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Distributed Algorithms in an Ergodic Markovian Environment

Francis Comets^{(a),*}, François Delarue^{(a),†} and René Schott^{(b),‡}

(a) Laboratoire de Probabilités et Modèles Aléatoires,
Université Paris 7, UFR de Mathématiques, Case 7012,
2, Place Jussieu, 75251 Paris Cedex 05 - FRANCE.

(b) IECN and LORIA, Université Henri Poincaré-Nancy 1,
54506 Vandoeuvre-lès-Nancy - FRANCE.

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Abstract

We provide a probabilistic analysis of the d -dimensional banker algorithm when transition probabilities may depend on time and space. The transition probabilities evolve, as time goes by, along the trajectory of an ergodic Markovian environment, whereas the spatial parameter just acts on long runs. Our model complements the one considered by Guillin-Plantard and Schott [10] where transitions are governed by a dynamical system, and appears as a new (small) step towards more general time and space dependent protocols.

Our analysis relies on well-known results in stochastic homogenization theory and investigates the asymptotic behaviour of the rescaled algorithm as the total amount of resource available for allocation tends to the infinity. In the two dimensional setting, we manage to exhibit three different possible regimes for the deadlock time of the limit system. To the best of our knowledge, the way we distinguish these regimes is completely new.

Keywords: distributed algorithms; random environment; stochastic homogenization; reflected diffusion.

AMS 2000 subject classifications. Primary: 68W15, 60K37. Secondary: 60F17, 60J10, 60H10.

1 Introduction

Many real-world phenomena involve time and space dependency. Think about option pricing: the behavior of traders is not the same when stock markets are opening and a few minutes before closure. Multi-agents problems, dam management problems are typical examples where (often random) decisions have to be made under time and space constraints.

In computer science such problems are usually called resource sharing problems. Their statement is as follows (see Maier [20], Maier and Schott [21]):

*comets@math.jussieu.fr; <http://www.proba.jussieu.fr/pageperso/comets/comets.html>

†delarue@math.jussieu.fr; <http://www.math.jussieu.fr/~delarue>

‡schott@loria.fr; <http://www.loria.fr/~schott>

Consider the interaction of q independent processes P_1, \dots, P_q , each with its own memory needs. The processes are allowed to allocate and deallocate r different, non-substitutable resources (types of memory): R_1, \dots, R_r . Resource limitations and resource exhaustions are defined as follows. At any time s , process P_i is assumed to have allocated some quantity $y_i^j(s)$ of resource R_j (both time and resource usage are taken to be discrete, so that $s \in \mathbb{N}$ and $y_i^j(s) \in \mathbb{N}$). Process P_i is assumed to have some maximum need m_{ij} of resource R_j , so that

$$0 \leq y_i^j(s) \leq m_{ij} \quad (1.1)$$

for all s . The numbers m_{ij} may be infinite; if finite, it is a hard limit that the process P_i never attempts to exceed. The resources R_j are limited, so that

$$\sum_{i=1}^q y_i^j(s) < \Lambda_j \quad (1.2)$$

for $\Lambda_j - 1$ the total amount of resource R_j available for allocation. Resource exhaustion occurs when some process P_i issues an unfulfillable request for a quantity of some resource R_j . Here ‘unfulfillable’ means that fulfilling the request would violate one of the inequalities (1.2).

The state space Q of the memory allocation system is the subset of \mathbb{N}^{qr} determined by (1.1) and (1.2). This polyhedral state space is familiar: it is used in the banker algorithm for deadlock avoidance. Most treatments of deadlocks (see Habermann [11]) assume that processes request and release resources in a mechanical way: a process P_i requests increasing amounts of each resource R_j until the corresponding goal m_{ij} is reached, then releases resource units until $y_i^j = 0$, and repeats (the r different goals of the process need not be reached simultaneously, of course). This is a powerful assumption: it facilitates a classification of system states into ‘safe’ and ‘unsafe’ states, the latter being those which can lead to deadlock. However it is an idealization. Assume that regardless of the system state, each process P_i with $0 < y_i^j < m_{ij}$ can issue either an allocation or deallocation request for resource R_j . The probabilities of the different sorts of request may depend on the current state vector (y_i^j) . In other words the state of the storage allocation system is taken as a function of time to be a finite-state Markov chain; this is an alternative approach which goes back at least as far as Ellis [5].

The goal is to estimate the amount of time τ until memory exhaustion occurs, if initially the r types of resources are completely unallocated: $y_i^j = 0$ for all i, j . The consequences of expanding the resource limits $\Lambda_j - 1$ and the per-process maximum needs m_{ij} (if finite) on the expected time to exhaustion are particularly interesting for practical applications. There has been little work on the exhaustion of *shared* memory, or on ‘multidimensional’ exhaustion, where one of a number of inequivalent resources becomes exhausted. D. E. Knuth [15], A. Yao [32], P. Flajolet [6], G. Louchard and R. Schott [18], have provided combinatorial or probabilistic analysis of some resource sharing problems under the assumption that transition probabilities are constant. Maier provided a large deviation analysis of colliding stacks for the more difficult case in which the transition probabilities are nontrivially state-dependent. More recently N. Guillotin-Plantard and R. Schott [10] analysed a model of exhaustion of shared resources where allocation and deallocation requests are modelled by time dependent dynamic random variables.

In this paper, we analyse such problems when the probability transitions are time and space dependent. We incorporate in the transitions of our model, the influence of an environment which

evolves randomly albeit in a stationary manner on the long run. Our analysis relies on well-known results in stochastic homogenization. We focus on the regime when memory exhaustion is caused by normal fluctuations (as in many of the above references), but not by large deviations (as in Maier [20]). This paper can be viewed as a (small) step towards the analysis of protocols where decisions are time and space dependent random variables. In addition to incorporating space dependence, we improve on the results of Guillotin-Plantard and Schott [10] in covering larger perturbations of time-homogeneous random walks.

The organization is as follows: Section 2 presents an example of resource sharing algorithm usually called banker algorithm. Precise statements of the main results are given in Section 3. Proofs are postponed to the three following parts of the paper: we investigate the asymptotic behaviour of the algorithm in Sections 4 and 5 and we discuss in Section 6 several different regimes for the limit process. As a conclusion, we indicate further conceivable extensions in Section 7.

2 Example: the Banker Algorithm

2.1 Practical Description of the Model in Dimension Two

We consider a simple distributed algorithm which involves only two customers C_1 and C_2 sharing a fixed quantity of a given resource M (money). There are fixed upper bounds m_1 and m_2 on how much of the resource each of the customers is allowed to use at any time. The banker decides to give to the customer C_i , $i = 1, 2$, the required resource units only if the remaining units are sufficient in order to fulfill the requirements of C_j , $j = 1, 2; j \neq i$. This situation is modelled (see Figure 1) by a random walk in a rectangle with a broken corner, i.e.,

$$\{(x_1, x_2) \in \mathbb{Z}^d, 0 \leq x_1 \leq m_1, 0 \leq x_2 \leq m_2, x_1 + x_2 \leq \Lambda\}$$

where the last constraint generates the broken corner. The random walk is reflected on the sides parallel to the axes and is absorbed on the sloping side.

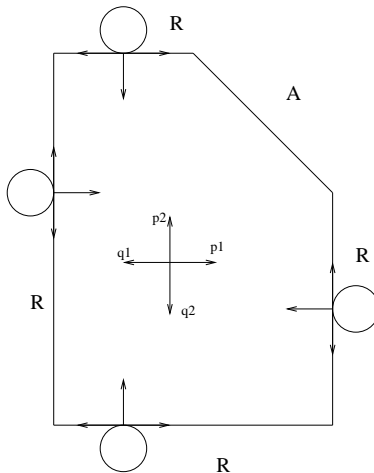


Figure 1: Banker algorithm

Again the hitting place and the hitting time of the absorbing boundary are the parameters of interest. This algorithm has been analysed by Louchard et al. [17, 18, 19] in the model where the transitions are constant and by Guillotin-Plantard and Schott [10] when the transitions are some time dependent dynamic random variables.

Our goal here is to analyse such distributed algorithms in an ergodic markovian environment where the transition probabilities are time and space dependent, in a sense that we specify below. The environment takes care of purely unpredictable behaviors in addition to periodic and quasi-periodic ones.

For the sake of simplicity, we assume from now on that $m_1 = m_2 = m$ and that Λ writes $\Lambda = \lambda m$, with $\lambda \in [1, 2)$. The size of the problem is then given by the parameter m , which therefore plays a crucial role in the time scale. For large m 's, we will prove approximations for the deadlock time by taking advantage of averaging properties of the environment.

2.2 Probabilistic Modelization

We generalize the situation described in Subsection 2.1 and assume that there are d customers instead of two. We then modelize the resources of these customers by a Markov chain with values in the broken corner $\{(x_1, \dots, x_d) \in \mathbb{Z}^d : 0 \leq x_i \leq m, 1 \leq i \leq d, x_1 + \dots + x_d \leq \lambda m\}$, $m \in \mathbb{N}^*$ and $\lambda \in [1, d)$.

Walk without Boundary Conditions.

Define first the transitions of the chain without taking care of the boundary conditions. Recall to this end that the transition matrix of a time-space homogeneous random walk to the nearest neighbours in \mathbb{Z}^d , $d \geq 1$, reduces to a probability $p(\cdot)$ on the $2d$ directions of the discrete grid,

$$\mathcal{V} \equiv \{e_1, -e_1, \dots, e_d, -e_d\},$$

where $(e_i)_{1 \leq i \leq d}$ denotes the canonical basis of \mathbb{R}^d . For given $i \in \{1, \dots, d\}$ and $u \in \mathcal{V}$, $p(u)$ simply denotes the probability of going from the current position x to $x + u$.

Assume for a moment that the transition matrices are space homogeneous but depend on time through some environment (evolving with time) with values in a finite space E , $N \equiv |E|$. We then need to consider, not a single transition probability, but a family $p(1, \cdot), \dots, p(N, \cdot)$ of N probabilities on \mathcal{V} . If the environment at time n is in the state $i \in E$, then the transition of the Markov chain at that time is governed by the probability $p(i, \cdot)$.

When the environment is given by a stochastic process $(\xi_n)_{n \geq 0}$ on E , the jumps $(J_n)_{n \geq 0}$ of the random walk are such that, for every $u \in \mathcal{V}$:

$$\mathbb{P}\{J_{n+1} = u | \mathcal{F}_n^{\xi, J}\} = p(\xi_n, u),$$

with $\mathcal{F}_n^{\xi, J} \equiv \sigma\{\xi_0, \dots, \xi_n, J_0, \dots, J_n\}$.

From now on, we assume that the environment $(\xi_n)_{n \geq 1}$ is a time-homogeneous Markov chain on E , and we denote by P its transition matrix, $P(k, \ell) \equiv \mathbb{P}(\xi_{n+1} = \ell | \xi_n = k)$ for $k, \ell \in E$. Then, the couple $(\xi_n, J_n)_{n \geq 1}$ is itself a time-homogeneous Markov chain on the product space $E \times \mathbb{Z}^d$ governed by the following transition:

$$\forall k \in E, \forall u \in \mathcal{V}, \mathbb{P}\{\xi_{n+1} = k, J_{n+1} = u | \mathcal{F}_n^{\xi, J}\} = P(\xi_n, k)p(\xi_n, u).$$

Define the position of the walker in \mathbb{Z}^d :

$$S_0 \equiv 0, \quad \forall n \geq 0, \quad S_{n+1} \equiv S_n + J_{n+1}. \quad (2.1)$$

In view of the applications mentioned above, the model is not fine enough. For this reason, we also assume that the steps $(J_n)_{n \geq 1}$ depend on the walker position in the following way: for all $k \in E$ and $u \in \mathcal{V}$,

$$\mathbb{P}\{\xi_{n+1} = k, J_{n+1} = u | \mathcal{F}_n^{\xi, J}\} = P(\xi_n, k)p(\xi_n, S_n/m, u), \quad (2.2)$$

where m denotes a large integer that refers to the size of the box in Subsection 2.1 and, for each $k \in E$ and $y \in \mathbb{R}^d$, $p(k, y, \cdot)$ a probability on \mathcal{V} . Note that the random walk $(S_n = S_n^{(m)})_{n \geq 0}$ depends on the parameter m . Nevertheless, for simplicity we will often forget the dependence on m in our notations. In other words, $(\xi_n, S_n)_{n \geq 0}$ defines a Markov chain with rates:

$$\forall k \in E, \quad \forall u \in \mathcal{V}, \quad \mathbb{P}\{\xi_{n+1} = k, S_{n+1} = u + S_n | \mathcal{F}_n^{\xi, S}\} = P(\xi_n, k)p(\xi_n, S_n/m, u),$$

where $\mathcal{F}_n^{\xi, S} = \sigma\{\xi_0, \dots, \xi_n, S_0, \dots, S_n\}$.

Walk with Reflection Conditions.

Our original problem with reflection on the hyperplanes $y_i = 0$, $i \in \{1, \dots, d\}$ and $y_i = m$, $i \in \{1, \dots, d\}$ follows from a slight correction of the former one. The underlying reflected walk $(R_n)_{n \geq 0}$ (also denoted by $(R_n^{(m)})_{n \geq 0}$ to specify the dependence on m) satisfies, with $\mathcal{F}_n^{\xi, R} \equiv \sigma\{\xi_0, \dots, \xi_n, R_0, \dots, R_n\}$:

$$\forall k \in E, \quad \forall u \in \mathcal{V}, \quad \mathbb{P}\{\xi_{n+1} = k, R_{n+1} = u + R_n | \mathcal{F}_n^{\xi, R}\} = P(\xi_n, k)q(\xi_n, R_n/m, u),$$

where q denotes the kernel:

$$\forall k \in E, \quad \forall y \in [0, 1]^d, \quad \forall \ell \in \{1, \dots, d\}, \quad q(k, y, \pm e_\ell) = p(k, y, \pm e_\ell) \text{ if } 0 < y_\ell < 1.$$

If $y_\ell = 1$, then $q(k, y, e_\ell) = 0$ and $q(k, y, -e_\ell) = p(k, y, e_\ell) + p(k, y, -e_\ell)$. If $y_\ell = 0$, then $q(k, y, -e_\ell) = 0$ and $q(k, y, e_\ell) = p(k, y, e_\ell) + p(k, y, -e_\ell)$.

