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On the strong consistency of approximated M -estimators

Djalil CHAFAÏ and Didier CONCORDET

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Abstract

The aim of this article is to provide a strong consistency Theorem for approximated M -estimators. It contains both Wald and Pfanzagl type results for maximum likelihood. The proof relies, in particular, on the existence of a sort of contraction of the parameter space which admits the true parameter as a fixed point. In a way, it can be seen as a simplification of ideas of Wang and Pfanzagl, generalised to approximated M -estimators. Proofs are short and elementary.

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1 Introduction

After the seminal work¹ of Fisher during the first third of the twentieth century, the asymptotic properties of maximum likelihood estimators, and in particular their consistency, were studied by various authors, including Doob [Doo34], Cramér [Cra46], and Huzurbazar [Huz48]. Nowadays, one of the most known result regarding consistency goes back to Wald, who gave in [Wal49] a short and elegant proof of strong consistency of parametric maximum likelihood estimators. Since that time, several authors studied various versions of such consistency problems, including among others, Le Cam

¹The interested reader may find a quite recent account in [Ald97] and references therein.

[LC53], Kiefer and Wolfowitz [KW56], Bahadur [Bah67], Huber [Hub67], Perlman [Per72], Wang [Wan85], and Pfanzagl [Pfa88, Pfa90].

Wald's original proof relies roughly on local compactness of the parameter space, on continuity and coercivity of the log-likelihood, on the law of large numbers, and last but not least on local uniform integrability of the log-likelihood. It does not require differentiability, and makes extensive use of likelihood ratios. One can find a modern presentation of the method of Wald for M -estimators in van der Vaart's monography [vdV98]. Kiefer and Wolfowitz have shown in [KW56] that a compactification trick allows to extend Wald's approach to semiparametric situations, provided that a local uniform integrability holds. In particular, they successfully address a semiparametric mixture model example. However, the local uniform integrability condition makes such results of difficult usage in many semiparametric and nonparametric situations.

Mixture models are very delicate inverse problems, which pose serious difficulties even in relatively simple cases, cf. for example [EL01], [MP00], [BL97], [Ish99], [Pat01], [JPM01], [CRI03], [BMV04], [FLS05], and references therein. It turns out that nonparametric mixture models are linear and thus concave with respect to the parameter. Pfanzagl gave in [Pfa88, Pfa90] a proof of consistency of approximated maximum likelihood estimators for nonparametric concave models, including nonparametric mixtures. His approach relies in particular on a simplification of an earlier work of Wang in [Wan85] based on uniform local bound of the likelihood ratio. Our Theorem can be seen as a unification of Pfanzagl's result with van der Vaart's formulation of Wald's result. We replace the log-likelihood ratio by contrast differences. We do not assume any coercivity of the contrast as in [Wal49]. However, we require the compactness of the space of the estimated parameter, as in [KW56] and [vdV98] for example. It turns out that such a compactness comes for free when dealing with fully nonparametric models. We do not make use of any Uniform Law of Large Numbers. In other words, our result does not belong to the Glivenko-Cantelli approaches of consistency, as in [Dud98], [Fio00], [AK94] for example, see also [vdV98] and [vdG03, vdG00] and references therein. The article ends up with an examination of some mixture models, following the historical example of Kiefer and Wolfowitz and the motivations of Pfanzagl. We emphasise the simplicity of the proofs.

Let Θ be a separable Hausdorff topological space with countable base. Let $(P_\theta)_{\theta \in \Theta}$ be a known family of Borel measures on a measurable space \mathcal{X} . Let $\theta^* \in \Theta$ be some unknown point of Θ such that $P^* := P_{\theta^*}$ is a probability

measure. Let $(X_n)_{n \in \mathbb{N}}$ be an i.i.d. sequence of observed random variables taking their values in \mathcal{X} with common law P^* . Let $(\hat{\theta}_n)_{n \in \mathbb{N}}$ be a random sequence of Θ such that $\hat{\theta}_n$ is \mathcal{F}_n -measurable, where $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$. We say that $(\hat{\theta}_n)_{n \in \mathbb{N}}$ is *strongly consistent* if and only if

$$P^* - \text{a.s.} \quad \lim_{n \rightarrow +\infty} \hat{\theta}_n = \theta^*. \quad (1)$$

We use in the sequel the abbreviations “a.s.” for *almost sure*, “a.a.” for *almost all*, and “a.e.” for *almost everywhere*. Let $\Theta \times \mathcal{X} \ni (\theta, x) \mapsto m(\theta, x) \in \mathbb{R}$ be a known function such that $m_\theta := m(\theta, \cdot)$ is measurable for any $\theta \in \Theta$. For any n , we define the random function $M_n : \Theta \rightarrow \mathbb{R}$ by

$$M_n(\theta) := \frac{1}{n} \sum_{i=1}^n m(\theta, X_i).$$

This can be written also $M_n(\theta) = \mathbb{P}_n m_\theta$ where $\mathbb{P}_n := \frac{1}{n}(\delta_{X_1} + \dots + \delta_{X_n})$ is the empirical measure. We say that $(\hat{\theta}_n)_n$ is a *sequence of approximated M -estimators* if and only if

$$P^* - \text{a.s.} \quad \overline{\lim}_{n \rightarrow +\infty} \left(\sup_{\Theta} M_n - M_n(\hat{\theta}_n) \right) = 0. \quad (2)$$

Suppose that for large enough n , there exists an \mathcal{F}_n -measurable $\hat{\theta}_n$ in Θ such that $M_n(\hat{\theta}_n) = \sup_{\Theta} M_n$, then such a random sequence $(\hat{\theta}_n)_{n \in \mathbb{N}}$ fulfils (2).

