



HAL
open science

A family of criteria for irrationality of Euler's constant

Marc Prévost

► **To cite this version:**

Marc Prévost. A family of criteria for irrationality of Euler's constant. *Journal of Computational and Applied Mathematics*, 2008, 219 (no. 2), pp.484–492. hal-00007476

HAL Id: hal-00007476

<https://hal.science/hal-00007476>

Submitted on 12 Jul 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A FAMILY OF CRITERIA FOR IRRATIONALITY OF EULER'S CONSTANT

MARC PRÉVOST

ABSTRACT. Following earlier results of Sondow, we propose another criterion of irrationality for Euler's constant γ . It involves similar linear combinations of logarithm numbers $L_{n,m}$. To prove that γ is irrational, it suffices to prove that, for some fixed m , the distance of $d_n L_{n,m}$ (d_n is the least common multiple of the n first integers) to the set of integers \mathbf{Z} does not converge to 0. A similar result is obtained by replacing logarithms numbers by rational numbers: it gives a sufficient condition involving only rational numbers. Unfortunately, the chaotic behavior of d_n is an obstacle to verify this sufficient condition. All the proofs use in a large manner the theory of Padé approximation.

1. INTRODUCTION

In [Sondow 2003], the author, using Beukers' integral [Beukers 1979], found a criterion for irrationality of Euler's constant γ . It depends on the limit of the fractional part of the following expression

$$L_n = 2 \sum_{k=1}^n \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \ln(n+k)$$

where H_n is the Harmonic number $H_n := \sum_{k=1}^n \frac{1}{k}$. In this paper, we establish the connexion

between Sondow's criteria and Padé approximant of the function $\frac{\ln u}{u-1}$. Moreover, following the same idea, we find a family of new criteria: *for each integer $m \leq n$, let us set*

$$L_{n,m} := \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1), \quad (1.1)$$

$d_n := \text{LCM}(1, \dots, n)$ and $\{x\} = x - \lfloor x \rfloor$ the fractional part of the real x . If, for some integer m , if the sequence $\{d_n (-1)^m L_{n,m}\}$ does not converge to 0 when n tends to infinity, then γ is irrational.

Using the property of the error term, a more precise criterion is proved here: *if, for some integer m , the sequence $\{d_{2^n} (-1)^m L_{2^n, m}\}$ is asymptotically non decreasing, when n tends to infinity, then γ is irrational.*

2000 *Mathematics Subject Classification.* Primary 11J72, Secondary 41A21.

Key words and phrases. Euler's constant, irrationality, Padé approximations.

2. SONDOW'S CRITERION WITH PADÉ APPROXIMANT

Sondow considers the double integral (so-called Beukers' integral)

$$I_n = \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)\ln(xy)} dx dy.$$

Applying Taylor expansion of $1/(1-xy)$ around 0, he proved the following identity

$$I_n = \binom{2n}{n} \gamma + L_n - A_n = \mathcal{O}(2^{-4n}n^{-1/2}) \quad (2.1)$$

where $A_n = \sum_{i=0}^n \binom{n}{i}^2 H_{n+i}$. After multiplication by d_{2n} , it arises

$$d_{2n} \binom{2n}{n} \gamma = d_{2n}(A_n - L_n) + o(1).$$

Sondow's criterion:

Since $d_{2n}A_n \in \mathbf{Z}$, if the sequence of fractional part $(\{d_{2n}L_n\})_n$ does not converge to 0 then $\gamma \notin \mathbf{Q}$.

Sebah computed this sequence for $1 \leq n \leq 2500$. Its cumulative average seems to converge $1/2$, but the mathematical proof remains to establish.

In the following, we will show that the sequence involved in the paper by Sondow can be recovered by means of Padé approximation.

Let us consider the function $(\ln u)/(u-1)$ and its Padé Approximant $[n-1/n]$ of degree $(n-1/n)$ at the point $u=1$:

$$\frac{\ln u}{u-1} = \frac{N_n(u)}{D_n(u)} + R_n(u) \quad (2.2)$$

where N_n and D_n are polynomials of respective degree $n-1$ and n , normalized by $N_n(1) = D_n(1) = 1$, and $R_n(u) = \mathcal{O}(u^{2n})$

