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ALMOST COMPLEX STRUCTURES ON THE COTANGENT BUNDLE

FLORIAN BERTRAND

ABSTRACT. We construct some lift of an almost complex structure to the cotangent bundle, using a connection on the base manifold. This generalizes the complete lift defined by I.Sato and the horizontal lift introduced by K.Yano and S.Ishihara. We study some geometric properties of this lift and its compatibility with symplectic forms on the cotangent bundle.

INTRODUCTION

In the recent paper [4], L.Lempert and R.Szöke defined an almost complex structure on the tangent bundle over an almost complex manifold. This lift of the ambient structure defined on the base manifold is characterized via a deformation property. This structure is also studied by K.Yano and S.Ishihara ([9]) as the *complete lift* of the ambient structure (see also [1] by P.Gauduchon and [3] by B.Kruglikov for related results). From a symplectic point of view, the cotangent bundle certainly plays a very important role. This is the phase space in mechanics and this carries a canonical symplectic structure induced by the Liouville form. The aim of this paper is to introduce some “natural lift” of an almost complex structure to the cotangent bundle and to study its compatibility with symplectic forms.

We construct this lift, which will be called the *generalized horizontal lift*, via a connection (not necessarily symmetric, minimal or almost complex). This generalizes the lift introduced by I.Sato ([6]) as a correction of the complete lift, and the horizontal lift constructed by K.Yano-S.Ishihara ([9]). We establish its geometric properties (Theorems 1.2 and 1.3). As an application, we prove that the structure defined by I.Sato may be characterized generically by the holomorphicity of the complex fiberwise multiplication (Corollary 4.2).

Finally we study the compatibility between lifts and symplectic forms on the cotangent bundle. The non existence of a lifted almost complex structure compatible with the canonical symplectic form on the cotangent bundle follows from the expression of such a lift in local coordinates (Proposition 5.1). The conormal bundle of a strictly pseudoconvex hypersurface is a totally real maximal submanifold in the cotangent bundle endowed with the structure defined by Sato in [6]. This was proved by S.Webster ([8]) for the standard complex structure, and by A.Spiro ([7]) for the almost complex case (see also [2]). One can search for a symplectic proof of this since every Lagrangian submanifold in a symplectic manifold is totally real for almost complex structures compatible with the symplectic form. We prove that for every almost complex manifold and every symplectic form on T^*M compatible with the generalized horizontal lift, the conormal bundle of a strictly pseudoconvex hypersurface is not Lagrangian (Proposition 1.1).

The structure of the paper is the following. In section one we introduce some notations and we present the results. In section two we recall some facts about almost complex manifolds, tensors and connections. In section three we construct the generalized horizontal lift to the

cotangent bundle of an almost complex manifold. This lift depends on the introduction of some connection; we study the dependence of the lift on the connection. We prove that the correction of the complete lift and the horizontal lift are particular cases of the generalized horizontal lift (Theorem 1.1). Section four contains proofs of Theorems 1.2 and 1.3; we give necessary and sufficient condition for a lift of a diffeomorphism to be holomorphic (Theorem 1.2). We also characterize the holomorphicity of the multiplication map on the cotangent bundle (Theorem 1.3). Finally in section five we are interested in the compatibility between the generalized horizontal structure and symplectic forms on the cotangent bundle.

1. NOTATIONS AND RESULTS

Let M be a real smooth manifold of even dimension n . We denote by TM and T^*M the tangent and cotangent bundles over M , by $\Gamma(TM)$ and $\Gamma(T^*M)$ the sets of sections of these bundles and by $\pi : T^*M \rightarrow M$ the fiberwise projection. Locally, we work with local coordinates systems $x = (x_1, \dots, x_n)$ in M and $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ in T^*M . We denote by δ_j^i the Kronecker symbol. For convenience we do not write any sum symbol; we use Einstein's summation.

We assume that M is endowed with an almost complex structure J and we denote by N_J the Nijenhuis tensor of J . Let ∇ be a connection on M and let T be the torsion of ∇ . Finally we define two tensors A and S by :

$$\begin{aligned} A(X, Y) &= \nabla_X JY - J\nabla_X Y \\ S(X, Y) &= -A(X, Y) + A(Y, X) + T(JX, Y) - JT(X, Y), \end{aligned}$$

for every $X, Y \in \Gamma(TM)$.

We first construct some general lift, called the *generalized horizontal lift* and denoted by $J^{G, \nabla}$ (subsection 3.1). Let \tilde{J} be the correction of the complete lift defined by I.Sato and let $J^{H, \nabla}$ be the horizontal lift introduced by K.Yano and S.Ishihara. The precise definitions of these two structures are reminded in Subsection 3.2. The link between $J^{G, \nabla}$, \tilde{J} and $J^{H, \nabla}$ is given by the following Theorem :

Theorem 1.1. *We have :*

- (1) $J^{G, \nabla} = \tilde{J}$ if and only if $S(X, Y) = -\frac{1}{2}JN_J$,
- (2) $J^{G, \nabla} = J^{H, \nabla}$ if and only if $T(J, \cdot) = T(\cdot, J)$ and,
- (3) For every almost complex and minimal connection, we have $J^{G, \nabla} = \tilde{J} = J^{H, \nabla}$.

Theorem 1.1 really explains in what sense the generalized horizontal lift generalizes the complete and the horizontal lifts.

We prove that the generalized horizontal lift $J^{G, \nabla}$ satisfies the following properties :

Theorem 1.2.

