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FREE AUGMENTED LD-SYSTEMS

PATRICK DEHORNOY

ABSTRACT. Define an augmented LD-system, or ALD-system, to be a set equipped with two binary operations, one satisfying the left self-distributivity law $x * (y * z) = (x * y) * (x * z)$ and the other satisfying the mixed laws $(x \circ y) * z = x * (y * z)$ and $x * (y \circ z) = (x * y) \circ (x * z)$. We solve the word problem of the ALD laws, and prove that every element in the parenthesized braid group B_\bullet of [2, 3, 5, 6] generates a free ALD-system of rank 1, thus getting a concrete realization of the latter structure.

Define an *LD-system* to be an algebraic system made of a set S equipped with a binary operation $*$ that satisfies the left self-distributivity law

$$(LD) \quad x * (y * z) = (x * y) * (x * z).$$

Classical examples include groups equipped with their conjugacy operation $x * y = xyx^{-1}$, and lattices with their inf or sup operation. Less classical examples have appeared in Set Theory with the iterations of elementary embeddings [12], and in Low Dimensional Topology where (LD) provides an algebraic translation of Reidemeister move III [10, 13, 9]. A rich theory has been developed for LD-systems [4]. In particular, it is known that there exists on Artin's braid group B_∞ an LD-operation $*$ such that the $*$ -closure of any braid is a free LD-system of rank 1—which provides a concrete realization of the latter structure.

Many examples of LD-systems turn out to be equipped with a second operation connected in various ways with the self-distributive operation. In the typical case of group conjugacy, using \circ for the group product, the following mixed identities are satisfied

$$(ALD_1) \quad x * (y * z) = (x \circ y) * z,$$

$$(ALD_2) \quad x * (y \circ z) = (x * y) \circ (x * z).$$

When we add the identity $x \circ y = (x * y) \circ x$, the associativity of \circ and the existence of a unit, one obtains the structure of an LD-monoid, which is investigated in Chapter XI of [4] (and in [7, 8] under the name of LD-algebra).

It is easy to verify that all LD-systems cannot be enriched into LD-monoids. In particular, this is the case for the above mentioned LD-structure on B_∞ , for which there can exist no second operation verifying (ALD₁). In [6], building on earlier approaches of [2, 3, 5], a new group B_\bullet extending both Artin's braid group B_∞ and R.Thompson's group F is investigated. This group is called the parenthesized braid group, as its elements can be naturally interpreted using braid diagrams in which the strands come grouped into blocks that can be encoded in parenthesized words. It is shown that the LD-structure of B_∞ extends to B_\bullet and that the latter can be completed with a second operation that satisfies the above identities (ALD₁) and (ALD₂)—but none of the further

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laws defining an LD-monoid. Such a structure is called an *augmented* LD-system, or *ALD-system*.

The aim of this note is to prove two new results about ALD-systems: firstly, we solve the associated word problem, and, secondly, we prove for the ALD-structure of the parenthesied braid group B_\bullet a result similar to that established in [4] for the LD-structure of ordinary braids, namely that every element of B_\bullet generates a free ALD-subsystem of B_\bullet . Being quite similar to those holding for *LD* and B_∞ , these results are not surprising. However, their proofs require a few new specific arguments that are the subject of this paper.

1. FREE AUGMENTED LD-SYSTEMS

The aim of this section is to solve the word problem for the ALD laws, *i.e.*, to describe an algorithm that enables one to decide whether two terms are or not equivalent up to ALD.

1.1. ALD-systems. The algebraic systems considered here are as follows:

Definition 1.1. An *ALD-system* is defined to be a set S equipped with two binary operations, $*$ and \circ that satisfy the identities (*LD*), (*ALD*₁), and (*ALD*₂).

Example 1.2. We already observed that any group G equipped with the conjugation operation $*$ and the product is an ALD-system—and even an LD-monoid. Another easy example is obtained by starting with an arbitrary binary system (S, \circ) and considering an \circ -endomorphism f . Then defining $x * y = f(y)$ turns $(S, *, \circ)$ into an ALD-system.

If L_x denotes the left $*$ -translation $y \mapsto x * y$, then (*LD*) and (*ALD*₂) express that, for each x in the considered domain, L_x is an endomorphism with respect to $*$ and \circ , respectively, while (*ALD*₁) expresses that \circ corresponds to a composition of translations: $L_{x \circ y} = L_x \circ L_y$. Thus, an ALD-system is an LD-system where the family of left translations is closed under composition—and in which (*ALD*₂) is satisfied. It may be noted that, in any case, the conjunction of (*LD*) and (*ALD*₁) implies some weak form of (*ALD*₂), as we can write

$$\begin{aligned} (x * (y \circ z)) * (x * u) &=_{LD} x * ((y \circ z) * u) =_{ALD_1} x * (y * (z * u)) \\ &=_{LD} (x * y) * ((x * z) * (x * u)) =_{ALD_1} ((x * y) \circ (x * z)) * (x * u), \end{aligned}$$

which follows from (*ALD*₂) and actually implies it if we may cancel $x * u$ on the right.

1.2. Terms and free ALD-systems. We consider in the sequel free ALD-systems. As usual, the latter can be introduced as quotients of absolutely free algebras, *i.e.*, of algebras consisting of terms subject to no relation. Our notation will be as follows.