From the theory of Padé approximation, it is well known that D_n is related with the shifted Legendre Polynomial orthogonal on the interval $[0,1]$ with respect to the Lebesgue weight function. Some of these expressions are

$$P_n^*(t) = \sum_{k=0}^n \binom{n}{k}^2 t^{n-k} (t-1)^k \quad (2.3)$$

$$= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} t^k \quad (2.4)$$

D_n has the following expression in terms of P_n^*

$$D_n(u) = P_n^* \left(\frac{1}{1-u} \right) (1-u)^n \binom{2n}{n}^{-1}. \quad (2.5)$$

Replacing P_n^* by its expressions (2.3,2.4), formula (2.5) becomes

$$\begin{aligned} D_n(u) &= \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n}{k}^2 u^k \\ &= \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (1-u)^{n-k} u^k. \end{aligned}$$

The numerator $N_n(u)$ of $[n - 1/n]$, is related with the associated polynomial of the denominator:

$$\begin{aligned} N_n(u) &= 2 \binom{2n}{n}^{-1} \sum_{k=1}^n \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) u^{k-1} \\ &= \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{i=0}^{k-1} (u-1)^{n-k+i} \frac{(-1)^i}{i+1}. \end{aligned}$$

Now, what is the link between Padé Approximation and Sondow's criterion?

The definition of γ is, primarily,

$$\gamma = \lim_n (H_n - \ln n).$$

An integral representation for Euler's constant is

$$\gamma = \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du. \quad (2.6)$$

Formula (2.2) can be rewritten as

$$\frac{N_n(u)}{\ln u} + \frac{D_n(u)}{1-u} = -\frac{R_n(u)D_n(u)}{\ln u}$$

and

$$\gamma = \int_0^1 \frac{1 - u^n N_n(u)}{\ln u} du + \int_0^1 \frac{1 - u^n D_n(u)}{1-u} du - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln u} du$$

By linearity, the second term is expanded as

$$\int_0^1 \frac{1 - u^n D_n(u)}{1 - u} du = \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n}{k}^2 \int_0^1 \frac{u^{n+k} - 1}{u - 1} du \quad (2.7)$$

$$= \binom{2n}{n}^{-1} \sum_{k=0}^n \binom{n}{k}^2 H_{n+k} \quad (2.8)$$

$$= \binom{2n}{n}^{-1} A_n. \quad (2.9)$$

The first integral can be computed as following:

$$\int_0^1 \frac{1 - u^n N_n(u)}{\ln(u)} du = 2 \binom{2n}{n}^{-1} \sum_{k=1}^n \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \int_0^1 \frac{1 - u^{n+k-1}}{\ln(u)} du \quad (2.10)$$

$$= -2 \binom{2n}{n}^{-1} \sum_{k=1}^n \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \ln(n+k) \quad (2.11)$$

$$= - \binom{2n}{n}^{-1} L_n. \quad (2.12)$$

From the theory of Padé approximation, in formula (2.2), the remainder term R_n has an integral representation

$$R_n(u) = \frac{(1-u)^n}{D_n(u)} \int_0^1 \frac{t^n D_n(1-1/t)}{1 - (1-u)t} dt.$$

Thanks to formulas (2.9,2.12), γ satisfies

$$\gamma = \frac{A_n - L_n}{\binom{2n}{n}} - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln(u)} du.$$

Thus another expression of the remainder term I_n of Sondow is

$$\begin{aligned} I_n &= \binom{2n}{n} \gamma - A_n + L_n = - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln(u)} du \\ &= \int_0^1 \int_0^1 \frac{u^n (1-u)^n}{\ln(u)} \frac{P_n^*(t)}{1 - (1-u)t} dt du \\ &= \int_0^1 \int_0^1 \frac{u^n (1-u)^{2n}}{\ln(u)} \frac{t^n (1-t)^n}{(1 - (1-u)t)^{n+1}} dt du \end{aligned}$$

thanks to integration by parts and Rodrigues formula for orthogonal polynomials.

Thus the approximation for Euler's constant γ (2.1) is a consequence of the Padé approximation to the function $(\ln u)/(1-u)$.