- (1) The projection $\pi : T^*M \rightarrow M$ is $(J^{G, \nabla}, J)$ -holomorphic.
- (2) The zero section $s : M \rightarrow T^*M$ is $(J, J^{G, \nabla})$ -holomorphic.
- (3) The lift of a diffeomorphism $f : (M_1, J_1, \nabla_1) \rightarrow (M_2, J_2, \nabla_2)$ to the cotangent bundle is $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphic if and only if f is a (J_1, J_2) -holomorphic map satisfying $f_*S_1 = S_2$.

We recall that the lift \tilde{f} of a diffeomorphism $f : M_1 \longrightarrow M_2$ to the cotangent bundle is defined by $\tilde{f} = (f, {}^t(df)^{-1})$. Locally the differential $d\tilde{f}$ is given by the matrix :

$$d\tilde{f} = \begin{pmatrix} df & 0 \\ (*) & {}^t(df)^{-1} \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}),$$

where $(*)$ denotes a $(n \times n)$ block of derivatives of f with respect to (x_1, \dots, x_n) .

Let Z denote the complex fiberwise multiplication on the cotangent bundle. This is defined, in local coordinates, by : $Z(x, p) = (a + ib)(x, p) = (x, (a + b{}^tJ(x))p)$ where $b \neq 0$.

Theorem 1.3. *The multiplication map Z is $J^{G,\nabla}$ -holomorphic if and only if $A(J, \cdot) = A(\cdot, J)$.*

We point out that if b is identically equal to zero then the map Z is always $J^{G,\nabla}$ -holomorphic.

The existence of a symplectic form on the cotangent bundle compatible with the structure defined by I.Sato and such that the conormal bundle of a strictly pseudoconvex hypersurface is Lagrangian would imply that the conormal bundle is totally real.

In this way, we have :

Proposition 1.1. *Assume (M, J, ∇) is an almost complex manifold equipped with a connection. Let ω' be a nondegenerate two-form on T^*M compatible with the generalized horizontal lift $J^{G,\nabla}$. If Γ is a strictly pseudoconvex hypersurface of M then there exist $X, Y \in TN^*(\Gamma)$ such that $\omega'(X, Y) \neq 0$.*

It is sufficient to consider nondegenerate two-forms for two reasons. Firstly, the fact that Lagrangian submanifolds are totally real for compatible almost complex structures does not depend on the closedness of the symplectic form. Secondly the existence of a symplectic form compatible with a prescribed almost complex structure is not guaranteed (see the case of \mathbb{S}^6 , the unit sphere in \mathbb{R}^7).

2. PRELIMINARIES

Let M be a real smooth manifold of even dimension n .

2.1. Almost complex structures.

Definition 2.1. *An almost complex structure on M is a tensor field J of type $(1, 1)$ which satisfies $J^2 = -Id$. The pair (M, J) is called an almost complex manifold.*

In local coordinates, J is given by $J_i^k dx^l \otimes \partial x_k$.

We say that a map $f : (M, J) \longrightarrow (M', J')$ between two almost complex manifolds is (J, J') -holomorphic if :

$$J'(f(x)) \circ d_x f = d_x f \circ J(x), \text{ for every } x \in M.$$

If $f : (M, J) \longrightarrow M'$ is a diffeomorphism, we define the direct image of J by f by :

$$f_* J(y) = d_{f^{-1}(y)} f \circ J(f^{-1}(y)) \circ d_y f^{-1}, \text{ for every } y \in M'.$$

The tensor field $f_* J$ is an almost complex structure on M' for which f is $(J, f_* J)$ -holomorphic.

We recall that the Nijenhuis tensor of the almost complex structure J is defined by :

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \text{ pour } X, Y \in \Gamma(TM).$$

It is important to notice that $N_J(X, JY) = -JN_J(X, Y)$.

2.2. Tensors and contraction. Let θ be the Liouville form on T^*M . This one-form is locally given by $\theta = p_i dx^i$. The two-form $\omega_{st} = d\theta$ is the canonical symplectic form on the cotangent bundle, with local expression $\omega_{st} = -dx^k \wedge dp_k$. We stress out that these forms do not depend on the choice of coordinates on T^*M .

We denote by $T_q^r M$ the space of q covariant and r contravariant tensors on M ; according to this notation we have $N_J \in T_2^1 M$. For positive q , we consider the following contraction map $\gamma : T_q^1 M \rightarrow T_{q-1}^1(T^*M)$ defined by :

$$\gamma(R) = p_k R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_{q-1}} \otimes \partial p^{i_q}$$

for $R = R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_q} \otimes \partial x_k$.

We also define a q -form on T^*M by $\theta(R) = p_k R_{i_1, \dots, i_q}^k dx^{i_1} \otimes \dots \otimes dx^{i_q}$ for a $(1, q)$ -tensor R on M . We notice that $\theta(R)(X_1, \dots, X_q) = \theta(R(d\pi(X_1), \dots, d\pi(X_q)))$ for $X_1, \dots, X_q \in \Gamma(T^*M)$.

Since the canonical symplectic form ω_{st} establishes a correspondence between q -forms and $(1, q-1)$ -tensors, one may define the contraction map γ using the Liouville form θ and the canonical symplectic form ω_{st} on T^*M by setting, for $X_1, \dots, X_q \in \Gamma(T^*M)$:

$${}^t(\theta(R))(X_1, \dots, X_q) = -\omega_{st}(X_1, \gamma(R)(X_2, \dots, X_q)),$$

where ${}^t(\theta(R))(X_1, \dots, X_q) = \theta(R)(X_2, \dots, X_q, X_1)$.

