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► **To cite this version:**

| Odinette Renée Abib. Geodesics in nilpotent Lie groups. 2007. hal-00005800v2

HAL Id: hal-00005800

<https://hal.science/hal-00005800v2>

Preprint submitted on 7 Dec 2007

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GEODESICS IN NILPOTENT LIE GROUPS

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Abstract: We study the sub-Riemannian geodesic problem in the $2n + 1$ -Heisenberg group. An independent set of $2n$ left-invariant vector fields determines a SR structure. We are interested in sub-Riemannian length minimizers for this structure. We use the hamiltonian formalism and apply Pontryagin maximum principle to write the equations for the extremals and describe some properties of the extremal curves. We consider geodesics in the group with a left-invariant Riemannian metric. We obtain their equations and prove that the set of directions of all rays issuing from zero is the sphere of unit horizontal directions.

Keywords: geometry, nonlinear control systems, distributions.

1. INTRODUCTION

Let L be the $2n+1$ -dimensional nilpotent Lie algebra with the following commutation rules in some basis $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n, T$:

$$\begin{aligned} [X_i, Y_j] &= 2\delta_{ij}T, \\ [X_i, X_j] &= [Y_i, Y_j] = \text{ad}_T = 0, \end{aligned}$$

and G be the connected, simply connected Lie group with the Lie algebra L . We consider X_i, Y_i, T as left-invariant vector fields on G . The set $\{X_1, Y_1, \dots, X_n, Y_n\}$ determines a left-invariant sub-Riemannian structure on G :

$$\begin{aligned} D &= \text{span}(X_1, Y_1, \dots, X_n, Y_n), \langle X_i, Y_j \rangle \\ &= \langle X_i, X_j \rangle = \langle Y_i, Y_j \rangle = \delta_{ij} \quad i, j = 1, 2, \dots, n, \end{aligned}$$

called the flat $(2n, 2n + 1)$ sub-Riemannian structure. Such structure is unique, up to isomorphism of Lie groups. We are interested in sub-Riemannian length minimizers for this sub-Riemannian structure, i.e., in solutions to the fol-

lowing optimal control problem:

$$\dot{\gamma} = \sum_{i=1}^n (u_i X_i(\gamma) + v_i Y_i(\gamma)) \quad (1)$$

$$\gamma \in G, u_i, v_i \in \mathbb{R},$$

$$\gamma(0) = q_0, \gamma(T) = q_1 \text{ fixed}, \quad (2)$$

$$l = \int_0^T \sqrt{\sum_{i=1}^n (u_i^2 + v_i^2)} \rightarrow \min. \quad (3)$$

Problem (1)-(3) was considered by Brockett(1981), and Liu and Sussmann (1995) for $n = 1$. G is a 2-step nilpotent Lie group and $L = D \oplus [L, L]$. In this note we consider also geodesic lines in the Lie group G with a left-invariant Riemannian metric.

The flat $(2n, 2n + 1)$ sub-Riemannian structure gives a local nilpotent approximation for an arbitrary sub-Riemannian structure with growth vector $(2n, 2n + 1)$, see Montgomery (2002). The dynamics of a classical electric charge in the plan un-

der the influence of a perpendicular magnetic field can be described by means of flat sub-Riemannian structure.

In general, if G is a 2-step nilpotent Lie group and $(D, \langle \cdot, \cdot \rangle)$ is a left-invariant sub-Riemannian structure such that $L = D \oplus [L, L]$, any minimizer is normal in some sub-group of G and hence any minimizer is smooth, see Abib (2002). This extends known results on the so-called Gaveau-Brockett problem, Brockett (1981) and Gaveau (1977).

2. MODEL

We choose the following model for the flat $(2n, 2n + 1)$ sub-Riemannian structure:

$G = \mathbb{R}^{2n+1}$ with coordinates $q = (x, z)$ where $x = (x_1, y_1, \dots, x_n, y_n), z \in \mathbb{R}$, and group law defined as follows:

$$(x, z)(x', z') = (x + x', z + z' + \sum_{i=1}^n x_i y'_i - \sum_{i=1}^n x'_i y_i)$$

$$X_i = \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial z}$$

$$Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}$$

$$D = \text{Ker } \theta$$

$$\theta = dz + \sum_{i=1}^n (y_i dx_i - x_i dy_i)$$

Control system (1) has the full rank and the state space G is connected, thus the systems is globally controllable on G . A standard existence theorem from optimal control theory, see Liu and Sussmann (1995), implies that any point can be connected with $q_0 \in G$ by an admissible length minimizer.