Definition 1.3. For $n \geq 1$, we denote by T_n^* (*resp.* T_n° , *resp.* $T_n^{*,\circ}$) the set of all binary terms constructed using the operator $*$ (*resp.* \circ , *resp.* $*$ and \circ) from n fixed variables x_1, \dots, x_n . We write T^* for the union of all T_n^* , and similarly with T° and $T^{*,\circ}$ —and x for x_1 .

The *size* of a term t is defined to be the number of occurrences of variables in t , *i.e.*, it is defined to be 1 when t is a variable, and to be the sum of the sizes of the left and the right subterms of t otherwise. By construction, $T^{*,\circ}n$ is an absolutely free algebra of rank n . The following is clear:

Lemma 1.4. *Let $=_{ALD}$ be the congruence on $T_n^{*,\circ}$ generated by all instances of the laws (LD) , (ALD_1) , and (ALD_2) ¹. Then, for each n , the system $T_n^{*,\circ}/=_{ALD}$ is a free ALD-system of rank n .*

We say that two terms t, t' are *ALD-equivalent* if $t =_{ALD} t'$ holds. Of course, there is a similar result for the free LD-system of rank n obtained as $T_n^*/=_{LD}$, where $=_{LD}$ is the congruence generated by the instances of the sole law (LD) .

It is helpful for intuition to associate with every term a finite binary rooted, labeled tree: the tree associated with a variable x consists of a single node labeled x ; for $\square = *$ or \circ , the tree associated with $t_1 \square t_2$ consists of a root labeled \square admitting as its left subtree the tree associated with t_1 , and as its right subtree the tree associated with t_2 .

As a preliminary remark, let us observe that the variety of ALD-systems is properly intermediate between LD-systems and LD-monoids.

Proposition 1.5. *(i) A free LD-system cannot be enriched into an ALD-system.*

*(ii) A free ALD-system does not obey the law $x \circ y = (x * y) \circ x$, and therefore is not an LD-monoid.*

Proof. (i) For t a term, let $ht_R(t)$ be the length of the rightmost branch in the associated tree, *i.e.*, define $ht_R(t)$ by $ht_R(x) = 0$ and $ht_R(t_1 \square t_2) = ht_R(t_2) + 1$ for $\square = *$ or \circ . Then the law (LD) preserves ht_R , and, therefore, ht_R induces a well defined parameter on each free LD-system. On the other hand, (ALD_1) changes ht_R , so there may exist no operation \circ satisfying (ALD_1) on a free LD-system.

(ii) The terms $x_1 \circ x_2$ and $(x_1 * x_2) \circ x_1$ are not ALD-equivalent, as none of the identities (LD) , (ALD_1) , (ALD_2) may apply to a term with only two occurrences of variables. \square

1.3. Two ALD-invariants. In order to subsequently solve the word problem of ALD, we shall associate with every term in $T^{*,\circ}$ two ALD-invariants, *i.e.*, two objects that depend only on the ALD-class of the term. The first invariant is a term in T_1° ; the second one is a finite sequence of LD-classes of terms in T^* . To introduce the latter, we first fix some notation for sequences.

Notation 1.6. Assume that $(S, *)$ is a binary system. The set of all finite, nonempty sequences of elements of S is denoted by \widehat{S} . An element of \widehat{S} is typically denoted \vec{s} ; its length is then denoted $\ell(\vec{s})$, and its successive elements $s_1, \dots, s_{\ell(\vec{s})}$. The concatenation of two sequences \vec{s}, \vec{t} , *i.e.*, the sequence of length $\ell(\vec{s}) + \ell(\vec{t})$ obtained by writing \vec{t} after \vec{s} , is denoted $\vec{s} \sim \vec{t}$. Next, we denote by $\vec{*}$ the binary operation on \widehat{S} defined by

$$(1.1) \quad \vec{s} \vec{*} \vec{t} = (s_1 * \dots * s_{\ell(\vec{s})} * t_1, \dots, s_1 * \dots * s_{\ell(\vec{s})} * t_{\ell(\vec{t})}),$$

where missing parentheses are to be added on the right: $x * y * z$ stands for $x * (y * z)$.

Lemma 1.7. *Assume that $(S, *)$ is an LD-system. Then $(\widehat{S}, \vec{*}, \sim)$ is an ALD-system.*

Proof. The only point that is not absolutely obvious is that (LD) holds. Now, for all s, t, u in \widehat{S} , the k th entry in $\vec{s} \vec{*} (\vec{t} \vec{*} \vec{u})$ is $s_1 * \dots * s_p * t_1 * \dots * t_q * u_k$, while that of $(\vec{s} \sim \vec{t}) \vec{*} (\vec{s} \sim \vec{u})$ is

$$(s_1 * \dots * s_p * t_1) * \dots * (s_1 * \dots * s_p * t_q) * s_1 * \dots * \dots * u_k.$$

Repeated applications of the LD law show that the expressions are equal. \square

¹*i.e.*, all pairs of terms of the form $(t_1 * (t_2 * t_3), (t_1 * t_2) * (t_1 * t_3)), (t_1 * (t_2 * t_3), (t_1 \circ t_2) * t_3)$, and $(t_1 * (t_2 \circ t_3), (t_1 * t_2) \circ (t_1 * t_3))$

We can now introduce the two mappings that give rise to ALD-invariants.