When R is a $(1, 2)$ -tensor, we have a matricial interpretation of the contraction γ ; if $R_{i,j}^k$ are coordinates of R then $\gamma(R)$ is given by :

$$\gamma(R) = \begin{pmatrix} 0 & 0 \\ a_j^i & 0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}), \text{ with } a_j^i = p_k R_{j,i}^k.$$

2.3. Connections. Let ∇ be a connection on M . We denote by $\Gamma_{i,j}^k$ its Cristoffel symbols defined by $\nabla_{\partial x_i} \partial x_j = \Gamma_{i,j}^k \partial x_k$. Let also $\Gamma_{i,j}$ defined in local coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ on T^*M by the equality $p_k \Gamma_{i,j}^k = \Gamma_{i,j}$.

Remark 2.1. *By the definition of ∇ , we have : $\nabla_{\partial x_i}(J\partial x_j) = \partial x_i J_j^k \partial x_k + \Gamma_{i,l}^k J_j^l \partial x_k$.*

The torsion T of ∇ is defined by :

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \text{ for every } X, Y \in \Gamma(TM).$$

There are ‘‘natural’’ families of connections on an almost complex manifold.

Definition 2.2. *A connection ∇ on M is called :*

- (1) *almost complex when $\nabla_X(JY) = J\nabla_X Y$ for every $X, Y \in \Gamma(TM)$,*
- (2) *minimal when its torsion T is equal to $\frac{1}{4}N_J$, and*
- (3) *symmetric when its torsion T is identically zero.*

A.Lichnerowicz, in [5], proved that the set of almost complex and minimal connections is nonempty. This fact is crucial in the following.

To the connection ∇ we associate three other connections :

- $\bar{\nabla} := \nabla - T$. The Cristoffel symbols $\bar{\Gamma}_{i,j}^k$ of $\bar{\nabla}$ are given by $\bar{\Gamma}_{i,j}^k = \Gamma_{j,i}^k$.
- $\tilde{\nabla} := \nabla - \frac{1}{2}T$. The connection $\tilde{\nabla}$ is a symmetric connection and its Cristoffel symbols $\tilde{\Gamma}_{i,j}^k$ are given by : $\tilde{\Gamma}_{i,j}^k = \frac{1}{2}(\Gamma_{i,j}^k + \Gamma_{j,i}^k)$.
- a connection on (M, T^*M) , still denoted by ∇ , and defined by :

$$(\nabla_X s)(Y) := X.s(Y) - s\nabla_X Y \text{ for every } X, Y \in \Gamma(TM) \text{ and } s \in \Gamma(T^*M).$$

Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. The horizontal distribution H^∇ of ∇ is defined by :

$$H_\xi^\nabla = \{d_x s(X), X \in T_x M, s \in \Gamma(T^*M), s(x) = \xi, \nabla_X s = 0\} \subseteq T_\xi T^*M.$$

We recall that $d_\xi \pi$ induces an isomorphism between H_ξ^∇ and $T_x M$. Moreover we have the following decomposition : $T_\xi T^*M = H_\xi^\nabla \oplus T_x^*M$. So an element $Y \in T_\xi T^*M$ decomposes as $Y = (X, v^\nabla(Y))$, where $v^\nabla : T_\xi T^*M \rightarrow T_x^*M$ is the projection on the vertical space T_x^*M parallel to H_ξ^∇ .

We introduce a tensor A on M , which measures the ‘‘lack of complexity’’ of the connection ∇ :

$$A(X, Y) = \nabla_X JY - J\nabla_X Y \text{ for every } X, Y \in \Gamma(TM).$$

Locally we have $A_{i,j}^k = \partial x_i J_j^k - J_l^k \Gamma_{i,j}^l + J_j^l \Gamma_{i,l}^k$. We denote by \tilde{A} the tensor associated to the symmetrization $\tilde{\nabla}$ of ∇ : $\tilde{A}(X, Y) = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y$.

3. GENERALIZED HORIZONTAL LIFT ON THE COTANGENT BUNDLE

Let (M, J) be an almost complex manifold. We define an almost complex lift of J to the cotangent bundle T^*M over M and we prove that this generalizes the complete lift ([6]) and the horizontal lift ([9]).

3.1. Generalized horizontal lift. Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. We locally have the following decomposition $T_\xi T^*M = T_x M \oplus T_x^*M$. From an algebraic point of view it seems natural to lift an almost complex structure J as a product structure, that is $J \oplus {}^t J$ with respect to $T_x M \oplus T_x^*M$. More generally if H is a distribution such that we have the decomposition $T_\xi T^*M = H_\xi \oplus T_x^*M$, one can define a lift of J by $J \oplus {}^t J$ with respect to $H_\xi \oplus T_x^*M$. We call such a lift the *generalized horizontal lift*. Since any such distribution determines a unique connection it is possible to define this lift using a connection; this point of view is inspired by the construction of an almost complex structure on the space of 1-jets of an almost complex manifold due to P.Gauduchon in [1].

