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Polymer pinning in a random medium as influence percolation

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Abstract: In this article we discuss a set of geometric ideas which shed some light on the question of directed polymer pinning in the presence of bulk disorder. Differing from standard methods and techniques, we transform the problem to a particular dependent percolative system and relate the pinning transition to a percolation transition.

1. Introduction

Motivating example. The totally asymmetric exclusion process (TASEP for short) is defined as follows on the set $\mathbb{Z}$: At time 0, a (possibly random) configuration of particles is given, in such a way that each site contains at most one particle. To each edge of the lattice is associated a Poisson clock of intensity 1. Whenever this clock rings, and there is a particle at the left-end vertex of this edge and no particle at the right-end vertex, the particle moves to the right; otherwise the ring of the clock is ignored. The product over $\mathbb{Z}$ of Bernoulli measures of density $\rho \in (0,1)$ is invariant by this dynamics; in that case, the average number of particles passing through the origin up to time t is equal to $\rho(1-\rho)t$, *i.e.* the flux through a given bond is exactly $\rho(1-\rho)$.

The process is modified at the origin, by imposing that the Poisson clock associated with the bond $e_0 = \langle 0,1\rangle$ is $\lambda > 0$. When $\lambda > 1$, one can still prove that the above expression for the flux holds asymptotically, although the Bernoulli measure is not an equilibrium measure anymore. From now on, we shall assume that $0 < \lambda \leq 1$.

One of the fundamental questions in driven flow is to understand under which conditions such a static obstruction results in the formation of a "platoon" starting at the origin and propagating leftward. A convenient quantitative criterion for platoon formation is to start the TASEP with step initial conditions, i.e. all sites $x \leq 0$ filled and all sites $x \geq 1$ empty, and to consider the average current, $j(\lambda)$, in the long time limit $t \to \infty$. $j(0) = 0$, j is non-decreasing, and $j(\lambda) = 1/4$ for $\lambda = 1$. Thus the issue is to determine the critical intensity λ_c, which is defined as the supremum of all the λ for which $j(\lambda) < \frac{1}{4}$. Estimates for the value of λ_c were given in [10] and [5], and the full hydrodynamical picture is proved in [15].

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The blockage problem for the TASEP has been studied numerically and by exact enumerations. On the basis of these data, in [8] the value $\lambda_c = 1$ is conjectured. Recently this result has been challenged ([6]) and $\lambda_c \cong 0.8$ is claimed.

One-dimensional driven lattice gases belong to the universality class of Kardar-Parisi-Zhang (KPZ) type growth models. In particular the asymmetric simple exclusion process can be represented as the so-called body-centered solid-on-solid version of (1+1)-dimensional polynuclear growth model, or as directed polymer subject to a random potential. The TASEP also has a well known representation in terms of last-passage percolation (or maximal increasing subsequences, known as Ulam's problem). In this article we choose to work in the setup of last passage percolation. The slow bond induces an extra line of defects relative to the disordered bulk. If $\lambda < \lambda_c$, the optimal path, *i.e.* the *geodesic*, is pinned to the line of defects. As $\lambda \to \lambda_c$, the geodesic wanders further and further away from the line of defects and the density of intersections with the line of defects tends to zero. For $\lambda > \lambda_c$, the fluctuations of the geodesic are determined by the bulk, and the line of defects is irrelevant.

In the present work we will not establish the actual value of λ_c (since we cannot) or settle the question as to whether it is equal to 1; the main goal of this paper is to describe a new way of looking at the problem which gives some insight about the precise behavior of the system. More precisely, we show how the problem can be studied through particular dependent percolative systems constructed in such a way that the pinning transition can be understood in terms of a percolation transition.

2. Interpretation as pinning in Ulam's problem

In this section we describe the representation of our initial problem in terms of Ulam's problem or polynuclear growth. For more detailed explanations, see *e.g.* [1] or [11].

2.1. The Model

Let $P^{(2)}$ be the distribution of a Poisson point process of intensity $\lambda^{(2)} = 1$ in the plane $\mathbb{R}^2$, and let $\Omega^{(2)}$ the set of all its possible configurations; for all $n > 0$, let $P_n^{(2)}$ be the law of its restriction to the square $\mathcal{Q}_n := [0, n] \times [0, n]$ and $\Omega_n^{(2)}$ be the configuration space of the restricted process.

The *maximal increasing subsequence problem*, or Ulam's problem, can be formulated in the following geometric way: Given a configuration $\omega^{(2)} \in \Omega^{(2)}$ and its restriction $\omega_n^{(2)}$ to $\mathcal{Q}_n$, look for an oriented path π (moving only upward and rightward) from $(0, 0)$ to (n, n) collecting as many points from $\omega_n^{(2)}$ as possible; as in the case of last-passage percolation described in the introduction, we shall call such a path a *geodesic*. Let $\mathcal{N}_n$ denote the number of collected points along such an optimal path (which need not be unique). It is a well known fact (see [1]) that

$$\lim_{n\to+\infty} \frac{E^{(2)}\mathcal{N}_n}{n} = 2. \tag{2.1}$$

In a remarkable paper [9], K. Johansson showed that for this model the transversal fluctuations of the geodesics are of order $n^{2/3}$. A closely related problem is considered in [3].

We now modify the original model in the following way: Assume in addition, that on the main diagonal $y = x$ there is an independent one-dimensional Poisson

point process of intensity $\lambda^{(1)} = \lambda$, and let $P^{(1)}$ be its distribution. We will denote by $\Omega^{(1)}$ (resp. $\Omega_n^{(1)}$) the configuration space of this process (resp. of its restriction to $\mathcal{Q}_n$). Finally if $\omega^{(1)}$ is a realization of $P^{(1)}$ and $\omega^{(2)}$ one of $P^{(2)}$, let $\omega = \omega^{(2)} \cup \omega^{(1)}$ be their union, and ω_n the restriction of ω to $\mathcal{Q}_n$.

What can now be said about an optimal directed path starting at $(0,0)$ and ending at (n,n)? If $\lambda \gg 1$, clearly the geodesic will stay close to the diagonal and

$$\lim_{n\to\infty} \frac{E^{(1)} \times E^{(2)}\mathcal{N}_n}{n} =: e(\lambda) > 2. \tag{2.2}$$

Instead of (2.2) in our context it will be more convenient to use a more geometric notion. We say that the directed polymer is pinned (with respect to the diagonal) if $P^{(1)} \times P^{(2)}$-almost surely, the number of visits made by geodesics to $\{(u,u)\colon 0 \le u \le t\}$ is of order γt, for some $\gamma > 0$, for all t large enough. One expects the "energy" notion (2.2) and geometric notion of pinning to be identical, but this is yet another point which remains to be proved.