For any probability measure P on \mathcal{X} , let $L_+^1(\mathcal{X}, P)$ (resp. $L_-^1(\mathcal{X}, P)$) be the set of random variables $Z : \mathcal{X} \rightarrow \mathbb{R}$ such that $Z^+ := \max(+Z, 0)$ (resp. $Z^- := \max(-Z, 0)$) is in $L^1(\mathcal{X}, P)$. On $E(\mathcal{X}, P) := L_-^1(\mathcal{X}, P) \cup L_+^1(\mathcal{X}, P)$, the expectation $P(Z) = P(Z^+) - P(Z^-)$ makes sense and takes its values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. For any $\theta \in \Theta$ such that $m_\theta \in E(\mathcal{X}, P^*)$, we define the *contrast* $M^*(\theta) \in \overline{\mathbb{R}}$ by

$$M^*(\theta) := P^* m_\theta. \quad (3)$$

In the sequel, we say that the model is *identifiable* when for any $\theta \in \Theta$, the condition $P_\theta = P^*$ implies that $\theta = \theta^*$.

Example 1.1 (Log-Likelihood). Assume that for some fixed Borel measure Q on \mathcal{X} , one has $P_\theta \ll Q$ for any $\theta \in \Theta$. Let $f_\theta := dP_\theta/dQ$ and assume that $f_\theta > 0$ on \mathcal{X} for any $\theta \in \Theta$. Define $m(\theta, x) := \log(f_\theta(x))$. Then $M_n : \Theta \rightarrow \mathbb{R}$ is the *log-likelihood* random functional given by $M_n(\theta) = \mathbb{P}_n m_\theta = \mathbb{P}_n \log(f_\theta)$.

We will speak about sequences of “approximated maximum likelihood estimators”. The log-likelihood ratio is

$$M_n(\theta_1) - M_n(\theta_2) = \mathbb{P}_n \log(f_{\theta_1}/f_{\theta_2}).$$

As usual for the log-likelihood, when $M^*(\theta^*)$ is finite, one can write for any θ

$$M^*(\theta) - M^*(\theta^*) = -\mathbf{Ent}(P_{\theta^*} | P_{\theta}),$$

where $\mathbf{Ent}(P_{\theta_1} | P_{\theta_2})$ is the Kullback-Leibler relative entropy of P_{θ_1} with respect to P_{θ_2} . In particular, $M^*(\theta) \leq M^*(\theta^*)$ with equality if and only if $P_{\theta} = P_{\theta^*}$, which implies $\theta = \theta^*$ if the model is identifiable. Notice that when Q is the Lebesgue measure on $\mathcal{X} = \mathbb{R}^n$, then $-M^*(\theta^*) = -\int_{\mathcal{X}} f_{\theta^*}(x) \log(f_{\theta^*}(x)) dx$ is the Shannon entropy of f_{θ^*} .

Example 1.2 (Beyond the log-likelihood). Assume that for some fixed Borel measure Q on \mathcal{X} , one has $P_{\theta} \ll Q$ for any $\theta \in \Theta$, with $P_{\theta}(\mathcal{X}) \leq 1$ and $f_{\theta} := dP_{\theta}/dQ$. Let $\Phi, \Psi : (0, +\infty) \rightarrow \mathbb{R}$ be two smooth functions. Assume that $\Psi(f_{\theta}) \in L^1(\mathcal{X}, Q)$ for any $\theta \in \Theta$. Define m_{θ} by

$$m_{\theta} = \Phi(f_{\theta}) - \int_{\mathcal{X}} \Psi(f_{\theta}) dQ + P_{\theta}(\mathcal{X}).$$

This gives rise the the following empirical contrast

$$M_n(\theta) = \mathbb{P}_n(\Phi(f_{\theta})) - \int_{\mathcal{X}} \Psi(f_{\theta}) dQ + P_{\theta}(\mathcal{X}).$$

In particular, if $\theta \in \Theta$ is such that $\Phi(f_{\theta}) \in L^1(\mathcal{X}, P^*)$ where here again $P^* := P_{\theta^*}$,

$$M^*(\theta) = P^*(\Phi(f_{\theta})) - \int_{\mathcal{X}} \Psi(f_{\theta}) dQ + P_{\theta}(\mathcal{X}).$$

