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Vincent Beffara, Vladas Sidoravicius

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Percolation theory

Vincent BEFFARA — CNRS, UMPA, ENS Lyon
Vladas SIDORAVICIUS — IMPA, Rio de Janeiro

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Introduction

Percolation as a mathematical theory was introduced by Broadbent and Hammersley [4], as a stochastic way of modeling the flow of a fluid or gas through a porous medium of small channels which may or may not let gas or fluid pass. It is one of the simplest models exhibiting a phase transition, and the occurrence of a critical phenomenon is central to the appeal of percolation. Having truly applied origins, percolation has been used to model the fingering and spreading of oil in water, to estimate whether one can build non-defective integrated circuits, to model the spread of infections and forest fires. From a mathematical point of view percolation is attractive because it exhibits relations between probabilistic and algebraic/topological properties of graphs.

To make the mathematical construction of such a system of channels, take a graph $\mathcal{G}$ (which originally was taken as $\mathbb{Z}^d$), with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and make all the edges independently *open* (or passable) with probability p or *closed* (or blocked) with probability $1-p$. Write P_p for the corresponding probability measure on the set of configurations of open and closed edges — that model is called *bond percolation*. The collection of open edges thus forms a random subgraph of $\mathcal{G}$, and the original question stated by Broadbent was whether the connected component of the origin in that subgraph is finite or infinite.

A *path* on $\mathcal{G}$ is a sequence $v_1, v_2, \dots$ of vertices of $\mathcal{G}$, such that for all $i \geq 1$, v_i and v_{i+1} are adjacent on $\mathcal{G}$. A path is called *open* if all the edges $\{v_i, v_{i+1}\}$ between successive vertices are open. The infiniteness of the cluster of the origin is equivalent to the existence of an unbounded open path starting from the origin.

There is an analogous model, called *site percolation*, in which all edges are assumed being passable, but the vertices are independently open or closed with probability p or $1-p$, respectively. An open path is then a path along which all vertices are open. Site percolation is more general than bond percolation in the sense that the existence of a path for bond percolation on a graph $\mathcal{G}$ is equivalent to the existence of a path for site percolation on the covering graph of $\mathcal{G}$. However, site percolation on a given graph may not be equivalent to bond percolation on any other graph.

All graphs under consideration will be assumed to be connected, locally finite and quasi-transitive. If $A, B \subset \mathcal{V}$, then $A \leftrightarrow B$ means that there exists an open path from some vertex of A to some vertex of B; by a slight abuse of notation, $u \leftrightarrow v$ will stand for the existence of a path between sites u and v, *i.e.* the event $\{u\} \leftrightarrow \{v\}$. The *open cluster* $C(v)$ of the vertex v is the set of all open vertices which are connected to v by an open path:

$$C(v) = \{u \in \mathcal{V} : u \leftrightarrow v\}.$$

The central quantity of the percolation theory is the *percolation probability*:

$$\theta(p) := P_p\{\mathbf{0} \leftrightarrow \infty\} = P_p\{|C(\mathbf{0})| = \infty\}.$$

The most important property of the percolation model is that it exhibits a *phase transition*, *i.e.* there exists a threshold value $p_c \in [0,1]$, such that the global behavior of the system is substantially different in the two regions $p < p_c$ and $p > p_c$. To make this precise, observe that θ is a non-decreasing function. This can be seen using Hammersley's joint construction of percolation systems for all $p \in [0,1]$ on $\mathcal{G}$: Let $\{U(v), v \in \mathcal{V}\}$ be independent random variables, uniform in $[0,1]$. Declare v to be p-open if $U(v) \leq p$, otherwise it is declared p-closed. The configuration of p-open vertices has the distribution P_p for each $p \in [0,1]$. The collection of p-open vertices is non-decreasing in p, and therefore $\theta(p)$ is non-decreasing as well. Clearly $\theta(0) = 0$ and $\theta(1) = 1$.

