfrak{f} \cap H^\pm \rangle = L$ . Thus we have to prove

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-))I(\mathfrak{f} \cap H^-) = 0.$$

The three transverse cones  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$ ,  $\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+)$  and  $\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-)$  coincide. The integrals add up:

$$I(\mathfrak{f}) = I(\mathfrak{f} \cap H^+) + I(\mathfrak{f} \cap H^-),$$

thus we get

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) (I(\mathfrak{f}) - I(\mathfrak{f} \cap H^+) - I(\mathfrak{f} \cap H^-))$$

which is equal to 0 as required.

- Case I.2: The relative interior of  $\mathfrak{f}$  is contained in the interior of, say,  $H^+$ . Then  $\mathfrak{f} = L \cap \mathfrak{a} \cap H^+$  is also a face of  $\mathfrak{a} \cap H^+$ , but  $L \cap \mathfrak{a} \cap H^-$  and  $L \cap \mathfrak{a} \cap H$  are smaller dimensional, or empty. This time we have to prove

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) = 0.$$

The transverse cones  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f})$  and  $\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f})$  coincide, so we get

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) (I(\mathfrak{f}) - I(\mathfrak{f} \cap H^+)).$$

As  $\mathfrak{f} = \mathfrak{f} \cap H^+$ , this is equal to 0 as required.

- Case I.3:  $\mathfrak{f}$  is contained in  $H$ , thus it is a face of all four cones, in other words

$$\mathfrak{f} = \mathfrak{f} \cap H^+ = \mathfrak{f} \cap H^- = \mathfrak{f} \cap H.$$

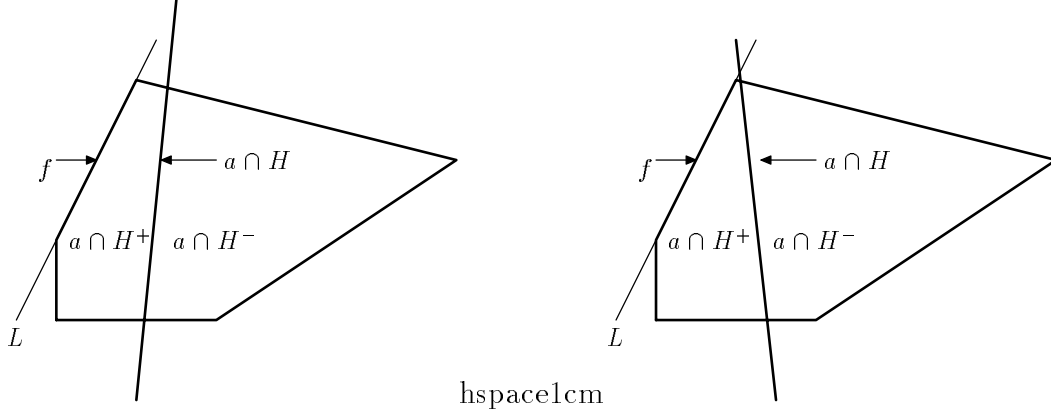


Figure 8: Case I.2

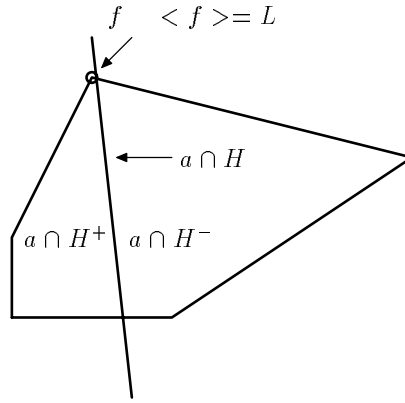


Figure 9: Case I.3

This time we have to prove

$$\begin{aligned} & \mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))I(\mathfrak{f}) - \mu(\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) \\ & - \mu(\mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-))I(\mathfrak{f} \cap H^-) + \mu(\mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f}))I(\mathfrak{f} \cap H) = 0. \end{aligned}$$

In this case, the intersection of the transverse cones is

$$\mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f}) \cap \mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f}) = \mathfrak{t}(\mathfrak{a} \cap H, \mathfrak{f}),$$

the union of the transverse cones is

$$\mathfrak{t}(\mathfrak{a}, \mathfrak{f}) = \mathfrak{t}(\mathfrak{a} \cap H^+, \mathfrak{f}) \cup \mathfrak{t}(\mathfrak{a} \cap H^-, \mathfrak{f}).$$

Thus we get

$$\left( \mu(\mathbf{t}(\mathfrak{a}, \mathfrak{f})) - \mu(\mathbf{t}(\mathfrak{a} \cap H^+, \mathfrak{f})) - \mu(\mathbf{t}(\mathfrak{a} \cap H^-, \mathfrak{f})) + \mu(\mathbf{t}(\mathfrak{a} \cap H, \mathfrak{f})) \right) I(\mathfrak{f}).$$

From the induction hypothesis applied to the space  $V/\text{lin}(L)$ , we deduce that this is equal to 0.

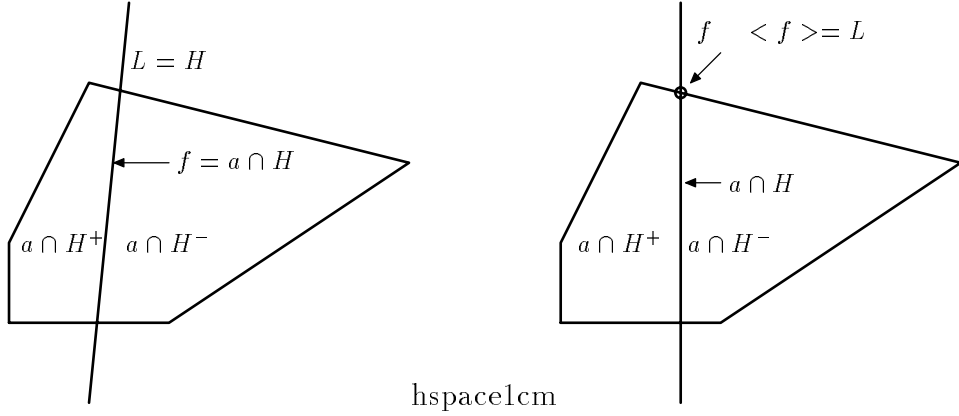


Figure 10: Case II

II) Next, we consider an affine subspace  $L$  such that  $L \cap \mathfrak{a}$  is not a face of  $\mathfrak{a}$  but, say,  $\mathfrak{f} = L \cap \mathfrak{a} \cap H^+$  is a face of  $\mathfrak{a} \cap H^+$ . Then we must have  $L \subset H$  so that  $\mathfrak{f}$  is a face of the three cones  $\mathfrak{a} \cap H$ ,  $\mathfrak{a} \cap H^+$  and  $\mathfrak{a} \cap H^-$ , but not a face of  $\mathfrak{a}$ . We have to show that

$$\begin{aligned} & \mu(\mathbf{t}(\mathfrak{a} \cap H^+, \mathfrak{f} \cap H^+))I(\mathfrak{f} \cap H^+) + \mu(\mathbf{t}(\mathfrak{a} \cap H^-, \mathfrak{f} \cap H^-))I(\mathfrak{f} \cap H^-) \\ & - \mu(\mathbf{t}(\mathfrak{a} \cap H, \mathfrak{f} \cap H))I(\mathfrak{f} \cap H) = 0, \end{aligned}$$

with

$$\mathfrak{f} \cap H^+ = \mathfrak{f} \cap H^- = \mathfrak{f} \cap H = \mathfrak{f}.$$