The deadlock time of the banker algorithm is then given by $T^{(m)} \equiv \inf\{n \geq 0, \sum_{\ell=1}^d \langle R_n, e_\ell \rangle \geq \Lambda\}$. This also writes: $T^{(m)} = \inf\{n \geq 0, R_n \in mF_0\}$, with $F_0 = \{y \in [0, 2]^d, \sum_{i \leq d} |y_i - 1| \leq (d - \lambda)\}$ (see Figure 2 below). This latter form will be useful in the proof of the main results.

2.3 Main Assumptions

In formula (2.2), the division by m indicates that the dependence of the transition kernel on the position of the walker takes place at scale m . For large m , the space dependence is mild, since we will assume all through the paper the following smoothness property:

Assumption (A.1). *The function p is twice continuously differentiable with respect to y with bounded derivatives. In particular, there exists a constant $K > 0$ such that:*

$$\forall x \in E, \quad u \in \mathcal{V}, \quad (y, y') \in (\mathbb{R}^d)^2, \quad |p(x, y, u) - p(x, y', u)| \leq K|y - y'|.$$

It is then readily seen that the transition kernel (2.2) weakly depends on the space position of the walker: a step of the walker modifies only slightly the transition kernel.

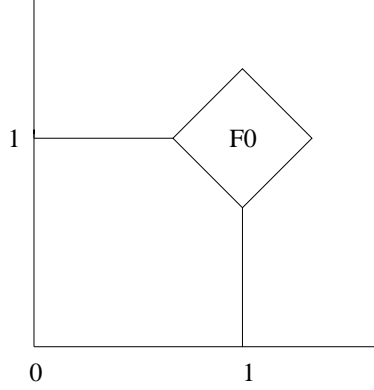


Figure 2: Absorption set.

We also assume the environment to be ergodic and to fulfill the so-called central limit theorem for Markov chains. We thus impose the following sufficient conditions:

Assumption (A.2). *The matrix P is irreducible on E . It is then well known that P admits a unique invariant probability measure, denoted by μ .*

Assumption (A.3) *The matrix P is aperiodic. In particular, it satisfies the Doeblin condition:*

$$\exists m \geq 0, \exists \eta > 0, \forall (k, \ell) \in E^2, (P^m)(k, \ell) \geq \eta.$$

Assumption (A.4) *Define for $k \in E$ and $y \in \mathbb{R}^d$, $g(k, y) \equiv \sum_{u \in \mathcal{V}} [p(k, y, u)u]$ (i.e., $g(k, y)$ matches the expectation of the measure $p(k, y, \cdot)$) and assume that $g(\cdot, y)$, seen as a function from E to \mathbb{R}^d , is centered with respect to the measure μ .*

Let us briefly describe the role of each of these conditions:

1. Thanks to Assumption **(A.2)**, the Markov chain $(\xi_n)_{n \geq 0}$ satisfies the ergodic theorem for Markov processes.
2. The general central limit theorem for Markov chains with finite state space follows from the Doeblin condition, given in Assumption **(A.3)**.
3. Assumption **(A.4)** permits to apply the previous central limit theorem to the function g .

2.4 Frequently Used Notations

For a square integrable martingale M , $\langle M \rangle$ denotes the bracket of M (do not mix up with the Euclidean scalar product, which is denoted by $\langle x, y \rangle$ for $x, y \in \mathbb{R}^d$).

We also denote by $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ the path space of right continuous and left limited functions from \mathbb{R}_+ into \mathbb{R}^d . A function in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is then said to be “càd-làg” for the French acronym “continue à droite-limite à gauche”.

3 Main Results

3.1 Asymptotic Behaviour of the Walk without Reflection

In light of Assumptions **(A.1-4)**, we expect the global effect of the environment process $(\xi_n)_{n \geq 0}$ to reduce for large time to a deterministic one. To this end, we view the process $(\bar{S}_t^{(m)} \equiv m^{-1} S_{[m^2 t]}^{(m)})_{t \geq 0}$ as a random element in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$:

Theorem 3.1 *The process $(\bar{S}_t^{(m)})_{t \geq 0}$ converges in law in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ endowed with the Skorokhod topology towards the (unique) solution of the martingale problem starting from the origin at time zero and associated to the operator:*

$$\mathcal{L} \equiv \frac{1}{2} \sum_{i,j=1}^d \bar{a}_{i,j}(y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d \bar{b}_i(y) \frac{\partial}{\partial y_i}.$$

The limit coefficients \bar{a} and \bar{b} are given by:

$$\begin{aligned} \forall y \in \mathbb{R}^d, \bar{a}(y) &\equiv \int_E [\alpha + gv^t + vg^t - 2gg^t](i, y) d\mu(i), \\ \bar{b}(y) &\equiv \int_E [(\nabla_y v - \nabla_y g)g](i, y) d\mu(i), \end{aligned}$$

where ∇ stands for the gradient and $\alpha(i, y)$ denotes the second order moment matrix of the measure $p(i, y, \cdot)$:

$$\forall i \in E, \forall y \in \mathbb{R}^d, \alpha(i, y) \equiv \sum_{u \in \mathcal{V}} [uu^t p(i, y, u)],$$

and $v(i, y)$ denotes the well-defined sum:

$$\forall i \in E, \forall y \in \mathbb{R}^d, v(i, y) \equiv \sum_{n \geq 0} \sum_{j \in E} [P^n(i, j)g(j, y)].$$

Moreover, for every $y \in \mathbb{R}^d$, the lowest eigenvalue of $\bar{a}(y)$ is greater or equal than the lowest eigenvalue of the matrix:

$$\int_E [\alpha(i, y) - gg^t(i, y)] d\mu(i).$$

In particular, if the covariance matrices of the measures $(p(i, y, \cdot))_{i \in E, y \in \mathbb{R}^d}$ are uniformly elliptic, the matrices $(\bar{a}(y))_{y \in \mathbb{R}^d}$ are also uniformly elliptic.

The existence of v , $\nabla_y v$ and $\nabla_y g$ will be detailed in the sequel of the paper.

We can denote by $\bar{\sigma}(x)$, for $x \in \mathbb{R}^d$, the nonnegative symmetric square root of $\bar{a}(x)$. Then, the SDE (stochastic differential equation) associated to \mathcal{L} writes:

$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)dB_t, \tag{3.1}$$

where $(B_t)_{t \geq 0}$ denotes a d -dimensional Brownian motion on a given filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Under Assumptions **(A.1-4)**, (3.1) is uniquely strongly solvable.

Of course, if $p(k, y, \cdot)$ does not depend on y (as in the beginning of Subsection 2.2), then the drift

\bar{b} vanishes in Theorem 3.1 since $b = 0$ and the matrix-valued function \bar{a} is constant so that \bar{X} is a non-standard Brownian motion.

Theorem 3.1 appears actually as an homogenization property: on the long run, the time inhomogeneous rescaled walk $\bar{S}^{(m)}$ behaves like an homogeneous diffusion. Underlying techniques to establish the asymptotic behaviour of $(\bar{S}_t^{(m)})_{t \geq 0}$ are well known in the literature devoted to this subject (see e.g. Bensoussan et al. [1] for a review on homogenization in periodic structures or Jikov et al. [13] for a monograph on stochastic homogenization). We will detail a few of them in the sequel of the paper.

We finally mention that Theorem 3.1 is very close to Theorem 3.1 in Guillotin-Plantard and Schott [10]. Indeed, any dynamical (E, \mathcal{A}, μ, T) , as considered by the previous authors, generates an homogeneous Markov chain with degenerate transitions: $\forall k \in E, P(k, Tk) = 1$ (but the state space E is then very large). In this framework, the condition $\int_E f_j d\mu = (2d)^{-1}$ in Theorem 3.1 in [10] implies that the expectation against μ of the transitions $p(T^i k, e_j) = f_j(T^i x)$ and $p(T_k^i, -e_j) = 1/d - f_j(T^i x)$ given in Subsection 3.1 in [10] vanishes, and thus amounts in some sense to Assumption **(A.4)** in our paper. In the same way, condition (H) in Theorem 3.1 in [10] refers more or less to a “degenerate” central limit theorem for the underlying dynamical system and thus to our standing Assumption **(A.3)**.

However, our own Theorem 3.1 does not recover exactly the result in Guillotin-Plantard and Schott [10]: due to the degeneracy of the transitions of a dynamical system, condition **(A.3)** cannot be satisfied. Actually, the reader must understand that condition **(A.3)** is a simple but very strong technical condition to ensure the validity of the so-called “central limit theorem” for Markov chains. It thus permits to draw up a clear framework in which the stochastic homogenization theory applies, but is obviously far from being optimal (refer to Olla [22] for a complete overview on central limit theorems for Markov chains).

Theorem 3.1 is proved in Section 4.

3.2 Asymptotic Behaviour of the Reflected Walk

The rescaled walk $(\bar{R}_t^{(m)} \equiv m^{-1} R_{\lfloor m^2 t \rfloor}^{(m)})_{t \geq 0}$ satisfies the following “reflected version” of Theorem 3.1:

Theorem 3.2 *The process $(\bar{R}_t^{(m)})_{t \geq 0}$ converges in law in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ endowed with the Skorokhod topology towards the (unique) solution of the martingale problem with normal reflection on the boundary of the hypercube $[0, 1]^d$, with zero as initial condition and with \mathcal{L} as underlying operator.*

The reflected SDE associated to the hypercube $[0, 1]^d$ and to the operator \mathcal{L} writes:

$$dX_t = \bar{b}(X_t)dt + dH_t - dK_t + \bar{\sigma}(X_t)dB_t. \quad (3.2)$$

We explain now the meaning of the different terms in the r.h.s of (3.2). The quantity $\bar{b}(X_t)dt$ refers to the drift $\bar{b}(\bar{X}_t)dt$ in (3.1) and $\bar{\sigma}(X_t)dB_t$ to the stochastic noise $\bar{\sigma}(\bar{X}_t)dB_t$ in (3.1). The new terms dH and dK stand for the differential elements of two continuous adapted processes with bounded variation that prevent X to leave the hypercube $[0, 1]^d$. More precisely, H^1 (the first coordinate of H) is an increasing process that pushes (when necessary) the process X^1 to the right to keep it above zero. In the same way, K^1 is an increasing process that pushes (again when necessary) the process X^1 to the left to keep it below one. For each $\ell \in \{2, \dots, d\}$, H^ℓ and K^ℓ act similarly in

the direction e_ℓ . Both processes H and K act in a minimal way. In particular, there is no reason to push the process X^ℓ when away from the boundary.

The least action principle for H and K can be summarized as follows. The process H^ℓ , for $\ell \in \{1, \dots, d\}$, does not increase when X^ℓ is different from zero, and the process K^ℓ , for $\ell \in \{1, \dots, d\}$, does not increase when X^ℓ is different from one, *i.e.*,

$$\int_0^{+\infty} \mathbf{1}_{\{X_t^\ell > 0\}} dH_t^\ell = 0, \quad \int_0^{+\infty} \mathbf{1}_{\{X_t^\ell < 1\}} dK_t^\ell = 0. \quad (3.3)$$

For a complete review on reflected SDE's, we refer the reader to Tanaka [31], Lions and Sznitman [16] and Saisho [28].

The following corollary describes the asymptotic behaviour of the deadlock time and the deadlock point of the algorithm. We assume the matrix $\alpha - gg^t$ to be uniformly elliptic to ensure \bar{a} to be so. In short, the ellipticity of \bar{a} avoids any singular behaviour (up to a \mathbb{P} -null set) of the trajectories of the limit process X .

Corollary 3.3 *Assume that $\lambda \in [1, d)$ and that the matrix $\alpha - gg^t$ is uniformly elliptic. Denote by T the deadlock time of the limit process X : $T \equiv \inf\{t \geq 0, X_t \in F_0\}$, where F_0 is given in Subsection 2.2. Then, the sequence of rescaled deadlock times $m^{-2}T^{(m)}$ converges in law towards T . In the same way, the sequence of rescaled deadlock points $m^{-1}R_{T^{(m)}}$ converges in law towards X_T .*

Theorem 3.2 and Corollary 3.3 are proved in Section 5.

3.3 Behaviour of the Limit Process in Dimension Two

A crucial question for numerical applications consists in estimating precisely the mean value of the deadlock time of the limit system. When the matrix \bar{a} is constant and diagonal and the drift \bar{b} reduces to zero, several explicit computations in terms of Bessel functions are conceivable (see again [10]).

In our more general framework, the story is quite different. In Delarue [4], we manage to establish in the two-dimensional case (*i.e.* $d = 2$) relevant estimates of the expectation of T (now denoted by $T_\lambda \equiv \inf\{t \geq 0, X_t^1 + X_t^2 \geq \lambda\}$ to take into account the parameter λ), and in particular to distinguish three different asymptotic regimes as the parameter λ tends to two, each of these regimes depending on the covariance matrix $\bar{a}(1, 1)$, and more precisely, on the sign of its off-diagonal components.

Since the matrix $\alpha - gg^t$ is assumed to be uniformly elliptic, the matrix $\bar{a}(1, 1)$ is positive and writes:

$$\bar{a}(1, 1) = \begin{pmatrix} \rho_1^2 & s\rho_1\rho_2 \\ s\rho_1\rho_2 & \rho_2^2 \end{pmatrix}, \quad (3.4)$$

with $\rho_1, \rho_2 > 0$ and $s \in]-1, 1[$. The matrix $\bar{a}(1, 1)$ admits two eigenvalues:

$$\lambda_1 = \frac{1}{2}[\rho_1^2 + \rho_2^2 + \delta], \quad \lambda_2 = \frac{1}{2}[\rho_1^2 + \rho_2^2 - \delta], \quad \text{where } \delta \equiv (\rho_1^4 + \rho_2^4 - 2(1 - 2s^2)\rho_1^2\rho_2^2)^{1/2}. \quad (3.5)$$

Denote by E_1 and E_2 the associated eigenvectors (up to a multiplicative constant). For $s \neq 0$,

$$E_1 = \begin{pmatrix} 1 \\ (2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2) \end{pmatrix}, \quad E_2 = \begin{pmatrix} -(2s\rho_1\rho_2)^{-1}(\delta + \rho_2^2 - \rho_1^2) \\ 1 \end{pmatrix}.$$

Since $\delta + \rho_2^2 - \rho_1^2 \geq 0$, the signs of the non-trivial coordinates of E_1 and E_2 are given by the sign of s . The main eigenvector (*i.e.* E_1) has two positive components for $s > 0$, and a positive one and a negative one for $s < 0$. Of course, if s vanishes, E_1 and E_2 reduce to the vectors of the canonical basis.