The *critical probability* is defined as

$$p_c := p_c(\mathcal{G}) = \sup\{p : \theta(p) = 0\}.$$

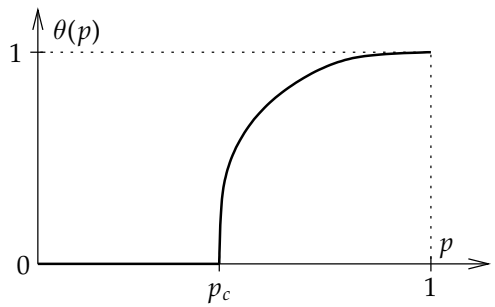

Figure 1: The behavior of $\theta(p)$ around the critical point (for bond-percolation)

By definition, when $p < p_c$ the open cluster of the origin is P_p-a.s. finite, hence all the clusters are also finite. On the other hand, for $p > p_c$ there is a strictly positive P_p-probability that the cluster of the origin is infinite. Thus, from Kolmogorov's zero-one law it follows that

$$P_p\{|C(v)| = \infty \text{ for some } v \in \mathcal{V}\} = 1 \text{ for } p > p_c.$$

Therefore, if the intervals $[0, p_c)$ and $(p_c, 1]$ are both non-empty, there is a phase transition at p_c.

Using a so-called Peierls argument it is easy to see that $p_c(\mathcal{G}) > 0$ for any graph $\mathcal{G}$ of bounded degree. On the other hand, Hammersley proved that $p_c(\mathbb{Z}^d) < 1$ for bond percolation as soon as $d \geq 2$, and a similar argument works for site percolation and various periodic graphs as well. But for some graphs $\mathcal{G}$ it is not so easy to show that $p_c(\mathcal{G}) < 1$. One says that the system is in the *subcritical (resp. supercritical) phase* if $p < p_c$ (resp. $p > p_c$).

It was one of the most remarkable moments in the history of percolation when Kesten proved [10] that the critical parameter for bond-percolation on $\mathbb{Z}^2$ is equal to $1/2$. Nevertheless the exact value of $p_c(\mathcal{G})$ is known only for a handful of graphs, all of them periodic and two-dimensional — see below.

1 Percolation in $\mathbb{Z}^d$

The graph on which most of the theory was originally built is the cubic lattice $\mathbb{Z}^d$, and it was not before the late 20th century that percolation was seriously considered on other kinds of graphs (such as *e.g.* Cayley graphs), on which specific phenomena can appear, such as the coexistence of multiple infinite clusters for some values of the parameter p. In all this section, the underlying graph is thus assumed to be $\mathbb{Z}^d$ for $d \geq 2$, although most of the results still hold in the case of a periodic d-dimensional lattice.

1.1 The sub-critical regime

When $p < p_c$, all open clusters are finite almost surely. One of the greatest challenges in percolation theory has been to prove that $\chi(p) := E_p\{|C(v)|\}$ is finite if $p < p_c$ (E_p stands for the expectation with respect to P_p). For that one can define another critical probability as the threshold value for the finiteness of the expected cluster size of a fixed vertex:

$$p_T(\mathcal{G}) := \sup\{p : \chi(p) < \infty\}.$$

It was an important step in the development of the theory to show that $p_T(\mathcal{G}) = p_c(\mathcal{G})$. The fundamental estimate in the subcritical regime, which is a much stronger statement than $p_T(\mathcal{G}) = p_c(\mathcal{G})$, is the following:

Theorem 1 (Aizenman and Barsky, Menshikov) *Assume that $\mathcal{G}$ is periodic. Then for $p < p_c$ there exist constants $0 < C_1, C_2 < \infty$, such that*

$$P_p\{|C(v)| \geq n\} \leq C_1 e^{-C_2 n}.$$

The last statement can be sharpened to a "local limit theorem" with the help of a subadditivity argument : For each $p < p_c$ there exists a constant $0 < C_3(p) < \infty$, such that

$$\lim_{n\to\infty} -\frac{1}{n} \log P_p\{|C(v)| = n\} = C_3(p).$$

1.2 The super-critical regime

Once an infinite open cluster exists, it is natural to ask how it looks like, and how many infinite open clusters exist. It was shown by Newman and Schulman that for periodic graphs, for each p, exactly one of the following three situations prevails: If $N \in \mathbb{Z}_+ \cup \{\infty\}$ is the number of infinite open clusters, then $P_p(N = 0) = 1$, or $P_p(N = 1) = 1$, or $P_p(N = \infty) = 1$.

