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▶ **To cite this version:**

Laurent Manivel. Configurations of lines and models of Lie algebras. Journal of Algebra, 2006, 304 (1), pp.457-486. hal-00005848

HAL Id: hal-00005848
https://hal.science/hal-00005848

Submitted on 6 Jul 2005

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CONFIGURATIONS OF LINES AND MODELS OF LIE ALGEBRAS

L. MANIVEL

Abstract. The automorphism groups of the 27 lines on the smooth cubic surface or the 28 bitangents to the general quartic plane curve are well-known to be closely related to the Weyl groups of E_6 and E_7. We show how classical subconfigurations of lines, such as double-sixes, triple systems or Steiner sets, are easily constructed from certain models of the exceptional Lie algebras. For $\mathfrak{e}_7$ and $\mathfrak{e}_8$ we are lead to beautiful models graded over the octonions, which display these algebras as plane projective geometries of subalgebras. We also interprete the group of the bitangents as a group of transformations of the triangles in the Fano plane, and show how this is related to an interpretation of the isomorphism $PSL(3,\mathbb{F}_2) \simeq PSL(2,\mathbb{F}_7)$ in terms of harmonic cubes.

1. Introduction

Such classical configurations of lines as the 27 lines on a complex cubic surface or the 28 bitangents to a smooth quartic plane curve have been extensively studied in the 19th century (see e.g. [21]). Their automorphism groups were known, but only at the beginning of the 20th century were their close connections with the Weyl groups of the root systems E_6 and E_7, recognized, in particular through the relation with Del Pezzo surfaces of degree three and two, respectively [8, 9]. Del Pezzo surfaces of degree one provide a similar identification of the diameters of the root system E_8, with the 120 tritangent planes to a canonical space curve of genus 4.

Can we go beyond the Weyl groups and find a connection with the Lie groups themselves? The 27 lines on the cubic surface are in natural correspondence with the weights of the minimal representation of E_6, from which the Lie group can be recovered as the stabilizer of a cubic form that already appears in Elie Cartan's thesis; Faulkner showed how to define this form in terms of the 45 tritangent planes. A similar phenomenon holds for the 28 bitangents to the quartic plane curve, which can be put in correspondence with pairs of opposite weights of the minimal representation of E_7. In both cases the connection between the Lie group and its Weyl group is particularly close because of the existence of a minuscule representation. For E_8 the minimal representation is the adjoint one and is no longer minuscule.

The first aim of this paper is to use these connections with the Lie groups, or rather the Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, to shed a new light on the work of the classical geometers on line configurations. Our main idea is that each time we consider a semisimple Lie subalgebra, the restriction of the minimal representation branches into a direct sum of subrepresentations, and consequently the weights split into special subsets forming interesting subconfigurations. In the case of $\mathfrak{e}_6$ and the 27 lines we get the following correspondence:

Subalgebra	*Subconfiguration*
$\mathfrak{spin}_{10}$	Line
$\mathfrak{spin}_8$	Tritangent plane
$\mathfrak{sl}_2 \times \mathfrak{sl}_6$	Double-six
$\mathfrak{sl}_3$	Steiner set
$\mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{sl}_3$	Triple system

In the case of $\mathfrak{e}_7$ and the 28 bitangents the notion of Steiner complexes of bitangents make a natural appearance. They are special sets of 12 bitangents which can be put in correspondence with positive roots, and also with points of a 5-dimensional projective space over $\mathbb{F}_2$. This space is endowed with a natural symplectic form, and indeed the Weyl group of E_7 is closely connected with the finite symplectic group $Sp(6, \mathbb{F}_2)$. This leads to a very interesting finite symplectic geometry whose lines are known to correspond to the so called *syzygetic triads* of Steiner sets. We prove that planes in this geometry are in correspondence with what we call *Fano heptads* of bitangents. The upshot is a finite geometry modeling the symplectic geometries related to the third line of Freudenthal Magic Square, whose last term is precisely $\mathfrak{e}_7$ [27].

Very interestingly, this leads to a beautiful model of $\mathfrak{e}_7$ and its minimal representation which, rather unexpectedly, turns out to be closely related with the Fano plane and the octonionic multiplication. Indeed, recall that $\mathbb{O}$, the Cayley algebra of octonions, can be defined as the eight-dimensional algebra with a basis $e_0 = 1, e_1, \ldots, e_7$, with multiplication rule encoded in an oriented Fano plane.

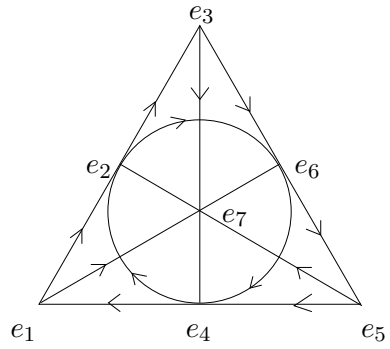


Figure 1. Octonionic multiplication

This means that $e_i e_j = \pm e_k$ if i, j, k are three distinct points on one of the projective lines in this plane, with a plus sign if and only if (ijk) gives the cyclic orientation fixed on the line.

We define an $\mathbb{O}$-grading on a Lie algebra $\mathfrak{g}$ to be a decomposition

$$\mathfrak{g} = \mathfrak{h}_0 e_0 \oplus \bigoplus_{1 \leq i \leq 7} \mathfrak{h}_i e_i$$

such that $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_k$ if $e_i e_j = \pm e_k$. In particular $\mathfrak{h}_0$ is a subalgebra and each $\mathfrak{h}_i$ is an $\mathfrak{h}_0$-module. More is true: for any point i and any line ℓ in the Fano plane, the direct sums

$$\mathfrak{g}_i = \mathfrak{h}_0 \oplus \mathfrak{h}_i, \qquad \mathfrak{g}_\ell = \mathfrak{h}_0 e_0 \oplus \bigoplus_{j \in \ell} \mathfrak{h}_j e_j$$

are subalgebras of $\mathfrak{g}$, so that we really have a configuration of Lie algebras defined by a plane projective geometry.

Our discussion of Fano heptads lead us to discover that $\mathfrak{e}_7$ has a natural structure of an $\mathbb{O}$-graded algebra, compatible with its action on the minimal representation V. Indeed, attach to each line ℓ of the Fano plane a two dimensional vector space A_ℓ. Then we can describe $\mathfrak{e}_7$ and V as follows:

$$\begin{aligned} \mathfrak{e}_7 &= \times_\ell \mathfrak{sl}(A_\ell) e_0 \oplus \bigoplus_{1\leq i\leq 7} (\otimes_{i\notin \ell} A_\ell)\, e_i, \\ V &= \bigoplus_{1\leq j\leq 7} (\otimes_{j\in \ell} A_\ell)\, e_j. \end{aligned}$$

Going a little deeper in the Lie algebra structure, we will discover a natural connection with the multiplication table of the Cayley algebra. This leads to an amusing interpretation of the isomorphism $PSL(3,\mathbb{F}_2) \simeq PSL(2,\mathbb{F}_7)$ in terms of *harmonic cubes*, and a permutation representation of the group of the bitangents on the triangles of the Fano plane.

A similar description of $\mathfrak{e}_8$ also exists, and the biggest two exceptional Lie algebras appear as plane projective geometries whose points are copies of $\mathfrak{so}_8$ and $\mathfrak{so}_8 \times \mathfrak{so}_8$, and whose lines are copies of $\mathfrak{so}_{12}$ and $\mathfrak{so}_{16}$, respectively. Moreover, this octonionic model of $\mathfrak{e}_8$ makes obvious the existence of the multiplicative orthogonal decomposition that was a key ingredient in Thompson's construction of the sporadic simple group denoted Th or F_3 (see [26], Chapters 3 and 13). It would certainly be interesting to use this octonionic model, suitably adapted, to construct forms of $\mathfrak{e}_8$ over arbitrary fields.

Classically, two unifying perspectives on the line configurations we are interested in have been particularly successful. We briefly discuss the connection with our present approach.

Theta characteristics. Bitangents to the plane quartic curve (a canonical curve of genus $g = 3$), as well as tritangent planes to the canonical curve of genus $g = 4$, can be interpreted as odd theta-characteristics. Since the theta-characteristics can be seen as points of an affine space over the half-periods of the curve, this leads to an interpretation in terms of finite symplectic geometries in dimension $2g$ over the field $\mathbb{F}_2$. This was developped in great detail by the classical geometers, in particular by Coble ([3], Chapter II). For example the theta-characteristics can be understood as the quadrics whose associated polarity is the natural symplectic form. Isotropic linear spaces also have natural geometric interpretations. For $g = 3$ one recovers the connection that we already mentionned between $W(E_7)$ and $Sp(6,\mathbb{F}_2)$. For $g = 4$, the Weyl group $W(E_8)$ is the automorphism group of the lines in the Del Pezzo surface of degree one, whose canonical model is a double covering of a quadratic cone, branched along a canonical sextic curve. As noticed by Schottky, there is a unique even theta-characteristic vanishing at the vertex of the cone, which explains why the automorphism group of the tritangent planes is an orthogonal group $O(8,\mathbb{F}_2)^+$ rather than a symplectic group.

Semi-regular polytopes. Gosset seems to have been the first, at the very beginning of the 20th century, to understand that the lines on the cubic surface can be interpreted as the vertices of a polytope, whose symmetry group is precisely the automorphism group of the configuration. Coxeter extended this observation to the 28 bitangents, and Todd to the 120 tritangent planes. Du Val and Coxeter provided systematic ways to construct the polytopes, which are denoted n_{21} for $n = 2, 3, 4$ and live in $n + 4$ dimensions [13, 6, 8]. They have the characteristic property of being semi-regular, which means that the automorphism group acts transitively on the vertices, and the faces are regular polytopes. In terms of Lie theory they are best understood as the polytopes in the weight lattices of the exceptional simple Lie algebras $\mathfrak{e}_{n+4}$, whose vertices are the weights of the minimal representations. Coxeter investigated in great detail their semi-regular sub-polytopes [6]. Algebraically, this amounts to identifying certain Lie subalgebras of the $\mathfrak{e}_{n+4}$. But Coxeter does not describe how the full polytopes are organized around these special sub-polytopes. In a sense this is what we will be doing in this paper, with the nice conclusion that it leads to a very natural, unified and easy-going description of (at least part of) the classical combinatorics of the line configurations, as well as new insights in the fascinating structure of the exceptional Lie algebras.

Acknowledgements: I thank I. Dolgachev and P.E. Chaput for their useful comments.

2. Models of the exceptional Lie algebras

The Reye configuration and triality. A classical elementary configuration of lines is the Reye configuration below, obtained from a cube in a three dimensional projective space (see [22] and [11]). This configuration can be understood as a central projection of the 24-cell, one of the regular polytopes in four dimensions. The vertices of this polytope are given by the roots of the root system D_4 (we use [2] as a general reference on root systems).