In this case, the union  $\mathbf{t}(\mathfrak{a} \cap H^+, \mathfrak{f}) \cup \mathbf{t}(\mathfrak{a} \cap H^-, \mathfrak{f})$  is the projection  $\pi_{\text{lin}(\mathfrak{f})}(\mathfrak{a})$  of  $\mathfrak{a}$  on  $V/\text{lin}(\mathfrak{f})$ ; it is not pointed, therefore, applying again the induction hypothesis to the space  $V/\text{lin}(L)$ , we have

$$\mu(\mathbf{t}(\mathfrak{a} \cap H^+, \mathfrak{f})) + \mu(\mathbf{t}(\mathfrak{a} \cap H^-, \mathfrak{f})) - \mu(\mathbf{t}(\mathfrak{a} \cap H, \mathfrak{f})) = \mu(\pi_{\text{lin}(\mathfrak{f})}(\mathfrak{a})) = 0$$

and the result follows.  $\square$

**Corollary 17** *Let  $\mathfrak{p}$  be a polytope in  $V$  and  $s \in V_{\mathbb{Q}}$ . Then*

$$\sum_{v \in \mathcal{V}(\mathfrak{p})} \mu(s + \text{dir}(\mathfrak{t}(\mathfrak{p}, v)))$$

*is equal to 1 if the point  $s$  is integral and 0 otherwise.*

**Proof.** This follows immediately from the valuation property and the relation (see [3]) between the characteristic functions:

$$\sum_{v \in \mathcal{V}(\mathfrak{p})} \chi(\text{dir}(\mathfrak{t}(\mathfrak{p}, v))) = \chi(\{0\}) \bmod \mathcal{L}$$

where  $\mathcal{L}$  denotes the space of linear combinations of characteristic functions of cones with lines.  $\square$

Now we show that our functions are analytic near 0.

**Proposition 18** *Let  $\mathfrak{a}$  be an affine cone in  $V$ . The function  $\mu(\mathfrak{a})$  is analytic near 0.*

**Proof.** The result is true when  $V = 0$  (and the explicit computation shows that it is true also when  $\dim V = 1$ ). We will prove it by induction on the dimension of  $V$ . Using the valuation property, it is enough to prove the analyticity when  $\mathfrak{a}$  is a solid simplicial unimodular affine cone in  $V$ . Let  $v_1, \dots, v_k$  be primitive integral generators of the edges of  $\text{dir } \mathfrak{a}$ . If  $\Phi$  is a meromorphic function on  $V^*$  such that the product

$$\left( \prod_{i=1}^k \langle \xi, v_i \rangle \right) \Phi(\xi)$$

is analytic, we denote by  $\text{Res}_{v_1}(\Phi)$  the residue of  $\Phi$  along  $v_1 = 0$ , that is to say the restriction to  $v_1^\perp \subset V^*$  of

$$\langle \xi, v_1 \rangle \Phi(\xi).$$

From the properties of the functions  $S$  and  $I$  (Lemma 11) and the induction hypothesis, it follows that the product

$$\left( \prod_{i=1}^k \langle \xi, v_i \rangle \right) \mu(\mathfrak{a})(\xi)$$

is analytic. Thus we want to show that  $\text{Res}_{v_1}(\mu(\mathbf{a})) = 0$ . Starting from the defining formula (9), we want to prove that, for  $\xi \in v_1^\perp$ , we have

$$(17) \quad \text{Res}_{v_1}(S(\mathbf{a}))(\xi) = \sum_{\mathbf{f}; \dim \mathbf{f} > 0} \mu(\mathbf{t}(\mathbf{a}, \mathbf{f}))(\xi) \text{Res}_{v_1}(I(\mathbf{f}))(\xi).$$

Let us denote by  $\pi$  the projection  $V \rightarrow V / \langle v_1 \rangle$ . The cone  $\pi(\mathbf{a})$  is also a simplicial unimodular cone with primitive integral generators  $\pi(v_2), \dots, \pi(v_d)$ . If  $\mathbf{a}$  is unimodular, then the parallelepiped  $s + \square(v_1, \dots, v_d) \subset V$  contains exactly one integral point. Therefore, the explicit computations (6) and (8) of  $I(\mathbf{a})$  and  $S(\mathbf{a})$  imply immediately that the residues along  $v_1 = 0$  of the functions  $S(\mathbf{a})$  and  $I(\mathbf{a})$  are given by:

$$\text{Res}_{v_1}(S(\mathbf{a})) = -S(\pi(\mathbf{a})),$$

$$\text{Res}_{v_1}(I(\mathbf{a})) = -I(\pi(\mathbf{a})).$$

In the sum (17), only the faces  $\mathbf{f}$  for which  $v_1$  is an edge of  $\text{dir } \mathbf{f}$  contribute, and these faces are in one to one correspondence with the faces of  $\pi(\mathbf{a})$ . For such a face  $\mathbf{f}$ , the transverse cone of  $\pi(\mathbf{a})$  along  $\pi(\mathbf{f})$  coincides with the transverse cone  $\mathbf{t}(\mathbf{a}, \mathbf{f})$  and we have also

$$\text{Res}_{v_1}(I(\mathbf{f})) = -I(\pi(\mathbf{f})),$$

whence (17), and the proposition.  $\square$

Next we will show that Formula (9) still holds when  $\mathbf{a}$  is replaced by any polyhedron  $\mathbf{p}$ . This will be an easy consequence of Brion's theorem and the valuation property of  $I$  and  $S$ . In the following three theorems, we collect the results of this section.

**Theorem 19** *Let  $V$  be a rational space and  $Q$  a rational scalar product on  $V^*$ . If  $W = V/L$  is a rational quotient space of  $V$ , we denote by  $\mathcal{C}(W)$  the set of affine cones in  $W$ . For  $\mathbf{a} \in \mathcal{C}(W)$ , let  $I(\mathbf{a})$  and  $S(\mathbf{a})$  be the meromorphic functions with rational coefficients on  $W^*$  defined in Propositions 5 and 7.*

*There exists a unique family of maps  $\mu_W$ , indexed by the rational quotient spaces  $W$  of  $V$ , such that the family enjoys the following properties:*

- (a)  $\mu_W$  maps  $\mathcal{C}(W)$  to  $\mathcal{H}(W^*)$ , the space of analytic functions on  $W^*$ , with rational Taylor coefficients.
- (b) If  $W = \{0\}$ , then  $\mu_W(\{0\}) = 1$ .



- (c) If the affine cone  $\mathfrak{a} \in \mathcal{C}(W)$  contains a straight line, then  $\mu_W(\mathfrak{a}) = 0$ .  
(d) For any affine cone  $\mathfrak{a}$  in  $W$ , one has

$$S(\mathfrak{a}) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{a})} \mu_{W/\text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f})) I(\mathfrak{f})$$

where the sum is over all faces of the cone  $\mathfrak{a}$ .

As in all this section, in Formula (d), the function  $\mu_{W/\text{lin}(\mathfrak{f})}(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}))$  is considered as a function on  $W^*$  itself by means of the orthogonal projection  $W^* \rightarrow (W/\text{lin}(\mathfrak{f}))^* = (\text{lin}(\mathfrak{f}))^\perp$  with respect to the scalar product on  $W^* \subset V^*$ .