$e(\lambda)$ is non-decreasing and there is a critical value λ_{c}, where e hits the value 2. The same arguments which predict $\lambda_{\mathrm{c}} = 1$ in the case of the TASEP yield $\lambda_{\mathrm{c}} = 0$ in the case of our model. Thus any extra Poisson points along the diagonal are expected to pin the directed polymer. Such a behavior is extremely delicate, and the answer depends on the nature and behavior of the geodesics in the initial, unperturbed system. Very little is known, even on a heuristic level, when the underlying measure governing the behavior of the polymer is not "nice" (with a kind of Markov property, for example a simple symmetric random walk). Our criteria presented below give partial but rigorous answers as to whether λ_{c} is strictly positive or not.

In passing let us note that for a symmetric environment pinning can be proved, at least on the level of $e(\lambda)$ [7]. Symmetric means that $P^{(2)}$ is concentrated on point configurations which are symmetric relative to the diagonal. In this case $\lambda_{\mathrm{c}} = 1$, *i.e.* $e(\lambda) = 2$ for $0 \le \lambda \le 1$ and $e(\lambda) > 2$ for $\lambda > 1$. Indeed in the symmetric case, the system is amenable to exact computations in terms of Fredholm determinants; a trace of the simplification can also be seen in our later discussion (see Section 2.6).

2.2. Construction of the broken-lines process

Hammersley gave a representation of the longest increasing subsequence problem for a random permutation in terms of broken lines built from a Poisson point process in the positive quadrant (we describe the construction in some detail below). The length of the longest increasing subsequence can then be seen to be the number of lines which separate the points $(0,0)$ and (n,n) from one another. The purpose of that representation is to obtain a superadditivity property which easily implies the existence of the limit in (2.1) — but doesn't specify its value. It is a very convenient formalism, which was used in [11, 13, 14] and [16, 17].

The broken line process Γ_S in a finite domain S can be defined as the space-time trace of some particle system with birth, death and immigration. For convenience we rotate the whole picture by an angle of $\pi/4$ clockwise, so that the geodesic is restricted to never have a slope which is larger than 1 in absolute value. In what follows we will consistently use the letters t and x for the first and second coordinates, respectively, in the rotated picture; we will refer to t as "time" (the reason for that will become clear shortly). The geodesics can then be seen as curves of space-type (using the usual language of general relativity).

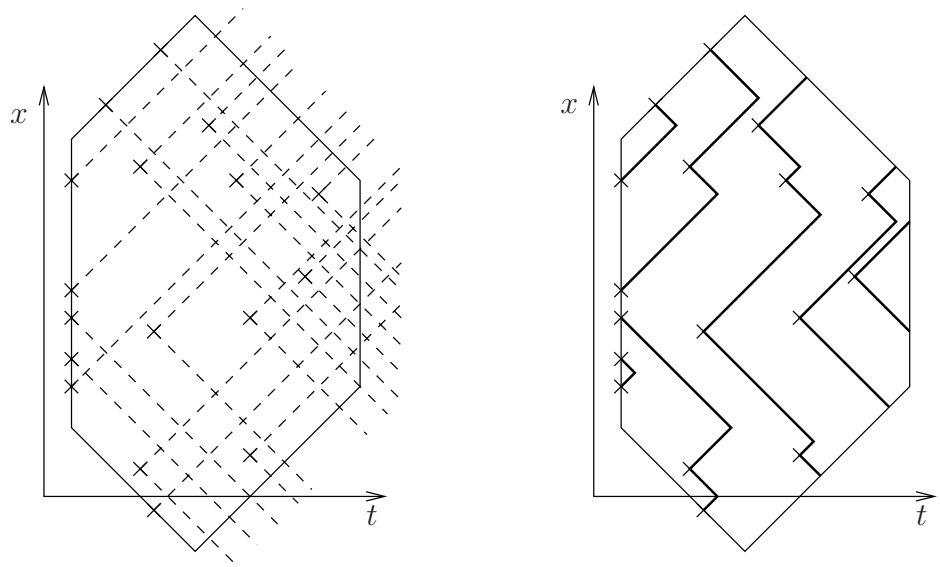

FIG 1. *The construction of a Hammersley process.*

Let S be the planar, bounded domain defined (cf. Figure 1) as

$$S := \{(t,x) : t_0 < t < t_1,\ g_-(t) < x < g_+(t)\}, \tag{2.3}$$

where $t_0 < t_1$ are given points and $g_-(t) < g_+(t)$, $t_0 < t < t_1$ are piecewise linear continuous functions such that, for some t_{01}^+ (resp. t_{01}^-) in the interval $[t_0, t_1]$, $g_+(t)$ (respectively $g_-(t)$) increases (respectively, decreases) on $(t_0,\, t_{01}^+)$ (resp. $(t_0,\, t_{01}^-)$) and decreases (respectively, increases) on $(t_{01}^+,\, t_1)$ (resp. $(t_{01}^-,\, t_1)$), always forming an angle of $\pm\pi/4$ with the t-axis.

Consider four independent Poisson processes $\Pi_{0,+}$, $\Pi_{0,-}$, Π_+ and Π_- on the boundary of S. The processes $\Pi_{0,\pm}$ are supported on the leftmost vertical boundary component $\triangle_0 S := \{(t_0, x) : g_{0,-} < x < g_{0,+}\}$ of S, where $g_{0,\pm} := g_\pm(t_0)$, and they both have intensity $\sqrt{\lambda^{(2)}/2}$. The process Π_+ is defined on the "northwest" boundary $\triangle_+ S := \{(t, g_{0,+}(t)) : t_0 < t < t_{01}^+\}$, and the process Π_- is defined on the "southwest" boundary $\triangle_- S := \{(t, g_{0,-}(t)) : t_0 < t < t_{01}^-\}$; their intensities (with respect to the length element of $\triangle_\pm S$) are both $\sqrt{\lambda^{(2)}}$. Finally, let Π_{in} be a Poisson point process in S, with intensity $\lambda^{(2)}$, and independent of the previous four.

Following the general strategy for the definition of Markov polygonal fields of [2], we define a broken line process as follows. Each point of the Poisson process Π_{in} is the point of birth of two particles which start moving in opposite directions, *i.e.* with velocities $+1$, -1. At each random point of $\Pi_{0,+}$, Π_- a particle is born having velocity $+1$. Similarly at each random point of $\Pi_{0,-}$, Π_+, a particle is born having velocity -1. All particles move with constant velocity until two of them collide, after which both colliding particles are annihilated (see Figure 1).