Let ∇ be a connection on M . We consider the connection induced by ∇ on (M, T^*M) , defined in subsection 2.3. For a vector $Y = (X, v^\nabla(Y)) \in T_\xi T^*M = H_\xi^\nabla \oplus T_x^*M$, we define :

$$J^{G,\nabla}(Y) = (JX, {}^t J(v^\nabla(Y))).$$

Let us explain the meaning of JX . Since the map $d_\xi \pi|_{H_\xi^\nabla}$ is a bijection between H_ξ^∇ and $T_x M$, we define JX as $(d_\xi \pi|_{H_\xi^\nabla})^{-1}(J(x)d_\xi \pi(X))$.

This almost complex structure is given by $J^{G,\nabla} = J \oplus {}^t J$ in the decomposition $T_\xi T^*M = H_\xi^\nabla \oplus T_x^*M$.

Definition 3.1. *The almost complex structure $J^{G,\nabla}$ is called the generalized horizontal lift of J associated to the connection ∇ .*

We first describe locally the horizontal distribution H^∇ :

Lemma 3.1. *We have $H_\xi^\nabla = \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_x M \right\}$.*

Proof. Let us prove that $H_\xi^\nabla \subseteq \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_x M \right\}$.

Let $Y \in H_\xi^\nabla$; Y is equal to $d_x s(X)$ where $X \in T_x M$ and s is a section of the cotangent bundle such that $\nabla_X s = 0$. Locally we have $s = s_i dx^i$, $X = X^i \partial x_i$ and so :

$$Y = \left(\begin{array}{c} X \\ X^j \partial x_j s_i \end{array} \right).$$

Since $\nabla_X s = 0$ we obtain :

$$0 = X^j \nabla_{\partial x_j} (s_i dx^i) = X^j s_i \nabla_{\partial x_j} dx^i + X^j \partial x_j s_i dx^i = -X^j s_i \Gamma_{j,k}^i dx^k + X^j \partial x_j s_k dx^k.$$

Therefore $X^j \partial x_j s_k = X^j s_i \Gamma_{j,k}^i = X^j \Gamma_{j,k}$. This proves the inclusion.

Moreover the following decomposition insures us the equality :

$$T_\xi T^*M = \left\{ \left(\begin{array}{c} X \\ \Gamma_{j,k} X^j \end{array} \right), X \in T_x M \right\} \oplus T_x^*M.$$

□

We study the dependence of $J^{G,\nabla}$ on the connection ∇ . We introduce the set $\mathcal{H} := \{L \in T_2^1 M, L(J, \cdot) = L(\cdot, J)\}$ of $(1, 2)$ -tensors L satisfying $L(J, \cdot) = L(\cdot, J)$.

Proposition 3.1. *Assume that ∇ and ∇' are two connections on M . Then $J^{G,\nabla} = J^{G,\nabla'}$ if and only if the tensor $\nabla' - \nabla$ belongs to \mathcal{H} .*

So let ∇' be another connection on M ; there exists a tensor $L \in T_2^1(M)$ such that $\nabla' = \nabla + L$. We notice that, considering the induced connections on (M, T^*M) , we have :

$$\nabla'_X s = \nabla_X s - s(L(X, \cdot)).$$

Moreover :

$$v^{\nabla'}(Y) = v^\nabla(Y) - \xi(L(d_\xi \pi(X), \cdot)),$$

where $Y = (X, v^\nabla(Y)) \in T_\xi T^*M$.

Proof of Proposition 3.1. A vector $Y \in T_\xi T^*M$ can be written $Y = (X, v^\nabla(Y))$ in the decomposition $H_\xi^\nabla \oplus T_x^*M$ of $T_\xi T^*M$ and $Y = (X', v^{\nabla'}(Y))$ in the decomposition $H_\xi^{\nabla'} \oplus T_x^*M$, with $d_\xi \pi(X) = d_\xi \pi(X')$. By construction we have $d_\xi \pi(JX) = d_\xi \pi(JX')$. Thus $J^{G,\nabla} = J^{G,\nabla'}$ if and only if $v^{\nabla'}(J^{G,\nabla'} Y) = v^\nabla(J^{G,\nabla} Y)$ for every $\xi \in T^*M$ and $Y \in T_\xi^*M$.

Let us compute $v^\nabla(J^{G,\nabla'}Y)$:

$$\begin{aligned} v^\nabla(J^{G,\nabla'}Y) &= v^{\nabla'}(J^{G,\nabla'}Y) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= {}^tJ(v^{\nabla'}(Y)) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= {}^tJ(v^\nabla(Y)) - {}^tJ\xi(L(d_\xi\pi(X), \cdot)) + \xi(L(Jd_\xi\pi(X), \cdot)) \\ &= v^\nabla(J^{G,\nabla}Y) - \xi(L(d_\xi\pi(X), J)) + \xi(L(Jd_\xi\pi(X), \cdot)). \end{aligned}$$

So $J^{G,\nabla'} = J^{G,\nabla}$ if and only if $L(d_\xi\pi(X), J) = L(Jd_\xi\pi(X), \cdot)$. Since $d_\xi\pi|_{H_\xi^\nabla}$ is a bijection between H_ξ^∇ and T_xM , we obtain the result. \square

A consequence of Proposition 3.1 is the following Corollary :

Corollary 3.1. *Let ∇ and ∇' be two minimal almost complex connections. One has $J^{G,\nabla'} = J^{G,\nabla}$.*

We see from corollary 3.1 that minimal almost complex connections are ‘‘natural’’ connections in almost complex manifolds, to construct generalized horizontal lifts.