**Theorem 20** *The analytic functions defined in Theorem 19 have the following properties:*

- (a) For any  $x \in \Lambda$ , one has  $\mu(x + \mathfrak{a}) = \mu(\mathfrak{a})$ .  
(b) The map  $\mathfrak{a} \mapsto \mu(\mathfrak{a})$  is equivariant with respect to lattice-preserving isometries. In other words, let  $g$  be an isometry of  $W$  which preserves the lattice  $\Lambda$ . Then  $\mu(g(\mathfrak{a}))(\mathfrak{t}g^{-1}\xi) = \mu(\mathfrak{a})(\xi)$ .  
(c) If  $W$  is an orthogonal sum  $W = W_1 \oplus W_2$  and  $\mathfrak{a}_i$  is an affine cone in  $W_i$  for  $i = 1, 2$ , then

$$\mu(\mathfrak{a}_1 + \mathfrak{a}_2) = \mu(\mathfrak{a}_1)\mu(\mathfrak{a}_2).$$

- (d) For a fixed  $s \in W_{\mathbb{Q}}$ , the map  $\mathfrak{c} \rightarrow \mu(s + \mathfrak{c})$  is a valuation on the set of cones in  $W$ .  
(e) Let  $\mathfrak{p} \subset W$  be a polyhedron, then

$$(18) \quad S(\mathfrak{p})(\xi) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p})} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi) I(\mathfrak{f})(\xi).$$

**Proof.** In Theorem 20, only point (e) has not yet been proven. If  $v$  is a vertex of  $\mathfrak{p}$ , let us denote by  $\mathcal{F}(\mathfrak{p}, v)$  the set of faces of  $\mathfrak{p}$  which contain  $v$ . For such a face  $\mathfrak{f} \in \mathcal{F}(\mathfrak{p}, v)$ , the intersection  $\mathfrak{d} = \langle \mathfrak{f} \rangle \cap \mathfrak{t}(\mathfrak{p}, v)$  is a face of the cone  $\mathfrak{t}(\mathfrak{p}, v)$  and this correspondence is a bijection between  $\mathcal{F}(\mathfrak{p}, v)$  and  $\mathcal{F}(\mathfrak{t}(\mathfrak{p}, v))$  with inverse given by  $\mathfrak{f} = \mathfrak{d} \cap \mathfrak{p}$ . Moreover the transverse cone  $\mathfrak{t}(\mathfrak{p}, v)$  of  $\mathfrak{p}$  along its face  $\langle \mathfrak{f} \rangle \cap \mathfrak{t}(\mathfrak{p}, v)$  coincides with the transverse cone  $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$  of  $\mathfrak{p}$  along  $\mathfrak{f}$ . Therefore we have

$$S(\mathfrak{n}(\mathfrak{p}, v)) = \sum_{\mathfrak{f} \in \mathcal{F}(\mathfrak{p}, v)} \mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})) I(\langle \mathfrak{f} \rangle \cap \mathfrak{t}(\mathfrak{p}, v)).$$

Replacing  $S(\mathbf{t}(\mathbf{p}, v))$  with the right-hand side of this equality in Brion's formula, we obtain

$$S(\mathbf{p}) = \sum_{v \in \mathcal{V}(\mathbf{p})} \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p}, v)} \mu(\mathbf{t}(\mathbf{p}, \mathbf{f})) I(< \mathbf{f} > \cap \mathbf{t}(\mathbf{p}, v)).$$

Then we reverse the order of summation and get

$$S(\mathbf{p}) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \mu(\mathbf{t}(\mathbf{p}, \mathbf{f})) \sum_{v \in \mathcal{V}(\mathbf{f})} I(< \mathbf{f} > \cap \mathbf{t}(\mathbf{p}, v)).$$

The last sum is equal to  $I(\mathbf{f})$ .  $\square$

**Theorem 21** *Assume  $V = \mathbb{R}^d$  with a fixed dimension  $d$ . Then, for  $m$  fixed, there exists a polynomial time algorithm which computes  $\mu(\mathbf{a})$  at order  $m$  for any affine cone  $\mathbf{a} \subset V$ .*

**Proof.** By [2], there exist polynomial time algorithms which compute the functions  $I(\mathbf{c})$  and  $S(\mathbf{c})$  at order  $m$  for any affine cone  $\mathbf{c}$  in  $\mathbb{R}^k$ , if  $k \leq d$ . Therefore by induction we get an algorithm which computes  $\mu(\mathbf{a})$  for any  $\mathbf{a} \subset V$ .  $\square$

Let  $\sigma$  be a cone in the dual space  $V^*$ . The dual cone  $\sigma^* \subset V$  contains the vector subspace  $< \sigma >^\perp$ . Let us denote by  $\pi_{< \sigma >^\perp}$  the projection  $V \rightarrow V / < \sigma >^\perp$ . For any  $s \in V_\mathbb{Q}$ , the projected cone  $\pi_{< \sigma >^\perp}(s + \sigma^*)$  is a pointed cone in  $V / < \sigma >^\perp$ . Thus  $\mu(\pi_{< \sigma >^\perp}(s + \sigma^*))$  is an analytic function on  $(V / < \sigma >^\perp)^* \cong < \sigma > \subset V^*$ . We consider it as a function on  $V^*$  by means of orthogonal projection, as before. We obtain a map  $\mu_s^* : \mathcal{C}(V^*) \rightarrow \mathcal{H}(V^*)$  defined by:

**Definition 22**

$$\mu_s^*(\sigma) = \mu(\pi_{< \sigma >^\perp}(s + \sigma^*)).$$

From the valuation behavior of  $\mu$ , it follows that  $\mu_s^*$  is a solid valuation. In other words, the following corollary holds.

**Corollary 23** *Let  $\sigma$  be a cone in  $V^*$ , and let  $\{\sigma'\}$  be a subdivision of  $\sigma$ . For any  $s \in V_\mathbb{Q}$ , we have*

$$\mu_s^*(\sigma) = \sum_{\dim \sigma' = \dim \sigma} \mu_s^*(\sigma').$$

**Proof.** As  $\{\sigma'\}$  is a subdivision of  $\sigma$ , we have

$$\chi(\sigma) = \sum_{\dim \sigma' = \dim \sigma} \chi(\sigma') + \sum_{\dim \sigma' < \dim \sigma} \pm \chi(\sigma').$$

Let  $L = \langle \sigma \rangle^\perp \subset V$  and let  $\pi$  denote the projection  $V \rightarrow V/L$ . The map  $\tau \mapsto \chi(\pi(s + \tau^*))$  is a valuation on the set of cones in  $V^*$  (see [3] for instance). Therefore

$$\chi(\pi(s + \sigma^*)) = \sum_{\dim \sigma' = \dim \sigma} \chi(\pi(s + \sigma'^*)) + \sum_{\dim \sigma' < \dim \sigma} \pm \chi(\pi(s + \sigma'^*)).$$

If  $\dim \sigma' < \dim \sigma$ , then the cone  $\pi(s + \sigma'^*) \subset V/L$  contains a straight line, thus  $\mu(\pi(s + \sigma'^*)) = 0$ , and the corollary follows from the valuation property of  $\mu$ .  $\square$

In a companion paper [6], we will prove the following theorem, which extends to equivariant homology a result of Pommersheim-Thomas [20] by which they answered a question of Danilov [14].

**Theorem 24** *Let  $\mathcal{E}$  be a fan in  $V^*$  and let  $X$  be the corresponding toric variety. For  $\sigma \in \mathcal{E}$ , let  $X(\sigma) \subseteq X$  be the corresponding orbit closure. Then the equivariant Todd class of  $X$  is equal to*

$$\sum_{\sigma \in \mathcal{E}} \mu_0^*(\sigma)[X(\sigma)]$$

*in the equivariant homology ring of  $X$ .*

## 5 Local Euler-Maclaurin formula

As in the previous section,  $V$  is a rational space and we fix a scalar product on  $V$ . Let  $\mathfrak{p}$  be a (convex rational) polyhedron in  $V$ . To each face  $\mathfrak{f}$  of  $\mathfrak{p}$ , we are going to associate a linear differential operator  $D(\mathfrak{p}, \mathfrak{f})$  on  $V$ .