3. EXTREMALS

A admissible curve for D is an absolutely continuous curve on G which is tangent to D almost everywhere; applying the Pontryagin maximum principle each minimizer parametrized by arc-length is a normal extremal or abnormal extremal, see Liu and Sussmann (1995).

3.1 Abnormal extremals

An abnormal extremal is a admissible curve for D which is the projection onto G of characteristic curve (in the symplectic sense) of the annihilator D^\perp of D in T^*G . Introduce the linear Hamiltonians corresponding to the basis fields:

$$h_i(\lambda) = \lambda(X_i), k_i(\lambda) = \lambda(Y_i), h(\lambda) = \lambda(T),$$

$$\lambda \in T^*G.$$

D^\perp is defined by equations $h_i = k_i = 0, i = 1, \dots, n$. Assume $\Gamma(t) \in T^*G - \{0\}$ characteristic curve of D^\perp , then $\dot{\Gamma}(t) \in \text{span} \{\vec{h}_i(\Gamma(t)), \vec{k}_i(\Gamma(t)); i = 1, 2, \dots, n\}$ and $h_i(\Gamma(t)) = k_i(\Gamma(t)) = 0$.

Abnormal extremals are exactly the constant curves.

3.2 Normal extremals

Normal biextremals are trajectories of the Hamiltonian system

$$\dot{\lambda} = \vec{H}(\lambda), \lambda \in T^*G \quad (4)$$

with the sub-Riemannian Hamiltonian $H = \frac{1}{2} \sum_{i=1}^n (k_i^2 + h_i^2)$. In the coordinates $(q, \lambda) = (x_1, \dots, x_n, y_1, \dots, y_n, z, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n, \zeta)$ of T^*G system (4) reads

$$\dot{x}_i = \xi_i + y_i \zeta \quad (5)$$

$$\dot{y}_i = \eta_i - x_i \zeta \quad (6)$$

$$\dot{z} = \sum_{i=1}^n (\dot{x}_i y_i - \dot{y}_i x_i) \quad (7)$$

$$\dot{\xi}_i = \dot{y}_i \zeta \quad (8)$$

$$\dot{\eta}_i = \dot{x}_i \zeta \quad (9)$$

$$\dot{\zeta} = 0 \quad (10)$$

If the constant ζ is 0, the curve $(x(t), y(t))$ is the straight line, along which the motion takes place with velocity constant. The function $z(t)$ is then computed by solving (7) and the curve $(x(t), y(t), z(t))$ is a straight line in G . If ζ is non zero constant, the equations (8)-(9) imply

$$\ddot{x}_i - 2\dot{y}_i \zeta = 0, \quad \ddot{y}_i + 2\dot{x}_i \zeta = 0.$$

So

$$\begin{aligned} \dot{x}_i &= A_i \cos 2\zeta t + B_i \sin 2\zeta t, \\ \dot{y}_i &= -A_i \sin 2\zeta t + B_i \cos 2\zeta t, \\ A_i, B_i & \end{aligned}$$

are constant. The condition that our curve $q(t) = (x(t), y(t), z(t))$ is parametrized by arc-length then says that $\sum_{i=1}^n (A_i^2 + B_i^2) = 1$. We obtain

$$\begin{aligned} x_i(t) &= \frac{r_i}{2\zeta} (\cos \theta_i - \cos(2\zeta t + \theta_i)), \\ y_i(t) &= \frac{r_i}{2\zeta} (\sin(2\zeta t + \theta_i) - \sin \theta_i) \end{aligned}$$

with $A_i = r_i \sin \theta_i, B_i = r_i \cos \theta_i$. The equation (7) imply

$$\ddot{z} = \zeta \sum_{i=1}^n \frac{d}{dt} (x_i^2 + y_i^2)$$

and then $z(t) = \sum_{i=1}^n r_i^2 \left(\frac{t}{2\zeta} - \frac{\sin 2\zeta t}{(2\zeta)^2} \right)$.

The Pontryagin maximum principle, applied to this case, implies that all the minimizers curve parametrized by arc-length are normal extremal $q(t)$.

4. CONTACT TRANSFORMATIONS

For $n = 1$, the Lie algebra of symmetries of the flat (2,3) distribution D is parametrized by arbitrary smooth functions of three variables and the Lie algebra of symmetries of the flat (2,3) sub-Riemannian structure is 4-dimensional, see Sachkov (1998).

For $n \geq 1$, let C be the sheaf of germs of infinitesimal automorphisms X such that $L_X \theta = f\theta$, the function f depending on X . C is the sheaf of contact transformations.