Proof of Corollary 3.1. Since ∇ and ∇' have the same torsion, the tensor $L := \nabla - \nabla'$ is symmetric. Moreover, since ∇ and ∇' are almost complex, we have $L(\cdot, J) = JL(\cdot, \cdot)$. Thus $L(J, \cdot) = JL(\cdot, \cdot) = L(\cdot, J)$. \square

The following Proposition gives local and tensorial expressions of the generalized horizontal lift. In order to obtain the tensorial expression it seems useful to consider the complete lift denoted by J^c and defined by I.Sato ([6]) as follows : let $\theta(J)$ be the one-form on T^*M with local expression $\theta(J) = p_k J_i^k dx^i$. We define J^c by the identity $d(\theta(J)) = \omega_{st}(J^c, \cdot)$. Then J^c is locally given by :

$$J^c = \begin{pmatrix} J_j^i & 0 \\ p_k(\partial x_j J_i^k - \partial x_i J_j^k) & J_i^j \end{pmatrix}.$$

Proposition 3.2.

(1) *With respect to local coordinates system $(x_1, \dots, x_n, p_1, \dots, p_n)$, $J^{G,\nabla}$ is equal to :*

$$J^{G,\nabla} = \begin{pmatrix} J_j^i & 0 \\ \Gamma_{l,i} J_j^l - \Gamma_{j,l} J_i^l & J_i^j \end{pmatrix}.$$

(2) *We have $J^{G,\nabla} = J^c + \gamma(S)$, where $S(X, Y) = -A(X, Y) + A(Y, X) + T(JX, Y) - JT(X, Y)$.*

Proof of Proposition 3.2. We first prove part (1). With respect to local coordinates system $(x_1, \dots, x_n, p_1, \dots, p_n)$, structure $J^{G,\nabla}$ is locally given by :

$$J^{G,\nabla} = \begin{pmatrix} J_j^i & 0 \\ a_j^i & J_i^j \end{pmatrix}.$$

Since $\begin{pmatrix} \delta_i^j \\ \Gamma_{i,j} \end{pmatrix} \in H_\xi^\nabla$, it follows from definition of $J^{G,\nabla}X$ where $X \in H^\nabla$ and from lemma 3.1, that for every $i \in \{1, \dots, n\}$:

$$J^{G,\nabla} \begin{pmatrix} \delta_i^j \\ \Gamma_{i,j} \end{pmatrix} = \begin{pmatrix} J_i^j \\ \Gamma_{k,j} J_i^k \end{pmatrix}.$$

Hence we have : $a_j^i = \Gamma_{l,i}J_j^l - \Gamma_{j,l}J_i^l$. This concludes the proof of part (1).

Then we prove part (2). Using the local expression of J^c , we have :

$$J^{G,\nabla} = J^c + \begin{pmatrix} 0 & 0 \\ -p_k \partial x_j J_i^k + p_k \partial x_i J_j^k + \Gamma_{l,i}J_j^l - \Gamma_{j,l}J_i^l & 0 \end{pmatrix}.$$

It follows from Remark 2.1 that :

$$-p_k \partial x_j J_i^k + p_k \partial x_i J_j^k + \Gamma_{l,i}J_j^l - \Gamma_{j,l}J_i^l = p_k dx_k [-\nabla_{\partial x_j}(J\partial x_i) + \bar{\nabla}_{\partial x_i}(J\partial x_j)].$$

We denote by $S'(X, Y) = -\nabla_X(JY) + \bar{\nabla}_Y(JX) = -\nabla_X(JY) + \nabla_Y JX + T(JX, Y)$ and we notice that $S'(\partial x_i, \partial x_j) = -\nabla_{\partial x_i}(J\partial x_j) + \bar{\nabla}_{\partial x_j}(J\partial x_i)$. We point out that S' is not a tensor. However with a correction term, we obtain the tensor S :

$$\begin{aligned} S(X, Y) &= S'(X, Y) + J[X, Y] \\ &= -\nabla_X(JY) + \nabla_Y(JX) + T(JX, Y) + J\nabla_X Y - J\nabla_Y X - JT(X, Y) \\ &= -A(X, Y) + A(Y, X) + T(JX, Y) - JT(X, Y). \end{aligned}$$

The components of S are given by $S(\partial x_i, \partial x_j) = S'(\partial x_i, \partial x_j)$ and so $J^{G,\nabla} = J^c + \gamma(S)$. \square

3.2. Generalization of the complete and horizontal lifts. The complete lift J^c defined by I.Sato is an almost complex structure on T^*M if and only if J is an integrable structure on M , that is if and only if M is a complex manifold. Introducing a correction term which involves the non integrability of J , I.Sato obtained an almost complex structure on the cotangent bundle; this latter one is given by :

$$\tilde{J} = J^c - \frac{1}{2}\gamma(JN_J).$$

For convenience we will also call \tilde{J} the *complete lift* of J .