The state space $\mathcal{X}_S$ of the process Γ_S is the set of all finite collections $(\gamma_1, \ldots, \gamma_k)$ (including the empty one) of disjoint "broken lines" γ_j inside S. By a broken line in S we mean the graph $\gamma = \{(t,x) \in S : t = \widetilde{\gamma}(x)\}$ in S of a continuous and piecewise linear function $\widetilde{\gamma}$, with slopes all ± 1. Let $\mathcal{P}_S$ denote the probability distribution on $\mathcal{X}_S$ corresponding to the broken line process Γ_S defined above.

Let $S' \subset S''$ be two bounded domains of the form of (2.3), and let $\mathcal{P}_{S'}$ and $\mathcal{P}_{S''}$ be the probability distributions of the broken line processes in S' and S'', respectively. The probability measure $\mathcal{P}_{S''}$ on $\mathcal{X}_{S''}$ induces a probability measure $\mathcal{P}_{S''}|_{S'}$ on $\mathcal{X}_{S'}$, which is the distribution of the restricted process $\Gamma_{S''} \cap S'$. Then, by the choice we made of the boundary conditions, the following consistency property holds (see [2]):

$$\mathcal{P}_{S''}|_{S'} = \mathcal{P}_{S'}. \tag{2.4}$$

This guarantees the existence of the broken line process $\Gamma \equiv \Gamma_{\mathbb{R}^2}$ on $\mathbb{R}^2$, its distribution $\mathcal{P}$ on $\mathcal{X}_{\mathbb{R}^2}$ being such that for every S of the above shape, $\mathcal{P}|_S = \mathcal{P}_S$. Moreover $\mathcal{P}$ is invariant with respect to the translations of $\mathbb{R}^2$.

Remark. The same description holds for the polynuclear growth (PNG) model, which describes a crystal growing layer by layer on a one-dimensional substrate through the random deposition of particles. They nucleate on existing plateaus of the crystal, forming new islands. In an idealization these islands spread laterally with constant speed by steady condensation of further material at the edges of the islands. Adjacent island of the same level coalesce upon meeting and on the top of the new levels further islands emerge.

Observe that a path π which can move only in the northeast-southeast cone (*i.e.* a path of space type) can collect at most one initial Poisson point from each broken line. In other words, the broken lines "factorize" the points of the configuration ω, in such a way that it tells us which points cannot be collected by the same path, and that the maximal number of points is bounded by the number of broken lines which lie in between start-point and end-point of the path π.

In fact, the lines also provide an explicit construction of a geodesic, as follows: Start at point (t_1, x) on the right boundary and move leftward until you meet a line, then follow this line until you arrive at a point, which you can collect. Then start moving leftward again until you collect a point in the second-to-last broken line, and so on. The number of collected points is then essentially equal to the number of broken lines, though a little care needs to be taken as far as boundary conditions are concerned if this comparison is to be made completely formal. This observation led to a new proof of (2.1) in [2]. It is also the starting point of our argument.

2.3. Essential and non-essential points

We now return to the question asked at the beginning of this section: How are extra added points affecting the initial system? We begin with a few purely deterministic observations and statements. We will need some extra notations: Given any configuration $\widetilde{\omega}_n$ (not necessarily sampled from a Poisson process) of points in $\mathcal{Q}_n$, let $H(\widetilde{\omega}_n)$ be the number of broken lines produced by the above construction. For $\mathbf{x} = (x, t)$ and $A = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ we will denote by $\widetilde{\omega}_n \cup \mathbf{x}$ or $\widetilde{\omega}_n \cup A$ the configuration obtained from $\widetilde{\omega}_n$ by the addition of the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$, and by $\Gamma_n(\widetilde{\omega}_n)$ the associated configuration of broken lines.

Proposition 2.1 (Abelian property, see [17]). *For any choice of $\widetilde{\omega}_n$, $\mathbf{x}_1$ and $\mathbf{x}_2$ we have*

$$\Gamma_n\left(\widetilde{\omega}_n \cup (\mathbf{x}_1 \cup \mathbf{x}_2)\right) = \Gamma_n\left((\widetilde{\omega}_n \cup \mathbf{x}_1) \cup \mathbf{x}_2\right) = \Gamma_n\left((\widetilde{\omega}_n \cup \mathbf{x}_2) \cup \mathbf{x}_1\right). \tag{2.5}$$

Proposition 2.2 (Monotonicity, see [17]). *For any choice of $\widetilde{\omega}_n$ and $\mathbf{x}_1$ we have*

$$H(\widetilde{\omega}_n) \;\le\; H(\widetilde{\omega}_n \cup \mathbf{x}_1) \;\le\; H(\widetilde{\omega}_n) + 1. \tag{2.6}$$

Definition 2.3. Given $\widetilde{\omega}_n$ and $\mathbf{x}$ we say that $\mathbf{x}$ is *essential for* $\widetilde{\omega}_n$ if $H(\widetilde{\omega}_n \cup \mathbf{x}_1) = H(\widetilde{\omega}_n) + 1$.

Remark. The above definition is domain-dependent: If $S_n \subset S_m$ are two domains in $\mathbb{R}^2$, and $\widetilde{\omega}_n, \widetilde{\omega}_m$ are the restrictions of a configuration $\widetilde{\omega}$ to S_n and S_m respectively, then an extra added point which is essential for $\widetilde{\omega}_n$ might not be essential for $\widetilde{\omega}_m$.

If an added point $\mathbf{x}$ is essential, its presence is felt on the boundary of the domain by the appearance of an extra broken line going outside of the area. If an added point $\mathbf{x}$ is not essential, its presence can be felt on the boundary or not, but in any case it will change the local geometry of existing broken lines.

Speaking informally, the configuration $\widetilde{\omega}_n$ determines a partition of the domain into two (possibly disconnected) regions E and B, such that any additional point chosen in E will be essential for $\widetilde{\omega}_n$, while if it is in B it will be not essential for $\widetilde{\omega}_n$. It is easy to construct examples of configurations $\widetilde{\omega}$ for which E is empty (*i.e.*, that are very insensitive to local changes). On the other hand B is never empty as soon as $\widetilde{\omega}$ is not empty. It is also easy to give examples of the following situations:

- $\mathbf{x}_1$ is not essential for $\widetilde{\omega}_n$ and $\mathbf{x}_2$ is not essential for $\widetilde{\omega}_n$, but $\mathbf{x}_1$ is essential for $\widetilde{\omega}_n \cup \mathbf{x}_2$ and $\mathbf{x}_2$ is essential for $\widetilde{\omega}_n \cup \mathbf{x}_1$;
- $\mathbf{x}_1$ is essential for $\widetilde{\omega}_n$ and $\mathbf{x}_2$ is essential for $\widetilde{\omega}_n$, but $\mathbf{x}_2$ is not essential for $\widetilde{\omega}_n \cup \mathbf{x}_1$.