Let $X \in C$ vanishing at 0, and ϕ_t be the one parameter group generated by X . Then $\phi_t(0) = 0$ and $\phi_t^* \theta = f_t \theta$. Let V_1 be the subspace of L defined by $\theta_0 = 0$. Then ϕ_t leaves V_1 invariant. Furthermore $\phi_t^* d\theta = df_t \wedge \theta + f_t d\theta$. This implies that the form $d\theta$ restricted to V_1 is preserved up to scalar factor. If we pass to the infinitesimal we see that the linear isotropy algebra of C is a subalgebra of the algebra g defined as follows:

let $u \in L$ not lying in V_1 and $g = K + N + M$ where K is the set of all $A \in \text{End}(L)$ with $Au = 0$ and $A/V_1 \in \text{sp}(V_1)$,

N is the of all B of the form $Bv_1 = 0$ for $v_1 \in V_1$ and $Bu \in V_1$,

M is the set of all multiples of the linear transformation C where $Cv_1 = v_1$ for $v_1 \in V_1$ and $Cu = 2u$.

The space $g^{(1)} \subset \text{Hom}(g, L)$ called the first prolongation of g , is the set of all $T \in \text{Hom}(g, L)$ which satisfy $(T(\omega))(v) = (T(v))(\omega)$ for all $\omega, v \in L$. In the case, $g^{(1)} = \text{sp}(V_1)^{(1)} + g$ and $g^{(k)} = \text{sp}(V_1)^{(k)} + g^{(k-1)}$ for all k . The algebra $\text{sp}(V_1)$ is of infinite type because $\text{sp}(V_1)^{(k)}$ can be identified with $S^{k+2}(V_1^*)$; then g is of infinite type.

Corresponding to the coordinates $z, x_1, \dots, x_n, y_1, \dots, y_n$, let us choose the basis $u = e_0, e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ of L . Then it is easy to verify that the following vector fields are all infinitesimal contact transformations (where

$$\theta = dz + \sum_{i=1}^n (y_i dx_i - x_i dy_i):$$

$$\alpha) \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}, \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z}, \frac{\partial}{\partial z}$$

$$\beta) \sum_{i,j=1}^n \left[A_{ij} x_i \frac{\partial}{\partial x_j} + A_{i+n,j} y_i \frac{\partial}{\partial x_j} + A_{i,j+n} x_i \frac{\partial}{\partial y_j} + A_{i+n,j+n} y_i \frac{\partial}{\partial y_j} \right]$$

$$\text{where } \sum_{r,s=1}^{2n} A_{rs} e_s \otimes e_r^* \in \text{sp}(V_1),$$

$$2z \frac{\partial}{\partial z} + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^n y_j \frac{\partial}{\partial y_j},$$

$$z \frac{\partial}{\partial x_i} + y_i \left[\sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) + z \frac{\partial}{\partial z} \right],$$

$$z \frac{\partial}{\partial y_i} + x_i \left[\sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) + z \frac{\partial}{\partial z} \right]$$

$$\gamma) z \left[\sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) + z \frac{\partial}{\partial z} \right]$$

From α) we see that C is transitive. From β) we see that the linear isotropy algebra of C is indeed g . A direct computation shows that the vector fields α), β) and γ) form a Lie algebra. C is an *non-flat, infinite, transitive* LAS (Lie algebra sheaf).

5. RIEMANNIAN CASE

We define the left-invariant metric on G by taking X_i, Y_i, T as the orthonormal frame. Let ∇ the Riemannian connection of left-invariant metric an $\{e_h, h = 1, 2, \dots, 2n + 1\} = \{X_1, Y_1, \dots, X_n, Y_n, T\}$. Then $\nabla_{e_i} e_j = \frac{1}{2} \sum_{k=1}^{2n} (c_{ij}^k + c_{ki}^j - c_{jk}^i) e_k$ where $[e_i, e_j] = \sum_{k=1}^{2n} c_{ij}^k e_k$.