The three different regimes can be distinguished as follows:

Positive Case.

If $s > 0$, the main eigenvector of $\bar{a}(1, 1)$ (*i.e.* E_1) is globally oriented from 0 to the neighbourhood of the corner $(1, 1)$ and tends to push the limit reflected diffusion towards the border line. The reflection on the boundary cancels most of the effects of the second eigenvalue and keeps on bringing back the diffusion along the main axis. As a consequence, the hitting time of the border line is rather small and the following asymptotic holds for the diffusion starting from 0:

$$\sup_{1 < \lambda < 2} \mathbb{E}(T_\lambda) < +\infty.$$

This phenomenon is illustrated below (see Figure 3) when \bar{b} reduces to 0 and \bar{a} is the constant matrix given by $\rho_1 = \rho_2 = 1$ and $s = 0,9$. We have plotted there a simulated trajectory of the reflected diffusion process, starting from 0 at time 0, and running from time 0 to time 10 in the box $[0, 1]^2$. The algorithm used to simulate the reflected process is given in Slomiński [29]. The eigenvector E_1 exactly matches $(1, 1)^t$.

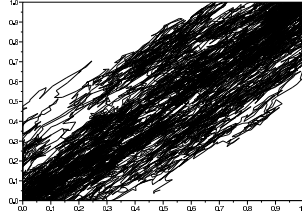


Figure 3: One trajectory of the reflected process over $[0, 10]$ for $s = 0,9$.

Negative Case.

If $s < 0$, the main eigenvector of $\bar{a}(1, 1)$ is globally oriented from $(1, 0)$ to the neighbourhood of the corner $(0, 1)$ and attracts the diffusion away from the border line. Again, the reflection on the boundary cancels most of the effects of the second eigenvalue, and thus, acts now as a trap: the diffusion stays for a long time along the main axis and hardly touches the boundary. The hitting time satisfies the following asymptotic behaviour when the diffusion starts from 0:

$$\exists c_1, c_2 \geq 1, \quad \forall \lambda \in]1, 2[, \quad c_1^{-1}(2 - \lambda)^{-c_2} - c_1 \leq \mathbb{E}(T_\lambda) \leq c_1(2 - \lambda)^{-c_2^{-1}} + c_1.$$

This point is illustrated by Figure 4 below when \bar{b} vanishes and \bar{a} reduces to the constant matrix $\rho_1 = \rho_2 = 1$ and $s = -0,9$ (again, the initial condition of the process is 0). The eigenvector E_1 is given, in this case, by $(1, -1)^t$.

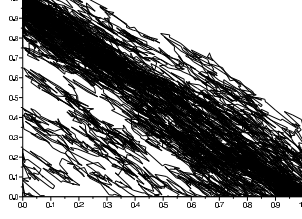


Figure 4: One trajectory over $[0, 10]$ for $s = -0,9$ (the trajectory is stopped before T).

Null Case.

The case $s = 0$ is intermediate. Eigenvectors are parallel to the axes and the behaviour of the diffusion is close to the behaviour of the two-dimensional Brownian motion. For the initial condition 0:

$$\exists c_1 \geq 1, \forall \lambda \in]1, 2[, -c_1^{-1} \ln(2 - \lambda) - c_1 \leq \mathbb{E}(T_\lambda) \leq -c_1 \ln(2 - \lambda) + c_1.$$

This is illustrated by Figure 5 below when \bar{b} vanishes and \bar{a} reduces to the identity matrix (the initial condition of the process is 0).

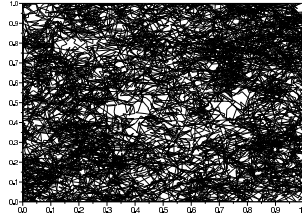


Figure 5: One Trajectory over $[0, 10]$ for $s = 0$.

The following theorem sums up these different cases (we refer to Delarue [4] for the whole proof, and just provide a sketch of it in Section 6):

Theorem 3.4 *Assume that $\alpha - gg^t$ is uniformly elliptic. Then, there exists a constant $C_{3.4} \geq 1$, depending only on known parameters and on the ellipticity constant of \bar{a} , such that:*

1. For $s > 0$, $\sup_{\ell \in]1, 2[} \mathbb{E}(T_\ell) \leq C_{3.4}$

2. For $s < 0$, set $\beta_- \equiv -s > 0$, $\beta_+ \equiv s(s - 3)(1 + s)^{-1} > 0$. Then,

$$\forall \ell \in]1, 2[, C_{3.4}^{-1}(2 - \ell)^{-\beta_-} - C_{3.4} \leq \mathbb{E}(T_\ell) \leq C_{3.4}(2 - \ell)^{-\beta_+} + C_{3.4}.$$

3. If $s = 0$, $\forall \ell \in]1, 2[$, $-C_{3.4}^{-1} \ln(2 - \ell) - C_{3.4} \leq \mathbb{E}(T_\ell) \leq -C_{3.4} \ln(2 - \ell) + C_{3.4}$.

Note that Theorem 3.4 leaves open many questions. For example, we do not know how to compute, for $s < 0$, the exact value of the “true” exponent $\beta \equiv \inf\{c > 0, \sup_{\ell \in]1, 2[} [(2 - \ell)^c \mathbb{E}(T_\ell)] < +\infty\}$. We are even unable to precise the asymptotic behaviour of β as $s \rightarrow -1$ (note indeed that $\lim_{s \rightarrow -1} \beta_- = 1$, $\lim_{s \rightarrow -1} \beta_+ = +\infty$).

We have very few ideas about the extension of Theorem 3.4 to the upper dimensional cases. The only accessible case for us is $\bar{a}(1, \dots, 1) = I_d$, I_d denoting the identity matrix of size d : in this case, the analysis derives from the transience properties of the Brownian motion in dimension $d \geq 3$.

4 Proof of Theorem 3.1

The strategy of proof of Theorem 3.1 is well-known. We aim at writing the sequence $(\bar{S}^{(m)})_{m \geq 1}$ as a sequence of martingales, or at least of semimartingales with relevant drifts. The asymptotic behaviour of the martingale parts then follows from a classical central limit argument.

4.1 Semimartingale Expansion of the Rescaled Walk

(First Step. A First Martingale Form.)

Denote by $(\mathcal{F}_n)_{n \geq 0}$ the filtration $(\mathcal{F}_n^{\xi, J})_{n \geq 0}$ and write $(S_n)_{n \geq 0}$ as follows:

$$\begin{aligned} S_{n+1} &= S_n + J_{n+1} \\ &= S_n + \mathbb{E}(J_{n+1} | \mathcal{F}_n) + J_{n+1} - \mathbb{E}(J_{n+1} | \mathcal{F}_n). \end{aligned}$$

The increment $Y_{n+1}^0 \equiv J_{n+1} - \mathbb{E}(J_{n+1} | \mathcal{F}_n)$ appears as a martingale increment. Referring to (2.2) and to the definition of g given in Assumption **(A.4)**, the underlying conditional expectation writes

$$\mathbb{E}(J_{n+1} | \mathcal{F}_n) = g(\xi_n, S_n/m). \quad (4.1)$$

Thus, S_{n+1} now writes:

$$S_{n+1} = S_n + g(\xi_n, S_n/m) + Y_{n+1}^0. \quad (4.2)$$

We deduce the following expression for the rescaled walk:

$$\bar{S}_t^{(m)} = m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} g(\xi_k, S_k/m) + m^{-1} \sum_{k=1}^{\lfloor m^2 t \rfloor} Y_k^0. \quad (4.3)$$

Note at this early stage of the proof that condition **(A.4)** is necessary to establish Theorem 3.1. Assume indeed that g does not depend on S_n/m . Then, the first term in the above right hand side reduces to

$$m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} g(\xi_k). \quad (4.4)$$

The ergodic theorem for Markov chains then yields:

$$m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} g(\xi_k) \rightarrow t \int_E g d\mu \text{ as } m \rightarrow +\infty.$$

In particular, if the expectation of g with respect to μ doesn't vanish, the term (4.4) explodes with m . The question is then to study the asymptotic behaviour of (4.4) under the centering condition **(A.4)**, or in other words to establish a central limit theorem for the sequence $(g(\xi_n))_{n \geq 0}$.

(Second Step. Auxiliary Problems.)

The basic strategy consists in solving the system of Poisson equations driven by the generator of the chain $(\xi_n)_{n \geq 0}$ and by the \mathbb{R}^d valued drift g . In our framework, this system extends to a family of systems since g does depend on the extra parameter $y \in \mathbb{R}^d$. We thus investigate:

$$(I - P)v = g(\cdot, y), \quad y \in \mathbb{R}^d. \quad (4.5)$$

For $y \in \mathbb{R}^d$, the solvability of this equation is ensured by the Doeblin condition **(A.3)**. Referring to Remark 2, Theorem 4.3.18 in Dacunha-Castelle and Duflo [3], we derive from **(A.3)** that there exist two constants Γ and c , $0 < c < 1$, depending only on known parameters appearing in Assumptions **(A.1-4)**, such that:

$$\forall n \geq 0, \forall i \in E, \sum_{j \in E} |P^n(i, j) - \mu(j)| \leq \Gamma c^n. \quad (4.6)$$

We then claim (see e.g. Theorem 4.3.18 in Dacunha-Castelle and Duflo [3]):

Proposition 4.1 *Under **(A.1-4)**, the equation (4.5) is solvable for every $y \in \mathbb{R}^d$. Moreover, there is a unique centered solution with respect to the measure μ . It is given by:*

$$\forall i \in E, v(i, y) = \sum_{n \geq 0} \sum_{j \in E} [P^n(i, j) g(j, y)].$$

Note carefully that $v(i, y)$ belongs to \mathbb{R}^d . Note also from **(A.4)** and (4.6) that $v(i, y)$ is well defined since

$$v(i, y) = \sum_{n \geq 0} \sum_{j \in E} [P^n(i, j) - \mu(j)] g(j, y).$$

The question of the regularity of g and v turns out to be crucial in the statement of Theorem 3.1. The following Lemmas directly follow from Assumption **(A.1)** and (4.6):

Lemma 4.2 *The function $g(i, y)$ is bounded by 1 and is twice continuously differentiable with respect to y with bounded derivatives. In particular, g is Lipschitz continuous in y , uniformly in i . The Lipschitz constant is denoted by $C_{4.2}$.*

Lemma 4.3 *The function $v(i, y)$ is bounded by $\Gamma(1 - c)^{-1}$ and is twice continuously differentiable in y with bounded derivatives. In particular, v is Lipschitz continuous in y , uniformly in i . The Lipschitz constant is denoted by $C_{4.3}$.*

(Third Step. Final Semimartingale Form.)

We claim:

Proposition 4.4 *Define for $(i, y) \in E \times \mathbb{R}^d$:*

$$b^{(m)}(i, y) \equiv \sum_{u \in \mathcal{V}} [p(i, y, u) ((v - g)(i, y + u/m) - (v - g)(i, y))].$$

and denote by $(Y_n)_{n \geq 1}$ the sequence of corrected martingale increments:

$$Y_n \equiv J_n + v(\xi_n, S_n/m) - \mathbb{E}[J_n + v(\xi_n, S_n/m) | \mathcal{F}_{n-1}].$$

Then, the walk $(S_n)_{n \geq 0}$ writes:

$$S_{n+1} + v(\xi_{n+1}, S_{n+1}/m) = S_n + v(\xi_n, S_n/m) + b^{(m)}(\xi_n, S_n/m) + Y_{n+1}. \quad (4.7)$$

In particular, the process $(S_n + v(\xi_n, S_n/m))_{n \geq 0}$ is a semimartingale. The term $b^{(m)}$ appears as a drift increment. It satisfies: $\forall m \geq 1, i \in E, y \in \mathbb{R}^d$, $|b^{(m)}(i, y)| \leq C_{4.4}/m$, where $C_{4.4}$ depends only on known parameters in **(A.1-4)**.