Remark. In order to simplify our explanations and make some concepts as well as the notations more transparent (and lighter), we will change from the consideration of point-to-point case to the point-to-hyperplane case, *i.e.* instead of looking for a geodesic connecting $(0,0)$ to $(t,0)$, we will be looking for an optimal path connecting $(t,0)$ to the line $x = 0$. This change is of purely "pedagogical" nature: All the ideas discussed above and below are easily transferred to the point-to-point case. Nevertheless we will not deny that it requires some amount of additional work due to boundary conditions.

The reader should also not be surprised with our taking of starting point as $(t,0)$ and moving backward to the x-axis in the point-to-hyperplane case (see the remark at the end of Section 2.2). Since our broken lines were constructed by drawing the space-time trajectories of particles in "forward time", the information provided by the broken lines about the underlying point configuration is useful in the backward direction, and thus it forces us to construct the geodesic this way. Conversely, in order to construct a forward geodesic, one could construct the broken lines on the same point configuration but backward in time.

Due to that, the area to which we will be restricting our process will be the triangular area S_n enclosed by segments connecting points $(0,-n)$, $(0,n)$ and $(n,0)$. If confusion doesn't arise, we will keep denoting related quantities by the same sub-index n as for the square case, for example $\widetilde{\omega}_n$ will stay from now on for the configuration of points in this triangular area.

The next proposition is the crucial point in our construction.

Proposition 2.4 (see [17]). *If* $\mathbf{x}$ *is essential for* $\widetilde{\omega}_n$ *then the point-to-plane geodesic in configuration* $\widetilde{\omega}_n \cup \mathbf{x}$ *has to collect point* $\mathbf{x}$.

Again this is a purely deterministic statement, and does not depend on the choice of $\widetilde{\omega}_n$ or $\mathbf{x}$. Further we will be considering only cases when extra points are added only along the t-axis.

2.4. Propagation of influence

Once an extra point $\mathbf{x}$ is added to the system, we need to update the configuration of broken lines. One way to do that is to redo the whole construction from scratch, *i.e.* to erase all the existing broken lines and redraw them using the algorithm we described previously, taking the new point into account. It is then natural to ask how much the new picture differs from the old one, which is not perturbed by addition of point $\mathbf{x}$. It turns out that there is a very simple algorithm allowing us to trace all the places of the domain where the addition of $\mathbf{x}$ will be felt, *i.e.* where local modification will be done.

Consider an augmented configuration $\widetilde{\omega}'_n = \widetilde{\omega}_n \cup \{\mathbf{x}\}$. In order to see where and how the broken lines of $\Gamma_n(\widetilde{\omega}_n)$ will be modified, we look at a new interacting particle system, starting from the points of $\widetilde{\omega}'_n$, but with new interaction rules:

1. Particles starting from the points of $\widetilde{\omega}_n$ are following the same rules as before, *i.e.* they are annihilated at the first time when they collide with any other particle. These particles will be called "regular particles";
2. The two particles which start from the newly added point $\mathbf{x}$ (also with velocities $+1$ and -1) will be called "superior particles", and they obey different rules:
   (a) Superior particles annihilate if and only if they collide with each other;
   (b) If a superior particle collides with some regular particle, the velocity of the superior particle changes to that of the incoming regular particle, which is annihilated while the superior particle continues to move.

We will denote by $p^+_{\mathbf{x}}$ (resp. $p^-_{\mathbf{x}}$) the superior particle which starts from $\mathbf{x}$ with initial velocity $+1$ (resp. -1); The space-time trajectories of $p^+_{\mathbf{x}}$ and $p^-_{\mathbf{x}}$ will be denoted by $\pi^+_{\mathbf{x}}$ and $\pi^-_{\mathbf{x}}$, respectively.

Observe that if any of these two trajectories leaves the triangular area S_n, it will never come back to it. (Notice that since superior particles can change their velocities during their evolution, both particles can leave the triangular area from the same side.) If $\pi^+_{\mathbf{x}}$ and $\pi^-_{\mathbf{x}}$ intersect inside of S_n, then, according to rule 2a the corresponding superior particles are annihilated.

The path $\pi^+_{\mathbf{x}}$ (resp. $\pi^-_{\mathbf{x}}$) can be represented as an alternating sequence of concatenated segments $\pi^+_{\mathbf{x}}(1,+)$, $\pi^+_{\mathbf{x}}(1,-)$, $\pi^+_{\mathbf{x}}(2,+)$, $\pi^+_{\mathbf{x}}(2,-),\dots$ (resp. $\pi^-_{\mathbf{x}}(1,-)$, $\pi^-_{\mathbf{x}}(1,+)$, $\pi^-_{\mathbf{x}}(2,-)$, $\pi^-_{\mathbf{x}}(2,+),\dots$), where each segment corresponds to the time interval between two consecutive velocity changes of the superior particle $p^+_{\mathbf{x}}$ (resp., $p^-_{\mathbf{x}}$), and during which its velocity is equal to $+1$ or -1 according to the sign given as second argument in the notation.

The trajectories of $p^+_{\mathbf{x}}$ and $p^-_{\mathbf{x}}$ are completely determined by $\widetilde{\omega}_n \cup \{\mathbf{x}\}$. Observe that each time $p^+_{\mathbf{x}}$ changes its velocity from $+1$ to -1 (resp. $p^-_{\mathbf{x}}$ changes its velocity from -1 to $+1$), it starts to move along a segment which also belongs to some broken line γ_i from $\Gamma_n(\widetilde{\omega}_n)$, and when it changes its velocity back to $+1$, it leaves this broken line, and moves until the next velocity flip, which happens exactly when the superior particle collides with the next broken line γ_{i+1} in $\Gamma_n(\widetilde{\omega}_n)$. This gives an extremely simple rule how to transform $\Gamma_n(\widetilde{\omega}_n)$ into $\Gamma_n(\widetilde{\omega}'_n)$ (see Figure 2):

- Erase all the $\pi^{\pm}_{\mathbf{x}}(j,\mp)$ (*i.e.*, all parts of the path of the superior particle which are contained in one of the original broken lines) to obtain an intermediate picture $\Gamma'_n(\widetilde{\omega}_n)$;
- Add all the $\pi^{\pm}_{\mathbf{x}}(j,\pm)$ thus obtaining the picture $\Gamma''_n(\widetilde{\omega}_n)$.