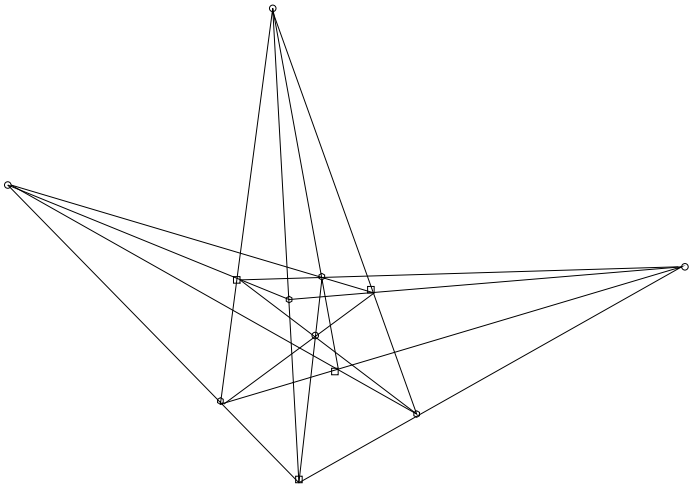


Figure 2. The Reye configuration

As one can easily see on Figure 2, there is a unique way to partition the points of the Reye configuration into three types, in such a way that each line contains exactly one point of each type. This decomposes the 24-cell into three 16-cells given by the vertices of three hypercubes. Each of these defines a root subsystem of D_4 of type A_1^4.

Restricting the adjoint representation of $\mathfrak{spin}_8$ to the corresponding subalgebra, a product of four copies of $\mathfrak{sl}_2$, we obtain the *four-ality model* [28]

$$\mathfrak{spin}_8 = \mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2) \times \mathfrak{sl}(A_3) \times \mathfrak{sl}(A_4) \oplus (A_1 \otimes A_2 \otimes A_3 \otimes A_4),$$

whose existence is indicated by the shape of the affine Dynkin diagram $\tilde{D}_4$. (Here A_1, A_2, A_3, A_4 are two dimensional complex vector spaces. Note that the construction works on the reals to give the split form $\mathfrak{so}_{4,4}$.) Four-ality reduces to the classical Cartan triality through the morphism $\mathcal{S}_4 \to \mathcal{S}_3$ induced by the permutation of the three partitions of four objects in two pairs. In terms of representations, this translates into the permutation of the three non equivalent eight dimensional representations of $\mathfrak{spin}_8$:

$$\begin{aligned} \Delta_1 &= (A_1 \otimes A_2) \oplus (A_3 \otimes A_4), \\ \Delta_2 &= (A_1 \otimes A_3) \oplus (A_2 \otimes A_4), \\ \Delta_3 &= (A_1 \otimes A_4) \oplus (A_2 \otimes A_3). \end{aligned}$$

Binary, ternary, and triality models. Among the models of the exceptional Lie algebras that we will meet in the sequel, most will be derived from the *triality model* first defined, in a more general context, by Allison in [1], and rediscovered in [29]. The idea is to associate to a (complexified) real normed algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, its *triality algebra* $\mathfrak{t}(\mathbb{A})$ with its three natural modules $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. Then for any pair $\mathbb{A}, \mathbb{B}$ of normed algebras, the direct sum

$$\mathfrak{g}(\mathbb{A}, \mathbb{B}) = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3)$$

has a natural Lie algebra structure. This leads to the famous Freudenthal Magic Square, whose fourth line $\mathfrak{g}(\mathbb{A}, \mathbb{O})$ is the series of exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$.

The Lie algebras on the second and third lines of the Magic Square are endowed each with a special module: $\mathfrak{g}(\mathbb{A}, \mathbb{C})$ with the cubic Jordan algebra $J_3(\mathbb{A})$, and $\mathfrak{g}(\mathbb{A}, \mathbb{H})$ with the Zorn algebra $\mathfrak{z}_2(\mathbb{A})$. The natural inclusions $\mathfrak{g}(\mathbb{A}, \mathbb{C}) \subset \mathfrak{g}(\mathbb{A}, \mathbb{H}) \subset \mathfrak{g}(\mathbb{A}, \mathbb{O})$ then lead to the *binary* and *ternary* models for the exceptional Lie algebras:

$$\begin{aligned} \mathfrak{g}(\mathbb{A}, \mathbb{O}) &= \mathfrak{sl}_2 \times \mathfrak{g}(\mathbb{A}, \mathbb{H}) \oplus (\mathbb{C}^2 \otimes \mathfrak{z}_2(\mathbb{A})), \\ \mathfrak{g}(\mathbb{A}, \mathbb{O}) &= \mathfrak{sl}_3 \times \mathfrak{g}(\mathbb{A}, \mathbb{C}) \oplus (\mathbb{C}^3 \otimes J_3(\mathbb{A})) \oplus (\mathbb{C}^3 \otimes J_3(\mathbb{A}))^*. \end{aligned}$$

This extends to $\mathfrak{spin}_8 = \mathfrak{t}(\mathbb{O})$, whose ternary model is the four-ality model related to the Reye configuration. Note also that the triality models can be interpreted as $\mathbb{H}$-graded Lie algebras, with a similar definition to the one we introduced for $\mathbb{O}$-gradings.

3. Lines on the cubic surface

The configuration of the 27 lines on a smooth cubic surface in $\mathbb{CP}^3$ have been thoroughly investigated by the classical algebraic geometers. It has

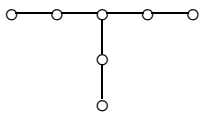
been known for a long time that the automorphism group of this configuration can be identified with the Weyl group of the root system of type E_6, of order 51 840 [8]. Moreover, the minimal representation J of the simply connected complex Lie group of type E_6 has dimension 27. This is a minuscule representation, meaning that the weight spaces are lines and that the Weyl group $W(E_6)$ acts transitively on the weights. In fact one can recover the lines configuration of the cubic surface by defining two weights to be incident if they are not orthogonal with respect to the unique (up to scale) invariant scalar product.

Conversely, one can recover the action of the Lie group E_6 on J from the line configuration. Faulkner defines a cubic form on J as the sums of signed monomials associated to the tritangent planes [18]. The stabilizer of that cubic form in $GL(J)$ is precisely E_6. Note that the polarization of this cubic form is a symmetric bilinear map $J \times J \to J^*$. Identifying appropriately J with J^* we get an algebra structure which is known to coïncide with the exceptional complex Jordan algebra $J_3(\mathbb{O})$.

The closed E_6-orbit in the projectivization $\mathbb{P}J_3(\mathbb{O})$ is known as the complex *Cayley plane* $\mathbb{OP}^2$ and should be thought of as the projective plane over the Cayley algebra of octonions. Being the orbit of any weight space it is circumscribed to the Schoute polytope 2_{21}, which appears as a discrete version of the Cayley plane. In particular the 10 lines incident to a given line correspond to the polar quadric or $\mathbb{O}$-line (whose Euler characteristic is 10). The property that two general $\mathbb{O}$-lines on the Cayley plane have a unique intersection point, thus mirrors the obvious fact that, two concurrent lines on the cubic surface being given, there exists a unique line meeting both.

It is well known that most of the interesting subgroups of $W(E_6)$ can be realized as stabilizers of some subconfigurations. It seems not to have been noticed before that most of them also have natural interpretations in terms of branching. By this we mean that we can find a subalgebra of $\mathfrak{e}_6$ such that the restriction of the representation in J splits in such a way that the relevant subconfiguration can immediately be read off.

There is a general recipe to identify semisimple subalgebras of a simple complex Lie algebra, that we illustrate with the case of $\mathfrak{e}_6$ (see [31], Chapter 6). One begins with the affine Dynkin diagram, which in the case we are interested in has a remarkable threefold symmetry:

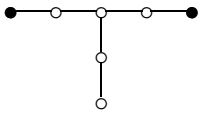


Then we choose a set of nodes, that we mark in black. Suppressing these nodes we get the Dynkin diagram (usually disconnected) of a semisimple Lie subalgebra $\mathfrak{h}$ of $\mathfrak{e}_6$ which is uniquely defined up to conjugation. The Weyl group W of this semisimple Lie algebra is a subgroup of $W(E_6)$, also uniquely defined up to conjugation. We get three types of data:

(1) *Combinatorial data*: W can be realized as the stabilizer of a certain subconfiguration of the 27 lines, encoded in the marked Dynkin diagram;

(2) *Representation theoretic data*: as an $\mathfrak{h}$-module, J splits into a direct sum of irreducible components;
(3) *Geometric data*: the $\mathfrak{h}$-components encode certain special subvarieties of the Cayley plane.

Example 1. We mark two of the three extreme nodes. Then $\mathfrak{h} = \mathfrak{spin}_{10}$ and $W = W(D_5) = \mathbb{Z}_2^4 \rtimes \mathcal{S}_5$.



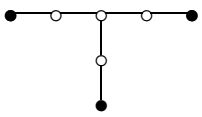
The index of W in $W(E_6)$ is 27: this subgroup is just the stabilizer of some line in the configuration. In fact $\mathfrak{h}$ is the semisimple part of the Lie algebra of the stabilizer of a one-dimensional weight space ℓ, which defines a point on the Cayley plane and can be identified with one of the lines of the configuration. The branching, i.e. the decomposition of J as an $\mathfrak{h}$-module, gives

$$J = \ell \oplus \Delta \oplus U.$$

The 16-dimensional half-spin representation Δ can be identified with the tangent space to the Cayley plane at ℓ; combinatorially, the sixteen weight spaces generating Δ give the sixteen lines which do not meet ℓ; geometrically, the intersection of the Cayley plane with its tangent space at ℓ is a cone over a ten dimensional spinor variety.

The 10-dimensional natural representation U encodes the normal space to the Cayley plane at ℓ; combinatorially, the ten weight spaces generating U give the ten incident lines to ℓ. Note that this representation is self-dual, so its weights occur in opposite pairs corresponding to incident pairs of incident lines to ℓ. Geometrically, the intersection of the Cayley plane with $\mathbb{P}U$ is the polar eight-dimensional quadric, a copy of the projective line $\mathbb{OP}^1$ over the Cayley algebra.

Example 2. We mark the three extreme nodes. In this case $\mathfrak{h} = \mathfrak{spin}_8$ and $W = W(D_8) = \mathbb{Z}_2^3 \rtimes \mathcal{S}_4$.



By restricting the previous case we get the branching

$$J = \ell_1 \oplus \ell_2 \oplus \ell_3 \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3,$$

where $\Delta_1, \Delta_2, \Delta_3$ are the three eight dimensional representations of $Spin_8$, which we deliberately avoid to distinguish since they are exchanged by Cartan's triality. The three lines ℓ_1, ℓ_2, ℓ_3 are pairwise incident, hence they are the three intersection lines of the cubic surface with a *tritangent plane*. Note that the index of W in $W(E_6)$ is $270 = 6 \times 45$. Since we have a sixfold ambiguity on the order of the three lines, we recover the classical fact that the cubic surface has exactly 45 tritangent planes.

Once we have fixed these three lines, each of them has eight more incident lines coming into four pairs, and corresponding to the pairs of opposite weights of one of the eight-dimensional representations of $\mathfrak{spin}_8$. Note that this exhausts the 27 lines.