Proof. Apply v to the couple $(\xi_{n+1}, S_{n+1}/m)$ for a given $n \geq 0$. In this perspective, note that:

$$\begin{aligned} \mathbb{E}(v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n) &= \sum_{i \in E, u \in \mathcal{V}} [P(\xi_n, i) p(\xi_n, S_n/m, u) v(i, (S_n + u)/m)] \\ &= \sum_{u \in \mathcal{V}} \left[p(\xi_n, S_n/m, u) \left(\sum_{i \in E} P(\xi_n, i) v(i, (S_n + u)/m) \right) \right]. \end{aligned}$$

According to Proposition 4.1, we have:

$$\sum_{i \in E} [P(\xi_n, i) v(i, (S_n + u)/m)] = (v - g)(\xi_n, (S_n + u)/m),$$

yielding:

$$\begin{aligned} &\mathbb{E}(v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n) \\ &= (v - g)(\xi_n, S_n/m) + \sum_{u \in \mathcal{V}} \left[p(\xi_n, S_n/m, u) \left((v - g)(\xi_n, (S_n + u)/m) - (v - g)(\xi_n, S_n/m) \right) \right]. \end{aligned}$$

Due to the definition of $b^{(m)}$, this writes in an equivalent form:

$$g(\xi_n, S_n/m) = v(\xi_n, S_n/m) - \mathbb{E}(v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n) + b^{(m)}(\xi_n, S_n/m). \quad (4.8)$$

From (4.2) and (4.8), derive that:

$$S_{n+1} = S_n + J_{n+1} + v(\xi_n, S_n/m) - \mathbb{E}(J_{n+1} + v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n) + b^{(m)}(\xi_n, S_n/m).$$

Thanks to the definition of $(Y_n)_{n \geq 1}$, we recover the required writing. The bound for $b^{(m)}$ derives from Lemmas 4.2 and 4.3 (Lipschitz continuity of g and v). \square

Investigate the bracket of the martingale part of $(S_n + v(\xi_n, S_n/m))_{n \geq 0}$:

Proposition 4.5 Denote by $(M_n)_{n \geq 0}$ the square integrable martingale: $M_n \equiv \sum_{k=1}^n Y_k$. Then, the bracket of $(M_n)_{n \geq 1}$ is given by:

$$\langle M \rangle_n = \sum_{k=1}^n a^{(m)}(\xi_{k-1}, S_{k-1}/m),$$

with (see the statement of Theorem 3.1 for the definition of α):

$$\begin{aligned} a^{(m)}(\xi_n, S_n/m) &\equiv [\alpha - (v + b^{(m)})(v + b^{(m)})^t](\xi_n, S_n/m) \\ &+ \sum_{u \in \mathcal{V}} \left[p(\xi_n, S_n/m, u) [P(vv^t) + u(Pv)^t + (Pv)u^t](\xi_n, (S_n + u)/m) \right]. \end{aligned} \quad (4.9)$$

Moreover, there exists a finite constant $C_{4.5}$, depending only on known parameters in **(A.1-4)**, such that: $\forall m \geq 1, i \in E, y \in \mathbb{R}^d, |a^{(m)}(i, y)| \leq C_{4.5}$.

The notation $(Pv)(i, y)$ in (4.9) (the analogue holds for $(P(vv^t))(i, y)$) stands for the d -dimensional vector $(Pv_1(i, y), \dots, Pv_d(i, y))^t$.

Proof. Recall that $|J_n|$ is almost surely bounded by 1 and that v is a bounded function (see Lemma 4.3). Hence, the variables $(Y_n)_{n \geq 1}$ are almost surely bounded and $(M_n)_{n \geq 1}$ is square-integrable. The bracket is given by:

$$\langle M \rangle_n = \sum_{k=1}^n \mathbb{V}[J_k + v(\xi_k, S_k/m) | \mathcal{F}_{k-1}],$$

where $\mathbb{V}[\cdot | \mathcal{F}_{k-1}]$ denotes the conditional covariance with respect to the σ -field \mathcal{F}_{k-1} . Due to (4.1) and (4.8), we have:

$$\mathbb{E}[J_{n+1} + v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n] = (v + b^{(m)})(\xi_n, S_n/m).$$

Since $\mathbb{V}[Z | \mathcal{F}_n] = \mathbb{E}[ZZ^t | \mathcal{F}_n] - \mathbb{E}[Z | \mathcal{F}_n]\mathbb{E}[Z^t | \mathcal{F}_n]$,

$$\begin{aligned} &\mathbb{V}[J_{n+1} + v(\xi_{n+1}, S_{n+1}/m) | \mathcal{F}_n] \\ &= \mathbb{E} \left[(J_{n+1} + v(\xi_{n+1}, S_{n+1}/m)) (J_{n+1} + v(\xi_{n+1}, S_{n+1}/m))^t | \mathcal{F}_n \right] \\ &\quad - [(v + b^{(m)})(v + b^{(m)})^t](\xi_n, S_n/m) \\ &= \sum_{i \in E} \sum_{u \in \mathcal{V}} \left[P(\xi_n, i) p(\xi_n, S_n/m, u) [u + v(i, (S_n + u)/m)] [u + v(i, (S_n + u)/m)]^t \right] \\ &\quad - [(v + b^{(m)})(v + b^{(m)})^t](\xi_n, S_n/m). \end{aligned}$$

Reducing the sum over $i \in E$:

$$\begin{aligned} \mathbb{V}[J_k + v(\xi_k, S_k/m) | \mathcal{F}_{k-1}] &= \sum_{u \in \mathcal{V}} [p(\xi_n, S_n/m, u) uu^t] \\ &\quad + \sum_{u \in \mathcal{V}} [p(\xi_n, S_n/m, u) u (Pv)^t(\xi_n, (S_n + u)/m)] \\ &\quad + \sum_{u \in \mathcal{V}} \left[p(\xi_n, S_n/m, u) (Pv)(\xi_n, (S_n + u)/m) u^t \right] \\ &\quad + \sum_{u \in \mathcal{V}} \left[p(\xi_n, S_n/m, u) [P(vv^t)](\xi_n, (S_n + u)/m) \right] \\ &\quad - [(v + b^{(m)})(v + b^{(m)})^t](\xi_n, S_n/m). \end{aligned}$$

This recovers the form of $a^{(m)}$. The bound for $a^{(m)}$ follows from Lemmas 4.2 and 4.3. \square

4.2 Scaling Procedure

(First Step. Semimartingale Form of the Rescaled Process.)

We deduce from Proposition 4.4 that for all $t \geq 0$:

$$\bar{S}_t^{(m)} + m^{-1}v(\xi_{\lfloor m^2 t \rfloor}, \bar{S}_t^{(m)}) = m^{-1}v(\xi_0, 0) + m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} b^{(m)}(\xi_k, S_k/m) + m^{-1} \sum_{k=1}^{\lfloor m^2 t \rfloor} Y_k.$$

Of course, if $\lfloor m^2 t \rfloor = 0$, the latter sums reduce to zero.

Since the function v is bounded (see Lemma 4.3), the supremum norm of the process $(m^{-1}v(\xi_{\lfloor m^2 t \rfloor}, \bar{S}_t^{(m)}))_{t \geq 0}$ tends almost surely towards 0 as m increases. In particular, $\bar{S}^{(m)}$ writes, up to a negligible term:

$$\forall t \geq 0, \bar{S}_t^{(m)} = m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} b^{(m)}(\xi_k, S_k/m) + m^{-1} \sum_{k=1}^{\lfloor m^2 t \rfloor} Y_k + O(1/m). \quad (4.10)$$

Set now for the sake of simplicity:

$$\bar{B}_t^{(m)} \equiv m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} b^{(m)}(\xi_k, S_k/m), \quad \bar{M}_t^{(m)} \equiv m^{-1} \sum_{k=1}^{\lfloor m^2 t \rfloor} Y_k. \quad (4.11)$$

(Second Step. Tightness Properties.)

Proposition 4.6 *The family $(\bar{S}^{(m)}, \bar{M}^{(m)}, \langle \bar{M}^{(m)} \rangle)_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d+d^2})$.*

Proof. We first establish the tightness of the processes $(\bar{B}^{(m)}, \bar{M}^{(m)})_{m \geq 1}$ in the space of càd-làg functions endowed with the Skorokhod topology. To this end, note from Proposition 4.4 (estimates for $(b^{(m)})_{m \geq 1}$) that for any $(s, t) \in (\mathbb{R}_+)^2$, $0 \leq s < t$:

$$\begin{aligned} |\bar{B}_t^{(m)} - \bar{B}_s^{(m)}| &\leq C m^{-2} (\lfloor m^2 t \rfloor - \lfloor m^2 s \rfloor) \\ &\leq C ((t - s) + m^{-2}). \end{aligned} \quad (4.12)$$

Therefore, the family $(\bar{B}^{(m)})_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ endowed with the Skorokhod topology: all the limit processes of the sequence have continuous trajectories (see e.g. Jacod and Shiryaev [12], Propositions VI 3.25 and VI 3.26).

We turn now to the process $\bar{M}^{(m)}$. It is a square integrable martingale with respect to the filtration $(\mathcal{F}_{\lfloor m^2 t \rfloor})_{t \geq 0}$. Moreover, from Proposition 4.5, the bracket of the martingale is given by:

$$\langle \bar{M}^{(m)} \rangle_t = m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} a^{(m)}(\xi_k, S_k/m). \quad (4.13)$$

Since the functions $(a^{(m)})_{m \geq 1}$ are uniformly bounded, the family $(\langle \bar{M}^{(m)} \rangle)_{m \geq 1}$ is also \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d})$, by the same argument as above.

Referring to Theorem VI 4.13 in Jacod and Shiryaev [12], we deduce that the family $(\bar{M}^{(m)})_{m \geq 1}$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. It is even \mathcal{C} -tight since the jumps of the martingale $\bar{M}^{(m)}$ are bounded in

absolute value by C/m .

Finally, we conclude that the family $(\bar{S}^{(m)}, \bar{M}^{(m)}, \langle \bar{M}^{(m)} \rangle)_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d+d^2})$ (see Corollary VI 3.33 in Jacod and Shiryaev [12]). \square

(Third Step. Extraction of a Converging Subsequence.)

Lemma 4.7 *Every limit (\bar{X}, \bar{M}, L) (in the sense of the weak convergence in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{2d+d^2})$) of the sequence of processes $(\bar{S}^{(m)}, \bar{M}^{(m)}, \langle \bar{M}^{(m)} \rangle)_{m \geq 1}$ is continuous. Moreover, \bar{M} is a square integrable martingale with respect to the right continuous filtration generated by the process (\bar{X}, \bar{M}, L) augmented with \mathbb{P} -null sets. Its bracket coincides with L .*

Proof. According to Proposition IX 1.17 in Jacod and Shiryaev [12], \bar{M} is a square integrable martingale. The same argument shows that $\bar{M}^2 - L$ is a martingale. It is readily seen from (4.13) that L is of bounded variation and satisfies $L_0 = 0$. Thus, $\langle \bar{M} \rangle = L$ (see Theorem I 4.2 in [12]). \square

We aim at expressing the process $\langle \bar{M} \rangle$. To this end, we have to study the asymptotic behaviour of $\langle \bar{M}^{(m)} \rangle$:

Lemma 4.8 *Referring to the definition of \bar{a} in the statement of Theorem 3.1:*

$$\sup_{0 \leq t \leq T} \left[\left| \langle \bar{M}^{(m)} \rangle_t - m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{a}(S_k/m) \right| \right] \xrightarrow{\mathbb{P}\text{-probability}} 0 \text{ as } m \rightarrow +\infty.$$

Proof. The function $a^{(m)}$ from (4.9) uniformly converges on $E \times \mathbb{R}^d$ towards the function a given by:

$$\begin{aligned} \forall i \in E, \forall y \in \mathbb{R}^d, a(i, y) &= \alpha(i, y) + [g(Pv)^t + (Pv)g^t + P(vv^t) - vv^t](i, y) \\ &= \alpha(i, y) + [gv^t + vg^t - 2gg^t + P(vv^t) - vv^t](i, y). \end{aligned} \quad (4.14)$$

This follows from the *a priori* estimates for $(b^{(m)})_{m \geq 1}$ (see Proposition 4.4) and from Lemma 4.3. Thus, $\langle \bar{M}^{(m)} \rangle$ behaves like:

$$\left(m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} a(\xi_k, S_k/m) \right)_{t \geq 0}. \quad (4.15)$$

Lemma 4.8 then follows from Proposition 4.9 below. \square

Proposition 4.9 *Let $f : E \times \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded and uniformly Lipschitz continuous with respect to the second variable. Then, we have for all $T \geq 0$:*

$$\sup_{0 \leq t \leq T} \left[m^{-2} \left| \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} f(\xi_k, S_k/m) - \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{f}(S_k/m) \right| \right] \xrightarrow{\mathbb{P}\text{-probability}} 0 \text{ as } m \rightarrow +\infty,$$

with $\forall y \in \mathbb{R}^d, \bar{f}(y) = \int_E f(i, y) d\mu(i)$.

The proof of Proposition 4.9 relies on the ergodic theorem for $(\xi_n)_{n \geq 0}$ and on the tightness of the family $(\bar{S}^{(m)})_{m \geq 1}$. It is postponed to the end of the section.

(Third Step. Identification of the Limit.)

Lemma 4.10 *Under the assumptions and notations of Lemmas 4.7 and 4.8 and of Proposition 4.9, the bracket of \bar{M} is given by $\forall t \geq 0$, $\langle \bar{M} \rangle_t = \int_0^t \bar{a}(\bar{X}_s) ds$, and the couple (\bar{X}, \bar{M}) satisfies:*

$$\forall t \geq 0, \bar{X}_t = \int_0^t \bar{b}(\bar{X}_s) ds + \bar{M}_t. \quad (4.16)$$

Proof. Focus first on the limit form of the bracket. Note to this end that for every $t \geq 0$:

$$m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{a}(S_k/m) = \int_0^t \bar{a}(\bar{S}_s^{(m)}) ds + O(1/m). \quad (4.17)$$

The convergence will then follow from the one of $(\bar{S}^{(m)})_{m \geq 0}$.

In fact, Proposition 4.9 applies to studying the asymptotic behaviour of the drift part. Derive indeed from Lemmas 4.2 and 4.3 and Proposition 4.4 that $mb^{(m)}$ uniformly converges on $E \times \mathbb{R}^d$ towards $(\nabla_y v - \nabla_y g)g$. Thus, the asymptotic behaviour of $\bar{B}^{(m)}$ in (4.11) just reduces to the one of:

$$m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} b(\xi_k, S_k/m), \quad b \equiv (\nabla_y v - \nabla_y g)g = \left(\sum_{j=1}^d \frac{\partial(v^i - g^i)}{\partial y_j} g_j \right)_{1 \leq i \leq d}.$$

According once again to Proposition 4.9, this just amounts to study:

$$\begin{aligned} m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{b}(S_k/m) &= \int_0^t \bar{b}(\bar{S}_s^{(m)}) ds + O(1/m), \\ \bar{b}(y) &\equiv \int_E b(i, y) d\mu(i) = \int_E [(\nabla_y v - \nabla_y g)g](i, y) d\mu(i). \end{aligned} \quad (4.18)$$

Note now from Lemma 4.7 and from the continuity of \bar{a} and \bar{b} (see Lemma 4.3 and the proof of Lemma 4.8) that the processes:

$$\left(\bar{S}^{(m)}, \bar{M}^{(m)}, \int_0^\cdot \bar{b}(\bar{S}_s^{(m)}) ds, \int_0^\cdot \bar{a}(\bar{S}_s^{(m)}) ds \right)_{m \geq 1} \quad (4.19)$$

converge in law in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{3d+d^2})$ towards $\left(\bar{X}, \bar{M}, \int_0^\cdot \bar{b}(\bar{X}_s) ds, \int_0^\cdot \bar{a}(\bar{X}_s) ds \right)$.