In the same manner, Pilehrood [Pilehrood 2004] found irrationality criteria for generalized Euler's constant. He defined the following linear form in logarithms

$$L_{(n_1, n_2)}(\alpha) = \sum_{m=1}^{n_1} \sum_{k=0}^{m-1} \binom{n_1}{k} \binom{n_2}{k} (H_{n_1-k} + H_{n_2-k} - 2H_k) \ln(m + n_1 + \alpha - 1) + \sum_{m=n_1+1}^{n_2} \sum_{k=m}^{n_2} (-1)^{k-1-n_1} / k \binom{n_2}{k} / \binom{k-1}{n_1} (\ln(m + n_1 + \alpha - 1))$$

Actually, following the same idea as for Sondow's criterion, it is possible to prove that Pilehrood's criterion comes from Padé approximations $[n_2 - 1, n_1] = \frac{R_{n_2-1}(u)}{S_{n_1}(u)}$ (normalized by $R_{n_2-1}(1) = S_{n_1}(1) = 1$ to the function $\ln(u)/(u-1)$ at the point $u = 1$). The linear form $L_{(n_1, n_2)}(\alpha)$ satisfies:

$$L_{(n_1, n_2)}(\alpha) = \int_0^1 (1 - u^{n_1+\alpha-1} R_{n_2-1}(u)) \frac{1}{\ln(u)} du$$

3. STATEMENT OF THE RESULTS

In order to simplify Sondow's criterion, it is convenient to choose a more simple approximation. This method leads to the following theorems.

Theorem 1. For $0 \leq m \leq n$, let us define

$$\begin{aligned} J_{n,m} &= \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du \\ L_{n,m} &= - \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} \right) du \\ A_{n,m} &= \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{1-u} \right) du \end{aligned}$$

then

$$\gamma = A_{n,m} - L_{n,m} + J_{n,m} \quad (3.1)$$

$$L_{n,m} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1) \quad (3.2)$$

$$A_{n,m} = 2H_n \quad (3.3)$$

Theorem 2. The following are equivalent:

- (a) The fractional part of $d_n L_{n,m}$ is given by $\{d_n (-1)^m L_{n,m}\} = d_n (-1)^m J_{n,m}$ (*) for some n or m .
- (b) The formula (*) holds for some m and for all sufficiently large n .
- (c) Euler's constant is a rational number.

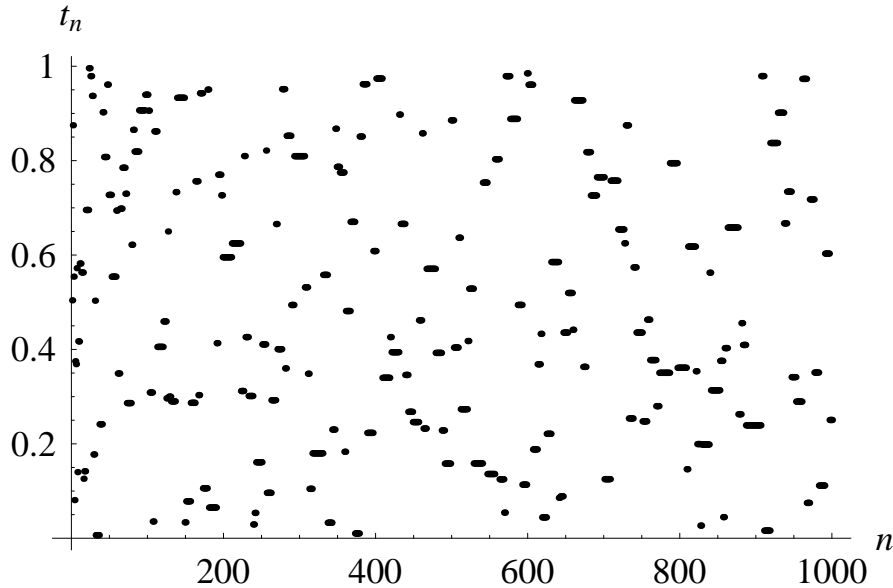


FIGURE 1. plot of values of $t_n = \{d_n(-1)^m L_{n,m}\}$, for $m = 0$

A sufficient condition which involves $d_n L_{n,m}$ but not $J_{n,m}$ is the following

Theorem 3. *If for some integer m , $\{d_n(-1)^m L_{n,m}\} \geq 0.707^n$ infinitely often, then γ is irrational.*

Computations (see Table 1) show that this condition is satisfied for $n \leq 1000$. Numerical results also suggest that for each m , $\{d_n(-1)^m L_{n,m}\}$ is dense in the interval $(0,1)$ and the cumulative average $n^{-1} \sum_{k=1}^{\infty} \{d_k(-1)^m L_{k,m}\}$ converges to 0.5 (see Figure 1 and 2).

To prove γ irrational, it just suffices to show that $\{d_n(-1)^m L_{n,m}\}$ does not converge to 0.

In section 6, we will prove the asymptotic formula

$$\gamma = A_{n,m} - L_{n,m} + \mathcal{O}(4^{-n}). \quad (3.4)$$

Actually, we will prove that the error term $J_{n,m}$ is a totally monotone sequence (i.e. a sequence of moments with respect a positive measure), converging to 0 as 4^{-n} .