Assume now that $u \mapsto u\Phi'(u)$ is locally integrable on \mathbb{R}_+ , and consider the case where Ψ is the Φ -transform given for any $u \in (0, +\infty)$ by

$$\Psi(u) = \int_0^u v\Phi'(v) dv.$$

For $\Phi : u \mapsto \log(u)$, one has $\Psi : u \mapsto u$ and we recover the log-likelihood contrast

$$M^*(\theta) = P^*(\log(f_{\theta})).$$

For $\Phi : u \mapsto u$, one has $\Psi : u \mapsto \frac{1}{2}u^2$, and we get the quadratic contrast

$$M^*(\theta) = -\frac{1}{2}\|f_\theta - f_{\theta^*}\|_{L^2(\mathcal{X}, Q)}^2 + \frac{1}{2}\|f_{\theta^*}\|_{L^2(\mathcal{X}, Q)}^2 + P_\theta(\mathcal{X}).$$

In both cases, the map $\theta \mapsto M^*(\theta)$ admits θ^* as unique maximum provided that the model is identifiable. More generally, define the Φ -transform $\Theta : (0, +\infty)^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} \Theta(u, v) &:= u\Phi(v) - \Psi(v) \\ &= u\Phi(v) - \int_0^v w\Phi'(w) dw. \end{aligned}$$

When θ and θ^* are such that both $\Theta(f_{\theta^*}, f_{\theta^*})$ and $\Theta(f_{\theta^*}, f_\theta)$ belong to $L^1(\mathcal{X}, Q)$,

$$M^*(\theta) = \int_{\mathcal{X}} (\Theta(f_{\theta^*}, f_\theta) - \Theta(f_{\theta^*}, f_{\theta^*})) dQ + \int_{\mathcal{X}} \Theta(f_{\theta^*}, f_{\theta^*}) dQ + P_\theta(\mathcal{X}).$$

Notice that Θ is linear in Φ . One can consider useful examples for which the function Φ is bounded, in such a way that m_θ is bounded for any $\theta \in \Theta$. For instance, let us examine the case where $\Phi : u \mapsto -(1+u)^{-2}$. Then, $\Psi : u \mapsto -u^2(1+u)^{-2}$, and the map $\theta \mapsto M^*(\theta)$ admits θ^* as unique maximum, provided identifiability, since for any $(u, v) \in \mathbb{R}_+^2$,

$$\Theta(u, v) = -\frac{u+v^2}{(1+v)^2} \quad \text{and} \quad \Theta(u, v) - \Theta(u, u) = -\frac{(v-u)^2}{(1+u)(1+v)^2}.$$

The function Ψ is additionally bounded here. The similar case $\Phi : u \mapsto -(1+u^2)^{-1}$ is also quite interesting. Notice that $\Theta(u, \cdot)$ is concave on $(0, +\infty)$ as soon as Φ is concave, non decreasing, with $\Phi'(v) + v\Phi''(v) \geq 0$ for any $v > 0$. Observe that this is not the approach of Pfanzagl in [Pfa90], which is more related to the log-likelihood ratio. Notice that in the case of the log-likelihood, one has $\Phi : u \mapsto \log(u)$, which gives $\Psi : u \mapsto u$ and $\Theta : (u, v) \mapsto -u \log(v) - v$, and thus $\Theta(u, v) - \Theta(u, u) = u \log(u/v) + u - v$. It might be possible to extensively analyse such “ Φ -estimators”, in the spirit of the “ Φ -calculus” developed in [Cha04, Cha05]. This will be hopefully the subject of a forthcoming article, with possible links with [BM93].

One can notice that the observation of Lindsay in [Lin83a, Lin83b] regarding the nature of maximum likelihood for nonparametric mixture models remains valid for more general models provided that m is concave.

2 Main result and Corollaries

With the settings given in the Introduction, the following Theorem holds.

Theorem 2.1. *Assume that Θ is compact and that the following assumptions hold.*

- (A1) *For P^* -a.a. $x \in \mathcal{X}$, the map $m(\cdot, x)$ is continuous on Θ ;*
- (A2) *There exists a continuous map $a^* : \Theta \rightarrow \Theta$ which may depend on θ^* such that for any $\theta \neq \theta^*$, there exists a neighbourhood $V \subset \Theta$ of θ for which $\sup_V (m - m_{a^*}) \in L^1_+(\mathcal{X}, P^*)$ and $P^*(m_\theta - m_{a^*(\theta)}) < 0$.*

Then any sequence $(\hat{\theta}_n)_n$ of approximated M -estimators is strongly consistent.

Proof. Postponed to section 4. □

The quantity $P^*(m_\theta - m_{a^*(\theta)})$ in (A2) has a meaning in $\overline{\mathbb{R}}$ since the first part of (A2) ensures that $m_\theta - m_{a^*(\theta)} \in L^1_+(\mathcal{X}, P^*)$. Moreover, $P^*(m_\theta - m_{a^*(\theta)})$ reads $M^*(\theta) - M^*(a^*(\theta))$ when the couple $(m_\theta, m_{a^*(\theta)})$ is in $L^1_-(\mathcal{X}, P^*) \times L^1_+(\mathcal{X}, P^*)$ or in $L^1_+(\mathcal{X}, P^*) \times L^1_-(\mathcal{X}, P^*)$.

Since θ^* is unknown in practice, each assumption in Theorem 2.1 must hold for any $\theta^* \in \Theta$ such that P_{θ^*} is a probability measure, in order to make the result useful.