Definition 1.8. For each term t in $T^{*,\circ}$, we define a term $I(t)$ in T_1° and a finite sequence of terms $J(t)$ in \widehat{T}^* using the inductive clauses

$$(1.2) \quad (I(t), J(t)) = \begin{cases} (x, t) & \text{if } t \text{ is a variable,} \\ (I(t_2), J(t_1) \vec{*} J(t_2)) & \text{for } t = t_1 * t_2, \\ (I(t_1) \circ I(t_2), J(t_1) \frown J(t_2)) & \text{for } t = t_1 \circ t_2, \end{cases}$$

For instance, for $t = x_1 * ((x_2 * x_3) \circ x_4)$, the reader can check the values $I(t) = x \circ x$, $J(t) = (x_1 * (x_2 * x_3), x_1 * x_4)$.

Lemma 1.9. *Assume that t, t' are ALD-equivalent terms in $T^{*,\circ}$. Then we have*

$$(1.3) \quad I(t) = I(t') \quad \text{and} \quad J(t) =_{LD} J(t'),$$

the latter meaning that the sequences $J(s)$ and $J(t)$ have equal lengths and pairwise LD-equivalent entries.

Proof. As ALD-equivalence is the congruence on $T^{*,\circ}$ generated by the pairs of terms occurring in the laws (LD), (ALD₁), and (ALD₂), it is sufficient to check that the relations $I(t) = I(t')$ and $J(t) =_{LD} J(t')$ are congruences on $T^{*,\circ}$, and that they include all instances of (LD), (ALD₁), and (ALD₂).

The fact that $I(t_1 * t_2)$ and $I(t_1 \circ t_2)$ are defined from $I(t_1)$ and $I(t_2)$ makes it clear that $I(t) = I(t')$ is a congruence, *i.e.*, that it is compatible with $*$ and \circ . The same argument works for $J(t) =_{LD} J(t')$, as the relation $=_{LD}$ on \widehat{T}^* is itself a congruence.

Let (t, t') be an instance of (LD), *i.e.*, assume that t and t' are of the form $t = t_1 * (t_2 * t_3)$ and $t' = (t_1 * t_2) * (t_1 * t_3)$. The definitions yields

$$I(t) = I(t_3) = I(t'), \\ J(t) = J(t_1) \vec{*} (J(t_2) \vec{*} J(t_3)), \quad J(t') = (J(t_1) \vec{*} J(t_2)) \vec{*} (J(t_1) \vec{*} J(t_3)),$$

and the latter are $=_{LD}$ -equivalent by Lemma 1.7. Similarly, for (t, t') an instance of (ALD₁), *i.e.*, for $t = t_1 * (t_2 * t_3)$ and $t' = (t_1 \circ t_2) * t_3$, we have

$$I(t) = I(t_3) = I(t') \quad \text{and} \quad J(t) = J(t_1) \vec{*} J(t_2) \vec{*} J(t_3) = J(t').$$

Finally, for (t, t') an instance of (ALD₂), *i.e.*, for $t = t_1 * (t_2 \circ t_3)$ and $t' = (t_1 * t_2) \circ (t_1 * t_3)$, we find

$$I(t) = I(t_2) \circ I(t_3) = I(t') \quad \text{and} \quad J(t) = (J(t_1) \vec{*} J(t_2)) \frown (J(t_1) \vec{*} J(t_3)) = J(t'),$$

which completes the proof. \square

Remark 1.10. The result that I is an ALD-invariant can also be deduced from applying the construction of Example 1.2 to the free algebra $(T_1^\circ, *)$ —as well as the result that $J \bmod (LD)$ is an ALD-invariant follows from the construction of Lemma 1.7.

1.4. Special terms. We shall now see that, for each term t in $T^{*,\circ}$, the pair $(I(t), J(t))$ determines the ALD-class of t .

Definition 1.11. For v is a term of size p in T_1° , and \vec{t} is a length p sequence of terms in T^* , we denote by $v[\vec{t}]$ the term obtained from v by substituting t_1, \dots, t_p to the variables of t enumerated from left to right. A term is called *special* if it is of the form $v[\vec{t}]$ with v, \vec{t} as above.

Saying that a term t is special means that, in the tree associated with t , no \circ symbol lies below an $*$ symbol (according to the convention that the root lies on the top). The following result shows non only that every term in $T^{*,\circ}$ is ALD-equivalent to a special term, but also that the pair $(I(t), J(t))$ determines the ALD-class of t .

Lemma 1.12. *For every term t in $T^{*,\circ}$ we have*

$$(1.4) \quad t =_{ALD} I(t)[J(t)].$$

Proof. If t is a variable, (1.4) is an equality. For an induction, it is sufficient to show that the following relations hold for all terms u, v in T_1° and all sequences \vec{s}, \vec{t} in \widehat{T}^*

$$(1.5) \quad u[\vec{s}] * v[\vec{t}] =_{ALD} v[\vec{s} \vec{*} \vec{t}],$$

$$(1.6) \quad u[\vec{s}] \circ v[\vec{t}] = (u \circ v)[\vec{s} \frown \vec{t}].$$

We establish (1.5) using induction on the sum of the sizes, say p and q , of u and v , which also are the lengths of \vec{s} and \vec{t} , respectively. We recall that missing parentheses are to be added on the right, *i.e.*, $x * y * z$ stands for $(x * y) * (x * z)$.