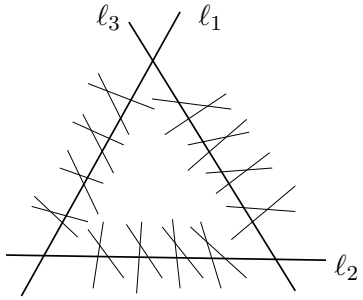
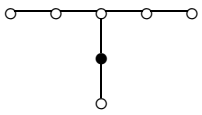


Figure 3. Triality from the 27 *lines*

The sum of the weights of ℓ_1, ℓ_2, ℓ_3 is zero, and this characterizes triples of lines on a tritangent plane.

Geometrically, we have three eight-dimensional quadrics on the Cayley plane, any two of them meeting exactly in one point. In terms of plane projective geometry, these three quadrics are projective lines which are the sides of a self-polar triangle.

Example 3. Now we mark a unique node, which is neither extremal nor central. Then $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{sl}_6$ and $W = \mathcal{S}_2 \times \mathcal{S}_5$. This leads to the binary model of $\mathfrak{e}_6$.



The index of W in $W(E_6)$ is 36. The branching gives

$$J = U \otimes A \oplus \Lambda^4 U,$$

where U denotes the six-dimensional natural representation of $\mathfrak{sl}_6$, and A the natural representation of $\mathfrak{sl}_2$. The twelve weights of $U \otimes A$ split into six pairs (ℓ_i, ℓ_i') where $\ell_1, \ldots, \ell_6$ have the same component over A. Of course

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 \\ \ell_1' & \ell_2' & \ell_3' & \ell_4' & \ell_5' & \ell_6' \end{pmatrix}$$

is a double-six, and there are exactly 36 double-sixes on the cubic surface.

Note that the node that we have marked is the node of the Dynkin diagram of E_6 that defines the adjoint representation. This explains the correspondence between double-sixes and pairs of opposite roots of E_6 ([10], 10.1.5).

Geometrically, such a pair of roots defines a point of the adjoint variety, the projectivization of the minimal nilpotent orbit in the Lie algebra $\mathfrak{e}_6$. The image of its action on J has minimal rank, namely 6 [24], and is a maximal linear space in the Cayley plane, with a weight basis given by one half of the double-six. The other half can be recovered from the similar action on J^*,

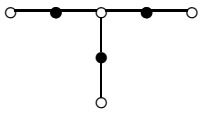
whose image is again a maximal linear space in (a dual copy of) the Cayley plane.

Explicitely, let $P(J)$ denote the set of weights of J (the $W(E_6)$-orbit of the fundamental weight ω_1, in the notation of [2]). Then the double-six D_α associated to a root α (up to sign), considered as a set of such weights, is

$$\begin{aligned} D_\alpha &= \{\gamma \in P(J),\ \gamma+\alpha \text{ or } \gamma-\alpha \in P(J)\} \\ &= \{\gamma \in P(J),\ (\gamma,\alpha) \neq 0\}. \end{aligned}$$

Now given two double-sixes D_α and D_β, they can have only two different relative positions, following that $(\alpha,\beta)=0$ or not. In the first case they are said to be *azygetic*, and $\#(D_\alpha \cap D_\beta) = 6$. Then $\alpha+\beta$ or $\alpha-\beta$ is again a root and defines a third double-six $D_{\alpha\pm\beta}$, which is azygetic to both D_α and D_β. There exist 120 such *azygetic triads of double-sixes*, corresponding to the 120 subsystems of type A_2 of the root system E_6.

In the latter case the double-sixes are *syzygetic*, and $\#(D_\alpha \cap D_\beta) = 4$. As indicated by the marked Dynkin diagram



we can complete such a pair of double-sixes into a *syzygetic tetrad* of double-sixes. If we let A_1, A_2, A_3, A_4 be the two-dimensional natural representations of the four copies of $\mathfrak{sl}_2$ corresponding to the white nodes of the diagram, the minimal representation of $\mathfrak{e}_6$ branches to

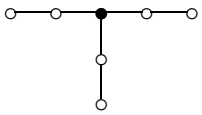
$$J = \mathbb{C}^3 \oplus \bigoplus_{i<j} A_i \otimes A_j,$$

and the associated tetrad of double-six is given by the weights of the four submodules

$$D_i = \bigoplus_{j\neq i} A_i \otimes A_j.$$

The number of syzygetic tetrads is the number of root subsystems of type A_1^4 in E_6, that is 135.

Example 4. Now we mark the central node. Then $\mathfrak{h} = \mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{sl}_3$ and $W = \mathcal{S}_3 \times \mathcal{S}_3 \times \mathcal{S}_3$. This leads to the ternary model of $\mathfrak{e}_6$.



The index of W in $W(E_6)$ is 240. The branching gives

$$J = (A^* \otimes B) \oplus (B^* \otimes C) \oplus (C^* \otimes A),$$

where A, B, C denote the natural representations of the three copies of $\mathfrak{sl}_3$ in $\mathfrak{h}$. The 27 weights are thus split into three bunches of nine.

Consider for example the nine weights $\epsilon'_i - \epsilon_j$ of $A^* \otimes B$, where the ϵ_j are the weights of A and the ϵ'_i those of B (note that both sets sum to zero).

Display these weights on a 3×3 square by putting $\epsilon'_{i+j-1} - \epsilon_{i+2j-2}$ in the box (i, j), the indices being considered modulo three. Then three weights on the same line or column have they sum equal to zero, and thus define a tritangent plane. We have obtained what is called a *Steiner set* – nine lines obtained as the intersections of two trihedra.

Moreover, we have split the 27 lines into three such sets forming a so-called *Steiner triple system*. Steiner sets are in correspondence with root subsystems of type A_2 of the root system E_6. The orthogonal to such a subsystem is the orthogonal product of two other A_2-subsystems. This is why, a Steiner set being given, there is a unique way to complete it into a triple system ([10], 10.1.7). Of course this is another manifestation of triality (see [17])!

Note also that the invariant cubic form on J can be characterized, up to scalar, as the unique $W(E_6)$-invariant cubic form whose restriction to each factor of type $A^* \otimes B$ is proportional to the determinant.

Geometrically, three Steiner sets correspond to three $\mathbb{P}^8$'s in $\mathbb{P}J$, each cutting the Cayley plane along a copy of the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$.

Remark. Define two Steiner sets to be incident if they have no common tritangent plane. Each Steiner set is incident to exactly 56 other sets, including the two special ones which complete it into a triple system. If we choose one of these, exactly 28 Steiner sets are incident to both, including the remaining set in the triple system. Contrary to what we could be tempted to believe, the configuration of the remaining 27 Steiner sets is not combinatorially equivalent to that of the 27 lines. Indeed, one can check that each of the 27 Steiner sets is incident to only eight of the other ones. Nevertheless, the Steiner sets define an interesting regular graph, which has the same number of edges and vertices than the graph defined by the diameters of the polytope 4_{21}, although it is not combinatorially equivalent.

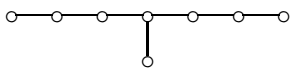
Coxeter already noticed in [7] that the 40 triple systems can be interpreted as hexagons on the polytope 4_{21}

4. Bitangents to the plane quartic curve

The 28 bitangents to a smooth plane quartic curve give rise to 56 lines on the Del Pezzo surface of degree two defined as the double cover of the projective plane, branched over the quartic [9]. This line configuration has for automorphism group the index two normal subgroup $W(E_7)^+$ of the Weyl group of E_7, which has order $2\,903\,040 = 2^{10} \times 3^4 \times 5 \times 7$.

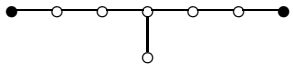
The Lie group of type E_7 has a minimal representation V of dimension 56, which is again minuscule. The invariant forms are a symplectic form – so that the 56 weights split into 28 pairs of opposite weights – and a quartic form which cannot be deduced, contrary to the case of the 27 lines, solely from the configuration. The weights of this representation form the *Gosset polytope* 3_{21}. This polytope appears as a discrete version of the minimal E_7-orbit in $\mathbb{P}V$, which we call the *Freudenthal variety*.

The affine Dynkin diagram of type E_7 has a two fold symmetry:



Again we deduce classical configurations of bitangents from markings of this diagram.

Example 1. We mark the two opposite extreme nodes. Then $\mathfrak{h} = \mathfrak{e}_6$, and $W = W(E_6)$ has index 28 in $W(E_7)$.

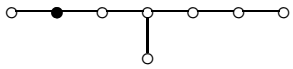


Indeed it is well-known that the stabilizer of any bitangent is a copy of the automorphism group of the 27 lines. Since the action of the latter is irreducible, the branching has to give an irreducible supplement, up to the sign of the weights. And indeed we have the decomposition into $\mathfrak{h}$-modules

$$V = \mathbb{C} \oplus J \oplus J^* \oplus \mathbb{C}.$$

Geometrically, the factor $J = J_3(\mathbb{O})$ appears as the tangent space to the Freudenthal variety, and J^* is the first normal space. Note that the intersection of the Freudenthal variety with its tangent space is a cone over the Cayley plane, whose discrete skeleton is precisely given by the 27 lines.

Example 2. Mark only one node next to one of the opposite extremal nodes. Then $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{spin}_{12}$, and W has index 63 in $W(E_7)$. This leads to the binary model of $\mathfrak{e}_7$.



Let A, W denote the natural representations of $\mathfrak{sl}_2$ and $\mathfrak{spin}_{12}$. Here the branching gives the very simple decomposition

$$V = A \otimes W \oplus \Delta,$$

where Δ is one of the half-spin representations, of dimension 32. The factor $A \otimes W$ corresponds to a set of twelve bitangents. Since W has an invariant quadratic form, its weights come into pairs of opposite weights. We thus get six pairs of bitangents forming a *Steiner complex* ([10], 6.1.2).

Since the node that we have marked defines the adjoint representation of $\mathfrak{e}_7$ on the Dynkin diagram of type E_7, the 63 Steiner complexes are in natural correspondence with the 63 pairs of opposite roots in the root system E_7. From this perspective they play the same role as the double-sixes of lines on the cubic surface.

Geometrically, the twelve weight spaces of a Steiner complex generate a $\mathbb{P}^{11}$ in $\mathbb{P}W$ whose intersection with the Freudenthal variety is a ten dimensional quadric. If we denote by $P(V)$ the set of weight of V (the $W(E_7)$-orbit of the fundamental weight ω_1, in the notation of [2]), the Steiner complex associated to the root α (up to sign) is

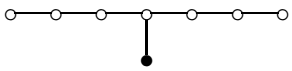
$$S_\alpha = \{\gamma \in P(V),\ \gamma + \alpha \text{ or } \gamma - \alpha \in P(V)\}.$$

Now if $\beta \neq \pm\alpha$ is another root, we can have $(\alpha, \beta) = 0$ or not, with respectively 30 and 32 possibilities for β, up to sign.