To any analytic function  $\Phi(\xi)$  on  $V^*$ , defined near 0, there corresponds a unique linear differential operator  $D(\Phi)$  (of infinite degree) with constant coefficients on  $V$  such that  $\Phi(\xi)$  is the symbol of  $D(\Phi)$ . More precisely, for  $\xi \in V^*$ , let us denote by  $e^\xi$  the function  $x \mapsto e^{\langle \xi, x \rangle}$  on  $V$ , then  $D(\Phi)$  is defined by the relation

$$D(\Phi) \cdot e^\xi = \Phi(\xi) e^\xi \text{ for } \xi \text{ small enough.}$$

Let  $W = V/L$  be a quotient space of  $V$  and let  $\mathbf{a}$  be a pointed affine cone in  $W$ . In the previous section, we constructed an analytic function  $\mu(\mathbf{a})$  on  $W^* = L^\perp \subset V^*$ . By orthogonal projection, we consider  $\mu(\mathbf{a})$  as a function on  $V^*$  and we introduce the corresponding differential operator  $D(\mu(\mathbf{a}))$  on  $V$ :

$$(19) \quad D(\mu(\mathbf{a})) \cdot e^\xi = \mu(\mathbf{a})(\xi) e^\xi.$$

Let  $\mathbf{p}$  be a polyhedron in  $V$ .

**Definition 25** *Let  $\mathbf{f}$  be a face of  $\mathbf{p}$ . We denote by*

$$D(\mathbf{p}, \mathbf{f}) = D(\mu(\mathbf{t}(\mathbf{p}, \mathbf{f})))$$

*the differential operator on  $V$  associated to the transverse cone  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  of  $\mathbf{p}$  along  $\mathbf{f}$ . We denote its constant term by  $\nu(\mathbf{p}, \mathbf{f})$ . Thus*

$$\nu(\mathbf{p}, \mathbf{f}) = D(\mathbf{p}, \mathbf{f}) \cdot 1 = \mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(0).$$

The operator  $D(\mathbf{p}, \mathbf{f})$ , as well as its constant term  $\nu(\mathbf{p}, \mathbf{f})$ , are **local** in the sense that they depend only on the class of  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  modulo integral translations. In particular, if  $\mathbf{p}$  has integral vertices, then  $D(\mathbf{p}, \mathbf{f})$  depends only on the cone of transverse feasible directions at a generic point of  $\mathbf{f}$ . The operator  $D(\mathbf{p}, \mathbf{f})$  involves only derivatives in directions orthogonal to the face  $\mathbf{f}$ .

We are now ready to state the local Euler-Maclaurin formula for any polytope.

**Theorem 26** *(Local Euler-Maclaurin formula)*

*Let  $\mathbf{p}$  be a polytope in  $V$ . For any polynomial function  $h(x)$  on  $V$ , we have*

$$(20) \quad \sum_{x \in \mathbf{p} \cap \Lambda} h(x) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \int_{\mathbf{f}} D(\mathbf{p}, \mathbf{f}) \cdot h$$

*where the integral on the face  $\mathbf{f}$  is taken with respect to the Lebesgue measure on  $\langle \mathbf{f} \rangle$  defined by the lattice  $\Lambda \cap \text{lin}(\mathbf{f})$ .*

**Proof.**

The method is to check equality (20) for a polynomial of the form  $h(x) = \langle \xi, x \rangle^k$ . Taking Taylor series, we may replace  $h(x)$  by  $e^{t\langle \xi, x \rangle}$  with  $t$  small. Then the equality (20) becomes the formula in Theorem 20, (c)

$$S(\mathbf{p})(\xi) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))(\xi) I_{\mathbf{f}}(\xi).$$

□

In dimension 1, when  $\mathfrak{p}$  is an interval  $[a_1, a_2]$ , applying Formulas (13) and (12) for  $\mu(a_1 + \mathbb{R}^+)$  and  $\mu(a_2 - \mathbb{R}^+)$ , we obtain

$$\begin{aligned} \sum_{a_1 \leq x \leq a_2, x \in \mathbb{Z}} h(x) &= \int_{a_1}^{a_2} h(t) dt - \sum_{n \geq 0} \frac{b((n+1), t_1)}{(n+1)!} \left( \left( \frac{d}{dt} \right)^n h \right)(a_1) \\ &\quad - \sum_{n \geq 0} (-1)^n \frac{b((n+1), t_2)}{(n+1)!} \left( \left( \frac{d}{dt} \right)^n h \right)(a_2), \end{aligned}$$

where  $t_1$  and  $t_2$  in  $[0, 1[$  are defined by  $t_1 = k_1 - a_1$ ,  $t_2 = a_2 - k_2$ , with  $k_1$  the smallest integer greater or equal than  $a_1$  and  $k_2$  the largest integer smaller or equal than  $a_2$  (Figure 5). Of course, when  $a_1$  and  $a_2$  are integers, we recover the historical Euler-Maclaurin formula.

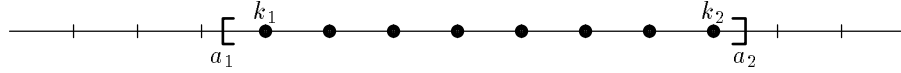


Figure 11: Euler-Maclaurin for the interval  $[a_1, a_2]$

## 6 Ehrhart polynomial

Let  $\mathfrak{p}$  be a rational polytope in a  $d$ -dimensional rational space  $V$  and let  $q$  be an integer such that  $q\mathfrak{p}$  has integral vertices. Let  $h(x)$  be a polynomial function of degree  $r$  on  $V$ . For any integer  $t$ , we consider the dilated polytope  $t\mathfrak{p}$  and the corresponding sum

$$S(t\mathfrak{p}, h) = \sum_{x \in t\mathfrak{p} \cap \Lambda} h(x).$$

As a function of  $t$ , it is given by a quasipolynomial: there exist functions  $t \mapsto E_i(\mathfrak{p}, h, t)$  on  $\mathbb{Z}$  which are periodic with period  $q$  such that

$$(21) \quad S(t\mathfrak{p}, h) = \sum_{i=0}^{d+r} E_i(\mathfrak{p}, h, t) t^i$$

whenever  $t$  is a positive integer and even in a slightly larger range including negative values.

**Definition 27** *The periodic functions  $E_i(\mathbf{p}, h, t)$  defined by Equation (21) are called the Ehrhart coefficients for the polytope  $\mathbf{p}$  and the polynomial  $h$ .*

*When  $h$  is the constant polynomial  $h(x) = 1$ , we denote the Ehrhart coefficients simply by  $E_i(\mathbf{p}, t)$ .*

Our local Euler-Maclaurin formula gives an expression of the coefficients  $E_i(\mathbf{p}, h, t)$  in terms of the functions  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}))$ , as we will now explain.

Let  $\mathbf{a}$  be a pointed affine cone with vertex  $s$  in  $V$ . If  $t$  is a nonzero integer, we define  $\mu(\mathbf{a}, t) = \mu(t\mathbf{a})$ . For  $t = 0$ , we define  $\mu(\mathbf{a}, 0) = \mu(\text{dir}(\mathbf{a}))$ . Then  $\mu(\mathbf{a}, t + q) = \mu(\mathbf{a}, t)$  for any integer  $q \in \mathbb{N}$  such that the point  $qs$  is integral. Let  $\mathbf{p}$  be a polyhedron and  $\mathbf{f}$  a face of codimension  $m$ . We define  $D(\mathbf{p}, \mathbf{f}, t) = D(\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}), t))$  for any integer  $t \in \mathbb{N}$ .