For $p = q = 1$, the terms u and v are variables, so we have $u[\vec{s}] = s_1$ and $v[\vec{t}] = t_1$, and (1.5) reduces to the equality $s_1 * t_1 = v[s_1 * t_1]$. Assume now $p + q > 2$. Then we have $p \geq 2$ or $q \geq 2$. Assume first $q \geq 2$. Write $v = v_1 \circ v_2$, and let r be the size of v_1 . We find

$$\begin{aligned} u[\vec{s}] * v[\vec{t}] &= u[\vec{s}] * (v_1[t_1, \dots, t_r] \circ v_2[t_{r+1}, \dots, t_q]) && \text{(by definition)} \\ &=_{ALD} (u[\vec{s}] * v_1[t_1, \dots, t_r]) \circ (u[\vec{s}] * v_2[t_{r+1}, \dots, t_q]) && (ALD_2) \\ &=_{ALD} v_1[\vec{s} \vec{*} t_1, \dots, \vec{s} \vec{*} t_r] \circ v_2[\vec{s} \vec{*} t_{r+1}, \dots, \vec{s} \vec{*} t_q] && \text{(by ind. hyp.)} \\ &= (v_1 \circ v_2)[\vec{s} \vec{*} \vec{t}] && \text{(by definition).} \end{aligned}$$

Assume now $p \geq 2$. Writing similarly $u = u_1 \circ u_2$, and letting r be now the size of u_1 , we find

$$\begin{aligned} u[\vec{s}] * v[\vec{t}] &= (u_1[s_1, \dots, s_r] \circ u_2[s_{r+1}, \dots, s_p]) * v[\vec{t}] && \text{(by definition)} \\ &=_{ALD} u_1[s_1, \dots, s_r] * (u_2[s_{r+1}, \dots, s_p] * v[\vec{t}]) && (ALD_1) \\ &=_{ALD} u_1[s_1, \dots, s_r] * v[s_{r+1} * \dots * s_p \vec{*} \vec{t}] && \text{(by ind. hyp.)} \\ &=_{ALD} v[s_1 * \dots * s_p * s_{r+1} * \dots * s_p \vec{*} \vec{t}] = v[\vec{s} \vec{*} \vec{t}] && \text{(by ind. hyp.).} \end{aligned}$$

As for (1.6), it follows from the definition directly. \square

1.5. The word problem of ALD. It is now easy to solve the word problem for ALD.

Proposition 1.13. *The word problem of ALD is decidable: if t, t' are terms in $T^{*,\circ}$, then $t =_{ALD} t'$ holds if and only if the terms $I(t)$ and $I(t')$ are equal, and the sequences $J(t)$ and $J(t')$ have the same length and consist of pairwise LD-equivalent terms of T^* .*

Proof. The condition is necessary by Lemma 1.9. It is sufficient by Lemma 1.12. Indeed, if \vec{s}, \vec{t} are length p sequences of pairwise LD-equivalent terms in T^* and if v is any size p term in T_1° , the terms $v[\vec{s}]$ and $v[\vec{t}]$ are ALD-equivalent. So, if t, t' are terms in $T^{*,\circ}$ satisfying $I(t) = I(t')$ and $J(t) =_{LD} J(t')$, we obtain

$$t =_{ALD} I(t)[J(t)] =_{ALD} I(t')[J(t')] =_{ALD} t',$$

hence $t =_{ALD} t'$. As the relation $=_{LD}$ is known to be decidable [4], so is $=_{ALD}$. \square

As for the complexity of the previous solution, the known upper bounds for the word problem of (LD) are a single exponential in the case of terms with one variable, and a double exponential in the general case. As the size of the sequence $J(t)$ may be exponential in the length of t since each application of (ALD_2) may double the length, the solution described in Proposition 1.13 has a (certainly not optimal) upper bound which is doubly exponential in the case of one variable, and triply exponential in the general case—the results of Section 2 below will give a better, simply exponential algorithm in the case of one variable.

2. PARENTHESIZED BRAIDS

The group of parenthesized braids B_\bullet was introduced in [2, 3, 5]—in a different framework—and further investigated in [6]. It is shown in the latter paper that B_\bullet can be equipped with two binary operations that make it an ALD-system. The aim of this section is to study this specific ALD-system, and in particular to show that it contains many copies of the free ALD-system on one generator.

2.1. The group B_\bullet . The simplest way to introduce B_\bullet is to start from a presentation:

Definition 2.1. We denote by B_\bullet the group generated by two infinite sequences $\sigma_1, \sigma_2, \dots, a_1, a_2, \dots$ subject to the relations

$$(2.1) \quad \begin{cases} \sigma_j \sigma_i = \sigma_i \sigma_j, & a_j \sigma_i = \sigma_i a_j & \text{for } j \geq i + 2, \\ a_j \sigma_i = \sigma_{i+1} a_j, & a_j a_j = a_{i+1} a_j & \text{for } j \leq i - 1, \\ \sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i, & \sigma_i \sigma_j a_i = a_j \sigma_i, & \sigma_j \sigma_i a_j = a_i \sigma_i, & \text{for } j = i + 1. \end{cases}$$

It is shown in [3] that B_\bullet is actually generated by σ_1, σ_2, a_1 , and a_2 , and that it admits a finite—but much less readable—presentation with respect to those generators. It is shown in [6] that the elements of B_\bullet admit a natural geometric interpretation in terms of parenthesized braid diagrams, which are similar to ordinary braid diagrams—*cf.* for instance [1, 4, 14]—but with non-uniform distances between the strands. As we shall use this interpretation here—nor do we either use the interpretation in terms of isotopy classes of homeomorphisms of a sphere with a Cantor set of punctures—we shall not go into details here and just refer to Figure 1 for a rough intuition.