Remark 3.1. By definition of $\theta(J)$, if Y is a vector field on T^*M locally defined by $Y = X^i \partial x_i + P^i \partial p_i$, then we have $\theta(J)(Y) = p_k J_i^k X_i$. This justifies the following local expression :

$$\theta(J)(Y) = \theta(J)(X^i \partial x_i) = \theta(J(X^i \partial x_i)).$$

The local expression of \tilde{J} is obtained by finding the coordinates of the tensor JN_J . Since $JN_J(X, Y)$ is equal to $J[JX, JY] + [X, JY] + [JX, Y] - J[X, Y]$, we find :

$$JN_J(\partial x_i, \partial x_j) = [-\partial x_j J_i^k + \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s] dx_k.$$

Thus we have the following expression :

$$\tilde{J} = \begin{pmatrix} J_j^i & 0 \\ B_j^i & J_i^j \end{pmatrix}, \text{ with } B_j^i = \frac{p_k}{2} [\partial x_j J_i^k - \partial x_i J_j^k + J_s^k J_i^q \partial x_q J_j^s - J_s^k J_j^q \partial x_q J_i^s].$$

We now recall the definition of the horizontal lift of an almost complex structure. Let ∇ be a connection on M and $\tilde{\nabla}$ its symmetrized. K.Yano and S.Ishihara defined in [9] the horizontal lift of J by :

$$J^{H,\nabla} = J^c + \gamma([\tilde{\nabla}J]),$$

where $[\tilde{\nabla}J]$ is a $(1, 2)$ -tensor given by :

$$[\tilde{\nabla}J](X, Y) = (\tilde{\nabla}_Y J)X - (\tilde{\nabla}_X J)Y = -\tilde{A}(X, Y) + \tilde{A}(Y, X), \text{ for every } X, Y \in \Gamma(TM).$$

They proved that $J^{H,\nabla}$ is an almost complex structure. It is important to notice that if we do not symmetrize ∇ , the horizontal lift of J is no more an almost complex structure. The horizontal lift is locally given by :

$$J^{H,\nabla} = \begin{pmatrix} J_j^i & 0 \\ \tilde{\Gamma}_{i,l}J_j^l - \tilde{\Gamma}_{j,l}J_i^l & J_i^j \end{pmatrix}.$$

From tensorial expressions of the generalized horizontal, the complete and the horizontal lifts, we can link these three lifted structures. When we consider ‘‘natural’’ connections on M we have the equality of the three structures. This is stated by the third part of Theorem 1.1.

Proof of Theorem 1.1. We compare the three lifted structures via the intrinsic expressions :

- $J^{G,\nabla} = J^c + \gamma(S)$ with $S(X, Y) = -A(X, Y) + A(Y, X) + T(JX, Y) - JT(X, Y)$,
- $\tilde{J} = J^c - \frac{1}{2}\gamma(JN_J)$ and,
- $J^{H,\nabla} = J^c + \gamma([\tilde{\nabla}])$ with $[\tilde{\nabla}](X, Y) = -\tilde{A}(X, Y) + \tilde{A}(Y, X)$.

Then part (1) is a direct lecture of the tensorial expressions of $J^{G,\nabla}$, \tilde{J} and $J^{H,\nabla}$.

To prove (2), it is enough to notice that :

$$\begin{aligned} [\tilde{\nabla}](X, Y) &= -\tilde{A}(X, Y) + \tilde{A}(Y, X) \\ &= -A(X, Y) + A(Y, X) + \frac{1}{2}T(X, JY) + \frac{1}{2}T(JX, Y) - JT(X, Y). \end{aligned}$$

Let us prove part (3). The equality $J^{G,\nabla} = \tilde{J}$ follows from the fact that $A = 0$ because ∇ is almost complex and from $-T(J, \cdot) + JT(\cdot, \cdot) = \frac{1}{4}JN_J + \frac{1}{4}JN_J = \frac{1}{2}JN_J$. Since $T = \frac{1}{4}N_J$ and $N_J(J, \cdot) = N_J(\cdot, J)$ we have $J^{G,\nabla} = J^{H,\nabla}$. \square

We end this section with the study of a distribution associated to the horizontal lift. We begin with an important Corollary of Theorem 1.1.

Corollary 3.2. *We have the equality $J^{H,\nabla} = J^{G,\tilde{\nabla}}$.*

Proof. We first notice that $J^{H,\nabla}$ is equal to $J^{H,\tilde{\nabla}}$ and since $\tilde{\nabla}$ is symmetric, Theorem 1.1 implies the equality $J^{G,\tilde{\nabla}} = J^{H,\nabla}$. \square

Remark 3.2. *Previous Corollary 3.2 can be also proved using the local expression of $J^{G,\tilde{\nabla}}$ and $J^{H,\nabla}$.*

Let $x \in M$ and let $\xi \in T^*M$ be such that $\pi(\xi) = x$. We consider the horizontal lift of ∂x_i and vertical lift of dx_i on the cotangent defined by K.Yano and S.Ishihara in [9] :

$$(\partial x_i)^{H,\nabla} = \begin{pmatrix} \delta_i^j \\ \tilde{\Gamma}_{i,j} \end{pmatrix} \text{ and } (dx_i)^V = \begin{pmatrix} 0 \\ \delta_i^j \end{pmatrix}.$$

The basis $\{(\partial x_1)^{H,\nabla}(\xi), \dots, (\partial x_n)^{H,\nabla}(\xi), (dx_1)^V(\xi), \dots, (dx_n)^V(\xi)\}$ of $T_\xi(T^*M)$ is called the *adapted frame* of ∇ . With respect to this basis we decompose :

$$T_\xi T^*M = \langle (\partial x_1)^{H,\nabla}(\xi), \dots, (\partial x_n)^{H,\nabla}(\xi) \rangle \oplus \langle (dx_1)^V(\xi), \dots, (dx_n)^V(\xi) \rangle.$$