**Remark 28** *If the affine span  $\langle \mathbf{f} \rangle$  of the face  $\mathbf{f}$  contains an integral point, then the vertex of the transverse cone  $\mathbf{t}(\mathbf{p}, \mathbf{f})$  is integral, therefore  $\mu(\mathbf{t}(\mathbf{p}, \mathbf{f}), t) = \mu(\text{dir}(\mathbf{t}(\mathbf{p}, \mathbf{f})))$  and  $D(\mathbf{p}, \mathbf{f}, t)$  do not depend on  $t$ . We have  $D(\mathbf{p}, \mathbf{f}, t) = D(\mathbf{p}, \mathbf{f})$ .*

Let  $w_1, \dots, w_m$  be an integral basis of the subspace  $\text{lin}(\mathbf{f})^{\perp_Q} \subset V$ . The operator  $D(\mathbf{p}, \mathbf{f}, t)$  has the following expression:

$$D(\mathbf{p}, \mathbf{f}, t) = \nu_0(\mathbf{p}, \mathbf{f}, t) + \sum_{A, |A|=1}^{\infty} \nu_A(\mathbf{p}, \mathbf{f}, t) \partial^A$$

where  $A = (a_1, \dots, a_m)$ , with  $a_i \in \mathbb{N}$  and  $\partial^A = D(w_1)^{a_1} \dots D(w_m)^{a_m}$ . The coefficients  $\nu_A(\mathbf{p}, \mathbf{f}, t) \in \mathbb{Q}$  are periodic with respect to  $t$ , with period equal to the smallest integer  $q_{\mathbf{f}}$  such that  $q_{\mathbf{f}} \langle \mathbf{f} \rangle$  contains an integral point.

**Proposition 29** *Let  $\mathbf{p}$  be a rational polytope and  $h$  a polynomial function of degree  $r$  on  $V$ . Then, for any integer  $t \geq 0$ , we have*

$$(22) \quad S(t\mathbf{p}, h) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \int_{t\mathbf{f}} D(\mathbf{p}, \mathbf{f}, t) \cdot h.$$

Furthermore we have

$$\int_{t\mathbf{f}} D(\mathbf{p}, \mathbf{f}, t) \cdot h = \sum_{i=\dim \mathbf{f}}^{\dim \mathbf{f} + r} E_i(\mathbf{p}, h, \mathbf{f}, t) t^i$$

where the coefficients  $E_i(\mathbf{p}, h, \mathbf{f}, t)$  are periodic with period  $q_{\mathbf{f}}$ .

Hence the Ehrhart coefficients are given by

$$E_i(\mathbf{p}, h, t) = \sum_{\mathbf{f}, \dim \mathbf{f} \leq i} E_i(\mathbf{p}, h, \mathbf{f}, t).$$

**Proof.** For  $t > 0$ , Formula (22) is just the Euler-Maclaurin formula of Theorem 26 and the second equation follows from obvious estimates on the polynomial behaviour of the integrals.

For  $t = 0$ , as both sides of (22) are quasipolynomials, they take the same value. We may also deduce the equality for  $t = 0$  from Corollary 17. Indeed, for  $t = 0$ , the left hand side is  $h(0)$  and the faces of dimension  $> 0$  give a zero contribution to the right hand side. The equality becomes

$$h(0) = \sum_{v \in \mathcal{V}(\mathbf{p})} (D(\text{dir}(\mathbf{t}(\mathbf{p}, v))) \cdot h)(0),$$

and follows immediately from Corollary 17.  $\square$

For instance, let  $h$  be a monomial  $h(x) = x^m$  with  $m \in \mathbb{N}^d$ . The coefficient of the highest degree term  $t^{d+|m|}$  is the integral  $\int_{\mathbf{p}} x^m dx$ . As the operator  $D(\mathbf{p}, \mathbf{p}, t)$  is equal to 1, the face  $\mathbf{p}$  does not contribute to the coefficients of lower degree. The coefficient of  $t^{d+|m|-1}$  involves only the faces of codimension 1, the coefficient of  $t^{d+|m|-2}$  involves only the faces of codimension 1 and 2, etc.

When we apply the last proposition to the function  $h(x) = 1$ , we obtain a formula for the number of integral points in  $t\mathbf{p}$ .

**Corollary 30** (a) *The Ehrhart quasipolynomial of the polytope  $\mathbf{p}$  is given by*

$$\text{Card}(t\mathbf{p} \cap \Lambda) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \nu_0(\mathbf{p}, \mathbf{f}, t) \text{vol}(\mathbf{f}) t^{\dim \mathbf{f}},$$

hence

$$E_k(\mathbf{p}, t) = \sum_{\mathbf{f}, \dim \mathbf{f} = k} \nu_0(\mathbf{p}, \mathbf{f}, t) \text{vol}(\mathbf{f}).$$

(b) *The rational number  $\nu_0(\mathbf{p}, \mathbf{f}, t)$  depends only on the class modulo lattice translations of the transverse cone  $t\mathbf{t}(\mathbf{p}, \mathbf{f})$ . Therefore, it is a periodic function of  $t$  with period at most equal to  $q_{\mathbf{f}}$ , the smallest integer such that  $q_{\mathbf{f}} < \mathbf{f} >$  contains integral points. In particular, if the affine span  $\langle \mathbf{f} \rangle$  contains integral points for every  $k$ -dimensional face  $\mathbf{f}$  of  $\mathbf{p}$ , then the Ehrhart coefficient  $E_k(\mathbf{p}, t)$  does not depend on  $t$ .*

(c) When  $\dim V$  is fixed, there exists a polynomial time algorithm which computes  $\nu_0(\mathfrak{p}, \mathfrak{f}, t)$ .

**Proof.** The last statement in (b) is due to Stanley [22]; it also follows immediately from Remark 28. The computability follows from the computability of the functions  $\mu(\mathfrak{a})$ .  $\square$

Barvinok [4] proved recently that, given an integer  $m$ , there exists a polynomial time algorithm which computes the  $m$  highest coefficients of the Ehrhart quasipolynomial of any rational simplex in  $\mathbb{R}^d$ , *when the dimension  $d$  is considered as an input*. We hope that our construction of the functions  $\mu$  leads to another polynomial time algorithm which would compute the  $m$  highest coefficients of the Ehrhart quasipolynomial for any simplex in  $\mathbb{R}^d$  and any polynomial  $h(x)$ , when the dimension  $d$  and the degree of the polynomial are considered as input.

## 7 Computations in dimension 2

In this section,  $V = \mathbb{R}^2$  with  $\Lambda = \mathbb{Z}^2$ . Let  $v_1$  and  $v_2$  be primitive integral vectors. Let  $s \in V_{\mathbb{Q}}$ . We are going to compute  $\mu(\mathfrak{a})$  for the affine cone

$$\mathfrak{a} = s + \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2.$$

We will use the following notations:

For  $t \in \mathbb{R}$ , we denote the smallest integer greater or equal than  $t$  by  $\text{ceil}(t)$  and we define  $[[t]] \in [0, 1[$  by  $[[t]] = \text{ceil}(t) - t$ .

We denote by  $B$  the function  $B(y, t) = \frac{e^{[[t]]y}}{1 - e^y} + \frac{1}{y}$ . Recall that the function  $\mu(\mathfrak{a})$  for a one-dimensional cone (half-line)  $\mathfrak{a}$  is expressed in terms of  $B$ .

$s = s_1 v_1 + s_2 v_2$  with  $s_i \in \mathbb{Q}$ .

$C_i = \frac{Q(v_1, v_2)}{Q(v_i, v_i)}$ , for  $i = 1, 2$ .

$q = \det(v_1, v_2)$ . We assume  $q > 0$ .

$w \in \mathbb{Z}^2$  is a vector such that  $\det(v_1, w) = 1$ ,

$p = \det(v_2, w)$ . Thus  $p$  and  $q$  are coprime integers.

$r = (qs_1 + [[qs_1]]) + p(qs_2 + [[qs_2]])$ .

$\zeta$  is a primitive  $q$ -th root of 1.