The assumptions (A1) and (A2) required by Theorem 2.1 are far to be as weak as possible. However, they permit a lightweighted presentation. The first part of (A2) is in a way an M -estimator version of the integrability condition considered by Kiefer and Wolfowitz for the log-likelihood. As stated in the following Corollary, Theorem 2.1 implies a version of Wald consistency Theorem for approximated M -estimators, cf. [Wal49] and [vdV98, Theorem 5.14].

Corollary 2.2 (van der Vaart-Wald). *Assume that Θ is compact, and that for P^* -a.a. $x \in \mathcal{X}$, the map $m(\cdot, x)$ is continuous on Θ . Assume that for any θ in Θ , there exists a neighbourhood V such that $\sup_V m \in L^1(\mathcal{X}, P^*)$. Assume in addition that M^* achieves its supremum over Θ at θ^* , and only at θ^* . Then, any sequence of approximated M -estimator is strongly consistent.*

Proof. One has $m_\theta \in L^1(\mathcal{X}, P^*)$ for any θ in Θ , and thus $M^* : \Theta \rightarrow \mathbb{R}$ is well defined. Moreover, (A2) holds with a constant map $a^* \equiv \theta^*$. Namely, for

any $\theta \neq \theta^*$, one has in one hand $P^*(m_\theta - m_{\theta^*}) < 0$ since $M^*(\theta) < M^*(\theta^*)$, and in the other hand

$$\sup_V (m - m_{a^*}) = -m_{\theta^*} + \sup_V m \in L^1(\mathcal{X}, P^*).$$

□

As stated in the following Corollary, Theorem 2.1 implies the main result of Pfanzagl in [Pfa88] for concave models, itself based on an earlier result of Wang in [Wan85]. This is typically the case for mixtures models, for which Θ is a convex set of probability measures on some measurable space, cf. section 3.

Corollary 2.3 (Pfanzagl-Wang). *Let Q be a reference Borel measure on \mathcal{X} . Consider the case where Θ is a convex compact subset of a linear space such that for any $\theta \in \Theta$, $P_\theta(\mathcal{X}) \leq 1$ and $P_\theta \ll Q$ with $f_\theta := dP_\theta/dQ > 0$ on \mathcal{X} . Suppose that Q -a.e. on \mathcal{X} , the map $\theta \mapsto f_\theta(x)$ is concave and continuous on Θ . Assume that the model is identifiable. Consider $m_\theta := \log(f_\theta)$ and the related log-likelihood M_n . Then any sequence of approximated log-likelihood estimators is strongly consistent.*

Proof. For an arbitrary $\lambda \in (0, 1)$, let us take $a^*(\theta) := \lambda\theta^* + (1 - \lambda)\theta$. The concavity of the model yields

$$m_{a^*(\theta)} - m_\theta = \log \left(\frac{f_{\lambda\theta^* + (1-\lambda)\theta}}{f_\theta} \right) \geq \log \left(\frac{\lambda f_{\theta^*} + (1 - \lambda)f_\theta}{f_\theta} \right) \geq \log(1 - \lambda).$$

One has $\log(1 - \lambda) \in L^1(\mathcal{X}, P^*)$ since $\lambda < 1$. Define the function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $\Phi(u) := u \log(\lambda u + (1 - \lambda))$. The concavity of the model yields

$$P^*(m_{a^*(\theta)} - m_\theta) \geq \int_{\mathcal{X}} f_{\theta^*} \log \left(\frac{\lambda f_{\theta^*} + (1 - \lambda)f_\theta}{f_\theta} \right) dQ = \int_{\mathcal{X}} \Phi \left(\frac{f_{\theta^*}}{f_\theta} \right) f_\theta dQ.$$

Let us show that the right hand side of the inequality above is strictly positive when $\theta \neq \theta^*$. One has $P_\theta(\mathcal{X}) > 0$ since $f_\theta > 0$. Define $\Psi(u) := u\Phi(1/u)$. Jensen's inequality for the probability measure $P_\theta(\mathcal{X})^{-1}P_\theta$ and the convex function Φ yields

$$\int_{\mathcal{X}} \Phi \left(\frac{f_{\theta^*}}{f_\theta} \right) f_\theta dQ \geq \Psi(P_\theta(\mathcal{X})). \quad (4)$$

It is enough to show that either (4) is strict or the right hand side of (4) is strictly positive. Since $\lambda > 0$, the function Φ is strictly convex. Thus

equality holds in (4) if and only if $P_\theta(f_{\theta^*} = \alpha f_\theta) = 1$ for some $\alpha \in \mathbb{R}_+$. The only admissible case is $\alpha = P_\theta(\mathcal{X})^{-1} > 1$ since $P_{\theta^*}(\mathcal{X}) = 1$ and since identifiability forbids $P_\theta(f_{\theta^*} = f_\theta) = 1$. Therefore, if $P_\theta(\mathcal{X}) = 1$, inequality (4) is necessarily strict. In the other hand, $\Psi(1) = 0$ and $\Psi(u) > 0$ when $u < 1$. Thus the right hand side of (4) is always non negative, and is strictly positive as soon as $P_\theta(\mathcal{X}) < 1$. We conclude that $P^*(m_{a^*(\theta)} - m_\theta) > 0$ as soon as $\theta \neq \theta^*$. This shows that **(A2)** holds with $V = \Theta$, and the proof is thus completed. \square