By substituting in $L_{n,m}$, $\ln(n+1+k-m)$ by some suitable Padé approximants, a sufficient condition, involving only rational numbers is the following

Corollary 1. *Let us define, for $n-m+1 = 2^p$, $p \in \mathbf{Z}$,*

$$\tilde{L}_{n,m} := p [n/n]_{t=1} + \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} [n/n]_{t=k/(n-m+1)} \quad (3.5)$$

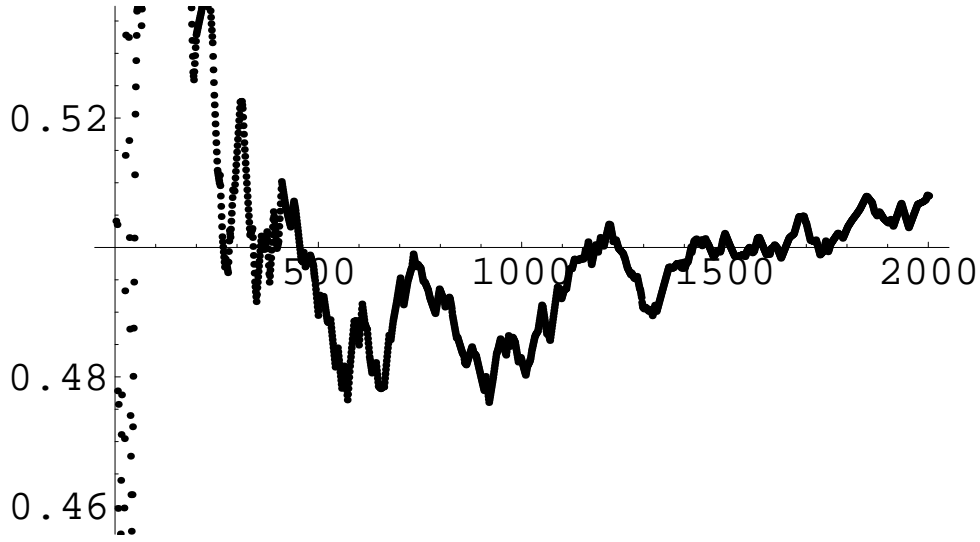


FIGURE 2. plot of values of $\langle t_n \rangle = n^{-1} \sum_{k=1}^n t_k$

where $[n/n]$ is the Padé approximant of $\ln(1+t)$ at $t=0$:

$$[n/n]_t = \frac{t \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\sum_{i=0}^{k-1} \frac{t^{i-k+n} (-1)^i}{i+1} \right)}{\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} t^{n-k}}$$

If for some integer m , $\{d_{2^p}(-1)^m \tilde{L}_{2^p+m-1,m}\}$ does not converge to 0 when p tends to infinity, then γ is irrational.

Another sufficient condition comes from the property of the error term in the asymptotic formula (3.4) and from the upper and lower bound of the $LCM(1, \dots, n)$:

Corollary 2. *If for some m , $\{d_{2^p}(-1)^m L_{2^p,m}\}$ is asymptotically non decreasing, then γ is irrational.*

4. TWO LEMMAS

Lemma 1. *The function $\frac{1}{\ln(1-u)} + \frac{1}{u}$ is a Markov-Stieltjes function. More precisely,*

$$\frac{1}{\ln(1-u)} + \frac{1}{u} = \int_0^1 \frac{1}{1-ut} w(t) dt \quad (4.1)$$

TABLE 1.

n	$\frac{.7^n}{\{d_n L_{n,0}\}}$	$\frac{.7^n}{\{-d_n L_{n,1}\}}$	$\frac{.7^n}{\{d_n L_{n,2}\}}$	$\frac{.7^n}{\{-d_n L_{n,3}\}}$
1	1.38868	1.81209	————	————
2	0.56003	0.58439	0.56609	————
3	0.61882	0.64252	0.63428	0.67030
4	2.97160	3.31151	3.23310	0.38225
5	0.44808	0.45886	0.45719	0.45913
6	0.31896	0.32064	0.32044	0.32061
7	0.14391	0.14467	0.14460	0.14465
8	0.41138	0.41543	0.41511	0.41528
9	0.09667	0.09689	0.09687	0.09688
10	0.06778	0.06781	0.06781	0.06781
11	0.03395	0.03398	0.03398	0.03398
12	0.02378	0.02379	0.02379	0.02379
13	0.01719	0.01721	0.01720	0.01720
14	0.01204	0.01204	0.01204	0.01204
15	0.00843	0.00843	0.00843	0.00843
16	0.02637	0.02637	0.02637	0.02637
17	0.01639	0.01639	0.01639	0.01639
18	0.01147	0.01147	0.01147	0.01147
19	0.00163	0.00163	0.00163	0.00163
20	0.001147	0.00114	0.00114	0.00114