We observe that the lattice in  $V/\mathbb{R}v_1$  is generated by  $\bar{w} = \frac{1}{q}\bar{v}_2$  where  $\bar{w}$ ,  $\bar{v}_2$  denotes the image of  $w$ ,  $v_2$  in  $V/\mathbb{R}v_1$ . Thus the transverse cone  $\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1) \subset V/\mathbb{R}v_1$  is given by

$$\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1) = (qs_2 + \mathbb{R}_+) \frac{1}{q} \bar{v}_2.$$



**Proposition 31** For  $\xi \in V^*$ , let  $y_i = \langle \xi, v_i \rangle$ , for  $i = 1, 2$ . Then the function  $\mu(\mathbf{a})(\xi)$  is given by

$$(23) \quad \mu(\mathbf{a})(\xi) =$$

$$\begin{aligned} & \frac{1}{q} e^{\frac{[qs_1]y_1}{q}} e^{\frac{[qs_2]y_2}{q}} \left( \frac{1}{(1 - e^{\frac{y_1}{q}})(1 - e^{\frac{y_2}{q}})} + \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k e^{\frac{y_1}{q}})(1 - \zeta^{kp} e^{\frac{y_2}{q}})} \right) \\ & + \frac{1}{y_1} B\left(\frac{y_2 - C_1 y_1}{q}, [[qs_2]]\right) + \frac{1}{y_2} B\left(\frac{y_1 - C_2 y_2}{q}, [[qs_1]]\right) - \frac{q}{y_1 y_2}. \end{aligned}$$

Its value at  $\xi = 0$  is equal to

$$\begin{aligned} (24) \quad \mu(\mathbf{a})(0) &= \frac{1}{q} \left( \left( \frac{1}{2} - [[qs_1]] \right) \left( \frac{1}{2} - [[qs_2]] \right) \right) \\ &+ \frac{Q(v_1, v_2)}{Q(v_1, v_1)} \left( \frac{1}{12} - \frac{1}{2} [[qs_2]] \right) + \frac{1}{2} [[qs_2]]^2 + \frac{Q(v_1, v_2)}{Q(v_2, v_2)} \left( \frac{1}{12} - \frac{1}{2} [[qs_1]] \right) + \frac{1}{2} [[qs_1]]^2 \\ &+ \frac{1}{q} \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k)(1 - \zeta^{kp})}. \end{aligned}$$

When  $\mathbf{a}$  is unimodular,  $\mu(\mathbf{a})(\xi)$  is given by

$$\begin{aligned} (25) \quad \mu(a)(\xi) &= \frac{\exp([s_1]y_1 + [s_2]y_2)}{(1 - e^{y_1})(1 - e^{y_2})} \\ &+ \frac{1}{y_1} B(y_2 - C_1 y_1, [[s_2]]) + \frac{1}{y_2} B(y_1 - C_2 y_2, [[s_1]]) - \frac{1}{y_1 y_2} \end{aligned}$$

and its constant term  $\mu(\mathbf{a})(0)$  is given by

$$\begin{aligned} (26) \quad \mu(\mathbf{a})(0) &= \left( \frac{1}{2} - [[s_1]] \right) \left( \frac{1}{2} - [[s_2]] \right) \\ &+ \frac{Q(v_1, v_2)}{Q(v_1, v_1)} \left( \frac{1}{12} - \frac{1}{2} [[s_2]] \right) + \frac{1}{2} [[s_2]]^2 + \frac{Q(v_1, v_2)}{Q(v_2, v_2)} \left( \frac{1}{12} - \frac{1}{2} [[s_1]] \right) + \frac{1}{2} [[s_1]]^2. \end{aligned}$$

**Remark 32** In actual Maple computations, we use only the unimodular case (25) which is computable in polynomial time at any given order. Thanks to the valuation property, for a non unimodular cone  $\mathbf{a}$ , we compute  $\mu(\mathbf{a})$  by performing first a signed decomposition of  $\mathbf{a}$  into unimodular cones, similar to Barvinok's decomposition. As a result, by our local Euler-Maclaurin formula, we have fast algorithms which compute, for a polygon  $\mathbf{p} \subset \mathbb{R}^2$  and a monomial  $h(x) = x_1^{m_1} x_2^{m_2}$ , the sum of values at integral points  $S(\mathbf{p}, h) = \sum_{x \in \mathbf{p} \cap \Lambda} h(x)$  and the coefficients of the corresponding Ehrhart quasipolynomial.

**Proof.** We use the defining relation of Proposition 12. First, we obtain a summation formula for  $S(\mathfrak{a})$  by using finite Fourier transform as in [9]. We observe that  $\Lambda \subset M = \frac{1}{q}(\mathbb{Z}v_1 + \mathbb{Z}v_2)$ . Let  $\tilde{\mathfrak{a}} \subset \mathfrak{a}$  be the cone

$$\tilde{\mathfrak{a}} = \tilde{s} + \mathbb{R}_+v_1 + \mathbb{R}_+v_2$$

with vertex

$$\tilde{s} = \frac{1}{q}(\text{ceil}(qs_1)v_1 + \text{ceil}(qs_2)v_2).$$

Then  $\tilde{\mathfrak{a}} \cap M = \mathfrak{a} \cap M$ . As  $\Lambda \subset M$ , we have also  $\tilde{\mathfrak{a}} \cap \Lambda = \mathfrak{a} \cap \Lambda$ . Consider the dual lattice  $M^* \subset \Lambda^*$ . Let  $x \in M$ . We have

$$\sum_{\gamma \in \Lambda^*/M^*} e^{2i\pi\langle \gamma, x \rangle} = \begin{cases} 0 & \text{if } x \notin \mathbb{Z}^2 \\ q & \text{if } x \in \mathbb{Z}^2. \end{cases}$$

Therefore we have

$$S(\tilde{\mathfrak{a}})(\xi) = \frac{1}{q} \sum_{\gamma \in \Lambda^*/M^*} \sum_{x \in \tilde{\mathfrak{a}} \cap M} e^{\langle 2i\pi\gamma + \xi, x \rangle}.$$

Since

$$\tilde{\mathfrak{a}} \cap M = \tilde{s} + \mathbb{Z}_+\frac{v_1}{q} + \mathbb{Z}_+\frac{v_2}{q},$$

we obtain

$$(27) \quad S(\mathfrak{a})(\xi) = S(\tilde{\mathfrak{a}})(\xi) = e^{\langle \xi, \tilde{s} \rangle} \frac{1}{q} \sum_{k=0}^{q-1} \frac{e^{\langle 2i\pi k\delta, \tilde{s} \rangle}}{(1 - e^{\langle 2i\pi k\delta + \xi, \frac{v_1}{q} \rangle})(1 - e^{\langle 2i\pi k\delta + \xi, \frac{v_2}{q} \rangle})}.$$

In this formula,  $\delta$  is a generator of the group  $\Lambda^*/M^* \cong \mathbb{Z}/q\mathbb{Z}$ . By using the basis  $(v_1, w)$  of  $\mathbb{Z}^2$ , we obtain:

$$\langle \delta, v_1 \rangle = 1, \quad \langle \delta, v_2 \rangle = p.$$

Let  $(v_1^*, v_2^*)$  be the dual basis of  $(v_1, v_2)$ . The orthogonal projection of  $\xi$  on  $(V/\mathbb{R}v_1)^* = \mathbb{R}v_2^*$  is equal to  $(-C_1y_1 + y_2)v_2^*$ , with

$$C_1 = -\frac{Q(v_1^*, v_2^*)}{Q(v_2^*, v_2^*)} = \frac{Q(v_1, v_2)}{Q(v_1, v_1)}.$$