From Lemmas 4.7 and 4.8 and (4.17), the bracket of \bar{M} is given by: $\forall t \geq 0$, $\langle \bar{M} \rangle_t = \int_0^t \bar{a}(\bar{X}_s) ds$, and the relation (4.10) yields (4.16). \square

We are now in position to complete the proof of Theorem 3.1. Thanks to Lemmas 4.2 and 4.3, the coefficients \bar{a} and \bar{b} given in the statement of Theorem 3.1 are once differentiable with bounded derivatives. It is then well known that \bar{X} in Lemma 4.10 turns to be the unique solution (starting from the origin at time zero) to the martingale problem associated to (\bar{b}, \bar{a}) (see e.g. Chapter VI in Stroock and Varadhan [30]).

We now prove that the diffusion matrix \bar{a} is elliptic when $\alpha - gg^t$ is uniformly elliptic. Referring to the definition of \bar{a} (see Theorem 3.1), it is sufficient to prove that the symmetric matrix:

$$\int_E [vg^t + gv^t - gg^t](i, y) d\mu(i),$$

is non-negative. Recall that $g = v - Pv$. Thus:

$$\begin{aligned} vg^t + gv^t - gg^t &= vv^t - v(Pv)^t + vv^t - (Pv)v^t - vv^t - (Pv)(Pv)^t + v(Pv)^t + (Pv)v^t \\ &= vv^t - (Pv)(Pv)^t. \end{aligned} \quad (4.20)$$

We see from Schwarz inequality that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned} \int_E \langle x, (Pv)(Pv)^t x \rangle(i, y) d\mu(i) &= \int_E \langle x, Pv \rangle^2(i, y) d\mu(i) \\ &= \int_E \left(\sum_{j \in E} P(i, j) \langle x, v(j, y) \rangle \right)^2 d\mu(i) \\ &\leq \int_E \sum_{j \in E} P(i, j) \langle x, v(j, y) \rangle^2 d\mu(i) \\ &= \int_E \langle x, v(i, y) \rangle^2 d\mu(i) \\ &= \int_E \langle x, (vv^t)x \rangle(i, y) d\mu(i). \end{aligned} \quad (4.21)$$

Thanks to (4.20) and (4.21), we complete the proof of Theorem 3.1. \square

4.3 Proof of Proposition 4.9

The proof follows the strategy in Pardoux and Veretennikov [24]. For this reason, we just present a sketch of it.

Let T be an arbitrary positive real number. Recall that $(\bar{S}^{(m)})_{m \geq 1}$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Hence, for a given $\varepsilon > 0$, there exists a compact set $A \subset \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ such that, for every $m \geq 1$, $\bar{S}^{(m)}$ belongs to A up to the probability ε :

$$\mathbb{P}\{\bar{S}^{(m)} \in A\} \geq 1 - \varepsilon. \quad (4.22)$$

Since A is compact for the Skorokhod topology, we know, for δ small enough, that every function $\ell \in A$ satisfies:

$$w'_{T+1}(\ell, \delta) < \varepsilon, \quad (4.23)$$

where w'_{T+1} denotes the usual modulus of continuity on $[0, T+1]$ of a càd-làg function (see e.g. Jacod and Shiryaev, VI 1.8 and Theorem VI 1.14). Moreover, there exist $\ell_1, \dots, \ell_{N_0}$ in A such that, for every $\ell \in A$, we can find an integer $i \in \{1, \dots, N_0\}$ and a strictly increasing continuous function λ_i from $[0, T]$ into $[0, T+1/2]$ such that:

$$\sup_{t \in [0, T]} [|t - \lambda_i(t)|] < \delta^2, \quad \sup_{t \in [0, T]} [|\ell(t) - \ell_i(\lambda_i(t))|] < \delta^2. \quad (4.24)$$

Since ℓ_i satisfies (4.23), there exists a step function $y_i : [0, T+1/2] \rightarrow \mathbb{R}$, with all steps but maybe the last one of length greater than δ , such that the supremum norm between y_i and ℓ_i is less than ε on $[0, T+1/2]$. The second term in (4.24) then yields: $\sup_{t \in [0, T]} [|\ell(t) - y_i(\lambda_i(t))|] < \delta^2 + \varepsilon$.

Choose now $\omega \in \Omega$ such that $\bar{S}^{(m)}(\omega) \in A$. Deduce that there exists $i \in \{1, \dots, N_0\}$ such that the supremum on $[0, T]$ of the distance between $\bar{S}^{(m)}$ and $y_i \circ \lambda_i$ is less than $\varepsilon + \delta^2$. In particular:

$$\sup_{k \in [0, [m^2 T - 1]]} |m^{-1} S_k - y_i(\lambda_i(k/m^2))| \leq \sup_{t \in [0, T]} |\bar{S}_t^{(m)} - y_i(\lambda_i(t))| \leq \varepsilon + \delta^2.$$

Since f is Lipschitz continuous,

$$\sup_{0 \leq t \leq T} \left[m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} |f(\xi_k, S_k/m) - f(\xi_k, y_i(\lambda_i(k/m^2)))| \right] \leq CT(\varepsilon + \delta^2). \quad (4.25)$$

Define now by $t_0 = 0, t_1, \dots, t_p = T$ a subdivision of $[0, T]$ with respect to the steps of y_i . Recall that $t_j - t_{j-1} \geq \delta$ for $j \in \{1, \dots, p-1\}$. According to (4.24), for $t \in G = \cup_{k=1}^{p-1} [t_{k-1} + \delta^2, t_k - \delta^2] \cup [t_{p-1} + \delta^2 \wedge T, T]$, $\lambda_i(t)$ and t belong to the same class of the subdivision, so that $y_i(\lambda_i(t)) = y_i(t)$. Hence,

$$\begin{aligned} & \limsup_{m \rightarrow +\infty} \left[m^{-2} \sum_{k=0}^{\lfloor m^2 T \rfloor - 1} |f(\xi_k, y_i(\lambda_i(k/m^2))) - f(\xi_k, y_i(k/m^2))| \right] \\ & \leq C \limsup_{m \rightarrow +\infty} \left[m^{-2} \sum_{k=0}^{\lfloor m^2 T \rfloor - 1} \mathbf{1}_{[0, T] \setminus G}(k/m^2) \right] \\ & = C(T - \mu(G)) \leq 2C\delta^2 p \leq 2CT\delta. \end{aligned} \quad (4.26)$$

It follows from (4.22), (4.25) and (4.26) that there exists a constant C (not depending on δ, ε, m and N_0) such that for m large enough:

$$\mathbb{P} \left(\bigcup_{i \in \{1, \dots, N_0\}} \left\{ \sup_{0 \leq t \leq T} \left[m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} |f(\xi_k, S_k/m) - f(\xi_k, y_i(k/m^2))| \right] \geq C(\delta + \varepsilon) \right\} \right) \leq \varepsilon. \quad (4.27)$$

Note that the same estimate holds with $|f(\xi_k, S_k/m) - f(\xi_k, y_i(k/m^2))|$ replaced by $|f(\xi_k, S_k/m) - f(\xi_k, y_i(k/m^2))| + |\bar{f}(S_k/m) - \bar{f}(y_i(k/m^2))|$. Moreover, for all $i \in E$:

$$\begin{aligned} \sum_{k=1}^{\lfloor m^2 T \rfloor} f(\xi_k, y_i(k/m^2)) &= \sum_{j=1}^p \sum_{k=\lfloor m^2 t_{j-1} \rfloor + 1}^{\lfloor m^2 t_j \rfloor} f(\xi_k, y_i(k/m^2)) \\ &= \sum_{j=1}^p \sum_{k=\lfloor m^2 t_{j-1} \rfloor + 1}^{\lfloor m^2 t_j \rfloor} f(\xi_k, y_i(t_{j-1})). \end{aligned}$$

The ergodic theorem for $(\xi_n)_{n \geq 0}$ yields:

$$m^{-2} \sum_{k=1}^{\lfloor m^2 T \rfloor} f(\xi_k, y_i(k/m^2)) \xrightarrow{\mathbb{P} \text{ a.s.}} \sum_{j=1}^p [(t_j - t_{j-1}) \bar{f}(y_i(t_{j-1}))] = \int_0^T \bar{f}(y_i(s)) ds.$$

Obviously, $m^{-2} \sum_{k=0}^{\lfloor m^2 T \rfloor - 1} \bar{f}(y_i(k/m^2)) \rightarrow \int_0^T \bar{f}(y_i(s)) ds$ as m tends to $+\infty$. Hence, for m large enough:

$$\mathbb{P} \left\{ \left| m^{-2} \sum_{k=0}^{\lfloor m^2 T \rfloor - 1} f(\xi_k, y_i(k/m^2)) - m^{-2} \sum_{k=0}^{\lfloor m^2 T \rfloor - 1} \bar{f}(y_i(k/m^2)) \right| \geq \varepsilon \right\} \leq N_0^{-1} \varepsilon^2.$$

This actually holds for every $t \in [0, T]$. Choosing t of the form $t = k\varepsilon$, $k \in \{0, \dots, \lfloor \varepsilon^{-1} T \rfloor\}$, it comes (for m large):

$$\mathbb{P} \left\{ \sup_{t \in \{0, \dots, \varepsilon \lfloor \varepsilon^{-1} T \rfloor\}} \left| m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} f(\xi_k, y_i(k/m^2)) - m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{f}(y_i(k/m^2)) \right| \geq \varepsilon \right\} \leq T N_0^{-1} \varepsilon.$$

Since f is bounded, it comes finally for a suitable constant $C > 0$ (and for m large):

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left| m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} f(\xi_k, y_i(k/m^2)) - m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} \bar{f}(y_i(k/m^2)) \right| \geq C\varepsilon \right\} \leq T N_0^{-1} \varepsilon, \quad (4.28)$$

The sum over $i \in \{1, \dots, N_0\}$ in (4.28) is bounded by ε . Combining (4.27) and (4.28), we complete the proof. \square

5 Proofs of Theorem 3.2 and Corollary 3.3

5.1 Proof of Theorem 3.2

The proof of Theorem 3.2 is rather similar to the one of Theorem 3.1. The main difference comes from the definition of the mean function g . In the new setting, it writes for $(i, y) \in E \times [0, 1]^d$ $h(i, y) \equiv \sum_{u \in \mathcal{V}} u q(i, y, u)$. In particular, for $\ell \in \{1, \dots, d\}$:

$$\begin{aligned} h_\ell(i, y) &= g_\ell(i, y), \text{ for } 0 < y_\ell < 1, \\ h_\ell(i, y) &= p(i, y, e_\ell) + p(i, y, -e_\ell), \text{ for } y_\ell = 0, \\ h_\ell(i, y) &= -(p(i, y, e_\ell) + p(i, y, -e_\ell)), \text{ for } y_\ell = 1, \end{aligned} \quad (5.1)$$

where g_ℓ (resp. h_ℓ) denotes the ℓ th coordinate of g (resp. h). The process R satisfies the analogue of (4.3) with g replaced by h . Due to (5.1), the drift h_ℓ , for $\ell \in \{1, \dots, d\}$, pushes upwards the ℓ th coordinate of R when matching 0, and downwards when matching 1.

The drifts of the processes R and S just differ in the extra term $(h - g)$. The following lemma derives from (5.1):

Lemma 5.1 *For $\ell \in \{1, \dots, d\}$ and $y \in [0, 1]^d$, $y_\ell = 0 \Rightarrow (h - g)_\ell \geq 0$, $y_\ell = 1 \Rightarrow (h - g)_\ell \leq 0$ and $y_\ell \in]0, 1[\Rightarrow (h - g)_\ell = 0$.*

(First Step. Semimartingale Form.)

Of course, the function h does not satisfy the centering condition **(A.4)**. However, for g as in Section 4, we can still consider the solution v to the family of Poisson equations (4.5). Following the proof of Proposition 4.4, it is plain to derive:

Proposition 5.2 Define $c^{(m)}$ as $b^{(m)}$ in Proposition 4.4, but with $p(i, y, u)$ replaced by $q(i, y, u)$. Then, the walk $(R_n)_{n \geq 0}$ satisfies:

$$R_{n+1} + v(\xi_{n+1}, R_{n+1}/m) = R_n + v(\xi_n, R_n/m) + (h - g)(\xi_n, R_n/m) + c^{(m)}(\xi_n, R_n/m) + Z_{n+1}, \quad (5.2)$$

where $\mathbb{E}(Z_{n+1} | \mathcal{F}_n^{\xi, R}) = 0$. The drift increment $c^{(m)}$ satisfies the same bound as $b^{(m)}$: $\forall m \geq 1, i \in E, y \in \mathbb{R}^d, |c^{(m)}(i, y)| \leq C_{4.4}/m$.