FIGURE 1. Diagram representation of the generators of B_\bullet : there are infinitely strands numbered by positive integer coefficients polynomials in an infinitely small variable ϵ ; the effect of σ_i is to let all strands with index $i + 1 + o(1)$ cross over all strands with index $i + o(1)$; the effect of a_i is to shrink all strands of the form $i + o(1)$ by a factor ϵ and to left translate all strands with index $\geq i + 1$ so as to avoid gaps.

Definition 2.2. We denote by ∂ the endomorphism of the group B_\bullet that maps σ_i to σ_{i+1} and a_i to a_{i+1} for each i .

It is shown in [6] that ∂ is injective—but not surjective: neither σ_1 nor a_1 belong to $\text{Im} \partial$.

Proposition 2.3 ([6]). (Figure 2) Let $*$, \circ be the binary operations on B_\bullet defined by

$$(2.2) \quad \beta * \gamma := \beta \cdot \partial\gamma \cdot \sigma_1 \cdot \partial\beta^{-1}, \quad \beta \circ \gamma := \beta \cdot \partial\gamma \cdot a_1.$$

Then $(B_\bullet, *, \circ)$ is an ALD-system.

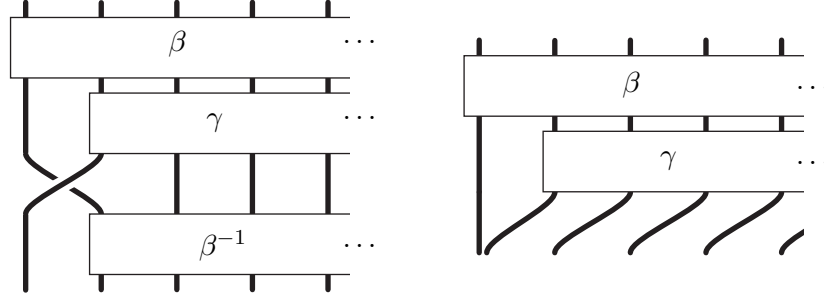


FIGURE 2. Diagram representation of the ALD operations on B_\bullet : the diagram of $\beta * \gamma$ (left) and $\beta \circ \gamma$ (right) from those of β and γ

2.2. A freeness criterion. Our aim is to show that the ALD-system $(B_\bullet, *, \circ)$ includes copies of the free ALD-system of rank 1. To prove the result, we need a criterion for recognizing such free ALD-systems.

Assume that $(S, *, \circ)$ is a double binary system generated by a single element γ . Then, there exists a surjective homomorphism π of $T_1^{*, \circ}$ onto S that maps x to γ : by definition, the value $\pi(t)$ is the *evaluation* of t at γ , and it will be denoted by $t(\gamma)$ —exactly as the evaluation of a polynomial P at γ would be denoted by $P(\gamma)$. Then, saying that $(S, *, \circ)$ is an ALD-system means that $t =_{ALD} t'$ implies $t(\gamma) = t'(\gamma)$, and saying that $(S, *, \circ)$ is a free ALD-system based on $\{\gamma\}$ means that $t =_{ALD} t'$ is equivalent to $t(\gamma) = t'(\gamma)$. In other words, in order to prove that some ALD-system S generated by an element γ is free, the point is to prove that $t(\gamma) \neq t'(\gamma)$ holds for all pairs of terms (t, t') satisfying $t \neq_{ALD} t'$. The criterion we shall establish new allows one to restrict to pairs of terms (t, t') of a restricted type.

Definition 2.4. For u, v in T_1° , we say that $u < v$ holds if we have either (i) $u = x$ and $v \neq x$, or (ii) $u = u_1 \circ u_2$ and $v = v_1 \circ v_2$ with $u_1 < v_1$, or (iii) $u = u_1 \circ u_2$ and $v = v_1 \circ v_2$ with $u_1 = v_1$ and $u_2 < v_2$.

Clearly, the relation $<$ is a strict linear order on T_1° .

Definition 2.5. (i) For s, t in T^* , we say that $s \sqsubset t$ holds if there exist $p \geq 1$ and terms t_1, \dots, t_p in T^* satisfying

$$t = (\dots((s * t_1) * t_2)\dots) * t_p.$$

(ii) For \vec{s}, \vec{t} in $\widehat{T^*}$, we say that $\vec{s} \vec{\sqsubset} \vec{t}$ holds if the lengths of \vec{s} and \vec{t} are equal and there exists $k \leq \ell(\vec{s})$ satisfying $s_i = t_i$ for $i < k$ and $s_k \sqsubset t_k$.

Proposition 2.6. Assume that S is an ALD-system generated by an element γ . Then a necessary and sufficient condition for S to be free based on $\{\gamma\}$ is that S satisfies no equality of the form

$$(2.3) \quad u[\vec{s}](\gamma) = v[\vec{t}](\gamma)$$

with u, v in T_1° and \vec{s}, \vec{t} in $\widehat{T_1^*}$ satisfying either $u < v$, or $u = v$ and $\vec{s} \vec{\sqsubset} \vec{t}$.