In the first case, the corresponding root spaces generate in $\mathfrak{e}_7$ a copy of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, and the Steiner complexes are *syzygetic*, which means that $\#(S_\alpha \cap S_\beta) = 4$. The roots which are orthogonal both to α and β form a reducible root system of type $A_1 \times D_4$. In particular, there is a third Steiner complex S_γ canonically associated to the pair (S_α, S_β), and syzygetic to both of them. The characteristic property of the triple $(S_\alpha, S_\beta, S_\gamma)$ is that its union is the whole set of bitangents (see Example 6 below). The number of such *syzygetic triads of Steiner complexes* is $63 \times 30/6 = 315$.

In the second case, the root spaces generate a copy of $\mathfrak{sl}_3$, and the Steiner complexes are *azygetic*: $\#(S_\alpha \cap S_\beta) = 6$. Then $\alpha + \beta$ (or $\alpha - \beta$) is again a root, and $S_{\alpha+\beta}$ is azygetic both to S_α and S_β: we obtain an *azygetic triad of Steiner complexes*. The number of such triads is $63 \times 32/6 = 336$.

Example 3. We mark the lowest node. Then $\mathfrak{h} = \mathfrak{sl}_8$, and $W = \mathcal{S}_8$ has index 36 in $W(E_7)$.



If we denote by U the natural representation of $\mathfrak{sl}_8$, the branching gives

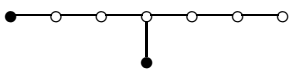
$$\begin{aligned} V &= \Lambda^2 U \oplus \Lambda^2 U^*, \\ \mathfrak{e}_7 &= \mathfrak{sl}_8 \oplus \Lambda^4 U. \end{aligned}$$

Once we fix a basis $u_1, \ldots, u_8$ of U, we can therefore identify each bitangent with a pair (ij), with $1 \leq i < j \leq 8$. Such a notation seems to have been first introduced by Hesse. Moreover, the Weyl group of E_7 is generated by the symmetric group $\mathcal{S}_8$ and the symmetries associated to the roots coming from the factor $\Lambda^4 U$. These symmetries are indexed by partitions $(pqrs|xyzt)$ of $[1, 8]$ into disjoints fourtuples. The induced action on the bitangents is given by

$$s_{(pqrs|xyzt)}(ij) = \begin{cases} (pqrs/ij) & \text{if} \quad \{ij\} \subset \{pqrs\}, \\ (xyzt/ij) & \text{if} \quad \{ij\} \subset \{xyzt\}, \\ (ij) & \text{otherwise.} \end{cases}$$

This is classicaly called a *bifid transformation*.

Example 4. We mark two extreme but not opposite nodes. Then $\mathfrak{h} = \mathfrak{sl}_7$, and $W = \mathcal{S}_7$ has index 288 in $W(E_7)$.



Obviously this example comes from the previous one: we have just passed from $\mathfrak{sl}_8$ to $\mathfrak{sl}_7$. If we denote by U the natural representation of $\mathfrak{sl}_7$, the branching gives

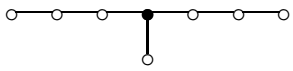
$$V = U \oplus \Lambda^2 U \oplus \Lambda^2 U^* \oplus U^*.$$

We have thus distinguished a set of seven bitangents forming what is called an *Aronhold set* ([10], 6.1.3). Geometrically, the seven weight spaces of

an Aronhold set generate a $\mathbb{P}^6$ which is a maximal linear space inside the Freudenthal variety.

Note that the Aronhold sets generate the remaining basic representation R of $\mathfrak{e}_7$, the one corresponding to the lowest extremal node of the Dynkin diagram. By this we mean that among the lines of $\Lambda^7 V$ generated by the weight vectors forming Aronhold sets, one has a dominant weight and generates a copy of R. We know that any irreducible representation of $\mathfrak{e}_7$ can then be constructed from V, R and the adjoint representation by natural tensorial operations.

Example 5. We mark the central node. Then $\mathfrak{h} = \mathfrak{sl}_4 \times \mathfrak{sl}_4 \times \mathfrak{sl}_2$, and $W = \mathcal{S}_4 \times \mathcal{S}_4 \times \mathcal{S}_2$ has index 1260 in $W(E_7)$.



Denote by C the two-dimensional natural representation, by A, B the two four dimensional ones. Then the branching gives

$$V = (\Lambda^2 A \oplus \Lambda^2 B) \otimes C \oplus (A \otimes B \oplus A^* \otimes B^*).$$

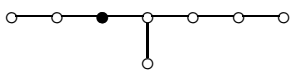
Note that $\Lambda^2 A$ and $\Lambda^2 B$ are self-dual, as well as C, so we have distinguished two sets of 6 bitangents forming a Steiner complex.

The remaining sixteen bitangents are indexed by the weights of $A \otimes B$. Recall that four bitangents form a *syzygetic tetrad* when their eight tangency points are the eight intersection points of the plane quartic with some conic ([10], 6.1.1). In terms of weights, this means that the four bitangents can be represented by weights summing to zero. Here we observe a phenomenon very similar to the property of a Steiner set of lines on the cubic surface: our sixteen bitangents can be split into four syzygetic tetrads in essentially twelve different ways. Indeed our tetrads must be of the form

$$T_\sigma = \{\epsilon_i + \epsilon_{\sigma(i)}, \quad 1 \leq i \leq 4\}$$

for some permutation σ, and we have to find four permutations σ_1, σ_2, σ_3, σ_4 such that $\sigma_j(i) \neq \sigma_k(i)$ for each i and each $j \neq k$. The twelve possibilities are given by the four-tuples of permutations of the form $(pqrs)$, $(qpsr)$, $(rspq)$, $(srqp)$ or $(pqrs)$, $(qpsr)$, $(rsqp)$, $(srpq)$.

Example 6. Now we mark a non extremal node next to the central one. Then $\mathfrak{h} = \mathfrak{sl}_3 \times \mathfrak{sl}_6$, and $W = \mathcal{S}_3 \times \mathcal{S}_6$ has index 336 in $W(E_7)$. This case leads to the ternary model of $\mathfrak{e}_7$.

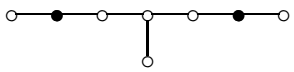


Let A, B denote the natural representations of $\mathfrak{sl}_3$ and $\mathfrak{sl}_6$. The branching gives the decomposition

$$V = A \otimes B \oplus \Lambda^3 B \oplus A^* \otimes B^*,$$

where the middle factor is self-dual. The factor $A \otimes B$ splits, following the A-component, into three sixers of bitangents. Aggregating them two by two we get an azygetic triad of Steiner sets.

Example 7. We mark the two nodes next to the two opposite extremal nodes. Then $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{spin}_8$, and W has index 2×315 in $W(E_7)$.



Let A, B denote the natural representations of the two copies of $\mathfrak{sl}_2$. Up to triality we may suppose that the natural (non spinorial) eight-dimensional representation W of $\mathfrak{spin}_8$ is given by the lowest node. Then the branching gives the decomposition

$$V = A \otimes \Delta_+ \oplus B \otimes \Delta_- \oplus 2W \oplus 2A \otimes B,$$

where the last two factors are two copies of the same module, put on duality by the symplectic form.

The symmetry of the picture suggests that we should introduce a supplementary copy of $\mathfrak{sl}_2$, hence three copies with natural representations A_1, A_2, A_3 such that

$$V = A_1 \otimes \Delta_1 \oplus A_2 \otimes \Delta_2 \oplus A_3 \otimes \Delta_3 \oplus A_1 \otimes A_2 \otimes A_3.$$

Indeed this is precisely what the trialitarian description of $\mathfrak{e}_7$ tells us (see [29], Theorem 4.1). We get a partition of the 28 bitangents into three groups of eight and a group of four. Adding the latter to the three former we get a *syzygetic triad of Steiner complexes*.

We have exactly 315 such triads, and this is also the number of syzygetic tetrads ([10], Corollary 6.1.4). Indeed the weights of $A_1 \otimes A_2 \otimes A_3$ define such a tetrad.

We recapitulate:

Proposition 1. *There are natural correspondences between:*

1. *Steiner complexes of bitangents and root subsystems of type A_1;*
2. *azygetic triples of Steiner complexes and subsystems of type A_2;*
3. *syzygetic triples of Steiner complexes and root subsystems of type D_4 of the root system E_7.*

Symplectic geometry. We have already mentionned that the Weyl group of E_7 can (almost) be identified with a classical group over a finite field (see [2], Exercice 3 of section 4, p. 229), namely

$$W(E_7)^+ \simeq Sp(6, \mathbb{F}_2).$$

This means that the incidence geometry of the 28 bitangents should be interpreted as a symplectic six-dimensional geometry over the field with two elements. Such symplectic geometries appear on the third line of Freudenthal's magic square, and the 28 bitangents give a finite model of these geometries (see e.g. [28]).

Recall that a symplectic five-dimensional projective geometry has three types of elements: points, isotropic lines and isotropic planes. In the E_7 geometry, points and isotropic lines are points and lines in the Freudenthal variety, while isotropic planes are in correspondence with maximal, ten dimensional quadrics on the Freudenthal variety.

In our finite geometry, we have seen that the points correspond to the 63 Steiner complexes. The 315 isotropic lines are in correspondence with the syzygetic triads of Steiner complexes, where we split the 28 bitangents into three sets of twelve, with four bitangents common to the three. It is obvious from that point of view that two syzygetic complexes can be uniquely completed into a syzygetic triad: indeed, they are syzygetic if they generate an isotropic line, and there is a unique other point on that line.

What are the 135 planes? Classically, they are called Göpel spaces (see [3], Chapter II, 22, and [12], Chapter IX) and play an important role in the study of the Schottky problem. But let us rather skip to our representation theoretic point of view.

A projective plane over $\mathbb{F}_2$ is a Fano plane. It has 7 points and 7 lines. So to get an isotropic plane we should be able to partition the 28 bitangents into 7 quadruples, in such a way that the complement of each of them can be split into three octuples defining syzygetic Steiner complexes. This looks like a combinatorial challenge but representation theory tell us what to do! We have already used the fact that $\mathfrak{e}_7$ has a maximal semisimple Lie algebra isomorphic to $\mathfrak{so}_8 \times \mathfrak{sl}_2^3$. Thanks to the four-ality model of $\mathfrak{so}_8$ we can take four two-dimensional spaces A_4, A_5, A_6, A_7 and decompose

$$\mathfrak{so}_8 = \mathfrak{sl}(A_4) \times \mathfrak{sl}(A_5) \times \mathfrak{sl}(A_6) \times \mathfrak{sl}(A_7) \oplus (A_4 \otimes A_5 \otimes A_6 \otimes A_7).$$

Then the three eight-dimensional representations decompose as

$$\begin{aligned} \Delta_1 &= (A_4 \otimes A_5) \oplus (A_6 \otimes A_7), \\ \Delta_2 &= (A_4 \otimes A_6) \oplus (A_5 \otimes A_7), \\ \Delta_3 &= (A_4 \otimes A_7) \oplus (A_5 \otimes A_6). \end{aligned}$$

Then we plug that in the decomposition of the 56-dimensional representation of E_7 given in Example 7 above. We obtain, if we denote $A_{ijk} = A_i \otimes A_j \otimes A_k$:

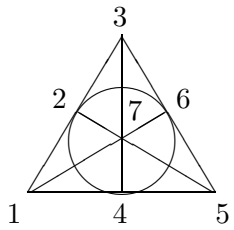
$$V = A_{123} \oplus A_{145} \oplus A_{167} \oplus A_{246} \oplus A_{257} \oplus A_{347} \oplus A_{356}.$$

The seven Steiner complexes that we are looking for are the sets of weights of the submodules

$$S_i = \bigoplus_{ijk} A_{ijk},$$

and the seven syzygetic triads they form are given by the weights of the three submodules S_i, S_j, S_k for (ijk) one of the seven triples in the decomposition of V.