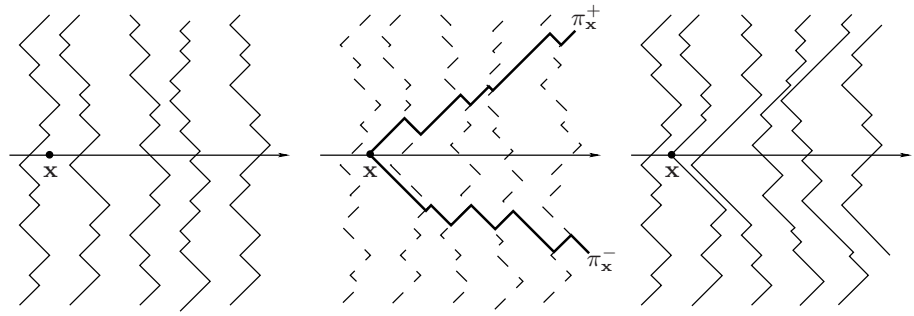

FIG 2. *Propagation of influence.*

It is not hard to conclude that $\Gamma_n''(\widetilde{\omega}_n) = \Gamma_n(\widetilde{\omega}_n')$. In other words the paths $\pi_{\mathbf{x}}^+$ and $\pi_{\mathbf{x}}^-$ show how the "influence" of $\mathbf{x}$ spreads along the configuration $\Gamma_n(\widetilde{\omega}_n)$. If both superior particles $p_{\mathbf{x}}^+$ and $p_{\mathbf{x}}^-$ collide inside the domain S_n, the trajectories $\pi_{\mathbf{x}}^+$ and $\pi_{\mathbf{x}}^-$ close into a loop, and outside of this loop the configuration $\Gamma_n(\widetilde{\omega}_n)$ was not modified, *i.e.* the presence of $\mathbf{x}$ was not felt at all.

Definition 2.5. Given a configuration ω of the underlying Poisson process, and an added point $\mathbf{x}$ on the time axis, we will denote by $\tau(\mathbf{x};\omega)$ (or simply $\tau(\mathbf{x})$ if there is no confusion possible) the *self-annihilation time* of the pair of particles created at $\mathbf{x}$, *i.e.* the time at which the paths $\pi_{\mathbf{x}}^-$ and $\pi_{\mathbf{x}}^+$ meet, if such a time exists; let $\tau(\mathbf{x}) = +\infty$ otherwise.

In the specific case we are looking at, $\tau(\mathbf{x})$ is almost surely finite if $\lambda^{(2)} > 0$, but it need not be the case for other underlying point processes.

2.5. Interaction and Attractors

Another important step in the analysis of the spread of influence, is to understand how the influence paths interact with each other if we add multiple points $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ to the initial configuration. Proposition 2.1 implies that we can obtain the full picture by adding the points one by one; to simplify the notations, in our description of the procedure we will also use the fact that the additional points will be placed along t-axis, though this is not essential.

Again, due to the presence of time orientation, the nature of the interaction between influence paths becomes exposed in a more transparent way if we proceed backward, *i.e.* if we begin to observe the modifications first when adding the rightmost point, and then continue progressively, adding the points one by one, moving leftward, each time checking the effect created by the newly added point. For notational convenience let us index the new points in the backward direction, *i.e.* $\mathbf{x}_i = (t_i, 0)$ with $t_1 > t_2 > \cdots$.

Applying the construction described in the previous subsection successively for each of the new points, we obtain the following rules for the updating a configuration with multiple points added: Take the initial configuration $\widetilde{\omega}_n$ and let $\widetilde{\omega}_n^{(r)} = \widetilde{\omega}_n \cup \{\mathbf{x}_i, 1 \le i \le r\}$ be the modified configuration. In order to see where and how the broken lines of $\Gamma_n(\widetilde{\omega}_n)$ will be updated, consider a new particle representation built on the configuration $\widetilde{\omega}_n^{(r)}$, and obeying the following rules:

1. Regular particles, starting from the points of $\widetilde{\omega}_n$, are annihilated as soon as they collide with any other particle;

2. The particles starting from the $(\mathbf{x}_i)_{1\le i\le r}$, with velocities $+1$ and -1 are denoted by $p^+_{\mathbf{x}_i}$ and $p^-_{\mathbf{x}_i}$ respectively; again we shall call them superior particles. They behave as follows:
   (a) Whenever two superior particle of different types collide ("$+-$" collision), they annihilate and both disappear;
   (b) If two superior particles of the same type collide ("$++$" or "$--$" collision), then they exchange their velocities (elastic interaction) and continue to move;
   (c) If a superior particle collides with a regular particle, the velocity of the superior particle changes that of the incoming regular particle; the regular particle is annihilated, while the superior particle survives and continues to move.

Denote the space-time trajectories of superior particles $p^+_{\mathbf{x}_i}$ and $p^-_{\mathbf{x}_i}$ by $\pi^+_{\mathbf{x}_i}$ and $\pi^-_{\mathbf{x}_i}$, respectively. As before, each pair of paths $\pi^+_{\mathbf{x}_i}$ and $\pi^-_{\mathbf{x}_i}$ can be represented as an alternating sequence of concatenated segments $\pi^+_{\mathbf{x}_i}(1,+)$, $\pi^+_{\mathbf{x}_i}(1,-)$, $\pi^+_{\mathbf{x}_i}(2,+)$, $\pi^+_{\mathbf{x}_i}(2,-),\dots$ or $\pi^-_{\mathbf{x}_i}(1,-)$, $\pi^-_{\mathbf{x}_i}(1,+)$, $\pi^-_{\mathbf{x}_i}(2,-)$, $\pi^-_{\mathbf{x}_i}(2,+),\dots$, respectively, with the same convention for the sign of the velocities. We are now ready to complete the set of rules which govern the transformation of $\Gamma_n(\widetilde{\omega}_n)$ in to $\Gamma_n(\widetilde{\omega}^{(r)}_n)$:

- Erase each segment $\pi^{\pm}_{\mathbf{x}_i}(j,\mp)$ from $\Gamma_n(\widetilde{\omega}_n)$, producing an intermediate picture $\Gamma'_n(\widetilde{\omega}_n)$;
- Add the segments $\pi^{\pm}_{\mathbf{x}_i}(j,\pm)$ to $\Gamma'_n(\widetilde{\omega}_n)$, thus producing $\Gamma''_n(\widetilde{\omega}_n)=\Gamma_n(\widetilde{\omega}^{(r)}_n)$.