Aizenman, Kesten and Newman showed that the third case is impossible on $\mathbb{Z}^d$. By now several proofs exist, perhaps the most elegant proof of that is due to Burton and Keane, who prove that indeed there cannot be infinitely many infinite open clusters on any amenable graph. However, there are some graphs, such as regular trees, on which coexistence of several infinite clusters is possible.

The geometry of the infinite open cluster can be explored in some depth by studying the behavior of a random walk on it. When $d = 2$, the random walk is recurrent, and when $d \geq 3$ is a.s. transient. In all dimensions $d \geq 2$ the walk behaves diffusively, and the Central Limit Theorem and the Invariance principle were established in both the annealed and quenched cases.

Wulff droplets

In the supercritical regime, aside from the infinite open cluster, the configuration contains finite clusters of arbitrary large sizes. These large finite open clusters can be thought of as droplets swimming in the areas surrounded by an infinite open cluster. The presence at a particular location of a large finite cluster is an event of low probability, namely, on $\mathbb{Z}^d$, $d \geq 2$, for $p > p_c$, there exist positive constants $0 < C_4(p), C_5(p) < \infty$, such that

$$C_4(p) \leq -\frac{1}{n^{(d-1)/d}} \log P_p\{|C(v)| = n\} \leq C_5(p)$$

for all large n. This estimate is based on the fact that the occurrence of a large finite cluster is due to a surface effect. The typical structure of the large finite cluster is described by the following theorem:

Theorem 2 *Let $d \geq 2$, and $p > p_c$. There exists a bounded, closed, convex subset W of $\mathbb{R}^d$ containing the origin, called the* normalized Wulff crystal *of the Bernoulli percolation model, such that, under the conditional probability $P_p\{\cdot \mid n^d \leq |C(\mathbf{0})| < \infty\}$, the random measure*

$$\frac{1}{n^d} \sum_{x \in C(\mathbf{0})} \delta_{x/n}$$

(where $\delta_{\mathbf{x}}$ denotes a Dirac mass at $\mathbf{x}$) converges weakly in probability towards the random measure $\theta(p)\mathbb{1}_W(x - M)dx$ (where M is the rescaled center of mass of the cluster $C(\mathbf{0})$). The deviation probabilities behave as $\exp\{-cn^{d-1}\}$ (i.e. *they exhibit large deviations of surface order).*

This result was proved in dimension 2 by Alexander, Chayes and Chayes [1], and in dimensions 3 and more by Cerf [5].

1.3 Percolation near the critical point

1.3.1 Percolation in slabs

The main macroscopic observable in percolation is $\theta(p)$, which is positive above p_c, 0 below p_c, and continuous on $[0,1] \setminus \{p_c\}$. Continuity at p_c is an open question in the general case; it is known to hold in two dimensions (cf. below) and in high enough dimension (at the moment $d \geq 19$ though the value of the critical dimension is believed to be 6) using lace expansion methods. The conjecture that $\theta(p_c) = 0$ for $3 \leq d \leq 18$ remains one of the major open problems.

Efforts to prove that led to some interesting and important results. Barsky, Grimmett and Newman solved the question in the half-space case, and simultaneously showed that the slab percolation and half-space percolation thresholds coincide. This was complemented by Grimmett and Marstrand showing that

$$p_c(slab) = p_c(\mathbb{Z}^d).$$

1.3.2 Critical exponents

In the sub-critical regime, exponential decay of the correlation indicates that there is a finite *correlation length* $\xi(p)$ associated to the system, and defined (up to constants) by the relation

$$P_p(0 \leftrightarrow n\mathbf{x}) \approx \exp\left(-\frac{n\varphi(x)}{\xi(p)}\right)$$

where φ is bounded on the unit sphere (this is known as *Ornstein-Zernike decay*). The phase transition can then also be defined in terms of the divergence of the correlation length, leading again to the same value for p_c; the behavior at or near the critical point then has no finite characteristic length, and gives rise to scaling exponents (conjecturally in most cases).