Note that these seven triples of indices have the crucial property that any pair of integers between one and seven, appear in one and only one of them. Otherwise said, they form a *Steiner triple system* $S(2, 3, 7)$ (see e.g. [5]). Up to isomorphism there is only one such system, given by the lines of the Fano plane, as one can see on the next picture:



In particular, by projective duality these lines can be represented by points of another Fano plane. We deduce that the stabilizer of our configuration in $W(E_7)$ is the product of 7 copies of A_2 by the automorphism group of the Fano plane, which is nothing else than the Klein group $PSL(3,\mathbb{F}_2) \simeq PSL_2(\mathbb{F}_7)$, with 168 elements. The index of this stabilizer is 135, as expected. We call the corresponding configurations *Fano heptads of Steiner complexes*. Let us recapitulate the correspondence:

Symplectic geometry	Number	Bitangents
Points	63	*Steiner complexes*
Lines	315	*syzygetic triads*
Planes	135	*Fano heptads*

The quartic form. The E_7-module V has two basic invariants tensors, such that E_7 can be described as the group of linear transformations of V preserving these tensors: a symplectic form and a quartic form. The existence of such invariant forms is clear on our previous decomposition of V. Indeed, each factor A_{ijk} has a symplectic form induced by the choice of two-forms on each factor A_l. Moreover, A_{ijk} has an invariant quartic form given by the *Cayley hyperdeterminant*, which is an equation of the dual variety of the Segre product $\mathbb{P}A_i \times \mathbb{P}A_j \times \mathbb{P}A_k \subset \mathbb{P}A_{ijk}$ [20].

Proposition 2. *The invariant quartic form on V is the unique $W(E_7)$-invariant quartic form whose restriction to each factor A_{ijk} is proportional to the Cayley hyperdeterminant.*

Proof. Since their is a unique invariant quartic form on a factor A_{ijk}, up to scalar, it certainly coïncides with the restriction of the invariant quartic form on V. Conversely, we know that up to the action of the Weyl group, the monomials in the invariant quartic form on V are of three types (see [30]): products of two, equal or distinct, products of two variables associated to opposite weights; other products of four variables associated to fourtuples of weights of sum zero (thus defining syzygetic tetrads of bitangents). Then we must give suitable relative coefficients to the sums of monomials of each type. This is fixed by restriction to a single factor A_{ijk} since the three types of monomials appear in the hyperdeterminant (see [20]). □

Reconstructing $\mathfrak{e}_7$. From the trialitarian construction of $\mathfrak{e}_7$ and the fourality for $\mathfrak{so}_8$ we deduce the model:

$$\mathfrak{e}_7 = \times_{i=1}^{7}\mathfrak{sl}(A_i) \oplus \bigoplus_{(ijkl)\in I} A_i \otimes A_j \otimes A_k \otimes A_l,$$

where I is the following set of 7 quadruples:

$$1247 \qquad 1256 \qquad 1346 \qquad 1357 \qquad 2345 \qquad 2367 \qquad 4567$$

These quadruples are in natural correspondence with lines: simply associate to a line the four points of its complement. Moreover, the action on V can be recovered geometrically: each quadruple $(ijkl)$ in I can be seen as a complete quadrangle in the Fano plane, with three pairs of opposite sides which are sent one to the other by the $\mathfrak{e}_7$-action restricted to $A_i \otimes A_j \otimes A_k \otimes A_l$.

Let us rather try to reconstruct the Lie bracket. Consider two factors $A_i \otimes A_j \otimes A_k \otimes A_l$ and $A_i \otimes A_j \otimes A_m \otimes A_n$: we have two indices i, j in common, and the third point of the line generated by $\alpha = (ijkl)$ and $\beta = (ijmn)$ is $\alpha + \beta = (klmn)$. The restriction of the Lie bracket defines a map

$$A_i \otimes A_j \otimes A_k \otimes A_l \times A_i \otimes A_j \otimes A_m \otimes A_n \longrightarrow A_k \otimes A_l \otimes A_m \otimes A_n$$

such that for some non zero constant $\theta_{\alpha,\beta}$,

$$[x_i \otimes x_j \otimes x_k \otimes x_l, y_i \otimes y_j \otimes y_m \otimes y_n] = \theta_{\alpha,\beta}\omega(x_i, y_i)\omega(x_j, y_j)x_k \otimes x_l \otimes y_m \otimes y_n.$$

Indeed this is the unique equivariant map up to scalar, and it must be non zero because of the properties of the Lie bracket in a semisimple Lie algebra. The skew symmetry of the Lie bracket then implies that

$$\theta_{\beta,\alpha} = -\theta_{\alpha,\beta}.$$

The Jacobi identity can be expressed in the following way: for each triangle (α, β, γ) in the Fano plane, we have the relation

$$\theta_{\alpha,\beta}\theta_{\alpha+\beta,\gamma} = \theta_{\beta,\gamma}\theta_{\beta+\gamma,\alpha} = \theta_{\gamma,\alpha}\theta_{\gamma+\alpha,\beta}.$$

The Fano plane has 28 triangles, hence 56 quadratic relations.

Lemma 3. *Let $\theta_{\alpha,\beta} = \pm 1$ according to the following rule: the multiplication table of the canonical basis $e_1, \dots, e_7$ of the imaginary octonions is given by*

$$e_\alpha e_\beta = \theta_{\alpha,\beta} e_{\alpha+\beta} \qquad for \quad \alpha \neq \beta.$$

Then the relations above are satisfied.

Proof. Denote by e_0, e_α, where $\alpha = 1, \dots, 7$, the canonical basis of the octonions, (see the Introduction). Our claim amounts to the identity

$$(e_\alpha e_\beta)e_\gamma = (e_\beta e_\gamma)e_\alpha$$

when α, β, γ are distinct and not aligned. To prove this we need to remember that the Cayley algebra, although not associative, is *alternative* [28]. This means that the associator $A(x, y, z) = (xy)z - x(yz)$ is an alternating function of the arguments. Using the fact that $e_\alpha e_\beta = -e_\beta e_\alpha$ when α, β are distinct, we deduce that

$$\begin{aligned}(e_\beta e_\gamma)e_\alpha - e_\beta(e_\gamma e_\alpha) &= A(e_\beta, e_\gamma, e_\alpha) \\ &= -A(e_\beta, e_\alpha, e_\gamma) = (e_\alpha e_\beta)e_\gamma - e_\beta(e_\gamma e_\alpha),\end{aligned}$$

which proves our claim. □

This means that the model that we have found for $\mathfrak{e}_7$ really has a very close connection with the octonions. We can reformulate our discussion as follows.

Theorem 4. *The exceptional complex Lie algebra $\mathfrak{e}_7$ has a natural structure of an $\mathbb{O}$-graded algebra, given in terms of points i and lines ℓ on the Fano plane by*

$$\mathfrak{e}_7 = \times_{i=1}^{7}\mathfrak{sl}(A_i)e_0 \ \oplus \bigoplus_{\ell}(\otimes_{i\notin\ell}A_i)\, e_\ell.$$

Moreover, its minimal representation decomposes as

$$V = \bigoplus_{\ell}(\otimes_{i\in\ell}A_i)\, e_\ell.$$

Note that there is a quaternionic analogue of this construction, where instead of the Fano plane we consider only one of its lines. This means that we glue three copies of $\mathfrak{so}_8$ along the product of four copies of $\mathfrak{sl}_2$. The resulting algebra is $\mathfrak{so}_{12}$.

We conclude that the Lie algebra $\mathfrak{e}_7$ is the support of a finite plane projective geometry whose points represent 7 copies of $\mathfrak{so}_8$, and whose lines represent 7 copies of $\mathfrak{so}_{12}$.

A sign problem and the isomorphism $PSL(3,\mathbb{F}_2) \simeq PSL(2,\mathbb{F}_7)$. We have just seen that the octonionic multiplication table gives a solution to the problem of finding a set of constants $\theta_{\alpha,\beta}$ satisfying the skewsymmetry condition and the 56 quadratic relations associated to the 28 triangles in the Fano plane. What are the other solutions such that $\theta_{\alpha,\beta} = \pm 1$?

Proposition 5. *There exist exactly sixteen such solutions, falling into two $PSL(3,\mathbb{F}_2)$-orbits. Each orbit can be identified, as a $PSL(2,\mathbb{F}_7)$-set, with a copy of the projective line $\mathbb{F}_7\mathbb{P}^1$.*

Proof. We proceed as follows. We first check that on a line the orientations must be coherent in the following sense: put an arrow from α to β if $\theta_{\alpha\beta} = +1$. Then the three arrows on a line, if we draw it as a circle, must go in the same direction. In particular there are only two possible choices of signs on a line, one for each cyclic orientation. We can switch from one to the other by changing one of the basis vectors in its opposite. Moreover, the possible solutions to our problem can now be interpreted as a coherent orientation of the seven lines in the plane.

Now we choose a triangle in the Fano plane. We have eight possible orientations for the three sides. We observe that once we choose one, the orientation of the line joining the three middle points of the sides of the triangle is fixed, and that there are only two possibilities for the three remaining lines, those passing through the center of the triangle. Moreover we pass from one to the other by changing the sign of the basis vector corresponding to the center.

Finally we check that once we fix a coherent orientation, we can transform it by $PSL_3(\mathbb{F}_2)$ to an arbitrarily chosen orientation on the triangle of reference. This implies that we have 16 possible orientations splitting in at most two orbits. But there cannot be a single orbit since 16 does not divide the order of $PSL_3(\mathbb{F}_2)$. To identify the two orbits with a projective line over $\mathbb{F}_7$, there just remains to observe that $PSL_3(\mathbb{F}_2)$, up to conjugation, has a unique subgroup of index 8 (see [4]). □

This suggests an interpretation of the isomorphism between $PSL(3,\mathbb{F}_2)$ and $PSL(2,\mathbb{F}_7)$. We obtained the following one which we could not find in the litterature. It is conveniently expressed in terms of cubes in a projective line, by which we simply mean a graph with eight vertices, which is topologically the incidence graph of a cube.

Definition. A cube in a projective line $\mathbb{P}^1$ is *harmonic* if each of its faces $(wxyz)$ is harmonic, that is, the opposite vertices (wy) and (xz) are in harmonic position.