By identification, we have the equality $\langle (dx_1)^V(\xi), \dots, (dx_n)^V(\xi) \rangle = T_x^*M$. K.Yano and S.Ishihara proved that in this basis, $J^{H,\nabla}$ can be written :

$$J^{H,\nabla} = \begin{pmatrix} J_j^i & 0 \\ 0 & J_i^j \end{pmatrix}.$$

Moreover, to the horizontal lift, we associate the distribution of horizontal lifted vectors fields $D = \{X^{H,\nabla}, X \in \Gamma(TM)\}$ of $T(T^*M)$ with $D_\xi = \langle (\partial x_1)^{H,\nabla}(\xi), \dots, (\partial x_n)^{H,\nabla}(\xi) \rangle$. The elements of the distribution D are characterized as follows : if $Y = \begin{pmatrix} X \\ P \end{pmatrix}$ is a vector of $T_\xi T^*M$ then $Y \in D_\xi$ if and only if $P_i = \tilde{\Gamma}_{i,k} X^k$.

Thus Lemma 3.1 implies $D_\xi = H_\xi^{\tilde{\nabla}}$, that is D is the horizontal distribution of $\tilde{\nabla}$. Finally $J^{H,\nabla} = J \oplus^t J = J^{G,\tilde{\nabla}}$ with respect to the decomposition $T_\xi(T^*M) = D_\xi \oplus T_x^*M = H_\xi^{\tilde{\nabla}} \oplus T_x^*M$.

4. PROOFS OF THEOREMS 1.2 AND 1.3

4.1. Properties of the generalized horizontal lift. Theorem 1.2 explains with geometric properties why the generalized horizontal lift of an almost complex structure is called a lift.

Proof of Theorem 1.2. The local expression of the generalized horizontal structure implies the first two points.

Let us prove property (3). Assume $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2) -holomorphic diffeomorphism satisfying $f_*S_1 = S_2$ and let \tilde{f} be its lift to the cotangent bundle. We recall that $J^{G,\nabla_i} = J^c + \gamma(S_i)$ for $i = 1, 2$ (Proposition 3.2). We denote by θ_i and $\omega_{i,st}$ the Liouville form and the canonical symplectic form of T^*M_i . The invariance by lifted diffeomorphisms of these forms insure us that $\tilde{f}_*\theta_1 = \theta_2$ and $\tilde{f}_*\omega_{1,st} = \omega_{2,st}$. We also recall that ${}^t(\theta_i(S_i)) = -\omega_{i,st}(\cdot, \gamma(S_i))$.

We want to establish the following equality $\tilde{f}_*(J_1^{G,\nabla_1}) = J_2^{G,\nabla_2}$. The first step consists in proving that the direct image of J_1^c by \tilde{f} is J_2^c . By the nondegeneracy of $\omega_{2,st}$, it is equivalent to obtain the equality $\omega_{2,st}(\tilde{f}_*J_1^c, \cdot) = \omega_{2,st}(J_2^c, \cdot)$:

$$\begin{aligned} \omega_{2,st}(\tilde{f}_*J_1^c, \cdot) &= \omega_{2,st}(d\tilde{f} \circ J_1^c \circ (d\tilde{f})^{-1}, \cdot) \\ &= \omega_{1,st}(J_1^c \circ (d\tilde{f})^{-1}, (d\tilde{f})^{-1}) \\ &= \tilde{f}_*(\omega_{1,st}(J_1^c, \cdot)) \\ &= \tilde{f}_*d(\theta_1(J_1)) \text{ and,} \\ \omega_{2,st}(J_2^c, \cdot) &= d(\theta_2(J_2)). \end{aligned}$$

So let us prove that the pull-back of $\theta_2(J_2)$ by \tilde{f} is $\theta_1(J_1)$. According to Remark 3.1 and to the local expression of $d\tilde{f}$ we have $\tilde{f}^*(\theta_2(J_2)) = \theta_2(J_2 \circ d\tilde{f})$ and then :

$$\tilde{f}^*(\theta_2(J_2)) = \theta_2(df \circ J_1) = (\tilde{f}^*\theta_2)(J_1) = \theta_1(J_1).$$

Thus we obtain $\tilde{f}_*d(\theta_1(J_1)) = d(\theta_2(J_2))$, that is $\tilde{f}_*J_1^c = J_2^c$.

To establish the result, it is enough to prove that the direct image of $\gamma(S_1)$ by \tilde{f} is $\gamma(S_2)$. We prove more generally that $f_*(S_1) = S_2$ if and only if $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$. It is equivalent to prove that $f_*(S_1) = S_2$ if and only if $\omega_{2,st}(\cdot, \tilde{f}_*(\gamma(S_1))) = \omega_{2,st}(\cdot, \gamma(S_2))$. We compute :

$$\begin{aligned}
\omega_{2,st}(\cdot, \tilde{f}_* \gamma(S_1) \cdot) &= \omega_{2,st}(\cdot, d\tilde{f} \circ \gamma(S_1) \circ (d\tilde{f})^{-1} \cdot) \\
&= \omega_{1,st}((d\tilde{f})^{-1} \cdot, \gamma(S_1) \circ (d\tilde{f})^{-1} \cdot) \\
&= \tilde{f}_*(\omega_{1,st}(\cdot, \gamma(S_1) \cdot)) \\
&= -\tilde{f}_*({}^t\theta_1(S_1)).
\end{aligned}$$