The most usual critical exponents are defined as follows, if $\theta(p)$ is the percolation probability, C the cluster of the origin, and $\xi(p)$ the correlation length:

$$\begin{aligned}
\frac{\partial^3}{\partial p^3} E_p[|C|^{-1}] &\approx |p - p_c|^{-1-\alpha} \\
\theta(p) &\approx (p - p_c)_+^{\beta} \\
\chi^f(p) := E_p[|C| \mathbb{1}_{|C|<\infty}] &\approx |p - p_c|^{-\gamma} \\
P_{p_c}[|C| = n] &\approx n^{-1-1/\delta} \\
P_{p_c}[x \in C] &\approx |x|^{2-d-\eta} \\
\xi(p) &\approx |p - p_c|^{\nu} \\
P_{p_c}[\mathrm{diam}(C) = n] &\approx n^{-1-1/\rho} \\
\frac{E_p[|C|^{k+1} \mathbb{1}_{|C|<\infty}]}{E_p[|C|^k \mathbb{1}_{|C|<\infty}]} &\approx |p - p_c|^{-\Delta}
\end{aligned}$$

These exponents are all expected to be universal, *i.e.* to depend only on the dimension of the lattice, although this is not well understood at the mathematical level; the following *scaling relations* between the exponents are believed to hold:

$$2 - \alpha = \gamma + 2\beta = \beta(\delta + 1),\ \Delta = \delta\beta,\ \gamma = \nu(2 - \eta).$$

In addition, in dimensions up to $d_c = 6$, two additional *hyperscaling relations* involving d are strongly conjectured to hold:

$$d\rho = \delta + 1, \quad d\nu = 2 - \alpha,$$

while above d_c the exponents are believed to take their mean-field value, *i.e.* the ones they have for percolation on a regular tree:

$$\alpha = -1,\ \beta = 1,\ \gamma = 1,\ \delta = 2,$$

$$\eta = 0,\ \nu = \frac{1}{2},\ \rho = \frac{1}{2},\ \Delta = 2.$$

Not much is known rigorously on critical exponents in the general case. Hara and Slade ([9]) proved that mean field behavior does happen above dimension 19, and the proof can likely be extended to treat the case $d \geq 7$. In the two-dimensional case on the other hand, Kesten ([13]) showed that, assuming that the exponents δ and ρ exist, then so do β, γ, η and ν, and they satisfy the scaling and hyperscaling relations where they appear.

1.3.3 The incipient infinite cluster

When studying long-range properties of a critical model, it is useful to have an object which is infinite at criticality, and such is not the case for percolation clusters. There are two ways to condition the cluster of the origin to be infinite when $p = p_c$: The first one is to condition it to have diameter at least n (which happens with positive probability) and take a limit in distribution as n goes to infinity; the second one is to consider the model for parameter $p > p_c$, condition the cluster of 0 to be infinite (which happens with positive probability) and take a limit in distribution as p goes to p_c. The limit is the same in both cases, it is known as the *incipient infinite cluster*.

As in the super-critical regime, the structure of the cluster can be investigated by studying the behavior of a random walk on it, as was suggested by de Gennes; Kesten proved that in two dimensions, the random walk on the incipient infinite cluster is sub-diffusive, *i.e.* the mean square displacement after n steps behaves as $n^{1-\varepsilon}$ for some $\varepsilon > 0$.

The construction of the incipient infinite cluster was done by Kesten in two dimensions [12], and a similar construction was performed recently in high dimension by Van der Hofstad and Jarai ([20]).

2 Percolation in two dimensions

As is the case for several other models of statistical physics, percolation exhibits many specific properties when considered on a two-dimensional lattice: Duality arguments allow for the computation of p_c in some cases, and for the derivation of *a priori* bounds for the probability of crossing events at or near the critical point, leading to the fact that $\theta(p_c) = 0$. On another front, the scaling limit of critical site-percolation on the two-dimensional triangular lattice can be described in terms of SLE processes.

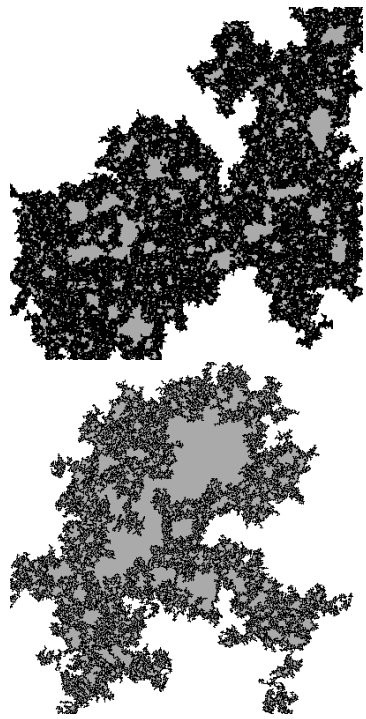

Figure 2: Two large critical percolation clusters in a box of the square lattice (first: bond-percolation, second: site-percolation)