where the weight function w is

$$w(t) := \frac{1}{t (\ln^2(1/t - 1) + \pi^2)}$$

Proof. After a change of variable ($u \rightarrow (1 - u)$ and $x = 1/t - 1$), formula (4.1) is equivalent to

$$\frac{1}{\ln(u)} + \frac{1}{1 - u} = \int_0^\infty \frac{1}{x + u} \frac{1}{\ln^2 x + \pi^2} dt \quad (4.2)$$

The weight function w can be found with the Stieltjes inversion formula (see [Widder 1941]). Another way to prove formula (4.2) is to apply residue theorem to the function

$$f(x) := \frac{1}{x + u} \frac{1}{\ln x + i\pi}.$$

Taking the determination of $\ln x$ on the complex plane cut along the positive real axis, the poles of f are $x = -u$ and $x = -1$.

Let us define γ_r a small semi-circle $z = re^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$, $r > 0$. D_r^+ the line $z = x + ir$, x running from 0 to R , Γ_R the circle $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ and D_r^- the line $z = x - ir$, for x from R to 0.

Now, we compute $\int_{\mathcal{C}} f(x) dx$ where \mathcal{C} is the union of D_r^+ , Γ_R , D_r^- and γ_r , with the theorem of residue to obtain

$$\begin{aligned} \int_0^\infty \frac{1}{x+u} \left(\frac{1}{\ln x + i\pi} + \frac{-1}{\ln x - i\pi} \right) dx &= \int_0^\infty \frac{1}{x+u} \left(\frac{-2i\pi}{\ln^2 x + \pi^2} \right) dx \\ &= -2i\pi \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \end{aligned}$$

□

Now, we are in position to prove a new formula for the Euler's constant γ .

Theorem 4. *The Euler's constant γ satisfies*

$$\gamma = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{-z})e^z}{z^2 + \pi^2} dz$$

Proof. In the integral representation of γ (2.6), let us substitute the integrand by the expression (4.1). This leads to

$$\begin{aligned} \gamma &= \int_0^1 -\frac{\ln(1-t)}{t} \frac{1}{t(\ln^2(1/t-1) + \pi^2)} dt \\ &= \int_{-\infty}^{\infty} \frac{\ln(1 + e^{-z})e^z}{z^2 + \pi^2} dz \end{aligned}$$

with the change of variable $t = (1 + e^z)^{-1}$.

□

Lemma 2. *For each fixed integer m , the sequence $((-1)^m J_{n,m})_n$ defined in Theorem 2 is totally monotonic. More precisely*

$$(-1)^m J_{n,m} = \int_0^{1/4} v^n \rho_m(v) dv$$

where the weight function is

$$\rho_m(v) = \int_{\frac{1-\sqrt{1-4v}}{2}}^{\frac{1+\sqrt{1-4v}}{2}} \left(\frac{u - u^2 - v}{u v} \right)^m \frac{1}{(u - u^2 - v) \left(\pi^2 + \ln^2 \left(\frac{-u v}{u^2 - u + v} \right) \right)} du.$$

Proof. $J_{n,m} = \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du$ appears as Legendre modified moments of the weight function $\left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du$.

For some particular cases of weight function, a sequence of modified moments can be itself a sequence of moments, with respect to a positive measure (see [Prévost 1994]). Using Rodrigues formula for orthogonal polynomials, Lemma 1, Fubini's theorem and after n integrations by parts, it arises

$$\begin{aligned} J_{n,m} &= \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{d^n}{du^n} (u^n(1-u)^n) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du \\ &= \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{d^n}{du^n} (u^n(1-u)^n) du \int_0^1 \frac{1}{1-(1-u)t} w(t) dt \\ &= \int_0^1 \int_0^1 \frac{(-1)^n}{n!} u^n(1-u)^n du \frac{d^n}{du^n} \left(\frac{u^{n-m}}{1-(1-u)t} \right) w(t) dt. \end{aligned}$$

The computation of $\frac{d^n}{du^n} \left(\frac{u^{n-m}}{1-(1-u)t} \right)$ needs the partial decomposition of the rational function $\frac{u^{n-m}}{1-(1-u)t} = q(u) + \left(\frac{t-1}{t} \right)^{n-m} \frac{1}{1-(1-u)t}$, where q is polynomial of degree $n-m-1$.