We are then able to control the variation of R :

Lemma 5.3 Define $\mathbf{e} = e_1 + \dots + e_d$. Then, there exists a constant $C_{5.3}$, depending only on known parameters in **(A.1-4)**, such that for $1 \leq n \leq p \leq q$,

$$\begin{aligned} |R_q - R_n|^2 + 2 \sum_{k=p}^{q-1} \langle m\mathbf{e} - R_n, (h - g)^-(\xi_k, R_k/m) \rangle + 2 \sum_{k=p}^{q-1} \langle R_n, (h - g)^+(\xi_k, R_k/m) \rangle \\ \leq |R_p - R_n|^2 + C_{5.3}(q - p + m) + 2 \sum_{k=p}^{q-1} \langle R_k - R_n, Z_{k+1} \rangle. \end{aligned} \quad (5.3)$$

Proof. For $k \in \{n, \dots, q - 1\}$, derive from (5.2):

$$\begin{aligned} |R_{k+1} - R_n|^2 &= |R_k - R_n|^2 + 2\langle R_{k+1} - R_k, R_k - R_n \rangle + |R_{k+1} - R_k|^2 \\ &= |R_k - R_n|^2 + 2\langle R_k - R_n, (h - g)(\xi_k, R_k/m) \rangle \\ &\quad - 2\langle R_k - R_n, v(\xi_{k+1}, R_{k+1}/m) - v(\xi_k, R_k/m) \rangle \\ &\quad + 2\langle R_k - R_n, c^{(m)}(\xi_k, R_k/m) \rangle + 2\langle R_k - R_n, Z_{k+1} \rangle \\ &\quad + |R_{k+1} - R_k|^2. \\ &\equiv |R_k - R_n|^2 + 2T(1, k) + 2T(2, k) + 2T(3, k) + 2T(4, k) + T(5, k). \end{aligned} \quad (5.4)$$

Apply Lemma 5.1, and derive that $-T(1, k)$ writes $\langle m\mathbf{e} - R_n, (h - g)^-(\xi_k, R_k/m) \rangle + \langle R_n, (h - g)^+(\xi_k, R_k/m) \rangle$. Summing $T(1, k)$, for k running from p to $q - 1$, we obtain the second and third terms in the l.h.s of (5.3).

Perform now an Abel transform to handle $\sum_{k=p}^{q-1} T(2, k)$. The boundary conditions are bounded by Cm since R takes its values in $[0, m]^d$ and v is bounded (see Lemma 4.3). Moreover, the sum $\sum_{k=p+1}^{q-1} \langle R_k - R_{k-1}, v(\xi_k, R_k/m) \rangle$ is bounded by $C(q - p)$ since the jumps of R are bounded by 1. Hence, the term $\sum_{k=p}^{q-1} T(2, k)$ is bounded by $C(q - p + m)$.

Due to the bound for $c^{(m)}$ in Proposition 5.2, the sum $\sum_{k=p}^{q-1} T(3, k)$ is bounded by $C(q - p)$. The sum of $T(4, k)$ over k provides the third term in the r.h.s of (5.3).

Finally, since the jumps of R are bounded by 1, the sum of $T(5, k)$ over k is bounded by $(q - p)$. \square

We turn now to the martingale part in (5.2). Following the proof of Proposition 4.5, we claim:

Proposition 5.4 Define the martingale N by $\forall n \geq 1, N_n \equiv \sum_{k=1}^n Z_k$. Its bracket $\langle N \rangle_n$ satisfies a suitable version of Proposition 4.5 with $a^{(m)}$ replaced by $\tilde{a}^{(m)}$, given by:

$$\begin{aligned} \tilde{a}^{(m)}(\xi_n, R_n/m) &= [\alpha - (v + h - g + c^{(m)})(v + h - g + c^{(m)})^t](\xi_n, R_n/m) \\ &+ \sum_{u \in \mathcal{V}} \left[q(\xi_n, S_n/m, u) [P(vv^t) + u(Pv)^t + (Pv)u^t](\xi_n, (R_n + u)/m) \right]. \end{aligned} \quad (5.5)$$

(Second Step. Tightness Properties.)

The scaling procedure then applies as follows. Define as in (4.11):

$$\begin{aligned} \bar{C}_t^{(m)} &= m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} c^{(m)}(\xi_k, R_k/m), \quad \bar{N}_t^{(m)} = m^{-1} N_{\lfloor m^2 t \rfloor}, \\ \bar{H}_t^{(m)} &= m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} (h - g)^+(\xi_k, R_k/m), \quad \bar{K}_t^{(m)} = m^{-1} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} (h - g)^-(\xi_k, R_k/m), \end{aligned}$$

Proposition 5.5 The family $(\bar{R}^{(m)}, \bar{N}^{(m)}, \langle \bar{N}^{(m)} \rangle, \bar{H}^{(m)}, \bar{K}^{(m)})_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{4d+d^2})$.

Proof. Following Proposition 4.6, the family $(\bar{N}^{(m)}, \langle \bar{N}^{(m)} \rangle)_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{d+d^2})$.

Apply now Lemma 5.3 to establish the same property for $(\bar{R}^{(m)})_{m \geq 1}$. Choose $n = p = \lfloor m^2 s \rfloor$ and $q = \lfloor m^2 t \rfloor$, with $s < t$. Since all the terms in the l.h.s of (5.3) are nonnegative, it comes:

$$|\bar{R}_t^{(m)} - \bar{R}_s^{(m)}|^2 \leq C(t - s) + 2m^{-2} \sum_{k=\lfloor m^2 s \rfloor}^{\lfloor m^2 t \rfloor - 1} \langle R_k, Z_{k+1} \rangle + O(1/m). \quad (5.6)$$

The second term in the r.h.s of the above inequality defines a family of martingales. It is plain to see that it is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and thus to derive the \mathcal{C} -tightness of the family $(\bar{R}^{(m)})_{m \geq 1}$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

We turn finally to the tightness of $(\bar{H}^{(m)}, \bar{K}^{(m)})_{m \geq 1}$. Choose p and q as above and $n = 0$ in (5.3), i.e. $R_n = 0$. It comes:

$$\begin{aligned} \langle \mathbf{e}, \bar{K}_t^{(m)} - \bar{K}_s^{(m)} \rangle &\leq -(\bar{R}_t^{(m)})^2 + (\bar{R}_s^{(m)})^2 \\ &+ C(t - s + 1/m) + 2m^{-2} \sum_{k=\lfloor m^2 s \rfloor}^{\lfloor m^2 t \rfloor - 1} \langle R_k, Z_{k+1} \rangle. \end{aligned}$$

Note that $|(\bar{R}_t^{(m)})^2 - (\bar{R}_s^{(m)})^2| \leq 2|\bar{R}_t^{(m)} - \bar{R}_s^{(m)}|$ since $\bar{R}^{(m)}$ belongs to $[0, 1]^d$. Due to (5.6), deduce that $(\bar{K}^{(m)})_{m \geq 1}$ is \mathcal{C} -tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Of course, the same holds for $(\bar{H}^{(m)})_{m \geq 1}$.

(Third Step. Extraction of a Converging Subsequence and Identification of the Limit.)

Proposition 5.6 Any weak limit (X, \tilde{N}, H, K) of the sequence $(\bar{R}^{(m)}, \bar{N}^{(m)}, \bar{H}^{(m)}, \bar{K}^{(m)})_{m \geq 1}$ satisfies:

$$\forall t \geq 0, X_t = \int_0^t \bar{b}(X_s) ds + H_t - K_t + \tilde{N}_t, \quad (5.7)$$

where \tilde{N} is a square integrable continuous martingale whose bracket writes $\langle \tilde{N} \rangle_t = \int_0^t \bar{a}(X_s) ds$, H and K are two nondecreasing continuous processes, matching 0 at zero and satisfying condition (3.3).

Proof. Note first that Proposition 4.9 still applies in the reflected setting, with S replaced by R . Focus now on the asymptotic behaviour of the drift $c^{(m)}$ in (5.2). As $mb^{(m)}$ does, $mc^{(m)}$ uniformly converges towards $(\nabla_y v - \nabla_y g)g$. Following (4.18), this provides the form of the limit drift.

Concerning the martingale part, we now prove the analogue of Lemma 4.8. The point is to study the asymptotic form of the bracket $\langle \bar{N}^{(m)} \rangle$. It will be seen from (5.5) that $\tilde{a}^{(m)}$ uniformly converges towards a (as in (4.14)) plus a corrector term that writes $(g-h)\gamma^t + \gamma(g-h)^t$ for a suitable bounded function γ . Of course, the part in a satisfies Proposition 4.9. The second part vanishes since:

$$m^{-2} \sum_{k=0}^{\lfloor m^2 t \rfloor - 1} |h - g|(\xi_k, R_k/m) \leq m^{-1} (\bar{H}_t^{(m)} + \bar{K}_t^{(m)}).$$

This provides the limit form of the brackets $(\langle \bar{N}^{(m)} \rangle)_{m \geq 1}$ and thus the form of the limit bracket.

We turn finally to the processes H and K . It is clear that $H_0 = K_0 = 0$ and that H and K are nondecreasing. It thus remains to verify (3.3). Focus on the case of H . Since $(\bar{R}^{(m)}, \bar{H}^{(m)})_{m \geq 1}$ converges weakly (up to a subsequence) in the Polish space $\mathbb{D}(\mathbb{R}, \mathbb{R}^{2d})$ towards (X, H) , we can assume without loss of generality from the Skorokhod representation theorem (see Theorem II.86.1 in Rogers and Williams [26]) that the convergence holds almost surely. In particular, for $T > 0$, for $\ell \in \{1, \dots, d\}$, for almost every $\omega \in \Omega$, $\langle \bar{H}^{(m)}, e_\ell \rangle$, seen as a distribution over $[0, T]$ converges weakly towards $\langle H, e_\ell \rangle$ seen as a distribution over $[0, T]$. Hence, for a continuous function f on $[0, T]$:

$$\lim_{n \rightarrow +\infty} \int_0^T f(t) d\langle \bar{H}^{(m)}, e_\ell \rangle_t = \int_0^T f(t) dH_t^\ell. \quad (5.8)$$

Since X is continuous, (5.8) holds with $f(X_t)$ instead of $f(t)$. Recall moreover from Proposition VI.1.17 in Jacod and Shiryaev [12] that $\sup_{0 \leq t \leq T} |\bar{R}_t^{(m)} - X_t|$ tends to 0 almost surely. Since the supremum $\sup_{m \geq 1} \bar{H}_T^{(m)}$ is almost surely finite, (5.8) yields:

$$\lim_{n \rightarrow +\infty} \int_0^T f(\bar{R}_t^{(m)}) d\langle \bar{H}^{(m)}, e_\ell \rangle_t = \int_0^T f(X_t) dH_t^\ell. \quad (5.9)$$

From Lemma 5.1, for a continuous function f with support included in $]0, 1]$, the term $\int_0^T f(X_t^\ell) dH_t^\ell$ vanishes. It is plain to derive the first equality in (3.3). \square

According to Proposition 5.4, (X, H, K) satisfies the martingale problem associated to (\bar{b}, \bar{a}) with normal reflection on $\partial[0, 1]^d$. Thanks to Theorem 3.1 in Lions and Sznitman [16], this martingale problem is uniquely solvable. This completes the proof of Theorem 3.2. \square

5.2 Proof of Corollary 3.3

The proof of Corollary 3.3 relies on the mapping Theorem 2.7, Chapter I, in Billingsley [2]. Indeed, consider the mapping $\Psi : x \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \mapsto \inf\{t \geq 0, x_t \in F_0\}$ ($\inf \emptyset = +\infty$). If x denotes a continuous function from \mathbb{R}_+ into \mathbb{R}^d and $(x_n)_{n \geq 1}$ a sequence of càd-làg functions from \mathbb{R}_+ into \mathbb{R}^d

converging towards x for the Skorokhod topology, we know from Proposition VI.1.17 in Jacod and Shiryaev [12] that $(x_n)_{n \geq 1}$ converges uniformly towards x on compact subsets of \mathbb{R}_+ . Assume now that $\Psi(x)$ is finite and that for every $\eta > 0$ there exists $t \in]\Psi(x), \Psi(x) + \eta[$ such that $x(t)$ belongs to the interior of F_0 (the point $x(\Psi(x))$ is then said to be regular). Then, $\Psi(x_n)$ tends to $\Psi(x)$. In particular, if almost every trajectory of X satisfies these conditions, $\Psi(\bar{R}^{(m)})$ converges in law towards $\Psi(X)$.

To prove that almost every trajectory of X satisfies the latter conditions, consider the solution \hat{X} to the martingale problem associated to the couple (\hat{b}, \hat{a}) , where \hat{b} and \hat{a} are periodic functions of period two in each direction of the space given by $(\hat{b}(y), \hat{a}(y)) = (N(y)\bar{b}(\Pi(y)), \bar{a}(\Pi(y)))$ for $y \in [0, 2[^d$, with $\Pi(y) = (\pi(y_1), \dots, \pi(y_d))$, $\pi(z) = z\mathbf{1}_{[0,1]}(z) + (2-z)\mathbf{1}_{]1,2]}(z)$ and $N(y) = \text{diag}(\text{sign}(1-y_1), \dots, \text{sign}(1-y_d))$ ($\text{diag}(x)$ denotes the diagonal matrix whose diagonal is given by x). Due to the boundedness of \hat{b} and to the continuity and to the ellipticity of \hat{a} , such a martingale problem is uniquely solvable (see Stroock and Varadhan [30], Chapter VII). Extend Π from a $2\mathbb{Z}^d$ -periodicity argument to the whole set \mathbb{R}^d , and derive from the Itô-Tanaka formula that $\Pi(\hat{X})$ satisfies the martingale problem associated to (\bar{b}, \bar{a}) with normal reflection on $\partial[0, 1]^d$. In particular, $\Pi(\hat{X})$ and X have the same law. It is then sufficient to prove that almost every trajectory of $\Pi(\hat{X})$ satisfies the conditions given in the above paragraph, and thus to prove that almost every trajectory of \hat{X} hits $F \equiv \cup_{k \in 2\mathbb{Z}^d} (k + F_0)$ and that every point of the boundary of F is regular for \hat{X} .

Due to the boundedness of \hat{b} and \hat{a} and to the uniform ellipticity of \hat{a} , deduce from Pardoux [23], Section 2, that the process \hat{X} , seen as a diffusion process with values in the torus $\mathbb{R}^d / (2\mathbb{Z}^d)$ is recurrent, and from Pinsky [25], Theorem 3.3, Section 3, Chapter II, that every point of the boundary of F is regular for \hat{X} .

The second assertion in Corollary 3.3 follows again from the mapping theorem. \square

6 Sketch of the Proof of Theorem 3.4

We now present several ideas to establish Theorem 3.4. Again, the whole proof is given in Delarue [4].