2.1 Duality, exact computations and RSW theory

Given a planar lattice $\mathcal{L}$, define two associated graphs as follows. The *dual lattice* $\mathcal{L}'$ has one vertex for each face of the original lattice, and an edge between two vertices if and only if the corresponding faces of $\mathcal{L}$ share an edge. The *star graph* $\mathcal{L}^*$ is obtained by adding to $\mathcal{L}$ an edge between any two vertices belonging to the same face ($\mathcal{L}^*$ is not planar in general; $(\mathcal{L}, \mathcal{L}^*)$ is commonly known as a *matching pair*). Then, a result of Kesten is that, under suitable technical conditions,

$$p_c^{\text{bond}}(\mathcal{L}) + p_c^{\text{bond}}(\mathcal{L}') = p_c^{\text{site}}(\mathcal{L}) + p_c^{\text{site}}(\mathcal{L}^*) = 1.$$

Two cases are of particular importance: The lattice $\mathbb{Z}^2$ is isomorphic to its dual; the triangular lattice $\mathcal{T}$ is its own star graph. It follows that

$$p_c^{\text{bond}}(\mathbb{Z}^2) = p_c^{\text{site}}(\mathcal{T}) = \frac{1}{2}.$$

The only other critical parameters that are known exactly are $p_c^{\text{bond}}(\mathcal{T}) = 2\sin(\pi/18)$ (and hence also p_c^{bond} for $\mathcal{T}'$, *i.e.* the hexagonal lattice), and p_c^{bond} for the bow-tie lattice which is a root of the equation $p^5 - 6p^3 + 6p^2 + p - 1 = 0$. The value of the critical parameter for site-percolation on $\mathbb{Z}^2$ might on the other hand never be known, it is even possible that it is "just a number" without any other signification.

Still using duality, one can prove that the probability, for bond-percolation on the square lattice with parameter $p = 1/2$, that there is a connected component crossing an $(n+1) \times n$ rectangle in the longer direction is exactly equal to $1/2$. This and clever arguments involving the symmetry of the lattice lead to the following result, proved independently by Russo and by Seymour and Welsh and known as the RSW theorem:

Theorem 3 (Russo [15]; Seymour-Welsh [17]) *For every $a, b > 0$ there exist $\eta > 0$ and $n_0 > 0$ such that for every $n > n_0$, the probability that there is a cluster crossing an $\lfloor na \rfloor \times \lfloor nb \rfloor$ rectangle in the first direction is greater than η.*

The most direct consequence of this estimate is that the probability that there is a cluster going around an annulus of a given modulus is bounded below independently of the size of the annulus; in particular, almost surely there is *some* annulus around 0 in which this happens, and that is what allows to prove that $\theta(p_c) = 0$ for bond-percolation on $\mathbb{Z}^2$.

2.2 The scaling limit

RSW-type estimates give positive evidence that a scaling limit of the model should exist; it is indeed essentially sufficient to show convergence of the crossing probabilities to a non-trivial limit as n goes to infinity. The limit, which should depend only on the ration a/b, was predicted by Cardy using conformal fields theory methods. A most celebrated result of Smirnov is the proof of Cardy's formula in the case of site-percolation on the triangular lattice $\mathcal{T}$:

Theorem 4 (Smirnov [18]) *Let Ω be a simply connected domain of the plane with four points a, b, c, d (in that order) marked on its boundary. For every $\delta > 0$, consider a critical site-percolation model on the intersection of Ω with $\delta\mathcal{T}$ and let $f_\delta(ab, cd; \Omega)$ be the probability that it contains a cluster connecting the arcs ab and cd. Then:*

1. *$f_\delta(ab, cd; \Omega)$ has a limit $f_0(ab, cd; \Omega)$ as $\delta \to 0$;*
2. *The limit is conformally invariant, in the following sense: If Φ is a conformal map from Ω to some other domain $\Omega' = \Phi(\Omega)$, and maps a to a', b to b', c to c' and d to d', then $f_0(ab, cd; \Omega) = f_0(a'b', c'd'; \Omega')$;*
3. *In the particular case when Ω is an equilateral triangle of side length 1 and vertices a, b and c, and if d is on (ca) at distance $x \in (0, 1)$ from c, then $f_0(ab, cd; \Omega) = x$.*

Point 3. in particular is essential since it allows to compute the limiting crossing probabilities in any conformal rectangle. In the original work of Cardy, he made his prediction in the case of a rectangle, for which the limit involves hypergeometric functions; the remark that the equilateral triangle gives rise to nicer formulae is originally due to Carleson.