Another expression of $J_{n,m}$ is then

$$J_{n,m} = \int_0^1 \int_0^1 u^n(1-u)^n \left(\frac{t-1}{t} \right)^{n-m} \frac{t^n}{(1-(1-u)t)^{n+1}} w(t) dt du.$$

We do the following change of variable

$$v = \frac{u(1-u)(1-t)}{1-(1-u)t} \in [0, 1/4] \Leftrightarrow t = \phi(v) = \frac{u^2 - u + v}{(u-v)(u-1)} \in [0, 1].$$

Let $\phi_1(v)$ and $\phi_2(v)$ denote the two roots of the quadratic equation $v = u - u^2$,
 $\phi_1(v) = \frac{1 + \sqrt{1-4v}}{2}$, $\phi_2(v) = \frac{1 - \sqrt{1-4v}}{2}$.

$$J_{n,m} = \int_0^{1/4} v^n dv \int_{\phi_1(v)}^{\phi_2(v)} (-1)^m \left(\frac{\phi(v)}{\phi(v)-1} \right)^m w(\phi(v)) \frac{u^2}{(u-1)(u-v)^2} \frac{(-1)^m}{1-(1-u)\phi(v)} du$$

which proves the lemma. □

5. PROOF OF THEOREM 1

We first prove the identity (3.1) linking Euler's constant γ , the linear combination of logarithms numbers $L_{n,m}$, the rational numbers $A_{n,m}$ and the integrals $J_{n,m}$. From formula (2.6), one substitute the integrand $\left(\frac{1}{\ln u} + \frac{1}{1-u} \right)$ by an approximation involving

Legendre Polynomials as follows:

$$\begin{aligned}\gamma &= \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du \\ &= \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} + \frac{1 - u^{n-m} P_n^*(u)}{1-u} \right) du + \\ &\quad \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) du.\end{aligned}$$

The expression (2.4) of P_n^* leads to analogous expressions $L_{n,m}$.

By linearity

$$\begin{aligned}L_{n,m} &= - \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \int_0^1 \left(\frac{1 - u^{k+n-m}}{\ln u} \right) \\ &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1)\end{aligned}$$

$A_{n,m}$ is treated quite differently:

P_n^* satisfies the following orthogonality relation

$$\int_0^1 P_n^*(u) q(u) du = 0, \text{ for all polynomial } q \text{ of degree less than } n.$$

Thus, by taking $q(u) = \frac{1 - u^{n-m}}{1-u}$, another expression for $A_{n,m}$ is

$$A_{n,m} = \int_0^1 \frac{1 - P_n^*(u)}{1-u} du = \int_0^1 \frac{P_n^*(1) - P_n^*(u)}{1-u} du \quad (5.1)$$

and so $A_{n,m}$ is independent of $0 \leq m \leq n$.

Let us now compute the integral in (5.1).

Legendre polynomials satisfy a three term recurrence relation which is

$$\begin{aligned}(n+1)P_{n+1}^*(u) &= (2n+1)(2u-1)P_n^*(u) - nP_{n-1}^*(u) \\ P_0^*(u) &= 1 \quad P_1^*(u) = 2u-1\end{aligned}$$

Thus, $A_{n,m}$'s also satisfy a similar recurrence relation

$$(n+1)A_{n+1,m} = (2n+1)(2u-1)A_{n,m}(u) - nA_{n-1,m}(u) \quad (5.2)$$

$$A_0 = 0 \quad A_1 = 2 \quad (5.3)$$

With (5.2) and (5.3), it is not difficult to prove that

$$A_{n,m} = 2H_n, 0 \leq m \leq n$$

6. PROOF OF THEOREM 2 AND THEOREM 3

All the arguments are based on the formula (3.1).

$$\gamma = A_{n,m} - L_{n,m} + J_{n,m} \iff d_n \gamma = d_n A_{n,m} - d_n L_{n,m} + d_n J_{n,m}$$

Thus,

$$d_n \gamma \in \mathbf{Z} \iff d_n L_{n,m} - d_n J_{n,m} \in \mathbf{Z} \quad (6.1)$$

$$\iff \{d_n (-1)^m L_{n,m}\} = \{d_n (-1)^m J_{n,m}\} \quad (6.2)$$

since $A_{n,m} = 2H_n$ and $(-1)^m J_{n,m}$ is positive.