Proposition 6. *There exist fourteen harmonic cubes in $\mathbb{F}_7\mathbb{P}^1$, made of 42 harmonic faces. They split uniquely into two $PSL(2,\mathbb{F}_7)$-orbits in such a way that each harmonic face belongs to exactly one cube of each family.*

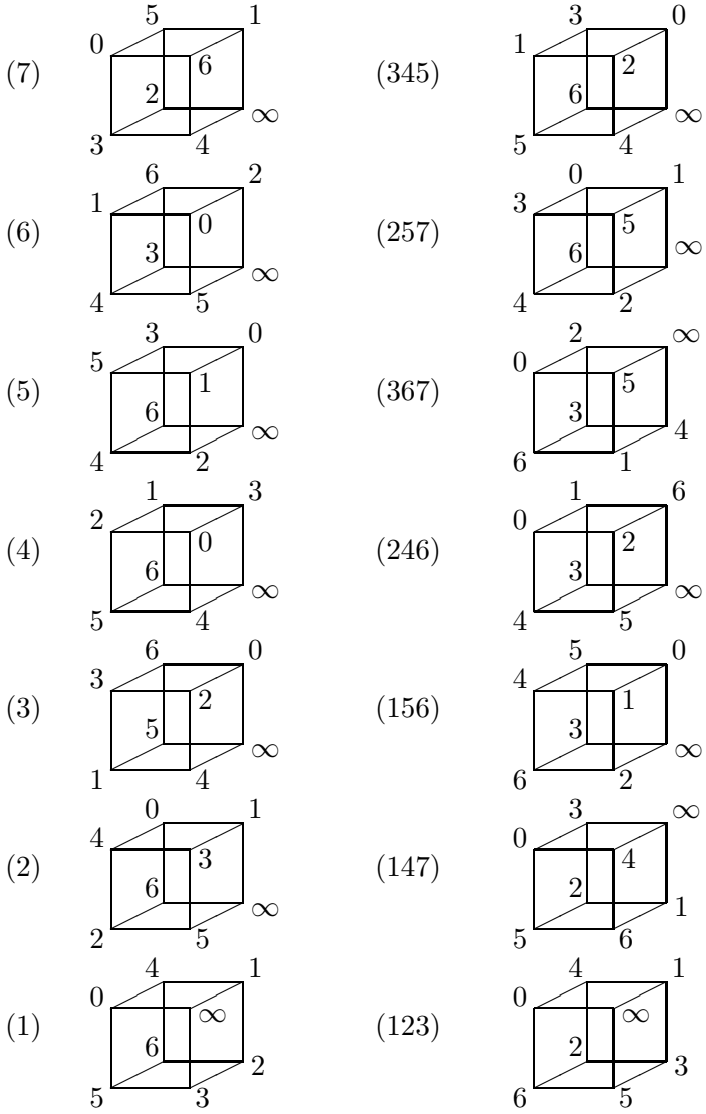


Figure 4. The fourteen harmonic cubes in $\mathbb{F}_7\mathbb{P}^1$

We thus get an interpretation of these two sets of harmonic cubes in $\mathbb{F}_7\mathbb{P}^1$ as points and lines in a Fano plane. We arbitrarily distinguish these two sets by calling them p-cubes and ℓ-cubes, respectively. (Note an interesting analogy with the construction of the projective space over $\mathbb{F}_2$ given in [32] from the Fano plane: there exists 30 unequivalent labelings of the seven vertices up to the $PSL(3,\mathbb{F}_2)$ action, and they are split into two families of 15 labelings by the property that each line appears only once in each family.)

The incidence relations can be defined as follows:

- Given a p-cube (respectively ℓ-cube), there exist exactly three ℓ-cubes (respectively p-cubes) sharing a pair of opposite faces with it.
- Given two p-cubes (respectively ℓ-cubes), there exists a unique p-cube (respectively ℓ-cube) such that the three cubes can be split each into two pairs of opposite edges forming squares with the same fourtuples of vertices.

Note that each pair (ij) is the diagonal of exactly one p-cube and one ℓ-cube.

A nice feature of the correspondence is that each pair of points in $\mathbb{F}_7\mathbb{P}^1$ defines an edge of exactly three cubes in each family, corresponding to the three vertices and to the three edges of a triangle in the Fano plane. Therefore:

Proposition 7. *There is an equivariant correspondence between the* 28 *triangles in the Fano plane, and the* 28 *pairs of points in the projective line over* $\mathbb{F}_7$.

Explicitely, this correspondence is as follows, where the triples in boldface are the vertices of a triangle in the Fano plane:

01	**256**	12	**145**	24	**235**	3∞	**146**
02	**346**	13	**234**	25	**247**	45	**456**
03	**457**	14	**136**	26	**126**	46	**157**
04	**124**	15	**357**	2∞	**567**	4∞	**347**
05	**167**	16	**467**	34	**267**	56	**134**
06	**237**	1∞	**127**	35	**125**	5∞	**236**
0∞	**135**	23	**137**	36	**356**	6∞	**245**

What about the orientations of the Fano planes that we were interested in? We can associate such an orientation, in a $PSL(2,\mathbb{F}_7)$-equivariant way, to each point $p \in \mathbb{F}_7\mathbb{P}^1$ as follows. For each point x in the Fano plane, consider the point $q_x \in \mathbb{F}_7\mathbb{P}^1$ such that pq_x is a diagonal of the harmonic cube corresponding to x. (This defines a bijection between $\mathbb{F}_2\mathbb{P}^2$ and $\mathbb{F}_7\mathbb{P}^1 - \{p\}$.) Then the line $\ell = (xyz)$ will we positively oriented if the cross-ratio

$$(pq_xq_yq_z) = 3.$$

On can easily check that in $\mathbb{F}_7$, this condition is invariant under a cyclic permutation of (xyz), so that this definition really makes sense! This makes explicit the identification that we obtained between the projective line over $\mathbb{F}_7$ and half of the coherent orientations of the Fano plane. Of course we obtain the other half by reversing all the arrows.

Our correspondence has the property to transform certain special configurations of bitangents into nice sets of triangles. We mention three instances of that.

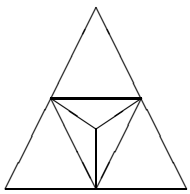
1. When we index the bitangents by pairs of points in a set with eight elements, we give a special role to the eight Aronhold sets A_i defined as the seven bitangents (ij), $j \neq i$. Indeed, this follows from the discussion of Example 4 above. We thus get eight sets of seven triangles T_i in the Fano plane (corresponding to the eight points in $\mathbb{F}_7\mathbb{P}^1$) with the following properties: given a line ℓ and two points on it, there is a unique triangle in T_i they are vertices of which; in particular, T_i is a copy of the Steiner triple system $S(2,3,7)$; from the three pairs of points in ℓ we thus deduce three triangles in T_i; the three vertices of these triangles which do not belong to ℓ are the vertices of a fourth triangle in T_i; this defines a natural bijection between the lines of the Fano plane and the triangles in T_i.

	123	**174**	**156**	**246**	**257**	**345**	**376**
$\mathbf{T_0}$	475	265	273	135	364	167	142
$\mathbf{T_1}$	467	265	234	375	163	172	154
$\mathbf{T_2}$	576	253	247	173	364	126	154
$\mathbf{T_3}$	475	356	234	173	146	276	125
$\mathbf{T_4}$	456	253	374	157	163	276	142
$\mathbf{T_5}$	456	236	247	375	134	167	125
$\mathbf{T_6}$	467	356	273	157	134	126	245
$\mathbf{T_\infty}$	576	236	374	135	465	172	245

This gives a remarkable configuration of eight Steiner triple systems formed on 28 triangles in such a way that each of them appears exactly twice.

Moreover, each point and each line in the Fano plane belongs to exactly three of the seven triangles in each system. And the centers of the seven triangles are the seven points of the plane.

2. We have seen in Example 5 that syzygetic tetrads of bitangents are defined by fourtuples of weights in the fundamental representation of $\mathfrak{e}_7$ summing to zero, but not in two opposite pairs. There are two types of such tetrads in Hesse's notations: 105 are permutations of $(01)(23)(45)(6\infty)$ and 210 are permutations of $(01)(23)(02)(13)$. Each of these tetrads defines a special configuration of four triangles, a typical one being



The four triangles in this picture are the great exterior triangle, plus the three triangles having the center of the picture for vertex, plus two others on the middle points of two of the sides of the first one.

3. A triple of points on the projective line over $\mathbb{F}_7$ defines three pairs, hence three triangles in the Fano plane. The following statement leads us back to the orientation problem which was the starting point of this long digression.

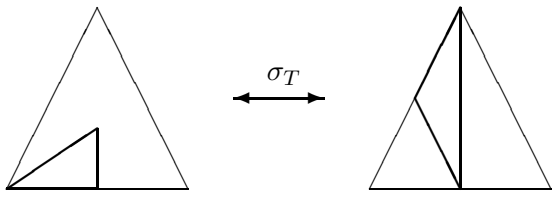
Proposition 8. *There is an induced equivariant correspondence between triples of points on the projective line over $\mathbb{F}_7$, and oriented triangles in the Fano plane.*

This goes as follows. Consider an oriented triangle in $\mathbb{F}_2\mathbb{P}^2$. For each vertex, consider the middle point on the opposite side, and then go to the next vertex following the orientation. We thus get three (non oriented) triangles, which can be checked to be in correspondence with three pairs $(ab), (bc), (ac)$ of a unique triple (abc) of points on $\mathbb{F}_7\mathbb{P}^1$.

Given a pair (pq) of points in $\mathbb{F}_7\mathbb{P}^1$, it defines a triangle in the Fano plane, hence two oriented triangles, hence two triples of points in the projective line. How can we obtain them directly? Simply by considering the unique harmonic cube in one of our two families having (pq) for diagonal. Then the three points on an edge of this cube passing to p (respectively q) give the two triples.

Another setting for the bitangents. The natural inclusion $SL(3, \mathbb{F}_2) \subset Sp(6, \mathbb{F}_2) = W(E_7)^+$ suggests to encode the 28 bitangents and their symmetry group, directly in the geometry of the Fano plane. This can indeed be done in a very natural way, once we have identified the bitangents with the 28 triangles in the Fano plane. Recall that the group of the bitangents is generated by the transpositions s_{ij} and the bifid tranformations $s_{(pqrs|xyzt)}$. We have checked that they have a simple geometric interpretation in terms of triangles.

Transpositions. Let T be the triangle associated to the pair (ij). There is a unique point in the Fano plane which does not belong to a side of T, and we call this point the center of the triangle. Up to the action of $SL(3, \mathbb{F}_2)$ we can draw the Fano plane in such a way that the exterior of the picture is precisely T. Then the involution σ_T on the set of triangles exchanges triangles as shown in the following picture:

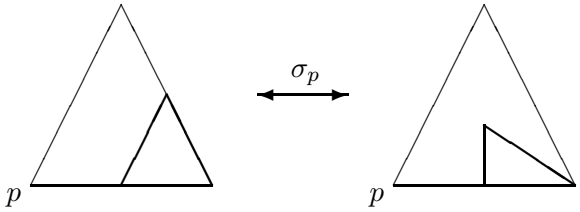


Otherwise said, a triangle whose vertices are a vertex v of T, the center c of T and the middle point p of a side of T, is mapped to the triangle whose vertices are p, its symmetric point w with respect to c, and the middle point of the side vw – and conversely, while the other triangles remain unchanged.