To precisely state the convergence of percolation to its scaling limit, define the random curve known as the *percolation exploration path* (see fig. 3) as follows: In the upper half-plane, consider a site-percolation model on a portion of the triangular lattice and impose the boundary conditions that on the negative real half-line all the sites are open, while on the other half-line the sites are closed. The exploration curve is then the common boundary of the open cluster spanning from the negative half-line, and the closed cluster spanning from the positive half-line; it is an infinite, self-avoiding random curve in the upper half-plane.

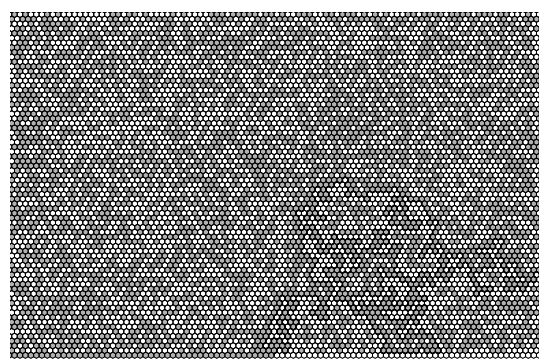

Figure 3: A percolation exploration path

As the mesh of the lattice goes to 0, the exploration curve then converges in distribution to the trace of an *SLE* process, as introduced by Schramm, with parameter $\kappa = 6$ — see fig. 4. The limiting curve is not simple anymore (which corresponds to the existence of pivotal sites on large critical percolation clusters), and it has Hausdorff dimension $7/4$. For more details on *SLE* processes, see *e.g.* the related entry in the present volume.

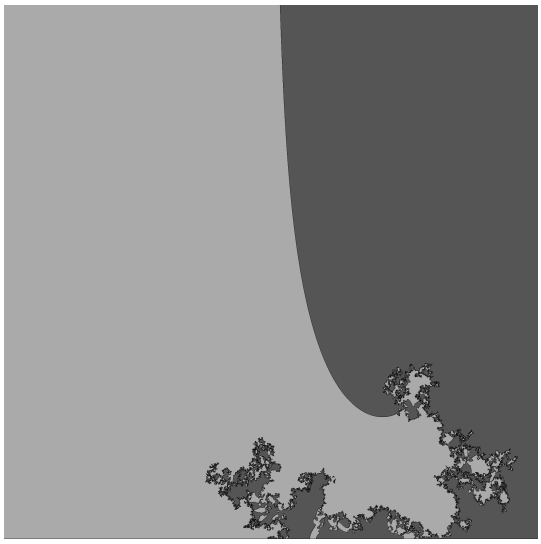

Figure 4: An SLE process with parameter $\kappa = 6$ (infinite time, with the driving process stopped at time 1)

As an application of this convergence result, one can prove that the critical exponents described in the previous section do exist (still in the case of the triangular lattice), and compute their exact values, except for α, which is still listed here for completeness:

$$\left[\alpha = -\frac{2}{3},\right] \beta = \frac{5}{36},\ \gamma = \frac{43}{18},\ \delta = \frac{91}{5},$$

$$\eta = \frac{5}{24},\ \nu = \frac{4}{3},\ \rho = \frac{48}{5},\ \Delta = \frac{91}{36}.$$

These exponents are expected to be *universal*, in the sense that they should be the same for percolation on any two-dimensional lattice; but at the time of this writing this phenomenon is far from being understood on a mathematical level.

The rigorous derivation of the critical exponents for percolation is due to Smirnov and Werner [19]; the dimension of the limiting curve was obtained by Beffara [2].