Proof. Assume that s, t are *ALD*-inequivalent terms in $T_1^{*,\circ}$. As was said above, the problem is to show that the evaluations $s(\gamma)$ and $t(\gamma)$ of s and t in S cannot be equal. By Lemma 1.12, there exist u, v in T_1° and \vec{s}, \vec{t} in \widehat{T}^* satisfying $s =_{ALD} u[\vec{s}]$ and $t =_{ALD} v[\vec{t}]$. As S is an *ALD*-system, we have $s(\gamma) = u[\vec{s}](\gamma)$ and $t(\gamma) = v[\vec{t}](\gamma)$, so it is sufficient to prove $u[\vec{s}](\gamma) \neq v[\vec{t}](\gamma)$. Now, by Lemma 1.4, the hypothesis $u[\vec{s}] \neq_{ALD} v[\vec{t}]$ implies $u \neq v$, or $u = v$ and $\vec{s} \neq_{LD} \vec{t}$. In the first case, we must have either $u < v$ or $v < u$ as $<$ is a linear ordering, hence, if no equality (2.3) holds, we deduce $s(\gamma) \neq t(\gamma)$. In the second case, as the sequences \vec{s} and \vec{t} have the same length, there exists an index $k \leq p$ such that we have $s_i =_{LD} t_i$ for $i < k$ and $s_k \neq_{LD} t_k$. By the results of [4], the latter relation implies the existence of terms s'_k, t'_k satisfying $s'_k =_{LD} s_k, t'_k =_{LD} t_k$ and either $s'_k \sqsubset t'_k$ or $t'_k \sqsubset s'_k$. Let \vec{s}' denote the sequence obtained from \vec{s} by replacing s_k by s'_k , and let \vec{t}' denote the sequence obtained from \vec{t} by replacing t_i with s_i for $i < k$, and by replacing t_k with t'_k . Then, as S is an *ALD*-system, we have $u[\vec{s}](\gamma) = u[\vec{s}'](\gamma)$ and $u[\vec{t}](\gamma) = u[\vec{t}'](\gamma)$, and, by construction, we have $\vec{s}' \sqsubset \vec{t}'$ or $\vec{t}' \sqsubset \vec{s}'$. If no equality (2.3) holds, we deduce $u[\vec{s}'](\gamma) \neq u[\vec{t}'](\gamma)$, hence $s(\gamma) \neq t(\gamma)$. \square

2.3. Term evaluation. In order to apply the criterion of Proposition 2.6 in the *ALD*-system $(B_\bullet, *, \circ)$, we need to be able to evaluate in B_\bullet expressions of the form $v[\vec{t}](\gamma)$ with v a term in T_1° and \vec{t} a sequence of terms in T_1^* . To this end, we shall use the following explicit formulas.

Lemma 2.7. *Assume that v is a term of size p in T_1° . Then, for each β in B_\bullet , we have*

$$(2.4) \quad v(1) \cdot \partial\beta = \partial^p\beta \cdot v(1).$$

Proof. We use induction on v . For $v = x$, we have $p = 1$ and $v(1) = 1$, so (2.4) is true. Otherwise, assume $v = v_1 \circ v_2$. By definition, $v(1)$ is $v_1(1) \cdot \partial v_2(1) \cdot a_1$. Let p_i be the size of v_i . Using the induction hypothesis, we find

$$\begin{aligned} v(1) \cdot \partial\beta &= v_1(1) \cdot \partial v_2(1) \cdot a_1 \cdot \partial\beta && \text{(by definition)} \\ &= v_1(1) \cdot \partial v_2(1) \cdot \partial^2\beta \cdot a_1 && \text{(by the relations of } B_\bullet) \\ &= v_1(1) \cdot \partial(v_2(1) \cdot \partial\beta) \cdot a_1 \\ &= v_1(1) \cdot \partial(\partial^{p_2}\beta \cdot v_2(1)) \cdot a_1 && \text{(by induction hypothesis)} \\ &= v_1(1) \cdot \partial(\partial^{p_2}\beta) \cdot \partial v_2(1) \cdot a_1 \\ &= \partial^{p_1}(\partial^{p_2}\beta) \cdot v_1(1) \cdot \partial v_2(1) \cdot a_1 && \text{(by induction hypothesis)} \end{aligned}$$

and the latter is $\partial^p\beta \cdot v(1)$. \square

Lemma 2.8. *Assume $t = v[\vec{t}]$, with v a size p term in T_1° and \vec{t} a length p sequence of terms in T_1^* . Then, for each γ in B_\bullet , we have*

$$(2.5) \quad t(\gamma) = t_1(\gamma) \cdot \partial t_2(\gamma) \cdot \dots \cdot \partial^{p-1}t_p(\gamma) \cdot v(1).$$