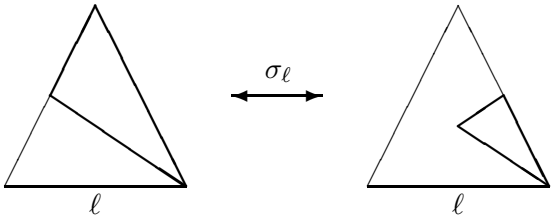
Bifid transformations. One can check that the 35 bifid transformations, when we interprete them as operations on the triangles, split into three

types which are naturally associated to the seven points p, the seven lines ℓ, and the 21 pairs of incident points and lines $p \in \ell$ in the Fano plane. We get the following transformations.

Associated to a point p is a transformation σ_p who takes a triangle with a side whose middle point is p, and changes the opposite vertex to the symmetric point with respect to p.

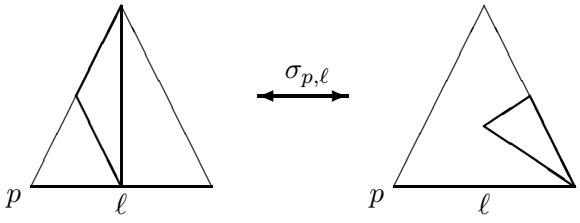


Associated to a line ℓ is a transformation σ_ℓ who takes a triangle with a unique vertex v on ℓ, and changes the two other vertices to the symmetric points with respect to v.



Note that these two types of transformations are exchanged by the polarity transformation on the set of triangles, which exchanges vertices and sides in the Fano plane with sides and vertices in the dual Fano plane.

Finally, associated to a pair $p \in \ell$ is a transformation $\sigma_{p,\ell}$ who takes a triangle with a unique vertex $v \neq p$ on ℓ, whose opposite side s has p for middle point, to the triangle with vertex the symmetric point of v with respect to p, and opposite side the symmetric of s with respect to ℓ.



The group of the bitangents is isomorphic with the group of permutations of the triangles generated by the elementary tranformations σ_T, σ_p, σ_ℓ, $\sigma_{p,\ell}$. It contains $SL(3, \mathbb{F}_2)$ as the group of collinations acting on the triangles. This makes clear the natural inclusions

$$SL(3, \mathbb{F}_2) \subset \mathcal{S}_8 \subset Sp(6, \mathbb{F}_2) = W(E_7)^+.$$

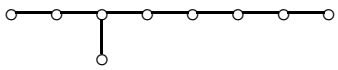
5. Del Pezzo surfaces of degree one

The bicanonical model of a Del Pezzo surface of degree one is the double cover of a quadratic cone, branched over a canonical space curve of genus 4 and degree 6 given by the complete intersection of the cone with a unique cubic surface ([9] V.1). The 240 lines on the Del Pezzo surface arise in pairs from the 120 tritangent planes to the canonical curve, which can be identified with its odd theta characteristics. Moreover, the fact that the unique quadric containing this curve is a cone distinguishes one of the 136 even theta characteristic by the property that it vanishes at the vertex of that cone.

The automorphism group of the 240 lines is the Weyl group $W(E_8)$ of the root system of type E_8. Its order is $696\,729\,600 = 128\times 27\times 5\times 8! = 2^{13}3^5 5^2 7$. The automorphism group of the 120 tritangent planes is the quotient by the normal subgroup $\{\pm 1\}$ and can be identified with the orthogonal group $O(8,\mathbb{F}_2)^+$ which preserves the quadratic form given by half the natural norm on the root lattice mod 2 ([2], Exercice 1 of section 4, p. 228). Among the $2^8 = 256$ points in $\mathbb{F}_2^8$, those with norm one can be identified with the odd theta-characteristics, and those with norm zero with the even theta-characteristics, including the special one which identifies with the origin.

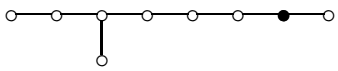
If we consider the action of the adjoint group E_8 on the projectivized adjoint representation $\mathbb{P}\mathfrak{e}_8$, the 240 root spaces can be interpreted as a kind of finite skeleton of the closed orbit, the adjoint variety of E_8. This variety parametrizes the so-called symplecta in Freudenthal's metasymplectic geometry (see [27]).

The affine Dynkin diagram of E_8 is



Relevant configurations will be provided by the simplest markings.

Example 1. We mark the node next to the rightmost extremal node. Then $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{e}_7$ and W has index 120 in $W(E_8)$.



The branching gives the binary model

$$\mathfrak{e}_8 = \mathfrak{sl}_2 \times \mathfrak{e}_7 \oplus A \otimes V,$$

where V is again the minimal 56-dimensional representation of $\mathfrak{e}_7$. In particular, this associates to each root α (up to sign) of $\mathfrak{e}_8$ a complex S_α of 56 tritangent planes. Two complexes S_α and S_β have two possible relative positions, distinguished by the fact that $(\alpha,\beta) = 0$ or not.

In the latter case, exactly as for $\mathfrak{e}_7$ the two complexes are *azygetic* and can be completed uniquely with a third complex S_γ, with $\gamma = \alpha \pm \beta$, azygetic

to both of them. There exists 1120 such *azygetic triads of complexes*. This leads to the ternary model of $\mathfrak{e}_8$,

$$\mathfrak{e}_8 = \mathfrak{sl}_3 \times \mathfrak{e}_6 \oplus (B \otimes J) \oplus (B \otimes J)^*.$$

The 120 tritangent planes are partitioned into the triple (α, β, γ), three sets $S_{\alpha\beta}$, $S_{\alpha\gamma}$, $S_{\beta\gamma}$ of cardinality 27, and their complement of cardinality 36, with

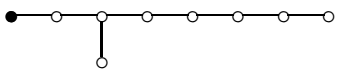
$$S_\alpha = \{\beta, \gamma\} \cup S_{\alpha\beta} \cup S_{\alpha\gamma}.$$

In the former case, the two complexes are *syzygetic*. Since their common orthogonal subsystem is a root system of type $A_1 \times A_1 \times D_4$, we can define a *syzygetic tetrad of complexes* to consist in four syzygetic complexes orthogonal to a D_4-subsystem of the root system E_8. Note that we have three syzygetic tetrads for each D_4-subsystem, making a total of 9450 such tetrads.

A pair of syzygetic complexes can be completed uniquely into a syzygetic tetrad $(\alpha, \beta, \gamma, \delta)$. The other 116 tritangent planes are then partitioned into a set $S_{\alpha\beta\gamma\delta}$ of cardinality 8, six sets $S_{\alpha\beta}$, $S_{\alpha\gamma}$, $S_{\alpha\delta}$, $S_{\beta\gamma}$, $S_{\beta\delta}$, $S_{\gamma\delta}$ of cardinality 16, and their complement of cardinality 12. Here

$$S_\alpha = S_{\alpha\beta\gamma\delta} \cup S_{\alpha\beta} \cup S_{\alpha\gamma} \cup S_{\alpha\delta}.$$

Example 2. We mark the leftmost node. Then $\mathfrak{h} = \mathfrak{spin}_{16}$ and W has index 135 in $W(E_8)$.



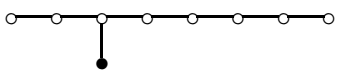
The branching gives another very nice model,

$$\mathfrak{e}_8 = \mathfrak{spin}_{16} \oplus \Delta,$$

where Δ is a half-spin representation, of dimension 128.

Geometrically, we get twelve dimensional quadrics in the adjoint variety.

Example 3. We mark the lowest node. Then $\mathfrak{h} = \mathfrak{sl}_9$ and $W = \mathcal{S}_9$ has index 1920.



The branching gives one of the prettiest models of $\mathfrak{e}_8$, namely

$$\mathfrak{e}_8 = \mathfrak{sl}_9 \oplus \Lambda^3 U \oplus \Lambda^6 U,$$

where U denotes the natural nine dimensional representation. The $\mathfrak{sl}_9$ factor defines what Du Val calls an α_8 configuration in the uniform polytope 4_{21}. (Note that there are 960 such configurations rather than 1920. This is because of the invariance by -1, the longest element in $W(E_8)$: indeed its restriction to the root system A_8 of $\mathfrak{sl}_9$ does not belong to $W(A_8)$ but defines an order two *outer* automorphism). It can be interpreted as a special system of then even theta-characteristics ([13], p. 51). One deduces a special set of 84, and the complementary set of 36 tritangent planes: the

odd theta-characteristics of the former set are obtained as sums of three of the distinguished even theta-characteristics (which is visible from the fact that they correspond to weights of a third wedge power), and the latter as sums of only two of them.

Geometrically, we obtain copies of $\mathbb{P}^7$ in the adjoint variety. From this point of view there is a close analogy with Aronhold sets of bitangents to the plane quartic curve.

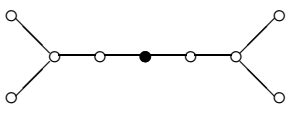
There is also a strong analogy with the decomposition of $\mathfrak{e}_7$ that we described in Example 2 above, from which we recovered Hesse's notation for the bitangents to the plane quartic, and the so called bifid transformations. Here the roots of $\mathfrak{e}_8$, up to sign, are split in two types by the model we are discussing. The roots that are weights of the factor $\Lambda^3 U$ are indexed by a triple (ijk) of integers between 1 and 9. The roots or $\mathfrak{sl}_9$ are indexed, up to sign, by a pair (ij) that we can identify with the triple $(0ij)$. The tritangent planes are then indexed by the 120 triples of integers between 1 and 10, a notation first introduced by Pascal (see [6]). Moreover, the group of the tritangent planes is generated by the symmetric group $\mathcal{S}_9$ plus the symmetries associated to the triples (ijk), which are again called *bifid transformations* (but beware that it does not contain the symmetric group $\mathcal{S}_{10}$). An easy computation shows that the bifid transformation associated to (ijk) exchanges $(ij\ell)$ with $(0k\ell)$ and sends a triple (abc) to (pqr) if $ijkabcpqr$ is a permutation of 123456789 – while the other triples are fixed.