3 Other lattices and percolative systems

Some modifications or generalizations of standard Bernoulli percolation on $\mathbb{Z}^d$ exhibit an interesting behavior and as such provide some insight into the original process as well; there are too many mathematical objects which can be argued to be percolative in some sense to give a full account of all of them, so the following list is somewhat arbitrary and by no means complete.

3.1 Percolation on non-amenable graphs

The first modification of the model one can think of is to modify the underlying graph and move away from the cubic lattice; phase transition still occurs, and the main difference is the possibility for infinitely many infinite clusters to coexist. On a regular tree, such is the case whenever $p \in (p_c, 1)$, the first non-trivial example was produced by Grimmett and Newman as the product of $\mathbb{Z}$ by a tree: There, for some values of p the infinite cluster is unique, while for others there is coexistence of infinitely many of them. The corresponding definition, due to Benjamini and Schramm, is then the following: If N is as above the number of infinite open clusters,

$$p_u := \inf\left\{p : P_p(N = 1) = 1\right\} \geq p_c.$$

The main question is then to characterize graphs on which $0 < p_c < p_u < 1$.

A wide class of interesting graphs is that of *Cayley graphs* of infinite, finitely generated groups. There, by a simultaneous result by Häggström and Peres and by Schonmann, for every $p \in (p_c, p_u)$ there are P_p-a.s. infinitely many infinite cluster, while for every $p \in (p_u, 1]$ there is only one — note that this does not follow from the definition since new infinite components could appear when p is increased. It is conjectured that $p_c < p_u$ for any Cayley graph of a non-amenable group (and more generally for any quasi-transitive graph with positive Cheeger constant), and a result by Pak and Smirnova is that every infinite, finitely generated, non-amenable group has a Cayley graph on which $p_c > p_u$; this is then expected not to depend on the choice of generators. In the general case, it was recently proved by Gaboriau that if the graph $\mathcal{G}$ is unimodular, transitive, locally finite and supports non-constant harmonic Dirichlet functions (*i.e.* harmonic functions whose gradient is in ℓ^2), then indeed $p_c(\mathcal{G}) < p_u(\mathcal{G})$.

For reference and further reading on the topic, the reader is advised to refer to the review paper by Benjamini and Schramm [3], the lecture notes of Peres[14], and the more recent article of Gaboriau [6].

3.2 Gradient percolation

Another possible modification of the original model is to allow the parameter p to depend on the location; the porous medium may for instance have been created by some kind of erosion, so that there will be more open edges on one side of a given domain than on the other. If p still varies

smoothly, then one expects some regions to look subcritical and others to look supercritical, with interesting behavior in the vicinity of the critical level set $\{p = p_c\}$. This particular model was introduced by Sapoval et al. [16] under the name of *gradient percolation*; see fig. 5.

Figure 5: Gradient percolation in a square. In black is the cluster spanning from the bottom side of the square.

The control of the model away from the critical zone is essentially the same as for usual Bernoulli percolation, the main question being how to estimate the width of the phase transition. The main idea is then the same as in scaling theory: If the distance between a point v and the critical level set is less than the correlation length for parameter $p(v)$, then v is in the phase transition domain. This of course makes sense only asymptotically, say in a large $n \times n$ square with $p(x, y) = 1 - y/n$ as is the case in the figure: The transition then is expected to have width of order n^a for some exponent $a > 0$.

3.3 First-passage percolation

First passage percolation (also known as Eden or Richardson model) was introduced by Hammersley and Welsh in 1965 [8] as a time-dependent model for the passage of fluid through a porous medium. To define the model, with each edge $e \in \mathcal{E}(\mathbb{Z}^d)$ is associated a random variable $T(e)$, which can be interpreted as being the time required for fluid to flow along e. $T(e)$ are assumed to be independent non-negative random variables having common distribution F. For any path π we define the passage time $T(\pi)$ of π as

$$T(\pi) := \sum_{e \in \pi} T(e).$$

The *first passage time* $a(x, y)$ between vertices x and y is given by

$$a(x, y) = \inf\{T(\pi) : \pi \text{ a path from } x \text{ to } y\};$$

and we can define

$$W(t) := \{x \in \mathbb{Z}^d : a(0, x) \leq t\},$$

the set of vertices reached by the liquid by time t. It turns out that $W(t)$ grows approximately linearly as time passes, and that there exists a non-random limit set B such that either B is compact and

$$(1 - \varepsilon)B \subseteq \frac{1}{t}\widetilde{W}(t) \subseteq (1 + \varepsilon)B, \text{ eventually a.s.}$$

for all $\varepsilon > 0$, or $B = \mathbb{R}^d$, and

$$\{x \in \mathbb{R}^d : |x| \leq K\} \subseteq \frac{1}{t}\widetilde{W}(t), \text{ eventually a.s.}$$

for all $K > 0$. Here $\widetilde{W}(t) = \{z + [-1/2, 1/2]^d : z \in W(t)\}$.