For G the matrice $(\nabla_{e_i}^{e_j})_{i,j}$ is:

$$\begin{pmatrix} 0 & T & 0 & \dots & 0 & -Y_1 \\ -T & 0 & 0 & \dots & 0 & X_1 \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & 0 & T & -Y_n \\ 0 & \dots & 0 & -T & 0 & X_n \\ -Y_1 & X_1 & \dots & -Y_n & X_n & 0 \end{pmatrix} \quad (11)$$

Let $\dot{c}(t) = \sum_{i=1}^n (u_i(t)X_i(t) + v_i(t)Y_i(t)) + \gamma(t)T$ geodesics issuing from $0 \in G$. Then $\nabla_{\dot{c}(t)}^{\dot{c}(t)} = 0$ and the table (11) give

$$\begin{aligned} \dot{u}_i + 2\gamma v_i &= 0, \\ \dot{v}_i - 2\gamma u_i &= 0, \\ \dot{\gamma} &= 0. \end{aligned}$$

Therefore, because the parameter t is natural we have $\sum_{i=1}^n (u_i^2 + v_i^2) + \gamma^2 = 1$ and we could take $\gamma(t) = \gamma$ where the constant $\gamma \in [-1, 1]$ is the cosine of the angle between $\dot{c}(0)$ and the T -axe.

For $\gamma \neq 0$, $u_i(t) = r_i \cos(2\gamma t + \theta_i)$ and $v_i(t) = r_i \sin(2\gamma t + \theta_i)$ where $r_i^2 = u_i^2 + v_i^2$.

In coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ the equations for geodesics $c(t) = (x_1(t), \dots, y_n(t), z(t))$ are:

$$\dot{x}_i = u_i, \dot{y}_i = v_i,$$

$$\dot{z} = \gamma + \sum_{i=1}^n v_i(t)x_i(t) - \sum_{i=1}^n u_i(t)y_i(t) \quad (12)$$

We have

$$\begin{aligned} x_i(t) &= -\frac{r_i}{2\gamma} \cos 2\gamma t + a_i \\ y_i(t) &= \frac{r_i}{2\gamma} \sin 2\gamma t + b_i \\ z(t) &= \frac{3\gamma^2 - 1}{2\gamma} t + \frac{b_i}{2\gamma} r_i \cos 2\gamma t \\ &\quad + \frac{a_i}{2\gamma} r_i \sin 2\gamma t + c \end{aligned}$$

for some numbers a_i, b_i, c which could be defined from the initial condition $c(0) = 0$.

If $\gamma = 0$, then they are horizontal and satisfy to the following:

$$x_i(t) = u_i(0)t, y_i(t) = v_i(0)t, z(t) = 0$$

The non-horizontal geodesics (i.e., with $\gamma \neq 0$) are not rays. Indeed. Let $|\gamma| \neq 1$. For every geodesic $c(t)$ their x_i, y_i coordinates are periodic functions. Their projection on $z = 0$ hyperspace is a circle. Take two points on $c(t)$ with difference of their parameters equals $\frac{\pi}{\gamma}$. Then, besides the geodesics $c(t)$ their is also the vertical geodesic $z(s) = (0, \dots, 0, s)$ connecting them. Its length is equal to the difference of z coordinates of considering points. But, as easy to check this is always strictly less than $\frac{\pi}{\gamma}$, i.e., the length of the interval of the $c(t)$ between them, because t is natural one by definition. Therefore, every geodesic $c(t)$ with $|\gamma| \neq 1$ is not minimal. Vertical geodesic $z(s)$ (with $|\gamma| = 1$) also is not minimal, because that the vector field $Z(s) = X_1 \cos s + Y_1 \sin$ is parallel along it and the sectional curvatures of G in 2-dimensional direction generated by $Z(s)$ and $\dot{z}(s)$ equals 1. This means that the index of every interval of $z(s)$ with a length greater than π is

positive, and z is not minimal.

Now let us check that all horizontal geodesics (with $\gamma = 0$) are rays. Suppose that the horizontal geodesic $a(t)$ intersects in some point with another geodesic $c(s)$ issuing the same point 0. Because horizontal geodesics do not intersect, $c(s)$ is non-horizontal one. For $c(t)$ from (12) we see that the length $\tilde{l}(c)$ of the projection curve $\tilde{c}(t) = (c_1(t), \dots, c_{2n}(t), 0)$, measured in euclidean coordinates, is strictly less than t , i.e., the length of the curve c in the left-invariant metric. Than in the submanifold $z = 0$ considered as euclidean space the length of every chord, connecting some points of \tilde{c} is less than $\tilde{l}(c)$. If the geodesic c issuing from 0 intersects some horizontal geodesic $a(t)$ also issuing from 0, then a is the chord of the corresponding \tilde{c} . For horizontal geodesics their lengths in the left-invariant metric coincide with the usual euclidean length in $z = 0$. The argument above means that the length of a is strictly less than that of c . Therefore, we have the following statement: the set of directions of all rays issuing from 0 in the group G is the sphere of unit horizontal directions in the point 0 i.e., $\{w = (w_1, \dots, w_{2n}, 0); \|w\| = 1\}$.

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