There is a connection with the model for $\mathfrak{e}_6$ discussed in Example 4 of section 3, as we can see by splitting the nine dimensional representation U into three supplementary spaces of dimension three. In fact the corresponding subroot system of type A_2^3 in E_8 is orthogonal to another A_2 subsystem, and we get another interesting model for $\mathfrak{e}_8$, which is graded over $\mathbb{Z}_3 \times \mathbb{Z}_3$:

$$\begin{aligned}\mathfrak{e}_8 = \quad & \mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2) \times \mathfrak{sl}(A_3) \times \mathfrak{sl}(A_4) \oplus \\ & \oplus A_{1234} \oplus A_{1^*234} \oplus A_{12^*34} \oplus A_{1234^*} \\ & \oplus A^*_{1234} \oplus A^*_{1^*234} \oplus A^*_{12^*34} \oplus A^*_{1234^*},\end{aligned}$$

where A_1, A_2, A_3, A_4 are three dimensional and we used the notation $A_{1^*234} = A_1^* \otimes A_2 \otimes A_3 \otimes A_4$. But this grading does not seem to be supported by any interesting geometry.

Example 2, continued. The trialitarian model of $\mathfrak{e}_8$ arises when we extract from $\mathfrak{spin}_{16}$ two orthogonal copies of $\mathfrak{spin}_8$. This can be seen from the affine Dynkin diagram of type D_8:



Indeed, we get

$$\mathfrak{e}_8 = \mathfrak{spin}_8 \times \mathfrak{spin}_8 \oplus (\mathbb{O}_1 \oplus \mathbb{O}_1') \oplus (\mathbb{O}_2 \oplus \mathbb{O}_2') \oplus (\mathbb{O}_3 \oplus \mathbb{O}_3'),$$

where $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ are the three eight-dimensional irreducible representations of $\mathfrak{spin}_8$. This models splits the 120 tritangent planes into two groups of 12 and three groups of 32. With which geometric characterization?

Then we can split each $\mathfrak{spin}_8$ using four copies of $\mathfrak{sl}_2$; we know that each of its eight-dimensional representation has a nice decomposition, and putting them together we obtain

$$\mathfrak{e}_8 = \times_{i=1}^{8} \mathfrak{sl}(A_i) \oplus \bigoplus_{(ijkl)\in I} A_i \otimes A_j \otimes A_k \otimes A_l,$$

where I is the following set of 14 quadruples:

1234	5678
1256	3478
1278	3456
1357	2468
1368	2457
1458	2367
1467	2358

As in the case of $\mathfrak{e}_7$ these fourteen quadruples form a *Steiner quadruple system* $S(3,4,8)$. In fact there exists a unique such system up to isomorphism.

Remark. Such a decomposition is induced by the choice of a root subsystem of type A_1^8 inside the root system E_8. This is equivalent to what Du Val calls a β_8 configuration in the polytope 4_{21}. There is also a correspondence with the 135 even theta-characteristics, or the 135 norm one vectors in the root lattice mod 2, which can be seen as follows. Consider such a vector, for example $\theta = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8$ if we denote by $\pm\omega_i$ the weights of A_i. Among the 127 projective lines through θ, 63 are contained in the quadric, 56 are tangents and only 8 are true bisecants; on each of these bisecants there is a unique point outside the quadric and we obtain a set of eight points defining a subsystem of type A_1^8.

Let us consider our Steiner quadruple system is some detail. We first note that each quadruple $(ijkl) \in I$ defines a copy of $\mathfrak{so}_8 \simeq \mathfrak{sl}(A_i) \times \mathfrak{sl}(A_j) \times \mathfrak{sl}(A_k) \times \mathfrak{sl}(A_l) \oplus (A_i \otimes A_j \otimes A_k \otimes A_l)$ inside $\mathfrak{e}_8$. We have thus constructed $\mathfrak{e}_8$ by gluing together fourteen copies of $\mathfrak{so}_8$ overlapping over eight copies of $\mathfrak{sl}_2$!

We can interprete the 14 quadruples in I as *points* of a configuration whose *lines* are triples of type $(ijkl), (klmn), (ijmn)$. A straightforward inspection shows that there are exactly 28 lines. Moreover, each line has three points and each point belongs to 6 lines. In other words, we have obtained a $(14_6, 28_3)$-configuration.

This configuration has the following interpretation. Consider $\mathbb{F}_2\mathbb{P}^3$, the three dimensional projective space over the field with two elements. It has fifteen points. Choose one, say p_∞, and throw it away. Since $\mathbb{F}_2\mathbb{P}^3$ contains 35 lines, 7 of which passing through p_∞, we remain with 14 points and 28 lines whose incidence configuration is the one we are interested in.

In particular, note the following properties:

(1) each point p has an antipodal point p^*, the unique point to which it is not joined by a line – in $\mathbb{F}_2\mathbb{P}^3$, this is the third point of the line $\overline{pp_\infty}$;
(2) the 6 lines passing through a point p split naturally into three pairs, in such a way that the four points different from p on each pair,

belong to a pair of lines passing through p^*; in $\mathbb{F}_2\mathbb{P}^3$, these three pairs are cut by the three planes containing the line $\overline{pp_\infty}$;

(3) the configuration is made of eight copies of the Fano plane, corresponding to the eight planes in $\mathbb{F}_2\mathbb{P}^3$ not containing p_∞; the other seven planes give sub-configurations of type $(6_2, 4_3)$ which are pointed Fano planes.

But the most satisfactory way to understand our configuration is probably to see it as a *doubled Fano plane*. By this we mean that we can associate to each point of the Fano plane a pair of antipodal quadruples, in such a way that the 28 lines of the configuration correspond four by four to the lines of the Fano plane. In terms of Lie subalgebras of $\mathfrak{e}_8$, this associates to each point of the Fano plane a copy of $\mathfrak{so}_8 \times \mathfrak{so}_8$, and to each line a copy of $\mathfrak{so}_{16}$. Moreover, there exists four copies of $\mathfrak{so}_{12}$ inside the $\mathfrak{so}_{16}$ defined by a line, meeting each $\mathfrak{so}_8 \times \mathfrak{so}_8$ corresponding to one of its points, along one of the two $\mathfrak{so}_8$ factors.

We have even more structure if we note that each integer i between 1 and 8 determines a copy of $\mathfrak{e}_7$ inside $\mathfrak{e}_8$. Two such copies meet along one of the $\mathfrak{so}_{12}$ indexed by the 28 lines. This has an interesting combinatorial interpretation. Consider the following 8×8 array:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

It is symmetric with respect to the main diagonal, and each number j between 0 and 7 appears once and only once in each line and in each column. If we associate to j the four pairs (lc) of numbers indexing the lines and columns of the boxes where j appears, this means that we obtain a partition of $[1, 8]$ into four pairs, for a total number of 28 distinct pairs:

$$\begin{array}{c}(12)(34)(56)(78)\\(13)(24)(57)(68)\\(14)(23)(58)(67)\\(15)(26)(37)(48)\\(16)(25)(38)(47)\\(17)(28)(35)(46)\\(18)(27)(36)(45)\end{array}$$

(Note that this array can also be obtained from the diagonals of each family of seven harmonic cubes of Figure 5.) These seven partitions index our seven lines in the following way: to each $(ij)(kl)(pq)(rs)$ are associated the six quadruples obtained by selecting two of the four pairs. The incidence is given by the following rule: two lines being given, they can always be indexed by partitions of type $(ij)(kl)(pq)(rs)$ $(ik)(jl)(pr)(qs)$, and then their intersection point is indexed by the pair of antipodal quadruples $(ijkl)$ and $(pqrs)$.

We recapitulate the main conclusions of our discussion:

Theorem 9. *The exceptional complex Lie algebra $\mathfrak{e}_8$ has a natural structure of an $\mathbb{O}$-graded Lie algebra, obtained by gluing seven copies of $\mathfrak{so}_8 \times \mathfrak{so}_8$ indexed by the points of a Fano plane.*

This occurs in such a way that the three copies indexed by three points of a same line are glued together in a copy of $\mathfrak{so}_{16}$.

First application. This construction of $\mathfrak{e}_8$ isolates eight tritangent planes corresponding to the roots of the eight copies of $\mathfrak{sl}_2$, the remaining 112 tritangent planes being split into fourteen groups of eight.

Define a *syzygetic tetrad* of tritangent planes by the property that their twelve tangency points are the twelve intersection points of the sextic canonical curve with some quadric hypersurface. This means that they can be defined by four roots summing to zero. We claim that the eight tritangent planes associated to each factor $A_i \otimes A_j \otimes A_k \otimes A_l$ in our decomposition of $\mathfrak{e}_8$, can be split in two syzygetic tetrads in exactly six ways. Indeed, the corresponding roots are the $\pm\epsilon_i \pm \epsilon_j \pm \epsilon_k \pm \epsilon_l$ and two syzygetic tetrads are given by the following two sign tables:

$$\begin{array}{cccc} + & + & + & + \\ + & + & - & - \\ - & - & + & - \\ - & - & - & + \end{array} \qquad \begin{array}{cccc} + & - & + & - \\ + & - & - & + \\ - & + & + & + \\ - & + & - & - \end{array}$$

The other five are obtained by permuting the three last columns of each table.

Second application. An *orthogonal decomposition* (OD) of a semisimple Lie algebra $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \oplus_{i=0}^{h} \mathfrak{t}_i$ into a direct sum of Cartan subalgebras. Such an OD is *multiplicative* if for each pair (i, j), there exists an integer k such that $[\mathfrak{t}_i, \mathfrak{t}_j] \subset \mathfrak{t}_k$. A trivial example is that of $\mathfrak{sl}_2 = \mathfrak{sl}(A)$, once we have chosen a basis (e, f) of A. If X, Y, H is the associated canonical basis of $\mathfrak{sl}_2(A)$, a multiplicative OD (or MOD) is given be the three lines generated by $H, X + Y$ and $X - Y$.

Multiplicative OD's have been used by Thompson to construct the finite sporadic simple group denoted Th or F_3. Indeed, his construction relied on the existence of a multiplicative OD for $\mathfrak{e}_8$.

In fact there exists, up to isomorphism, a unique multiplicative OD of $\mathfrak{e}_8$ ([26], Chapter 3). Our construction provides it for free! Indeed we just need to notice that each component $A_i \otimes A_j \otimes A_k \otimes A_l$ can be split into the direct sum of two Cartan subalgebras $\mathfrak{t}^{\pm}_{ijkl}$ by choosing a basis (e_i, f_i) of each A_i, and letting

$$\mathfrak{t}^{\pm}_{ijkl} = \langle x_i \otimes x_j \otimes x_k \otimes x_l \pm y_i \otimes y_j \otimes y_k \otimes y_l \rangle,$$

where (x_i, y_i) is (e_i, f_i) or (f_i, e_i). This gives 28 Cartan subalgebras of $\mathfrak{e}_8$, and we add three others, say $\mathfrak{t}_0$, $\mathfrak{t}_+$ and $\mathfrak{t}_-$, by putting together the OD's of the $\mathfrak{sl}(A_i)$'s associated to the basis of the A_i's that we have chosen. We obtain:

Theorem 10. *The direct sum decomposition*

$$\mathfrak{e}_8 = \mathfrak{t}_0 \oplus \mathfrak{t}_+ \oplus \mathfrak{t}_- \oplus \bigoplus_{(ijkl)\in I} (\mathfrak{t}^+_{ijkl} \oplus \mathfrak{t}^-_{ijkl})$$

is a multiplicative OD of $\mathfrak{e}_8$.

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