Then the computation in dimension one (11) gives

$$\mu(\mathfrak{t}(\mathfrak{a}, \mathfrak{f}_1))(\xi) = B\left(\frac{-C_1y_1 + y_2}{q}, [[qs_2]]\right)$$

and similarly

$$\mu(\mathbf{t}(\mathbf{a}, \mathbf{f}_2))(\xi) = B\left(\frac{y_1 - C_2 y_2}{q}, [[qs_1]]\right)$$

with

$$C_2 = \frac{Q(v_1, v_2)}{Q(v_2, v_2)}.$$

We have

$$\begin{aligned} I(\mathbf{f}_i)(\xi) &= e^{y_1 s_1 + y_2 s_2} \left(\frac{-1}{y_i}\right), \\ I(\mathbf{a})(\xi) &= e^{y_1 s_1 + y_2 s_2} \frac{q}{y_1 y_2}. \end{aligned}$$

Therefore, by (12), we have

$$\begin{aligned} (28) \quad \mu(\mathbf{a})(\xi) &= e^{-(y_1 s_1 + y_2 s_2)} S(\mathbf{a})(\xi) \\ &+ \frac{1}{y_1} B\left(\frac{-C_1 y_1 + y_2}{q}, [[qs_2]]\right) + \frac{1}{y_2} B\left(\frac{y_1 - C_2 y_2}{q}, [[qs_1]]\right) \\ &- \frac{q}{y_1 y_2}. \end{aligned}$$

In Formula (28), we replace  $S(\mathbf{a})$  with the right hand side of (27), taking in account the equality  $\text{ceil}(qs_i) - qs_i = [[qs_i]]$ . This gives (23).  $\square$

If  $\mathbf{a}$  is not unimodular, then  $\mu(\mathbf{a})(\xi)$  involves the “extended” Fourier-Dedekind sum

$$\frac{1}{q} \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k e^{\frac{y_1}{q}})(1 - \zeta^{kp} e^{\frac{y_2}{q}})},$$

and  $\mu(\mathbf{a})(0)$  involves the Fourier-Dedekind sum

$$D(q, 1, p, r) = \frac{1}{q} \sum_{k=1}^{q-1} \frac{\zeta^{kr}}{(1 - \zeta^k)(1 - \zeta^{kp})}.$$

One has (see for instance [5])

$$D(q, 1, p, r) = \sum_{k=0}^{q-1} \left((- \frac{kp + r}{q})\right) \left(\left(\frac{k}{q}\right)\right) - \frac{1}{4q},$$

where the “sawtooth” function  $((a))$  is defined by

$$((a)) = a - [a] - \frac{1}{2}.$$

The Dedekind sum  $D(q, 1, p, r)$  can also be computed in polynomial time by means of reciprocity relations (see for example [5]), but here we do not use this fact.

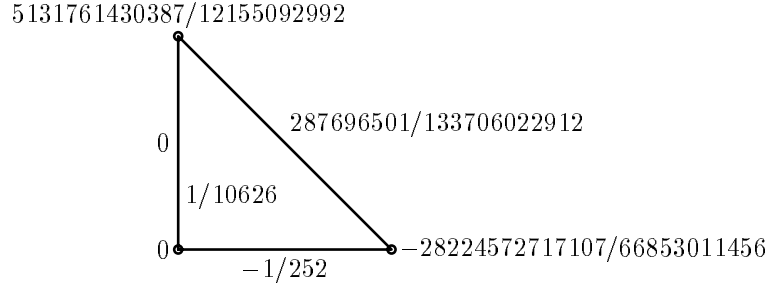


Figure 12: Le savant Cosinus

**Example 33** (Figure 12) We compute the right hand side of Euler-Maclaurin formula in the case of the “dull triangle” with vertices  $(0, 0), (1, 0), (0, 1)$ , and the polynomial  $h(x) = x_1^{20}x_2$ . As expected, the contributions of the various faces of  $\mathfrak{p}$  add up to  $0^1$ .

Contribution of vertices:  $0, -\frac{28224572717107}{66853011456}, \frac{5131761430387}{12155092992}$ .  
Contribution of edges:  $-\frac{1}{252}, \frac{287696501}{133706022912}, 0$ .  
Integral over triangle:  $\frac{1}{10626}$ .

**Example 34** Triangle with vertices  $s_1 = (\frac{1}{3}, \frac{1}{5}), s_2 = (\frac{16}{3}, \frac{1}{7}), s_3 = (\frac{37}{5}, \frac{92}{7})$ .

Number of integral points: 31

Contribution of vertices:  $\frac{89133678169939}{66088208614500}, -\frac{4281800310619}{2106396270216}, -\frac{401172431621091}{457987274773000}$

Contribution of edges :  $\frac{1}{210}, -\frac{1}{210}, \frac{1}{1050}$

Area of triangle:  $\frac{34187}{1050}$

**Example 35** Quadrangle with vertices  $(\frac{1}{3}, \frac{1}{5}), (\frac{16}{3}, \frac{1}{7}), (\frac{37}{5}, \frac{92}{7}), (3, 10)$ .

Number of integral points: 49

Contribution of vertices:  $\frac{210849514883}{127956322980}, -\frac{4281800310619}{2106396270216}, -\frac{179008247}{706816180}, -\frac{4382929}{6869864}$

Contribution of edges :  $\frac{1}{210}, -\frac{1}{210}, \frac{11}{35}, \frac{1}{30}$

Area:  $\frac{699}{14}$

---

<sup>1</sup>This computation delighted us, and it would have delighted Dr. Pancrace Eusèbe Zéphyrin Brioché alias “Dr. Cosinus” [13]

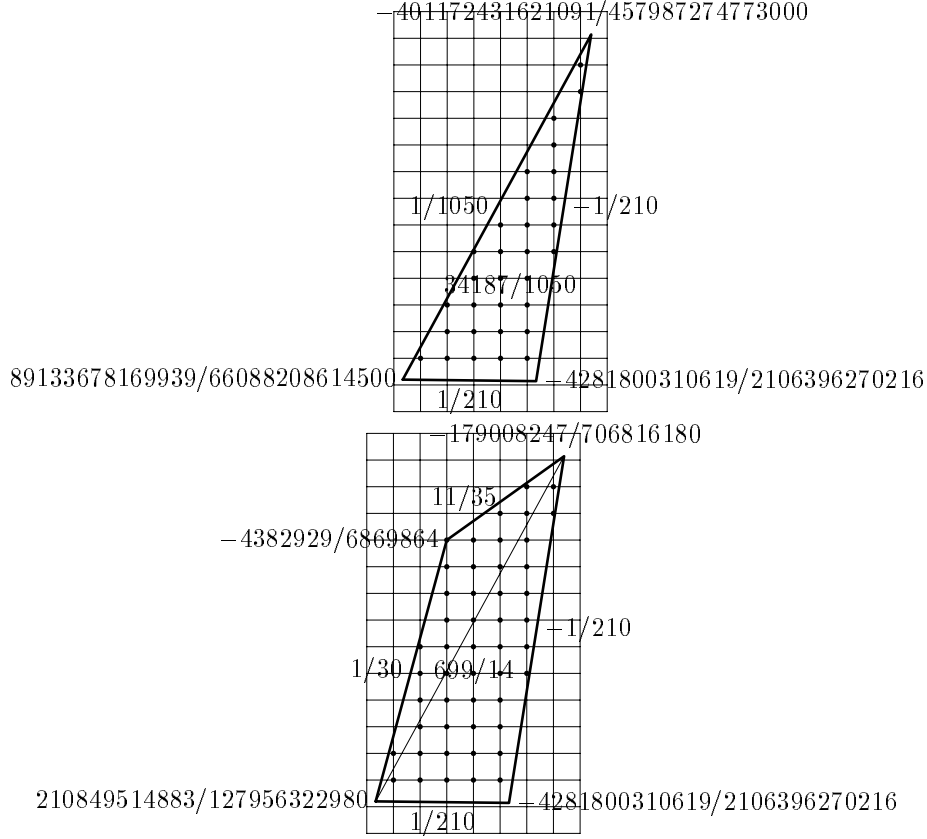


Figure 13: Triangle with vertices  $(\frac{1}{3}, \frac{1}{5})$ ,  $(\frac{16}{3}, \frac{1}{7})$ ,  $(\frac{37}{5}, \frac{92}{7})$  and quadrangle with extra vertex  $(3, 10)$

Remark that, as expected, the contributions of the bottom right vertex  $(\frac{16}{3}, \frac{1}{7})$  in the triangle or the trapezoid of Figure 13 are the same, as this vertex have the same tangent cone in both polygons.