Proof. We use induction on v . For $v = x$, we have $p = 1$ and $t = t_1$, so the result is clear. Otherwise, assume $v = v_1 \circ v_2$. Let q be the size of v_1 . Then we have

$$t = v_1[t_1, \dots, t_q] \circ v_2[t_{q+1}, \dots, t_p],$$

and, using the induction hypothesis twice, we deduce

$$\begin{aligned}
t(\gamma) &= v_1[t_1, \dots, t_q](\gamma) \cdot \partial v_2[t_{q+1}, \dots, t_p](\gamma) \cdot a_1 \\
&= t_1(\gamma) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot v_1(1) \cdot \partial v_2[t_{q+1}, \dots, t_p](\gamma) \cdot a_1 && \text{(ind. hyp.)} \\
&= t_1(\gamma) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot \partial^q v_2[t_{q+1}, \dots, t_p](\gamma) \cdot v_1(1) \cdot a_1 && (2.4) \\
&= t_1(\gamma) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot \partial^q t_{q+1}(\gamma) \cdot \dots \cdot \partial^{p-1} t_p(\gamma) \cdot \partial^q v_2(1) \cdot v_1(1) \cdot a_1 && \text{(ind. hyp.)} \\
&= t_1(\gamma) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot \partial^q t_{q+1}(\gamma) \cdot \dots \cdot \partial^{p-1} t_p(\gamma) \cdot v_1(1) \cdot \partial v_2(1) \cdot a_1, && (2.4)
\end{aligned}$$

which gives (2.5) since we have $v(1) = v_1(1) \cdot \partial v_2(1) \cdot a_1$. \square

2.4. Monogenerated subsystems of B_\bullet . It is shown in [6] that the evaluation mapping $v \mapsto v(1)$ of T_1° into B_\bullet is injective. We shall need the following strengthening of this result:

Lemma 2.9. *If u, v are distinct terms in T_1° , then, in B_\bullet , the quotient $u(1)^{-1}v(1)$ does not belong to $\text{Im}\partial$.*

Proof. Let $x^{[N]}$ denote the term of T_1° inductively defined by $x^{[1]} = x$ and $x^{[N]} = x \circ x^{[N-1]}$ for $N \geq 2$. The subgroup of B_\bullet generated by the elements a_i is isomorphic to Thompson's group F , and it gives rise to a partial action on T_1° corresponding to applying the associativity law [6]: the action of a_i on a term v is defined provided v can be expressed as $v_1 \circ \dots \circ v_{i+2}$, i.e., we have $\text{ht}_R(v) \geq i + 2$, and, in this case, one defines $v \cdot a_i = v_1 \circ \dots \circ v_{i-1} \circ (v_i \circ v_{i+1}) \circ v_{i+2}$. Then, an easy induction shows that, for each term v of size p in T_1° , the element $v(1)$ of B_\bullet maps any sufficiently large term $x^{[N]}$ to the term $v \circ x^{[N-p]}$. Hence $u(1)^{-1}v(1)$ maps $u \circ x^{[N-p]}$ to $v \circ x^{[N-q]}$, where p is the size of u . Now any element of $\text{Im}\partial$ maps a term of the form $u \circ \dots$ to another term of the form $u \circ \dots$, since only a_1 may change the left subterm of the initial term. Hence $u(1)^{-1}v(1) \in \text{Im}\partial$ is impossible for $v \neq u$. \square

Proposition 2.10. *For any γ in B_\bullet , the closure of $\{\gamma\}$ under $*$ and \circ is free ALD-system.*

Proof. We apply the criterion of Proposition 2.6. Assume that u, v are terms in T_1° and \vec{s}, \vec{t} are sequences of terms in T_1^* . Let $s = u[\vec{s}]$ and $t = v[\vec{t}]$. Our aim is to prove $s(\gamma)^{-1}t(\gamma) \neq 1$ both for $u < v$, and for $u = v$ with $\vec{s} \vec{c} \vec{t}$. Applying Lemma 2.8, we find

$$(2.6) \quad s(\gamma)^{-1}t(\gamma) = u(1)^{-1} \cdot \partial^{p-1} s_p(\gamma)^{-1} \cdot \dots \cdot s_1(\gamma)^{-1} \cdot t_1(\gamma) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot v(1).$$

We shall consider three cases, which cover the cases $u < v$, and $u = v$ with $\vec{s} \vec{c} \vec{t}$, and prove in each of them that the right hand side of (2.6) is not 1.

Assume first that there exists $k \leq \inf(p, q)$ such that $s_i =_{LD} t_i$ holds for $i < k$, and $s_k \neq_{LD} t_k$ holds. Then we have $s_i(\gamma) = t_i(\gamma)$ for $i < k$, and (2.6) becomes

$$s(\gamma)^{-1}t(\gamma) = u(1)^{-1} \cdot \partial^{p-1} s_p(\gamma)^{-1} \cdot \dots \cdot \partial^{k-1} (s_k(\gamma)^{-1} t_k(\gamma)) \cdot \dots \cdot \partial^{q-1} t_q(\gamma) \cdot v(1).$$

By the results of [4], the hypothesis $s_k \neq_{LD} t_k$ implies either $s_k \sqsubset_{LD} t_k$ or $t_k \sqsubset_{LD} s_k$, and the explicit definition of operation $*$ on B_\bullet then implies that the braid $s_k(\gamma)^{-1} t_k(\gamma)$ admits an expression where the generator σ_1 appears but σ_1^{-1} does not, or σ_1^{-1} appears but σ_1 does not. It follows that $s(\gamma)^{-1}t(\gamma)$ admits an expression in which σ_k appears but neither σ_k^{-1} nor any $\sigma_i^{\pm 1}$ with $i < k$ does, or vice versa exchanging σ_k and σ_k^{-1} . By [6], Proposition 4.6, this guarantees $s(\gamma) < t(\gamma)$ in the canonical ordering of B_\bullet , hence $s(\gamma) \neq t(\gamma)$.