Remark 2.4 (About the map a^*). Let $a^* : \Theta \rightarrow \Theta$ be a map which satisfies the condition $P^*(m_\theta - m_{a^*(\theta)}) < 0$ for any $\theta \neq \theta^*$ of **(A2)**. Then,

- a^* never meets the diagonal, and in particular, it cannot be the identity map;
- if a^* is constant, then $a^* \equiv \theta^*$;
- θ^* is the unique fixed point of a^* , and for any θ , the sequence $(\theta_n)_{n \in \mathbb{N}}$ defined by $\theta_0 := \theta$ and $\theta_{n+1} = a^*(\theta_n)$ converges towards θ^* provided that Θ is compact. One can notice that the existence of a fixed point can be related to Brouwer like fixed point Theorems, cf. [Goe02].

In some sense, the map a^* is a contraction around θ^* , and that is clear for instance on the specific a^* maps considered in the proofs of Corollaries 2.2 and 2.3.

Remark 2.5 (Infinite values of m). Theorem 2.1 does not allow m to take the value $-\infty$. This limitation is due to the fact that differences of the form $m_\theta - m_{\theta'}$ do not make sense if m is allowed to take the value $-\infty$. The consistency proof of Wald does not suffer from such a limitation since it does not rely on m differences, but it requires however strong uniform integrability assumptions. A careful reading of the proof of Theorem 2.1 shows that only differences of the form $m_\theta - m_{a^*(\theta)}$ are involved. In the other hand, according to Remark 2.4, a^* never meet the diagonal. Consequently, one may allow, in Theorem 2.1, the map $m(\theta, x)$ to take the value $-\infty$ for at most one value of θ . For the log-likelihood, $m_\theta = \log(f_\theta)$ and one has $m_\theta(x) = -\infty$ if and only if $f_\theta(x) = 0$. One may allow $f_\theta \equiv 0$ for at most one value of θ in Corollary 2.3.

Remark 2.6. Let $\theta \in \Theta$ such that $m_\theta \in E(\mathcal{X}, P^*)$. Then, the law of large numbers applies and gives that P^* -a.s., $\lim_n M_n(\theta) = M^*(\theta) \in \overline{\mathbb{R}}$, and the

a.s. subset of \mathcal{X} may depend on θ . In particular $M_n(\theta) = M^*(\theta) + o_P(1)$. For a sequence $(\hat{\theta}_n)_n$ satisfying (2), one can write for any $\theta \in \Theta$ with finite $M_n(\theta)$

$$\begin{aligned} M_n(\hat{\theta}_n) &= M_n(\hat{\theta}_n) - M_n(\theta) + M_n(\theta) \\ &\geq -\left(\sup_{\Theta} M_n - M_n(\hat{\theta}_n)\right) + M_n(\theta) \\ &= o_P(1) + M(\theta) \end{aligned}$$

where the last step follows by (2) and the law of large numbers.

3 Log-Likelihood and mixtures models

For any topological space \mathcal{Z} equipped with its Borel σ -field, we denote by $\mathcal{M}_1(\mathcal{Z})$ the set of probability measures on \mathcal{Z} , and by $\mathcal{C}_b(\mathcal{Z})$ the set of bounded real valued continuous functions on \mathcal{Z} . The Prohorov topology on $\mathcal{M}_1(\mathcal{Z})$ is defined as follows: $\theta_n \rightarrow \theta$ in $\mathcal{M}_1(\mathcal{Z})$ if and only if $\int_{\mathcal{Z}} f d\theta_n \rightarrow \int_{\mathcal{Z}} f d\theta$ for any $f \in \mathcal{C}_b(\mathcal{Z})$. It is known that a subset of $\mathcal{M}_1(\mathcal{Z})$ is compact if and only if it is tight. As a consequence, $\mathcal{M}_1(\mathcal{Z})$ is not compact in general. Following [Pfa88, section 5 page 149], the set sub-probabilities provides a compactification which allows the following consistency result for approximated log-likelihood estimators of nonparametric mixture models.

Corollary 3.1 (Pfanzagl). *Let \mathcal{Z} be a locally compact Hausdorff topological space with countable base. Let Q be a measure on a measurable space \mathcal{X} . Let $k : \mathcal{X} \times \mathcal{Z} \rightarrow (0, +\infty)$ be such that $\int k(x, z) dQ(x) = 1$ for any $z \in \mathcal{Z}$ and $k(x, \cdot) \in \mathcal{C}_b(\mathcal{Z})$ for any $x \in \mathcal{X}$. Let $\Theta := \mathcal{M}_1(\mathcal{Z})$ and consider the family $(P_\theta)_{\theta \in \Theta}$ of probability measures on \mathcal{X} defined by $dP_\theta = f_\theta dQ$ with $f_\theta(x) := \int k(x, z) d\theta(z)$. Assume that the model is identifiable. Let $m : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$ be the map defined by $m(\theta, x) := \log f_\theta(x)$, and M_n be the corresponding log-likelihood. Then any sequence of approximated maximum likelihood estimators is strongly consistent for the Prohorov topology.*

A mixture model can always be seen as a conditional model. The observed random variables X with values in \mathcal{X} is the first component of the couple (X, Z) with values in $\mathcal{X} \times \mathcal{Z}$. The component Z is not observed. However, the conditional law $\mathcal{L}(X | Z = z)$ is known, and has density $k(\cdot, z)$ with respect to Q on \mathcal{X} . If $\theta = \mathcal{L}(Z)$, then $\mathcal{L}(X)$ has density f_θ with respect to Q on \mathcal{X} .