**Example 36** We compute the Ehrhart quasipolynomial  $E_2t^2 + E_1(t)t + E_0(t)$  for the number of integral points of the triangle of Example 34. The highest coefficient is the area of the triangle,  $E_2 = \frac{34187}{1050}$ . The coefficient  $E_1(t)$  is the sum of the contributions of the edges. The coefficient  $E_0(t)$  is the sum of the contributions of the vertices.

On this example, we can observe the periods of the contributions of the edges and vertices to the Ehrhart coefficients (Corollary 30). The period of a

vertex contribution is equal to the lcm of the denominators of its coordinates. For an edge starting from a vertex  $(a_1, a_2)$  and parallel to the primitive vector  $(v_1, v_2)$ , the period is the least integer  $q$  such that  $q(a_1v_2 - a_2v_1)$  is an integer.

Contribution of edges (periods 3, 5, 7 respectively):

$$-\frac{1}{105} \bmod(t, 3) + \frac{1}{70}, \quad -\frac{1}{105} \bmod(4t, 7) + 1/30, \quad -\frac{1}{525} \bmod(2t, 5) + \frac{1}{210}.$$

Contribution of vertex  $s_1 = (\frac{1}{3}, \frac{1}{5})$  (period 15):

$$\begin{aligned} & \frac{1}{75} \bmod(t, 5) \bmod(t, 15) - \frac{1}{45} \bmod(t, 3) \bmod(t, 15) - \frac{1}{15} \bmod(t, 5) \bmod(2t, 3) \\ & + \frac{1765457034769}{293725371620} - \frac{44}{5} \bmod(t, 5) - \frac{747989}{11987225} \bmod(2t, 5) + \frac{1}{30} \bmod(7t, 15) \\ & + \frac{3}{5} \bmod(8t, 15) - \frac{1}{15} \bmod(t, 15) + \frac{44}{25} (\bmod(t, 5))^2 \\ & - \frac{2}{45} (\bmod(8t, 15))^2 + \frac{1}{225} (\bmod(t, 15))^2 + \frac{1}{3} \bmod(2t, 3) \\ & + \frac{15227}{183774} \bmod(t, 3) - \frac{1}{9} (\bmod(2t, 3))^2 + \frac{2567}{91887} (\bmod(t, 3))^2 \\ & - \frac{901467}{119872250} (\bmod(2t, 5))^2 - \frac{1}{150} (\bmod(7t, 15))^2 + \frac{1}{75} \bmod(2t, 5) \bmod(7t, 15) \\ & + \frac{1}{45} \bmod(8t, 15) \bmod(2t, 3) \end{aligned}$$

Contribution of vertex  $s_2 = (\frac{16}{3}, \frac{1}{7})$  (period 21):

$$\begin{aligned} & -\frac{32132693735}{4776408776} - \frac{13}{14} (\bmod(t, 7))^2 + \frac{13}{2} \bmod(t, 7) + \frac{1}{63} \bmod(t, 3) \bmod(2t, 21) \\ & - \frac{15227}{183774} \bmod(t, 3) - \frac{1}{21} \bmod(2t, 21) - \frac{2567}{91887} (\bmod(t, 3))^2 + \frac{1}{9} (\bmod(2t, 3))^2 \\ & - \frac{2}{3} \bmod(2t, 3) + \frac{1}{63} \bmod(2t, 21) \bmod(2t, 3) \\ & + \frac{1}{21} \bmod(2t, 3) \bmod(4t, 7) - \frac{30189}{545804} \bmod(4t, 7) \\ & - \frac{8797}{3820628} (\bmod(4t, 7))^2 \end{aligned}$$

Contribution of vertex  $s_3 = (\frac{37}{5}, \frac{92}{7})$  (period 35):

$$\begin{aligned}
& \frac{9}{1225} (\text{mod}(16t, 35))^2 + \frac{3}{1225} (\text{mod}(23t, 35))^2 + \frac{641856910509}{373867163080} \\
& + \frac{30189}{545804} \text{mod}(4t, 7) + \frac{1}{1225} (\text{mod}(34t, 35))^2 \\
& + 1/10 \text{mod}(3t, 5) - 1/35 \text{mod}(34t, 35) - \frac{1}{2450} (\text{mod}(18t, 35))^2 \\
& + \frac{2}{1225} (\text{mod}(9t, 35))^2 + \frac{901467}{119872250} (\text{mod}(2t, 5))^2 \\
& + \frac{1}{70} \text{mod}(18t, 35) + \frac{1}{50} (\text{mod}(3t, 5))^2 + \frac{1}{1225} \text{mod}(34t, 35) \text{mod}(18t, 35) \\
& + \frac{1}{1225} f \text{mod}(23t, 35) \text{mod}(16t, 35) \\
& + \frac{1}{175} \text{mod}(3t, 5) \text{mod}(23t, 35) - \frac{1}{1225} \text{mod}(34t, 35) \text{mod}(16t, 35) \\
& - \frac{1}{175} \text{mod}(2t, 5) \text{mod}(16t, 35) \\
& - 1/35 \text{mod}(3t, 7) \text{mod}(3t, 5) \\
& - 1/35 \text{mod}(3t, 5) \text{mod}(4t, 7) - \frac{1}{245} \text{mod}(9t, 35) \text{mod}(3t, 7) \\
& - \frac{1}{175} \text{mod}(3t, 5) \text{mod}(9t, 35) \\
& - \frac{1}{1225} \text{mod}(16t, 35) \text{mod}(18t, 35) + \frac{747989}{11987225} \text{mod}(2t, 5) \\
& + \frac{8797}{3820628} (\text{mod}(4t, 7))^2 - 1/35 \text{mod}(9t, 35) - \frac{4}{35} \text{mod}(23t, 35) \\
& - \frac{8}{35} \text{mod}(16t, 35) + 1/49 (\text{mod}(3t, 7))^2
\end{aligned}$$

**Example 37 (Computation time).**

We computed the full Ehrhart quasipolynomial corresponding to the triangle of Example 34 and the polynomial  $h(x_1, x_2) = x_1^{k_1} x_2^{k_2}$ , with increasing exponents  $k_1$  and  $k_2$ . Allowing a computation time of about one hour, we reached  $k_1 = k_2 = 24$ . The result is of course too big to write here.

The sum of values  $x_1^{48} x_2^{48}$  at the integral points of the triangle of Example 34 dilated by the factor  $N = 11^5$  took about the same time. The result is the following number

55969247458735493271268368615238071121335974262337882261418363621  
89704055956429496253759473056373507451253522021344188115187647607  
84555431172202923756940824265247663088847763429436570335188702325  
06644969965841257822711805056447218921550669146263582661876630783  
21357671611262065293901983868557252464459832189159990869820527095  
53646871654914800005753059422066576204781923454823934475242960034  
42199041253798398004263030681714027295470241663946228744550160085  
43856624239377702107746492579014275563017167813144052693763385569  
75239252588060279466314599314734680953729093269435217987689840619  
0740089242444014302.

As experiments showed, our method for the computation of  $\sum_{x \in \text{pr} \cap \mathbb{Z}^d} h(x)$  is very efficient for this small dimension, compared to other available softwares. Furthermore, as Example 36 shows, the Ehrhart polynomial is written

as a sum of canonical contributions of all faces, once the scalar product is fixed. We will come back soon to the computational and complexity aspects of this problem for higher dimensions.

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