On the other hand, Lemma 2 implies that the sequence $(J_{n,m})_n$ converges to 0 as 4^{-n} . The numbers d_n converges to infinity as e^n . Thus $d_n (-1)^m J_{n,m}$ is decreasing to 0. So, for all sufficiently large n , $\{d_n (-1)^m J_{n,m}\} = d_n (-1)^m J_{n,m}$.

If γ is a rational number, then $d_N \gamma \in \mathbf{Z}$ for some N and for all $n \geq N$, $d_n \gamma \in \mathbf{Z}$. (c) \implies (b) follows from the previous arguments. Thus (c) \implies (b) \implies (a) \implies (c).

In (6.2), we substitute d_n by an upper bound : $d_n \leq e^{1.039 n}$ [Rosser et al. 1962]. Thus $\forall n$, $d_n (-1)^m J_{n,m} \leq e^{1.039 n} 4^{-n} < 0.707^n$ and Theorem 3 is proved.

7. PROOF OF COROLLARIES

1) In the numerical computation of formula

$$L_{n,m} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1),$$

the problem is the evaluation of logarithmic functions.

A mean to avoid this drawback is the substitution of $\ln(n-m+k+1)$ by some suitable approximations, enough good to keep the irrationality criteria. We will show now that Padé approximants satisfy this condition:

another expression of $L_{n,m}$ is

$$L_{n,m} = \ln(n-m+1) + \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln \left(1 + \frac{k}{n-m+1} \right).$$

The Padé error for the logarithmic function is

$$\ln(1+x) - [n/n]_x = \frac{(-1)^n x^{n+1}}{P_n^*(-1/x)} \int_0^1 \frac{t^n (1-t)^n}{(1+xt)^{n+1}} dt \quad (7.1)$$

Let us set

$$L'_{n,m} := \ln(n-m+1) + \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} [n/n]_{t=k/(n-m+1)}$$

We have to evaluate the difference $\delta_{n,m} := L_{n,m} - L'_{n,m}$. For sake of simplicity, we set $\zeta_k = \frac{k}{n-m+1}$.

$$\begin{aligned}
\delta_{n,m} &= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} (\ln(1 + \zeta_k) - [n/n]_{t=\zeta_k}) \\
&= \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k \frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})} \int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} dt
\end{aligned}$$

Since $\zeta_k \in [0, 1]$ and P_n^* has all its roots in $[0, 1]$, $\left| \frac{\zeta_k^n}{P_n^*(-\zeta_k^{-1})} \right| \leq \frac{1}{|P_n^*(-1)|}$. On the other hand, the integral

$$\begin{aligned}
\int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} dt &\leq 4^{-n} \int_0^1 \frac{1}{(1 + \zeta_k t)^{n+1}} dt \\
&\leq 4^{-n} \frac{1}{n \zeta_k}.
\end{aligned}$$

So,

$$\begin{aligned}
|\delta_{n,m}| &\leq \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left| \frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})} \right| \left| \int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} dt \right| \\
&\leq \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{\zeta_k}{|P_n^*(-1)|} 4^{-n} \frac{1}{n \zeta_k} \\
&\leq \frac{1}{|nP_n^*(-1)|} 4^{-n} \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = \frac{1}{|nP_n^*(-1)|} 4^{-n} |P_n^*(-1)| = (n 4^n)^{-1}
\end{aligned}$$

The goal is partly reached since the error between $L_{n,m}$ and its approximation is less than $J_{n,m}$. Now, let us consider the approximation of $\ln(n - m + 1)$. It is difficult to approximate this number (which tends to infinity) with an error less than 4^{-n} . So, we consider sequences of integers n , such that $n - m + 1$ is a power of 2: $n - m + 1 = 2^p$. With this hypothesis, $\ln(n - m + 1) = p \ln 2$

In (7.1), if $x = 1$, $\ln 2 - [n/n]_{x=1} = \frac{(-1)^n}{P_n^*(-1)} \int_0^1 \frac{t^n (1-t)^n}{(1+t)^{n+1}} dt$. The asymptotics for Legendre polynomials are well known

$$P_n(\alpha) \sim (\alpha + \sqrt{\alpha^2 - 1})^n, \text{ for } \alpha \in \mathbf{R} \setminus [-1, 1].$$