(Here as previously, two $\pm$ in the same formula are taken to be equal signs, while $\pm$ and $\mp$ in the same formula stand for opposite signs.)

Recall that we are working in the bounded triangular domain S_n with the configuration $\widetilde{\omega}_n\cup\{\mathbf{x}_i, 1\le i\le r\}$, where r is the number of added points. By $\mathbf{f}^+_i=(t_{\mathbf{f}^+_i},x_{\mathbf{f}^+_i})$ and $\mathbf{f}^-_i=(t_{\mathbf{f}^-_i},x_{\mathbf{f}^-_i})$ we shall denote the end-points of the influence paths $\pi^+_{\mathbf{x}_i}$ and $\pi^-_{\mathbf{x}_i}$ — they can be points where the corresponding paths exit the triangular domain, or points where a "$+-$" collision happens, in which case the two corresponding end-points are equal.

Besides, let $\mathbf{r}^+_i=(t_{\mathbf{r}^+_i},0)$ and $\mathbf{r}^-_i=(t_{\mathbf{r}^-_i},0)$ be the points of first return to the t-axis of the paths $\pi^+_{\mathbf{x}_i}$ and $\pi^-_{\mathbf{x}_i}$, respectively, and define

$$\widehat{t}_i := \min\{t_{\mathbf{f}^+_i}, t_{\mathbf{f}^-_i}, t_{\mathbf{r}^+_i}, t_{\mathbf{r}^-_i}\}. \tag{2.7}$$

Last, let $\mathbf{e}^+_i := \pi^+_{\mathbf{x}_i}\cap\{(\widehat{t}_i,x),\, x\in\mathbb{R}\}$ and $\mathbf{e}^-_i := \pi^-_{\mathbf{x}_i}\cap\{(\widehat{t}_i,x),\, x\in\mathbb{R}\}$, and denote by $\widehat{\pi}^+_{\mathbf{x}_i}$, $\widehat{\pi}^-_{\mathbf{x}_i}$ the parts of $\pi^+_{\mathbf{x}_i}$ and $\pi^-_{\mathbf{x}_i}$ lying between $\mathbf{x}_i$ and $\mathbf{e}^+_i$, and respectively between $\mathbf{x}_i$ and $\mathbf{e}^-_i$.

Definition 2.6. Let J_i be the (random) Jordan curve starting at $\mathbf{x}_i$, following the path $\widehat{\pi}^-_{\mathbf{x}_i}$ until the point $\mathbf{e}^-_i$, then the vertical line $t=\widehat{t}_i$ up to $\mathbf{e}^+_i$, and then the path $\widehat{\pi}^+_{\mathbf{x}_i}$ backward until it comes back to $\mathbf{x}_i$. The domain bounded by J_i will be called the *attractor of the point* $\mathbf{x}_i$ and denoted by $\mathcal{A}_i$ (see Figure 3). The part of its boundary which is contained in $\widehat{\pi}^+_{\mathbf{x}_i}$ (resp. $\widehat{\pi}^-_{\mathbf{x}_i}$) will be called the *upper* (resp. *lower*) *boundary* of the attractor.

It is important to understand how attractors are affected by one another, in order to give a convenient description of the whole augmented process. The key remark is the following: Informally speaking, a superior particle of a given type is

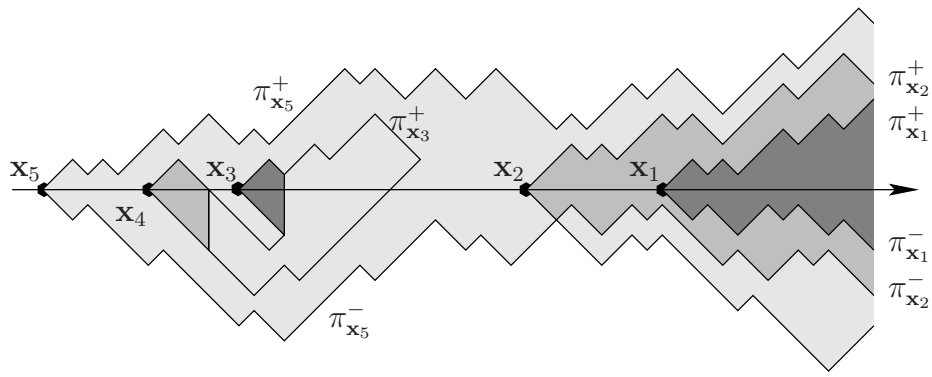

FIG 3. *Attractors (shaded).*

not affected by an older one (*i.e.* one with larger index) of the same type. Indeed this is a consequence of the previous construction, and especially of the Abelian property (see Proposition 2.1).

Of course this does not mean that the attractors $\mathcal{A}_j$ for $j < j_0$ do not change when $\mathbf{x}_{j_0}$ is added, since it remains possible that a "$+-$"-collision happens. In that case, we get some kind of a monotonicity property, for the statement of which some additional notation will be needed. Recall that $\tau(\mathbf{x};\omega)$ stands for the self-annihilation time of a particle *in the underlying scenery*, *i.e.* ignoring the effect of the other new particles, both younger and older. We will denote by $\tau(\mathbf{x},\mathbf{y};\omega)$ the annihilation time of the set of superior particles born from x and y, *i.e.* the last time at which any of the corresponding four superior particles is still alive. Such an annihilation can happen in one of four ways:

- Flat: $\mathbf{x}$ is born, the two particles thus created annihilate, then $\mathbf{y}$ is born and its two particles collide;
- Embedded: $\mathbf{x}$ is born, then $\mathbf{y}$ appears between $p^-_{\mathbf{x}}$ and $p^+_{\mathbf{x}}$, then the particles issued from $\mathbf{y}$ collide, then so do those issued from $\mathbf{x}$;
- Parallel: $\mathbf{x}$ is born, then $\mathbf{y}$ appears outside of $(p^-_{\mathbf{x}}, p^+_{\mathbf{x}})$, then the particles issued from $\mathbf{y}$ collide, then so do those issued from $\mathbf{x}$;
- Crossed: $\mathbf{x}$ is born, then $\mathbf{y}$ (also outside of $(p^-_{\mathbf{x}}, p^+_{\mathbf{x}})$), then one particle issued from $\mathbf{x}$ annihilates the particle of the other type coming from $\mathbf{y}$, then the remaining two collide.