Proof. As explained above, $\Theta = \mathcal{M}_1(\mathcal{Z})$ is not compact for the Prohorov topology, and one must consider a well chosen compact over-set. Let $\mathcal{C}_0(\mathcal{Z})$ be the set of real valued continuous functions on \mathcal{Z} which vanish at infinity. Let $\overline{\Theta}$ be the set of Borel measures θ on \mathcal{Z} such that $\theta(\mathcal{Z}) \leq 1$ (i.e. sub-probabilities), equipped with the vague topology related to $\mathcal{C}_0(\mathcal{Z})$. Namely, $\theta_n \rightarrow \theta$ in $\overline{\Theta}$ if and only if $\int_{\mathcal{Z}} f d\theta_n \rightarrow \int_{\mathcal{Z}} f d\theta$ for any $f \in \mathcal{C}_0(\mathcal{Z})$. The injection $\Theta \subset \overline{\Theta}$ is continuous; $\overline{\Theta}$ is a compact metrisable topological space, and thus has a countable base. Moreover, $\overline{\Theta}$ is convex, and for any $\theta \in \overline{\Theta}$, there exists $\theta' \in \Theta$ and $\alpha \in [0, 1]$ such that $\theta = \alpha\theta'$.

We extend the set of probability measures $(P_\theta)_{\theta \in \Theta}$ on \mathcal{X} by the set of sub-probability measures $(P_\theta)_{\theta \in \overline{\Theta}}$ on \mathcal{X} , where $dP_\theta = f_\theta dQ$ and $f_\theta(x) := \int k(x, z) d\theta(z)$. One has by virtue of Fubini-Tonelli Theorem that $P_\theta(\mathcal{X}) = \theta(\mathcal{Z})$, and thus $P_\theta \in \mathcal{M}_1(\mathcal{X})$ if and only if $\theta \in \Theta := \mathcal{M}_1(\mathcal{Z})$. Notice that θ^* is taken in Θ .

Let $\theta \in \overline{\Theta}$ such that $P_\theta = P_{\theta^*}$. Since θ^* is taken in Θ , one has that $P_\theta \in \mathcal{M}_1(\mathcal{X})$, therefore $\theta \in \Theta$ and thus $\theta = \theta^*$ by identifiability in Θ . Notice that $\overline{\Theta}$ is the convex envelope of $\Theta \cup \{0\}$. The set $\overline{\Theta}$ contains the null measure 0, for which $f_0 \equiv 0$ and thus $m_0 \equiv -\infty$. If $\theta \in \overline{\Theta}$ with $\theta \neq 0$, then $f_\theta > 0$ on \mathcal{X} since $k > 0$, and thus $m_\theta(x) := \log f_\theta(x)$ is finite for any $x \in \mathcal{X}$. For any $x \in \mathcal{X}$, the map $\theta \in \overline{\Theta} \mapsto m_\theta(x)$ is continuous since $k(x, \cdot)$ is in $\mathcal{C}_0(\mathcal{Z})$.

For any $\theta \in \overline{\Theta}$ with $\theta \neq 0$, one can write $\theta = \alpha\theta'$ with $\theta' \in \Theta$ and $\alpha := \theta(\mathcal{Z}) \in [0, 1]$. One has then $f_\theta = \alpha f_{\theta'}$ and thus $m_\theta = \log \alpha + m_{\theta'}$. Therefore,

$$M_n(\theta) = \log \alpha + M_n(\theta') \leq M_n(\theta').$$

As a consequence, $\sup_{\theta \in \Theta} M_n(\theta) = \sup_{\theta \in \overline{\Theta}} M_n(\theta)$, and one may substitute Θ by $\overline{\Theta}$ in the definition (2). Now, let $(\hat{\theta}_n)_{n \in \mathbb{N}}$ be a sequence in Θ of approximated maximum likelihood estimators. Corollary 2.3 and Remark 2.5 for $(P_\theta)_{\theta \in \overline{\Theta}}$ apply and give the P^* -a.s. convergence for the vague topology of $(\hat{\theta}_n)_{n \in \mathbb{N}}$ towards θ^* . Since both the sequence and the limit are in Θ , the convergence holds for the Prohorov topology, and the desired result is established. \square