We denote by $(\bar{\mathbf{A}})$ the assumption satisfied by \bar{a} and \bar{b} under the statement of Theorem 3.4. Briefly, \bar{a} and \bar{b} are bounded and Lipschitz continuous, and \bar{a} is uniformly elliptic.

6.1 Description of the Method

As a starting point of the proof, recall that the two-dimensional Brownian motion B never hits zero at a positive time, but hits infinitely often any neighbourhood of zero with probability one. The proof of this result (see e.g. Friedman [8]) relies on the differential form of the Bessel process of index 1, *i.e.* of the process $|B|$. In short, for B different from zero, $d|B|$ writes $d|B_t| = 1/(2|B_t|)dt + d\tilde{B}_t$, where \tilde{B} denotes a one-dimensional Brownian motion.

The common strategy to investigate the recurrence and transience properties of $|B|$ then consists in exhibiting a Lyapunov function for the process $|B|$ (see again Friedman [8] for a complete review on this topic). In dimension two, *i.e.* in our specific setting, the function \ln is harmonic for the process $|B|$ (the Itô formula yields for B different from zero: $d \ln(|B_t|) = |B_t|^{-1} d\tilde{B}_t$). In particular, for

$|B_0| = 1$, for a large $N \geq 1$ and for the stopping time $\bar{\tau}_N \equiv \tau_{N-1} \wedge \tau_2$, with $\tau_x \equiv \inf\{t \geq 0, |B_t| = x\}$:

$$\begin{aligned} 0 &= \mathbb{E}(\ln(|B_0|)) = \mathbb{E}(\ln(|B_{\bar{\tau}_N}|)) \\ &= -\ln(N)\mathbb{P}\{\tau_{N-1} < \tau_2\} + \ln(2)\mathbb{P}\{\tau_2 < \tau_{N-1}\}. \end{aligned}$$

Hence, $\lim_{N \rightarrow +\infty} \mathbb{P}\{\tau_{N-1} < \tau_2\} = 0$, so that $\mathbb{P}\{\tau_0 < \tau_2\} = 0$. Of course, for every $r \geq 2$, the same property holds: $\mathbb{P}\{\tau_0 < \tau_r\} = 0$. Letting r tend towards $+\infty$, it comes $\mathbb{P}\{\tau_0 < +\infty\} = 0$, as announced above.

More generally, if Y denotes a Bessel process of index $\gamma \geq -1$, $\gamma \neq 1$, *i.e.* Y satisfies the SDE $dY_t = (\gamma/2)Y_t^{-1} + d\tilde{B}_t$, where \tilde{B} denotes a one-dimensional Brownian motion, the function $\phi(x) = x^{1-\gamma}$ is harmonic for the process Y . A similar argument to the previous one permits to derive the attainability or non-attainability of the origin for the process Y , depending on the sign of $1 - \gamma$.

Roughly speaking, the strategy used in Delarue [4] to establish Theorem 3.4 aims to reduce the original limit absorption problem to the attainability problem for a suitable process. For this reason, we translate the absorption problem in F_0 to the neighbourhood of zero. This amounts to investigate the behaviour of $\mathbb{E}(\tilde{T}_\ell)$, for $\ell \in]0, 1]$, with $X_0 = (1, 1)^t$ and $\tilde{T}_\ell \equiv \inf\{t \geq 0, X_t^1 + X_t^2 \leq \ell\}$. As guessed by the reader, our final result then writes in terms of $\bar{a}(0)$ and not of $\bar{a}(1, 1)$ as in Theorem 3.4.

A crucial point is then to define the analogue of $|B|$ for the reflected diffusion X . In order to take into account the influence of the diffusion matrix $\bar{a}(0)$, we focus on the norm of X with respect to the scalar product induced by the inverse of the matrix $\bar{a}(0)$ (which is non-degenerate under the assumptions of Theorem 3.4):

$$\begin{aligned} \forall t \geq 0, Q_t &\equiv \langle X_t, \bar{a}^{-1}(0)X_t \rangle \\ &= (1 - s^2)^{-1} [\rho_1^{-2}(X_t^1)^2 + \rho_2^{-2}(X_t^2)^2 - 2s\rho_1^{-1}\rho_2^{-1}X_t^1X_t^2]. \end{aligned} \quad (6.1)$$

The process Q is devised to mimic the role played by $|B|^2$ in the non-reflected Brownian case. Its differential form writes:

Proposition 6.1 *There exist a constant $C_{6.1}$, depending only on known parameters in Assumption **(A)**, as well as a measurable function $\Gamma_{6.1} : \mathbb{R}^2 \rightarrow \mathbb{R}$, bounded by $C_{6.1}$, such that for $N \geq 1$ and $t \in [0, \zeta_N]$, $\zeta_N \equiv \inf\{t \geq 0, |X_t| \leq N^{-1}\}$:*

$$\begin{aligned} \forall t \in [0, \zeta_N], dQ_t^{1/2} &= \Gamma_{6.1}(X_t)dt + \frac{1}{2}Q_t^{-1/2}dt \\ &\quad - \frac{s}{\sqrt{1-s^2}} [\rho_1^{-1}dH_t^1 + \rho_2^{-1}dH_t^2] - Q_t^{-1/2} [\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + Q_t^{-1/2} \langle \bar{\sigma}(X_t) \bar{a}^{-1}(0)X_t, dB_t \rangle, \end{aligned} \quad (6.2)$$

with:

$$\kappa_t^1 \equiv \frac{1}{1-s^2} [\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^2], \quad \kappa_t^2 \equiv \frac{1}{1-s^2} [\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1}X_t^1]. \quad (6.3)$$

The proof of Proposition 6.1 is put aside for the moment.

At this stage of the sketch, the most simple situation to focus on seems to be the case “ $s = 0$ ”. In this framework, the dH term in (6.2) reduces to zero so that Q behaves like a standard Itô process in the neighbourhood of zero:

(1) For X close to 0, the scalar product $Q_t^{-1/2} \langle \bar{\sigma}(X_t) \bar{a}^{-1}(0) X_t, dB_t \rangle$ driving the martingale part of $Q^{1/2}$ looks like the process $Q_t^{-1/2} \langle \bar{\sigma}^{-1}(0) X_t, dB_t \rangle$, which is, thanks to Lévy's theorem, a Brownian motion.

(2) For X close to 0, the term $\Gamma_{6.1}(X_t)$ is negligible in front of $Q_t^{-1/2}$.

Thus, from (1) and (2), the differential form of the process $Q_t^{1/2}$ satisfies, for $s = 0$, the Bessel equation of index 1, at least for X in the neighbourhood of zero.

For the sake of simplicity, we assume from now on that $\bar{a}(x)$ reduces for all $x \in [0, 1]^2$ to $\bar{a}(0)$, and in the same way, that $\Gamma_{6.1}(x)$ vanishes for $x \in [0, 1]^2$. Up to the differential term dK , $Q^{1/2}$ then appears as a Bessel process. For this reason, we expect the function \ln to be a kind of Lyapunov function (in a sense to be precised) for $Q_t^{1/2}$ in the case $s = 0$. Itô's formula applied to (6.2) yields for Q different from zero:

$$d \ln(Q_t^{1/2}) = d\tilde{B}_t - Q_t^{-1} [\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2]. \quad (6.4)$$

It now remains to handle the dK term in (6.4). Since s vanishes in this first analysis, the processes κ^1 and κ^2 driven by the differential terms dK^1 and dK^2 are always bounded from above and from below by positive constants. Recall moreover that $Q_t^{-1} dK_t^1 = Q_t^{-1} \mathbf{1}_{\{X_t^1=1\}} dK_t^1$ (the same holds for dK_t^2). We deduce that there exists a constant $C > 0$ such that:

$$d\tilde{B}_t - C[dK_t^1 + dK_t^2] \leq d \ln(Q_t^{1/2}) \leq d\tilde{B}_t - C^{-1}[dK_t^1 + dK_t^2]. \quad (6.5)$$

The strategy then consists in comparing (6.5) to a bounded real functional of X . The most simple one is $|X|^2$. For $\bar{b} = 0$, it is plain to derive from Itô's formula (note again that $\langle X_t, dH_t \rangle = X_t^1 dH_t^1 + X_t^2 dH_t^2$ vanishes since $dH_t^1 = \mathbf{1}_{\{X_t^1=0\}} dH_t^1$ and $dH_t^2 = \mathbf{1}_{\{X_t^2=0\}} dH_t^2$):

$$\begin{aligned} d|X_t|^2 &= \text{trace}(\bar{a}(0))dt - 2(X_t^1 dK_t^1 + X_t^2 dK_t^2) + 2\langle X_t, \bar{\sigma}(0) dB_t \rangle \\ &= \text{trace}(\bar{a}(0))dt - 2(dK_t^1 + dK_t^2) + 2\langle X_t, \bar{\sigma}(0) dB_t \rangle. \end{aligned} \quad (6.6)$$

Add (6.5) and (6.6) (up to multiplicative constants) and deduce:

$$\begin{aligned} d(-\ln(Q_t^{1/2}) + \frac{C}{2}|X_t|^2) &\leq \frac{C}{2} \text{trace}(\bar{a}(0))dt - d\tilde{B}_t + C\langle X_t, \bar{\sigma}(0) dB_t \rangle \\ d(-\ln(Q_t^{1/2}) + \frac{C^{-1}}{2}|X_t|^2) &\geq \frac{C^{-1}}{2} \text{trace}(\bar{a}(0))dt - d\tilde{B}_t + C^{-1}\langle X_t, \bar{\sigma}(0) dB_t \rangle. \end{aligned} \quad (6.7)$$

Recall that $X_0 = (1, 1)^t$. Take the expectation between 0 and $\tau_N \equiv \inf\{t \geq 0, Q_t^{1/2} = 1/N\}$, for $N \geq 1$. Since the expectation of the martingale parts vanishes and since the matrix $\bar{a}(0)$ is non-degenerate, it comes for a new positive constant C' :

$$(C')^{-1}(\ln(N) - 1) \leq \mathbb{E}(\tau_N) \leq C'(\ln(N) + 1).$$

Of course, $\mathbb{E}(\tau_N)$ does match exactly $\mathbb{E}(\tilde{T}_{1/N})$. However, it is well seen from the definition of $Q^{1/2}$ that there exists a constant $c > 0$, such that $\tau_{c^{-1}N} \leq \tilde{T}_{1/N} \leq \tau_{cN}$. This proves the third point in Theorem 3.4. \square

6.2 Non-Zero Cases

As easily guessed by the reader, the cases “ $s < 0$ ” and “ $s > 0$ ” are more difficult. We derive first several straightforward consequences from Proposition 6.1:

- (3) If $s > 0$, the dH term is always nonpositive. Up to a slight modification of the dK term (that does not play any role in the neighbourhood of the origin), the differential form of $Q^{1/2}$ writes as the differential form of $Q^{1/2}$ in the zero case plus a nonincreasing process. This explains why the process $Q^{1/2}$ reaches the neighbourhood of the origin in the case “ $s > 0$ ” faster than in the case “ $s = 0$ ”. We thus expect the case “ $s > 0$ ” to be sub-logarithmic (again, in a sense to be precised).
- (4) On the opposite, if $s < 0$, the dH term is always nonnegative. With a similar argument to the previous one, we expect the case “ $s < 0$ ” to be super-logarithmic.

The strategy now consists in correcting $Q^{1/2}$ to get rid of the dH term and to reduce the analysis to the one stated for “ $s = 0$ ”. It is then rather natural to focus on the auxiliary process $\forall t \geq 0$, $A_t \equiv Q_t^{1/2} + Z_t$, with:

$$\forall t \geq 0, Z_t \equiv \frac{s}{\sqrt{1-s^2}} \langle (\rho_1^{-1}, \rho_2^{-1})^t, X_t \rangle = \frac{s}{\sqrt{1-s^2}} [\rho_1^{-1} X_t^1 + \rho_2^{-1} X_t^2].$$

For s close to 0, the processes A and $Q^{1/2}$ are equivalent in the following sense: there exists a constant $C > 0$, such that $C^{-1}Q_t^{1/2} \leq A_t \leq CQ_t^{1/2}$. The value of the constant C can be precised for s in the neighbourhood of zero. Indeed, the process Z can be expressed as: $Z_t = s\gamma_t(s)Q_t^{1/2}$, with $1 - \varepsilon \leq \gamma_t(s) \leq \sqrt{2} + \varepsilon$, for s in the neighbourhood of zero.

Actually, we prove in Delarue [4] that the equivalence property still holds true for all $s \in]-1, 1[$. As a first consequence, we derive that the processes $Q^{1/2}$ and A admit the same asymptotic behaviour. We thus focus on the second one and in particular on the differential form of A (the proof derives from Proposition 6.1 and is thus left to the reader):

Proposition 6.2 *There exist a constant $C_{6.2}$, depending only on known parameters in $(\bar{\mathbf{A}})$, and a function $\Gamma_{6.2}$, bounded by $C_{6.2}$, such that:*

$$\begin{aligned} \forall t \in [0, \zeta_N], dA_t &= \Gamma_{6.2}(X_t)dt + \frac{1}{2}Q_t^{-1/2}dt \\ &\quad - [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + \langle Q_t^{-1/2} \bar{\sigma}(X_t) \bar{a}^{-1}(0) X_t + \frac{s}{\sqrt{1-s^2}} \bar{\sigma}(X_t) (\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle, \end{aligned} \tag{6.8}$$

with:

$$\begin{aligned} \forall t \geq 0, \bar{\kappa}_t^1 &= Q_t^{-1/2} \kappa_t^1 + \frac{s}{\sqrt{1-s^2}} \rho_1^{-1}, \\ \bar{\kappa}_t^2 &= Q_t^{-1/2} \kappa_t^2 + \frac{s}{\sqrt{1-s^2}} \rho_2^{-1}. \end{aligned}$$

From now on, we assume that $\bar{\sigma}$ and \bar{a} do not depend on x , as done in the previous subsection to investigate the case “ $s = 0$ ”. In the same way, we assume for the sake of simplicity that $\Gamma_{6.2}$

reduces to zero. Equation (6.8) becomes:

$$\begin{aligned} \forall t \in [0, \zeta_N], \quad dA_t &= \frac{1 + s\gamma_t(s)}{2} A_t^{-1} dt \\ &\quad - [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] \\ &\quad + \langle Q_t^{-1/2} \bar{\sigma}^{-1}(0) X_t + \frac{s}{\sqrt{1-s^2}} \bar{\sigma}(0) (\rho_1^{-1}, \rho_2^{-1})^t, dB_t \rangle, \end{aligned} \quad (6.9)$$

We try to mimic the arguments given in Subsection 6.1. For s close to zero, we deduce from Proposition 6.1 that processes κ^1 and κ^2 are positive. Hence, from Proposition 6.2, the same holds for $\bar{\kappa}^1$ and $\bar{\kappa}^2$. Since the general case $s \in]-1, 1[$ is really more difficult to handle, we restrict our analysis to s in the neighbourhood of zero.