The combinatorics become much more involved when more particles are added; nevertheless, it is possible (if a bit technical if a formal proof is needed, see [4]) to show the following:

Proposition 2.7 (Monotonicity of the influence). *For any two added points* $\mathbf{x}$ *and* $\mathbf{y}$*, we have the following inequality:*

$$\tau(\mathbf{x},\mathbf{y};\omega) \geq \mathrm{Max}(\tau(\mathbf{x},\omega), \tau(\mathbf{y};\omega));$$

and more generally the annihilation time of the union of two finite families of added points is at least equal to the larger of the two annihilation times of the parts.

In the flat, embedded and parallel cases, the monotonicity extends to the shapes of the attractors (the attractor of $\mathbf{y}$ in the presence of $\mathbf{x}$ contains the one without); there is true reinforcement in the embedded case, in that the inclusion is strict as

soon as there is a "++"- or "−−"-collision. This is not always the case in crossed configurations (cf. Fig. 3, where the attractor of $\mathbf{x}_3$ is shortened by the addition of $\mathbf{x}_4$), which leads us to the following definition:

Definition 2.8. We say that two attractors $\mathcal{A}_i$ and $\mathcal{A}_j, i > j$ are *connected* if there exists a sub-sequence $j = i_0 < i_1 < i_2 < \cdots < i_k = i$ such that $\mathbf{x}_{i_r} \in \mathcal{A}_{i_{r+1}}$ for all $0 \le r < k$. We will call $(i_j)_{0\le j\le k}$ a *connecting subsequence* between $\mathcal{A}_i$ and $\mathcal{A}_j$.

Observe that if $i > j > k$, if $\mathcal{A}_i$ is connected to $\mathcal{A}_j$ and $\mathcal{A}_j$ is connected to $\mathcal{A}_k$, then $\mathcal{A}_i$ is connected to $\mathcal{A}_k$. Nevertheless, due to the presence of orientation in the temporal direction, the above implication generally does not hold without the condition $i > j > k$.

Our construction immediately implies the following:

Proposition 2.9. *If $\mathcal{A}_i$ is connected to $\mathcal{A}_j$, $i > j$, then $\widehat{t}_i \ge \widehat{t}_j$.*

Corollary 2.10. *If $\mathcal{A}_i$ is connected to $\mathcal{A}_j$, $i > j$, and the end-points $\mathbf{e}_j^+$ and $\mathbf{e}_j^-$ belong respectively to the south-east and north-east boundaries of the triangular domain S_n, then so do $\mathbf{e}_i^+$ and $\mathbf{e}_i^-$.*

Corollary 2.11. *Assume that $\mathcal{A}_i$ is connected to $\mathcal{A}_j$, $i > j$, with connecting subsequence $(i_s)_{0\le s\le k}$: If $\mathbf{x}_j$ is essential for the configuration $\widetilde{\omega}_n$, then so are all the $\mathbf{x}_{i_s}$, $1 \le s \le k$.*

2.6. Pinning of the geodesics

We now return to our original problem. Observe that if, for some fixed configuration $\widetilde{\omega}_n$ in S_n, we pick a realization of the points $(\mathbf{x}_i)_{1\le i\le v}$ in such a way that $\mathcal{A}_1$ and $\mathcal{A}_v$ are connected (say), with connecting subsequence $(i_j)_{0\le j\le k}$, then all the $\mathbf{x}_{i_j}$ must be essential for $\widetilde{\omega}_n$, and therefore the point-to-plane geodesics for the configuration $\widetilde{\omega}_n \cup \{\mathbf{x}_i\}$ has to visit all the $\mathbf{x}_{i_j}$.

In the new formalism, the original question of whether, for any given density $\lambda^{(1)} > 0$ of the one-dimensional Poisson point process, the limiting value in (2.1) is increased, becomes equivalent to the following: Is there a positive δ such that, with high probability as n goes to infinity, at least a fraction δ of the newly added points are essential for $\omega^{(2)}$?

This question is more complicated than simply whether there exists a chain of pairwise connected attractors spanning from the left to the right boundary of the domain: Indeed, such a chain does not necessarily have positive density. In the next Section we also mention some of the interesting mathematical questions that arise in the construction.

It is not an easy task to understand how the attractors behave. The fact that the structure of the influence paths $\widehat{\pi}^+_{\mathbf{x}_{i+1}}$ and $\widehat{\pi}^-_{\mathbf{x}_{i+1}}$ depends on $\Gamma_n(\widetilde{\omega}_n \cup \{\mathbf{x}_j, 1 \le j \le i\})$, but not on the $\mathbf{x}_j$ for $j > i+1$, reduces the problem to checking whether none of the influence paths $\widehat{\pi}^+_{\mathbf{x}_{i+1}}$, $\widehat{\pi}^-_{\mathbf{x}_{i+1}}$ hits the t-axis before $\mathbf{x}_i$, in which case $\mathcal{A}_{i+1}$ is connected to $\mathcal{A}_i$ (see Figure 3).

For a single point $\mathbf{x}$ added to the initial configuration, each influence path $\pi^+_{\mathbf{x}}$, $\pi^-_{\mathbf{x}}$ has the same statistical properties as what is known as a "second-class particle" in the framework of exclusion processes. Since in the definition of an attractor an important role is played by the (possible) return times of the influence paths to the t-axis, several things must be settled:

1. The first return time to the t-axis of a single influence path. It is believed (but remains a challenging open problem) that in the case of a one-dimensional

exclusion process, a second-class particle behaves super-diffusively. Though some bounds are available, and we know the mean deviations of the second class particle [12], they do not provide good control on return times;

2. The joint behavior of the influence paths $\widehat{\pi}^+_{\mathbf{x}}$ and $\widehat{\pi}^-_{\mathbf{x}}$. Generally, it is a complicated question too, but for our purposes we need to have such control only up to the first times when one of $\widehat{\pi}^+_{\mathbf{x}}$, $\widehat{\pi}^-_{\mathbf{x}}$ returns to the t-axis. Before such a time, both paths stay apart from each other, and some good mixing properties of the system come into play; so the question reduces to how efficiently we control point 1.