Assume now $p < q$ with $s_i =_{LD} t_i$ for $i \leq p$. In this case, (2.6) reduces to

$$s(\gamma)^{-1}t(\gamma) = u(1)^{-1} \cdot \partial^p(t_{p+1}(\gamma) \cdot \dots \cdot \partial^{q-p-1}t_q(\gamma)) \cdot v(1).$$

By Lemma 2.7, we have $u(1)^{-1} \cdot \partial^p z = \partial z u(1)^{-1}$ for each z in B_\bullet , so we get

$$s(\gamma)^{-1} \cdot t(\gamma) = \partial(t_{p+1}(\gamma) \cdot \dots \cdot \partial^{q-p-1}t_q(\gamma)) \cdot u(1)^{-1}v(1).$$

This cannot be 1, as the first factor belongs to $\text{Im}\partial$, while, according to Lemma 2.9, $u(1)^{-1}v(1)$ does not unless $u = v$ holds.

Assume finally $p = q$ with $s_i =_{LD} t_i$ for $i \leq p$, and $u < v$. Then (2.6) reduces to

$$s(\gamma)^{-1}t(\gamma) = u(1)^{-1} \cdot v(1),$$

and, by Lemma 2.7, the above expression cannot be 1. \square

Remark 2.11. It is shown in [6] that the parenthesized braid group B_\bullet comes can be equipped with a distinguished linear ordering that extends both the linear ordering of braids and the natural ordering on Thompson's group induced by the lexicographical ordering of finite trees. Let us define a relation $<_{ALD}$ on special terms in $T_1^{*,\circ}$ as follows: first say that $u[\vec{s}] < v[\vec{t}]$ holds if we have either $\vec{s} \preceq \vec{t}$, or \vec{s} is a proper prefix of \vec{t} , or we have $\vec{s} = \vec{t}$ and $u < v$ holds; then say that $s <_{ALD} t$ holds if there exist special terms $u[\vec{s}], v[\vec{t}]$ satisfying $u[\vec{s}] < v[\vec{t}]$, $s =_{ALD} u[\vec{s}]$, and $t =_{ALD} v[\vec{t}]$. Then the relation $<_{ALD}$ induces a linear ordering on the free ALD-system $T_1^{*,\circ}/=_{ALD}$, and what actually shows the proof of Proposition 2.10 is that, for each parenthesized braid γ in B_\bullet , the evaluation mapping $t \mapsto t(\gamma)$ is increasing.

2.5. The converse direction. According to Proposition 2.3, the operations of (2.2) define operations on B_\bullet that make it an ALD-system. We conclude with the easy observation that, conversely, the operations defined on a group G by formulas of the type (2.2) give rise to an ALD-system only if G is closely connected to B_\bullet :

Proposition 2.12. *Assume that G is a group, ∂ is an endomorphism of G , and a, σ are fixed elements of G . Write σ_i for $\partial^{i-1}(\sigma)$ and a_i for $\partial^{i-1}(a)$. Then defining*

$$(2.7) \quad x * y = x \cdot \partial y \cdot \sigma \cdot \partial x^{-1}, \quad x \circ y = x \cdot \partial y \cdot a$$

yields an ALD-system on the subgroup H generated by the elements σ_i 's and the a_i 's—i.e., on the smallest subgroup of G containing σ and a and closed under ∂ —if and only if the elements σ_i and a_i obey the relations (2.1), i.e., if and only if H is a homomorphic image of B_\bullet .

Proof. Assume that $(G, *, \circ)$ is an ALD-system. The instance $1 * (1 * z) = (1 * 1) * (1 * z)$ of (LD) expands into

$$(2.8) \quad \partial^2 z \cdot \sigma_2 \sigma_1 = \sigma_1 \cdot \partial^2 z \cdot \sigma_2 \sigma_1 \sigma_2^{-1}.$$

For $z = 1$, we obtain the braid relation

$$(2.9) \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2,$$

and, then, (2.8) gives

$$(2.10) \quad \partial^2 z \cdot \sigma_1 = \sigma_1 \cdot \partial^2 z$$

for each z . Similarly, the instance $1 * (1 * z) = (1 \circ 1) * z$ of (ALD₁) expands into

$$(2.11) \quad \partial^2 z \cdot \sigma_2 \sigma_1 = a_1 \cdot \partial z \cdot \sigma_1 a_2^{-1}.$$

For $z = 1$, we deduce

$$(2.12) \quad a_1 \sigma_1 = \sigma_2 \sigma_1 a_2,$$

and, then, (2.11) gives

$$(2.13) \quad \partial^2 z \cdot a_1 = a_1 \cdot \partial z$$

for each z . Finally, the instance $1 * (1 \circ 1) = (1 * 1) \circ (1 * 1)$ of (ALD_2) expands into

$$(2.14) \quad a_2 \sigma_1 = \sigma_1 \sigma_2 a_1.$$

Conversely, it is easy to verify that the conjunction of (2.8), (2.11) (for each z), and (2.9), (2.12), and (2.14) guarantees that $(G, *, \circ)$ be an ALD-system. When we restrict to the subgroup H , this amounts to saying that the elements σ_i and a_i satisfy the defining relations (2.1) of B_\bullet . \square

The previous result shows that there is no flexibility or randomness in the construction of an ALD-system using the formulas of (2.7). However, what was not explained here—nor was it in [6] either—is where do these formulas come from. Actually, the group B_\bullet and the formulas (2.7) arise naturally when investigating the so-called geometry monoid of the ALD laws. This will be explained in a forthcoming paper.

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