Thus shifted Legendre Polynomials satisfy

$$P_n^*(t) \sim ((2t - 1) + 2\sqrt{t^2 - t})^n, \text{ for } t \in \mathbf{R} \setminus [0, 1].$$

The maximum of the fraction $\left(\frac{t(1-t)}{1+t}\right)$ for $t \in [0, 1]$ is obtained for $t = \sqrt{2} - 1$, and its value is $(3 - 2\sqrt{2})$. Thus

$$|\ln 2 - [n/n]_{x=1}| \leq \frac{(3 - 2\sqrt{2})^n}{(3 + 2\sqrt{2})^n} \ln 2$$

For $n - m + 1 = 2^p$, $\ln(2^p) - p [n/n]_{x=1} \leq p (3 - 2\sqrt{2})^{2^n}$ which is a $o(4^{-n}/n)$. At last, the error $|L_{n,m} - \tilde{L}_{n,m}|$ satisfies

$$|L_{n,m} - \tilde{L}_{n,m}| \leq (4^{-n}/n)$$

and the corollary 1 is proved.

2) For the proof of Corollary 2, we exploit the property of totally monotonic sequences (TMS).

A sequence u_n is called TMS if there exists a non negative measure $d\mu$ with infinitely many points of increase such that

$$\forall n \in \mathbf{N}, u_n = \int_0^\infty x^n d\mu(x).$$

If the support of the measure $d\mu$ is the interval $[0, 1/R]$, then $\forall n, u_{n+1}/u_n \leq R$ and $\lim_n \frac{u_{n+1}}{u_n} = R$. If $R = 1$, it is equivalent to

$$\forall n \in \mathbf{N}, \forall k \in \mathbf{N}, (-1)^k \Delta^k(u_n) > 0$$

where $\Delta^0(u_n) := u_n$ and $\Delta^{k+1}u_n = \Delta^k u_{n+1} - \Delta^k u_n$. (see [Widder 1941], p. 108).

The previous properties can be applied to the sequence $J_{n,m}$ for which we prove some convergence properties. If they are not satisfied by $\{d_n(-1)^m L_{n,m}\}$ then γ is irrational.

First we will prove that $J_{n,m}$ satisfies $d_{2n}(-1)^m J_{2n,m} < d_n(-1)^m J_{n,m}$: the numbers d_n and $J_{n,m}$ satisfy $2^n \leq d_n < e^{1.039n}$ (see [Tenenbaum 1990], p.12-13 for the lower bound and [Rosser et al. 1962] for the upper one) $\frac{J_{n+1,m}}{J_{n,m}} < 1/4$ (property of totally monotonic sequence [Widder 1941], p.135).

$$\frac{d_n J_{n,m}}{d_{2n} J_{2n,m}} > \frac{2^n}{e^{1.039 \times 2n}} 4^n > 1.0014$$

Thus, for all integer m , $(d_{2^p}(-1)^m J_{2^p,m})_{p \in \mathbf{N}}$ is a positive decreasing sequence, converging to 0. So, if $(\{d_{2^p}(-1)^m L_{2^p,m}\})_p$ is non decreasing for p greater than any integer, then γ is irrational.

I would thank my colleague S. Eliahou for reference ([Pilehrood 2004])

BIBLIOGRAPHY

- [Beukers 1979] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$* , Bull. London Math. Soc. **11**, 268-272 (1979).
- [Rosser et al. 1962] J. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6**, 64-94 (1962).
- [Sondow 2003] J. Sondow, *Criteria for irrationality of Euler's constant*, Proc. Amer. Math. Soc. **131** (2003), 3335-3344.
- [Pilehrood 2004] T. Hessami Pilehrood, Kh. Hessami Pilehrood, *Criteria for irrationality of generalized Euler's constant*, J. of Number Theory, 108 (2004) 169-185.
- [Prévost 1994] M. Prévost, *Acceleration of some logarithmic sequences*, J. Comput. Appl. Math. **55**, No.3, 357-367 (1994).
- [Tenenbaum 1990] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Université de Nancy (1990).
- [Widder 1941] D. V. Widder, *The Laplace transform*, Princeton Mathematical Series, Princeton University Press, N. J. X, 406 p. (1941).

E-mail address: `prevost@lmpa.univ-littoral.fr`

LMPA JOSEPH LIOUVILLE, CENTRE UNIVERSITAIRE DE LA MI-VOIX, BÂT. H. POINCARÉ, 50 RUE F. BUISSON, BP 699, 62228 CALAIS CEDEX, FRANCE