The fact that the influence lines of "younger" points (with smaller indices, *i.e.* sitting more to the right) repeal the influence lines of "older" points, leads to the following observation: Once the attractor $\mathcal{A}_{i+m}$ of an older point reaches the younger point $\mathbf{x}_i$, then it cannot end before the attractor $\mathcal{A}_i$ ends. If the attractor $\mathcal{A}_i$ ends before reaching the next point $\mathbf{x}_{i-1}$, then $\mathcal{A}_{i+m}$ can still go forward, and possibly itself reach $\mathbf{x}_{i-1}$. Observe that at the time $\mathcal{A}_i$ ends, the boundaries of $\mathcal{A}_{i+m}$ are necessarily at a positive distance from the t-axis.

That, together with the fact that the evolution of $p^+_{\mathbf{x}_i}$ and $p^-_{\mathbf{x}_i}$ in the slab $(t_i, t_{i-1}) \times \mathbb{R}$ depends only on $\Gamma_n(\widetilde{\omega}_n)$ brings some notion of week dependence to the system from one side, and the idea of a "re-start point" from another. This reduces the study of percolation of attractors to a more general problem of one-dimensional, long-range, dependent percolation which we formulate in the next section. There we also mention some of the interesting mathematical questions that have arisen.

3. Stick percolation

In this section we introduce two "stick percolation" models, which will serve as toy models in the study of the propagation of influence in the broken-line model. In spite of their apparent simplicity, these models can be very useful studying effects of columnar defects and establishing (bounds for) critical values for asymptotic shape changes for some well known one-dimensional growth systems (see [4]).

3.1. Model 1: overlapping sticks

Let $(x_i)_{i\in\mathbb{N}}$ be a Poisson point process of intensity $\lambda > 0$ on the positive real line. We call the points of this process "seeds", and assume that they are ordered, x_0 being the point closest to the origin. To each seed x_i we associate a positive random variable S_i (a "length") and assume that the $(S_i)_{i\in\mathbb{N}}$ are i.i.d. with common distribution function F.

The system we consider is then the following: For every $i \in \mathbb{N}$, construct the segment $\widehat{S}_i = [x_i, x_i + S_i]$ (which we will call the i-th "stick"). We say that the sticks $\widehat{S}_i$ and $\widehat{S}_j$, $i < j$ are *connected* if $x_j < x_i + S_i$, *i.e.* if they have non-empty intersection; the system *percolates* if and only if there is an infinite chain of distinct, pairwise connected sticks, which (with probability 1) is equivalent to saying that the union of the sticks contains a half-line.

It is easy to see that the system percolates with probability 0 or 1 (it is a tail event for the obvious filtration); and in fact there is a complete characterization of both cases:

Proposition 3.1 ([4]). *Let $R(x) = P(S_1 > x)$ be the tail of the stick length distribution, and let $\varphi(x) = \int_0^x R(u)du$. Then the system percolates with probability*

1 *if and only if*

$$\int_0^{+\infty} e^{-\lambda\varphi(x)}dx < +\infty. \tag{3.1}$$

This leads us to the following definition:

Definition 3.2. The distribution F governing the stick process is said to be *cluster-stable* if the system percolates for every positive value of λ.

Example 1 (Return times of random walks). One natural distribution for the length of the sticks, in view of the previous construction, is the following: At time x_i, start two random walks with no drift (the specifics, *e.g.* whether they are discrete or continuous time walks, will not matter at this point — for that matter we could also take two Brownian motions), one from $+1$ and the other from -1. Then, let $x_i + S_i$ be the first time when these two walks meet. It is well known that, up to multiplicative constants, $P(S_i > t)$ behaves as $t^{-1/2}$ for large t, as soon as the walks are irreducible and their step distribution have finite variance. It is easy to check that the obtained distribution is cluster-stable.

Example 2. Change the previous example a little, as follows: For every value of i, start a two-dimensional Brownian motion (or random walk) starting at $(x_{i,1})$, and let $x_i + S_i$ be the first hitting point of the axis by this Brownian motion; but erase the stick if $S_i < 0$. Then the distribution F is Cauchy restricted to be positive, and its tail is equivalent to $P(S_i > t) \sim c/t$ as t goes to infinity. In that case, the stick length distribution is not cluster-stable for model 1, and there exists a critical value $\lambda_c = 1/c$, such that for $\lambda > \lambda_c$ the system percolates while it does not if $\lambda < \lambda_c$.

3.2. Model 2: reinforced sticks

There are several ways to mimic the interaction between "funnels". One of the most simple possibilities is to add extra rules to Model 1 to account for the interaction. The basic idea behind this modification, is that if two or more sticks from Model 1 overlap, then there is a certain reinforcement of the system, which depends on how many sticks overlap, and then the whole connected component is enlarged correspondingly.

One way to do that can be described informally as the following dynamic process. First, see each stick $\widehat{S}_i$ as the flight time of a particle π_i born at time x_i. We want to model the fact that if π_i wants to land when a younger one (say π_j) is still flying, instead π_i "bounces" on π_j; π_j on the other hand should not be affected by π_i, if the process is to look like the propagation of influence described in he previous section.

Assign to each particle a "counter of chances" N_i which is set to 1 when the particle is born (formally it should be a function from $\mathbb{R}_+$ to $\mathbb{N}$, and we let $N_i(x_i) = 1$). Then, two things can happen. If $x_i + S_i < x_{i+1}$, there is no interaction and π_i dies at time $x_i + S_i$. If on the other hand $x_i + S_i \geq x_{i+1}$, at time x_{i+1} the particle π_i gets a "bonus", so that $N_i(x_{i+1}) = 2$; and similarly, it gets a bonus each time it passes above a seed point, so that $N_i(x_i + S_i-) - 1$ is the number of seed points in $\widehat{S}_i$. Now when π lands, its counter is decreased by 1, but if it is still positive the particle bounces on the axis and restarts using an independent copy of S. If all the chances are exhausted before the closest seed is reached, π_i gets killed.

Again, we may ask a similar question: Given $\lambda > 0$, for which distribution functions F is the probability for a given particle to survive up to infinity positive? It is

obvious that even with $E(S) < +\infty$ we can obtain infinite trajectories for certain (large) values of λ. We will call F *cluster-stable for the reinforced process* if this happens for every positive λ.

Example. The Cauchy distribution used in the previous example cluster-stable for the reinforced model.

It is easy to see (*e.g.* by a coupling argument) that if the original system percolates, the reinforced version (for the same value of λ and the same length distribution F) percolates too. In particular a cluster-stable distribution is cluster-stable for the reinforced problem. Nevertheless it is an interesting open problem to give full characterization of distributions which are cluster-stable for the reinforced models.

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