Studies of first passage percolation brought many fascinating discoveries, including Kingman's celebrated sub-additive ergodic theorem. In recent years interest has been focused on study of fluctuations of the set $\widetilde{W}(t)$ for large t. In spite of huge effort and some partial results achieved, it still remains a major task to establish rigorously conjectures predicted by Kardar-Parisi-Zhang theory about shape fluctuations in first passage percolation.

3.4 Contact processes

Introduced by Harris and conceived with biological interpretation, the *contact process* on $\mathbb{Z}^d$ is a continuous-time process taking values in the space of subsets of $\mathbb{Z}^d$. It is informally described as follows: Particles are distributed in $\mathbb{Z}^d$ in such a way that each site is either empty or occupied by one particle. The evolution is Markovian: Each particle disappears after an exponential time of parameter 1, independently from the others; at any time, each particle has a possibility to create a new particle at any of its empty neighboring sites, and does so with rate $\lambda > 0$, independently of everything else.

The question is then whether, starting from a finite population, the process will die out in finite time or whether it will survive forever with positive probability. The outcome will depend on the

value of λ, and there is a critical value λ_c, such that for $\lambda \leq \lambda_c$ process dies out, while for $\lambda > \lambda_c$ indeed there is survival, and in this case the shape of the population obey a shape theorem similar to that of first-passage percolation.

The analogy with percolation is strong, the corresponding percolative picture being the following: In $\mathbb{Z}_+^{d+1}$, each edge is open with probability $p \in (0,1)$, and the question is whether there exists an infinite *oriented* path π (*i.e.* a path along which the sum of the coordinates is increasing), composed of open edges. Once again, there is a critical parameter customarily denoted by $\vec{p}_c$, at which no such path exists (compare this to the open question of the continuity of the function θ at p_c in dimensions $3 \leq d \leq 18$). This variation of percolation lies in a different universality class than the usual Bernoulli model.

3.5 Invasion percolation

Let $X(e) : e \in \mathcal{E}$ be independent random variables indexed by the edge set $\mathcal{E}$ of $\mathbb{Z}^d$, $d \geq 2$, each having uniform distribution in $[0,1]$. One constructs a sequence $C = \{C_i, i \geq 1\}$ of random connected subgraphs of the lattice in the following iterative way: The graph C_0 contains only the origin. Having defined C_i, one obtains C_{i+1} by adding to C_i an edge e_{i+1} (with its outer lying end-vertex), chosen from the outer edge boundary of C_i so as to minimize $X(e_{i+1})$. Still very little is known about the behavior of this process.

An interesting observation, relating $\theta(p_c)$ of usual percolation with the invasion dynamics, comes from C.M. Newman:

$$\theta(p_c) = 0 \;\Leftrightarrow P\{x \in C\} \to 0 \text{ as } |x| \to \infty.$$

Further reading

For a much more in-depth review of percolation on lattices and the mathematical methods involved in its study, and for the proofs of most of the results we could only point at, we refer the reader to the standard book of Grimmett [7]; another excellent general reference, and the only place to find some of the technical graph-theoretical details involved, is the book of Kesten [11]. More information in the case of graphs that are not lattices can be found in the lecture notes of Peres [14].

For curiosity, the reader can refer to the first mention of a problem close to percolation, in the problem section of the first volume of the American Mathematical Monthly [21]. References on more specific topics are given at the end of each section.

See also

Introductory article: Statistical mechanics; 2D Ising model; Wulff droplets; Stochastic Loewner evolutions.

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Vincent BEFFARA
UMPA – ENS Lyon
46 Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Vladas SIDORAVICIUS
IMPA
Estrada Dona Castorina 110
Rio de Janeiro 22460-320
BRASIL