An emblematic mixture model example is the location scale Gaussian mixture, for which $\mathcal{X} = \mathbb{R}^p$, and $k(\cdot, z)$ is the probability density function of the Gaussian distribution $\mathcal{N}(\alpha(z), \text{diag}(\beta(z)^2))$ on \mathbb{R}^p , where α and β are known functions from \mathcal{Z} to \mathbb{R}^p . Let us give a more general example. Let $\alpha : \mathcal{Z} \rightarrow \mathbb{R}^p$ and $\beta : \mathcal{Z} \rightarrow \text{GL}_p(\mathbb{R})$ be two continuous functions, where $\text{GL}_p(\mathbb{R})$

denotes the linear group of \mathbb{R}^p . Let g be a probability density function on \mathcal{X} with respect to some fixed Borel measure Q . The support of g may be strictly smaller than \mathcal{X} . Consider the mixture model with kernel k given for any $(x, z) \in \mathcal{X} \times \mathcal{Z}$ by

$$k(x, z) = g(\beta(z)^{-1}(x - \alpha(z)))\det(\beta(z))^{-1}.$$

The associated model is often referred as a “location scale mixture”, with “location function” α , “scale function” β , “base density” g , and “Markov kernel” k . It can also be seen as a particular elliptic model. Geometrically speaking, one can see k as the result of the action on g of the similitude group of \mathcal{X} equipped with a law constructed from an unknown law on \mathcal{Z} via α and β . When β is constant, the corresponding mixture models are far more simple. However, such pure location models are less realistic in many non trivial applications. Recall that the triple α, β, k is known, and that $k(\cdot, z)$ is the probability density function with respect to Q of the conditional law $\mathcal{L}(X | Z = z)$. One can write X in terms of Z as

$$X = \alpha(Z) + \beta(Z)(\varepsilon),$$

where ε an independent random variable on \mathcal{X} , independent of Z , with probability density g . We set typically $\Theta := \mathcal{M}_1(\mathcal{Z})$, and we are interested for instance in the nonparametric estimation of the law of Z , from the observation of an i.i.d. sample of X . The random variables Z and ε are not observed, and we end up actually with a stochastic inverse problem, cf. [CL04]. It turns out that the method of Kiefer and Wolfowitz in [KW56] allows to derive the consistency of the associated nonparametric maximum likelihood estimator (NPMLE), provided quite reasonable assumptions on α, β and g . Pfanzagl’s result in [Pfa88] shows in particular that such a consistency remains valid for approximate NPMLE. Finally, our result allows to switch to more general approximate M -estimators. Lindsay developed an alternative approach based on convexity, cf. [Lin95, LL95], for which the result of Pfanzagl ensures consistency. The approach of Kiefer and Wolfowitz permits to consider semiparametric extensions of mixture models, where α and β may depend on an extra finite dimensional parameter, which is jointly estimated. However, these approaches do not lead to any asymptotic normality for the infinite dimensional part of the parameter. The consistency rates of convergence are also of difficult access. A true alternative approach, based on the seminal work of Dudley, cf. [Dud98] and [vdVW00], is to consider Uniform Law

of Large Numbers (ULLN). It relies roughly on empirical processes analysis based on Glivenko-Cantelli-Donsker like functional classes, and allows the derivation of Hellinger consistency of the NPMLE, under conditions on k which are stronger than the one considered by Kiefer and Wolfowitz, cf. [vdG00, Ex. 4.2.4]. To the authors knowledge, the role of Hellinger distance for maximum likelihood goes back to Le Cam, cf. for instance [LC86, LCY00]. The reader may find an elegant presentation in the survey [vdG03] and in the monography [vdG00] by van de Geer. Despite the genericity loss on k , the ULLN approach permits to derive the asymptotic normality for some linear functionals of the infinite dimensional parameter. Notice that the finite dimensional part of the parameter in semiparametric mixture models may often be analysed alone by considering the infinite dimensional part as a nuisance, cf. for instance [vdV98].

4 Proof of main result

One can replace “= 0” by “ ≤ 0 ” in (2) by definition of the supremum. Actually, (2) is equivalent to state that P^* – a.s. –, for any sequence $(\theta_n)_n$ of Θ

$$\overline{\lim}_{n \rightarrow +\infty} \left(M_n(\theta_n) - M_n(\widehat{\theta}_n) \right) \leq 0. \quad (5)$$

The necessity comes from $M_n(\theta_n) \leq \sup_{\Theta} M_n$ by definition of the supremum. The sufficiency is due to the fact that by definition of the supremum, one can take θ_n such that $\sup_{\Theta} M_n \leq M_n(\theta_n) + 1/n$.

Lemma 4.1. *Assume the following separation property: P^* – a.s. –, for any neighbourhood U of θ^* , for any sequence $(\theta_n)_n$ in U^c , there exists a sequence $(\theta'_n)_n$ in Θ such that*

$$\underline{\lim}_{n \rightarrow +\infty} (M_n(\theta'_n) - M_n(\theta_n)) > 0. \quad (6)$$

Then, any approximated M-estimator sequence $(\widehat{\theta}_n)_n$ is strongly consistent.