It now remains to investigate the martingale part in (6.8). Its bracket writes:

$$\begin{aligned} \frac{d\langle A \rangle_t}{dt} &= \left| Q_t^{-1/2} \bar{\sigma}^{-1}(0) X_t + \frac{s}{\sqrt{1-s^2}} \bar{\sigma}(0) (\rho_1^{-1}, \rho_2^{-1})^t \right|^2 \\ &= \left[1 + 2Q_t^{-1/2} Z_t + \frac{s^2}{1-s^2} \langle (\rho_1^{-1}, \rho_2^{-1})^t, \bar{a}(0) (\rho_1^{-1}, \rho_2^{-1})^t \rangle \right] \\ &= [1 + 2s\gamma_t(s) + 2s^2(1-s)^{-1}] \end{aligned}$$

Apply now the function x^β to A , for a given $\beta \neq 0$ (recall that this function is harmonic for the Bessel process of index $1 - \beta$). Itô's formula yields:

$$\begin{aligned} dA_t^\beta &= \frac{\beta}{2} A_t^{\beta-2} \Theta_t(\beta, s) dt - \beta A_t^{\beta-1} [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] + dM_t, \\ \text{with } \Theta_t(\beta, s) &\equiv [(\beta - 1)(1 + 2s\gamma_t(s) + 2s^2(1-s)^{-1}) + 1 + s\gamma_t(s)] \end{aligned}$$

where M denotes a martingale.

We are now in position to complete the proof:

1. If $s > 0$ (and small), then the quantity $\Theta_t(\beta, s)$ tends to $-s\gamma_t(s) - 2s^2(1-s)^{-1}$ when β tends to zero. In particular, there exists $\beta > 0$ (β close to zero) such that $\Theta_t(\beta, s) \leq 0$. For this β , $dA_t^\beta \leq -\beta A_t^{\beta-1} [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] + dM_t$. We thus recover the right hand side in (6.5) with respect to the function x^β and to the process A . We then derive the second inequality in (6.7), with $-\ln(x)$ replaced by $-x^\beta$, and $Q^{1/2}$ by A .

This proves the first point in Theorem 3.4.

2. If $s < 0$ (and small), we can follow the previous argument and thus prove that there exists $\beta < 0$ (β close to zero) such that $\Theta_t(\beta, s) \geq 0$. For this β , $dA_t^\beta \leq -\beta A_t^{\beta-1} [\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] + dM_t$. Again, we recover the right hand side in (6.5) with respect to the function x^β and to the process A , but with $-C^{-1} > 0$, *i.e.* $C < 0$. We then derive the second inequality in (6.7), with $-\ln(x)$ replaced by $-x^\beta$, $Q^{1/2}$ by A and C by $-C$.

This proves the lower bound in the second point of Theorem 3.4 (with respect to a possibly different β_-).

To obtain the upper bound, note for $\beta \rightarrow -\infty$ that $\Theta_t(\beta, s) \sim \beta(1 + 2s\gamma_t(s) + 2s^2(1-s)^{-1})$ and deduce (at least for s close to zero) that there exists $\beta < 0$ such that $\Theta_t(\beta, s) \leq 0$. For

this β , $dA_t^\beta \geq -\beta A_t^{\beta-1}[\bar{\kappa}_t^1 dK_t^1 + \bar{\kappa}_t^2 dK_t^2] + dM_t$. Again, we follow the argument given in the case “ $s = 0$ ” and then derive the upper bound in the second point of Theorem 3.4 (the resulting β_+ may differ from the one given in the statement). \square

6.3 Proof of Proposition 6.1

Write first the SDE satisfied by the process Q :

Lemma 6.3 *Keep the notations introduced in the statement of Proposition 6.1. Then, there exists a constant $C_{6.3}$, depending only on known parameters in $(\bar{\mathbf{A}})$, as well as a measurable function $\Gamma_{6.3}$, bounded by $C_{6.3}$, such that:*

$$\begin{aligned} \forall t \geq 0, \quad dQ_t &= Q_t^{1/2} \Gamma_{6.3}(X_t) dt + 2dt \\ &\quad - \frac{2s}{\sqrt{1-s^2}} Q_t^{1/2} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2] - 2[\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + 2\langle \bar{\sigma}(X_t) \bar{a}^{-1}(0) X_t, dB_t \rangle. \end{aligned} \quad (6.10)$$

Assume for the moment that Lemma 6.3 holds and complete the proof of Proposition 6.1. Apply Itô's formula to $Q^{1/2}$ and derive from the differential form given in (6.10) (note that Q does not vanish for $t \in [0, \zeta_N]$):

$$\begin{aligned} \forall t \in [0, \zeta_N], \quad dQ_t^{1/2} &= \frac{1}{2} \Gamma_{6.3}(X_t) dt + Q_t^{-1/2} dt \\ &\quad - \frac{s}{\sqrt{1-s^2}} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2] - Q_t^{-1/2} [\kappa_t^1 dK_t^1 + \kappa_t^2 dK_t^2] \\ &\quad + Q_t^{-1/2} \langle \bar{\sigma}(X_t) \bar{a}^{-1}(0) X_t, dB_t \rangle \\ &\quad - \frac{1}{2} Q_t^{-3/2} \langle X_t, \bar{a}^{-1}(0) \bar{a}(X_t) \bar{a}^{-1}(0) X_t \rangle dt \\ &\equiv [\Delta(1, X_t) + Q_t^{-1/2}] dt + d\Delta_t(2) + d\Delta_t(3) - \frac{1}{2} Q_t^{-3/2} \Delta(4, X_t) dt. \end{aligned} \quad (6.11)$$

Focus on $\Delta(4, \cdot)$ in (6.11):

$$\begin{aligned} \forall t \in [0, \zeta_N], \quad \Delta(4, X_t) &= Q_t + \langle X_t, \bar{a}^{-1}(0) [\bar{a}(X_t) - \bar{a}(0)] \bar{a}^{-1}(0) X_t \rangle \\ &\equiv Q_t + \Delta(5, X_t). \end{aligned} \quad (6.12)$$

Thanks to Assumption $(\bar{\mathbf{A}})$ (boundedness, ellipticity and Lipschitz continuity of \bar{a}), the function $\Delta(5, \cdot)$ satisfies:

$$\forall t \in [0, \zeta_N], \quad |\Delta(5, X_t)| \leq C Q_t^{3/2}. \quad (6.13)$$

where the constant C depends only on known parameters. Plug now (6.12) into (6.11):

$$\forall t \in [0, \zeta_N], \quad dQ_t^{1/2} = [\Delta(1, X_t) - \frac{1}{2} Q_t^{-3/2} \Delta(5, X_t) + \frac{1}{2} Q_t^{-1/2}] dt + d\Delta_t(2) + d\Delta_t(3). \quad (6.14)$$

Set finally $\Gamma_{6.1}(X_t) = \Delta(1, X_t) - \frac{1}{2} Q_t^{-3/2} \Delta(5, X_t)$. Thanks to (6.11), (6.13) and to (6.14), this completes the proof. \square

Proof of Lemma 6.3. Thanks to Itô's formula, $d\langle X, \bar{a}^{-1}(0)X \rangle$ writes:

$$\begin{aligned}
\forall t \geq 0, d\langle X_t, \bar{a}^{-1}(0)X_t \rangle &= 2\langle X_t, \bar{a}^{-1}(0)\bar{b}(X_t) \rangle dt + \text{trace}[\bar{\sigma}^{-1}(0)\bar{a}(X_t)\bar{\sigma}^{-1}(0)] dt \\
&\quad + 2\langle X_t, \bar{a}^{-1}(0)(dH_t - dK_t) \rangle + 2\langle \bar{a}^{-1}(0)X_t, \bar{\sigma}(X_t)dB_t \rangle \\
&= 2dt \\
&\quad + 2\langle X_t, \bar{a}^{-1}(0)\bar{b}(X_t) \rangle dt + \text{trace}[\bar{\sigma}^{-1}(0)(\bar{a}(X_t) - \bar{a}(0))\bar{\sigma}^{-1}(0)] dt \\
&\quad + 2\langle X_t, \bar{a}^{-1}(0)(dH_t - dK_t) \rangle + 2\langle \bar{a}^{-1}(0)X_t, \bar{\sigma}(X_t)dB_t \rangle
\end{aligned} \tag{6.15}$$

Set $\Gamma_{6.3}(X_t) = Q_t^{-1/2}[2\langle X_t, \bar{a}^{-1}(0)\bar{b}(X_t) \rangle + \text{trace}[\bar{\sigma}^{-1}(0)(\bar{a}(X_t) - \bar{a}(0))\bar{\sigma}^{-1}(0)]]$.

Focus now on the dH and dK terms in the above r.h.s. Due to (3.3), note first that $X_t^1 dH_t^1 = 0$ and $X_t^2 dH_t^2 = 0$ and that $\rho_2^{-1} X_t^2 dH_t^1 = (1-s^2)^{1/2} Q_t^{1/2} dH_t^1$ and $\rho_1^{-1} X_t^1 dH_t^2 = (1-s^2)^{1/2} Q_t^{1/2} dH_t^2$. Derive that:

$$\begin{aligned}
\langle X_t, \bar{a}^{-1}(0)dH_t \rangle &= X_t^1 (\bar{a}^{-1}(0))_{1,1} dH_t^1 + X_t^2 (\bar{a}^{-1}(0))_{2,2} dH_t^2 \\
&\quad + (\bar{a}^{-1}(0))_{1,2} [X_t^2 dH_t^1 + X_t^1 dH_t^2] \\
&= -\frac{s}{1-s^2} \rho_1^{-1} \rho_2^{-1} [X_t^2 dH_t^1 + X_t^1 dH_t^2] \\
&= -\frac{s}{\sqrt{1-s^2}} Q_t^{1/2} [\rho_1^{-1} dH_t^1 + \rho_2^{-1} dH_t^2].
\end{aligned} \tag{6.16}$$

Note again from (3.3) that $X_t^1 dK_t^1 = dK_t^1$ and $X_t^2 dK_t^2 = dK_t^2$. Thus:

$$\begin{aligned}
\langle X_t, \bar{a}^{-1}(0)dK_t \rangle &= X_t^1 (\bar{a}^{-1}(0))_{1,1} dK_t^1 + X_t^2 (\bar{a}^{-1}(0))_{2,2} dK_t^2 \\
&\quad + (\bar{a}^{-1}(0))_{1,2} [X_t^1 dK_t^2 + X_t^2 dK_t^1] \\
&= \frac{1}{1-s^2} [\rho_1^{-2} - s\rho_1^{-1}\rho_2^{-1} X_t^2] dK_t^1 + \frac{1}{1-s^2} [\rho_2^{-2} - s\rho_1^{-1}\rho_2^{-1} X_t^1] dK_t^2.
\end{aligned} \tag{6.17}$$

Plug now (6.16) and (6.17) into (6.15) and complete the proof of (6.10). \square

Note from (6.11), (6.12), (6.14) and (6.15) that $\Gamma_{6.1}$ reduces to zero if \bar{b} vanishes and \bar{a} is constant. This explains why we can assume in Subsections 6.1 and 6.2 $\bar{a}(x) = \bar{a}(0)$, $\bar{b} = 0$ and $\Gamma_{6.1} = \Gamma_{6.2} = 0$.

7 Conclusion

Probabilistic models depending on an extra random environment have been widely developed for twenty years. From a practical point of view, the environment expresses external constraints that satisfy some statistical properties. For example, the behaviour of an investor that borrows a certain amount of money may depend on the values of various assets as well as some quasi-periodic parameters.

From a more theoretical point of view, our work shows that earlier results (see e.g. Louchard [17], Louchard et al. [18] and [19] and Guillin and Schott [10]) for the diffusive behaviour of the banker algorithm extend to a more general framework. This is a new small step in the robustness of the underlying model. The theory of stochastic homogenization is also relevant for other distributed algorithms such as the problem of colliding stacks.

The reader may object that our own paper exclusively focuses on the trend-free case (see Assumption **(A.4)**), whereas so-called contracting and expanding regimes for distributed algorithms have been investigated in the previous references. We reasonably guess that the strategy presented here applies to the expanding case, after a suitable subtraction of the trend, as done for example in Louchard and Schott [18]. On the opposite, the contracting framework would require further developments, both based on homogenization techniques and on large deviations arguments as in Maier [20]. We leave this for further research.

In a different perspective, two questions follow from the study of the limit reflected diffusion. On the one hand, our global result writes as a double asymptotic property: we let first m tend to $+\infty$ in $m^{-2}T^{(m)}$ and then λ to 2 in $\mathbb{E}(T_\lambda)$ (at least in the two dimensional setting). A natural question would consist in investigating $m^{-2}\mathbb{E}(T_{\Lambda_m}^{(m)})$ (with obvious notations) as m tends to $+\infty$ and for $(m^{-1}\Lambda_m)_{m \geq 1}$ converging towards 2. On the other hand, the extension of Theorem 3.4 to the d -dimensional case appears as a challenging problem. Nevertheless, numerical methods offer a possible alternative to obtain an empirical description of the asymptotic behaviour of the limit diffusion or even to compute with a Monte-Carlo procedure an estimate of underlying quantities of interest.

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