Proof. Let $(\widehat{\theta}_n)_n$ be a sequence of approximated M-estimators. Assume condition (6) holds and that $(\widehat{\theta}_n)_n$ is not strongly consistent. Then, for any measurable subset $A \subset \mathcal{X}$ with $P^*(A) = 1$, there exists a neighbourhood U of θ^* and a subsequence $(\widehat{\theta}_{n_k})_k$ in U^c on A . For such a subsequence, (6) holds P^* –a.s., and contradicts (5) which holds P^* –a.s. too. \square

Proof of Theorem 2.1. The desired result follows from Lemma 4.1. Let us show that (6) is a consequence of the following property: there exists a map $a^* : \Theta \rightarrow \Theta$ such that for any $\theta \neq \theta^*$, there exists a neighbourhood U_θ of θ such that

$$P^* - \text{a.s.}, \quad \varliminf_{n \rightarrow +\infty} \inf_{U_\theta} (M_n(a^*) - M_n) > 0. \quad (7)$$

Namely, let U be an open neighbourhood of θ^* . For any $\theta \in U^c$, let U_θ be the neighbourhood of θ given by (7), on the P^* -a.s. set A_θ . Then, $U^c = \cup_{\theta \in U^c} U_\theta$ is compact as a closed subset of the compact set Θ . One can thus extract a finite sub-covering $U^c = \cup_{i=1}^k U_{\theta_i}$, and write

$$\begin{aligned} \varliminf_n \inf_{U^c} (M_n(a^*) - M_n) &= \varliminf_n \min_{1 \leq i \leq k} \inf_{U_{\theta_i}} (M_n(a^*) - M_n) \\ &\geq \min_{1 \leq i \leq k} \varliminf_n \inf_{U_{\theta_i}} (M_n(a^*) - M_n). \end{aligned}$$

By virtue of (7) we get from the above that

$$P^* - \text{a.s.}, \quad \varliminf_n \inf_{U^c} (M_n(a^*) - M_n) > 0,$$

where the P^* -a.s. set is $A_U := \cap_{i=1}^k A_{\theta_i}$. Let $(U_k)_k$ be a countable base for θ^* , then the result holds on the P^* -a.s. subset $A := \cap_{i=1}^\infty A_{U_k}$, which does not depend on U . This gives (6) with $(\theta'_n)_{n \in \mathbb{N}} = (a^*(\theta_n))_{n \in \mathbb{N}}$ since

$$M_n(a^*(\theta_n)) - M_n(\theta_n) \geq \inf_{U^c} (M_n(a^*) - M_n)$$

as soon as $\theta_n \in U^c$ by definition of the infimum. This shows as announced that (6) follows from (7). Let us show now that (7) is a consequence of **(A2)**. Let $\theta \neq \theta^*$ and let a^* and V as in **(A2)**. Let $V_k \searrow \{\theta\}$ be a decreasing local base with $V_0 \subset V$. Let $Z_k := \sup_{V_k} (m - m_{a^*})$ and $Z := \sup_V (m - m_{a^*}) \in L_+^1(P)$. Notice that **(A2)** implies that $m_{a^*(\theta)} - m_\theta \in L_+^1(\mathcal{X}, P^*)$ and $P^*(m_{a^*(\theta)} - m_\theta) > 0$. By **(A1)** and the continuity of a^* and the separability of Θ , we get that $Z_k : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is measurable, and that P^* -a.s.

$$Z \geq Z_k \searrow Z_\infty = m_\theta - m_{a^*(\theta)}.$$

Fatou Lemma 4.2 for the sequence $(-Z_k)_k$ gives that

$$\varliminf_k P^* \left(\inf_{V_k} (m_{a^*} - m) \right) \geq P^*(m_{a^*(\theta)} - m_\theta) > 0.$$

One has then $P^*(\inf_{V_k}(m_{a^*} - m)) > 0$ for some k (actually for k large enough). Let us denote $U_\theta := V_k$. Now, by the law of large numbers

$$P^* - \text{a.s.}, \lim_n \mathbb{P}_n \left(\inf_{U_\theta} (m_{a^*} - m) \right) = P^* \left(\inf_{U_\theta} (m_{a^*} - m) \right) > 0.$$

This gives finally (7) since for any n

$$\inf_{U_\theta} (M_n(a^*) - M_n) = \inf_{U_\theta} \mathbb{P}_n(m_{a^*} - m) \geq \mathbb{P}_n \left(\inf_{U_\theta} (m_{a^*} - m) \right).$$

□

Lemma 4.2 (Fatou Lemma). *Let μ be a non-negative Borel measure on a measurable space \mathcal{X} . Let $(Z_n)_n$ be a sequence of measurable real valued functions. Assume that there exists $Z \in L^1_-(\mathcal{X}, \mu)$ such that μ -a.s., $Z \leq Z_n$ for any n . Then $\int \underline{\lim}_n Z_n d\mu$ makes sense in $[0, +\infty]$ and $\underline{\lim}_n \int Z_n d\mu \geq \int \underline{\lim}_n Z_n d\mu$.*

Proof. First, one can replace $Z \in L^1_-(\mathcal{X}, \mu)$ by $Z \in L^1(\mathcal{X}, \mu)$ since $Z \geq -Z^-$. The desired result follows from standard Fatou Lemma for the non-negative sequence $(Z_n - Z)_{n \in \mathbb{N}}$. Notice that $\underline{\lim}_n Z_n - Z \geq 0$ and thus $\int \underline{\lim}_n Z_n d\mu = \int \underline{\lim}_n (Z_n - Z) d\mu + \int Z d\mu$. □

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