Let us check that $f_*(S_1) = S_2$ if and only if $\tilde{f}_*{}^t(\theta_1(S_1)) = {}^t(\theta_2(S_2))$. We have :

$$\tilde{f}^*(\theta_2(S_2)) = \theta_2(S_2(df, df)) \text{ and } \theta_1(S_1) = (\tilde{f}^*\theta_2)(S_1) = \theta_2(df \circ S_1).$$

According to this fact and the definition of $\theta(R)$, where $R \in T_2^1 M$ given in the section 2.2, it follows that $f_*S_1 = S_2$ if and only if $\theta_2(S_2(df, df)) = \theta_2(df \circ S_1)$. So $f_*(S_1) = S_2$ if and only if $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$. Finally we have proved that if $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ is a (J_1, J_2) -holomorphic diffeomorphism satisfying $f_*S_1 = S_2$ then \tilde{f} is $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphic.

Reciprocally if \tilde{f} is $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphic then f is (J_1, J_2) -holomorphic. Indeed the zero section $s_1 : M_1 \longrightarrow T^*M_1$ is (J_1, J_1^{G, ∇_1}) -holomorphic by (2), the projection $\pi_2 : T^*M_2 \longrightarrow M_2$ is (J_2^{G, ∇_2}, J_2) -holomorphic by (1) and we have the equality $f = \pi_2 \circ \tilde{f} \circ s_1$. Since f is (J_1, J_2) -holomorphic we have $\tilde{f}_*J_1^c = J_2^c$. Moreover the $(J_1^{G, \nabla_1}, J_2^{G, \nabla_2})$ -holomorphicity of \tilde{f} implies the equality $\tilde{f}_*(\gamma(S_1)) = \gamma(S_2)$, that is $f_*S_1 = S_2$. \square

Properties of Theorem 1.2 can also be established for the complete and the horizontal lifts by considering special connections. This is stated by the following Corollary.

Corollary 4.1.

- (1) The projection $\pi : T^*M \longrightarrow M$ is (J', J) -holomorphic for $J' = \tilde{J}, J^{H, \nabla}$.
- (2) The zero section $s : M \longrightarrow T^*M$ is (J, J') -holomorphic $J' = \tilde{J}, J^{H, \nabla}$.
- (3) The lift of a diffeomorphism $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ to the cotangent bundle is $(\tilde{J}_1, \tilde{J}_2)$ -holomorphic if and only if f is (J_1, J_2) -holomorphic.
- (4) The lift of a diffeomorphism $f : (M_1, J_1, \nabla_1) \longrightarrow (M_2, J_2, \nabla_2)$ to the cotangent bundle is $(J_1^{H, \nabla_1}, J_2^{H, \nabla_2})$ -holomorphic if and only if f is a (J_1, J_2) -holomorphic map satisfying $f_*[\nabla_1 J_1] = [\nabla_2 J_2]$.

Theorem 1.2 and Corollary 4.1 characterize the complete lift via the lift of diffeomorphisms. Indeed the generalized horizontal lift and the horizontal lift need a connection to be constructed whereas this is not essential in the construction of the complete lift. Hence it is natural to require more conditions if we want to lift some diffeomorphisms in case of generalized horizontal and horizontal structures. And generically there is no way for an almost complex diffeomorphism to send the tensors S_1 (resp. $[\nabla_1 J_1]$) on S_2 (resp. $[\nabla_2 J_2]$).

Proof of Corollary 4.1. By local expressions of the complete lift and of the horizontal lift we have the first two properties.

To prove property (3) we consider almost complex and minimal connections ∇_1 and ∇_2 on M_1 and M_2 . Hence $\tilde{J}_1 = J^{G, \nabla_1} = J_1^c + \gamma(S_1)$ and $\tilde{J}_2 = J^{G, \nabla_2} = J_2^c + \gamma(S_2)$. We have $S_1 = -\frac{1}{2}J_1 N_{J_1}$ and $S_2 = -\frac{1}{2}J_2 N_{J_2}$. We notice that if $f : (M_1, J_1) \longrightarrow (M_2, J_2)$ is a (J_1, J_2) -holomorphic diffeomorphism then $f_*N_{J_1} = N_{J_2}$ and then $f_*J_1 N_{J_1} = J_2 N_{J_2}$. According to

Theorem 1.2 the lift of a diffeomorphism f to the cotangent bundle is $(\tilde{J}_1, \tilde{J}_2)$ -holomorphic if and only if f is (J_1, J_2) -holomorphic.

The property (4) follows from the equality $J^{G, \tilde{\nabla}} = J^{H, \nabla}$ obtained in Corollary 3.2 and from Theorem 1.2. \square

4.2. Fiberwise multiplication. We consider the multiplication map $Z : T^*M \longrightarrow T^*M$ by a complex number $a + ib$ with $b \neq 0$ on the cotangent bundle. This is locally defined by $Z(x, p) = (x, (a + b^t J(x))p)$. For $(x, p) \in T^*M$ we have $d_{(x,p)}Z = \begin{pmatrix} Id & 0 \\ C & aId + b^t J \end{pmatrix}$, where $C_j^i = bp_k \partial x_j J_i^k$.

Proof of Theorem 1.3. Let us evaluate $d_{(x,p)}Z \circ J^{G, \nabla}(x, p) - J^{G, \nabla}(x, ap + b^t Jp) \circ d_{(x,p)}Z$. This is equal to :

$$\begin{pmatrix} 0 & 0 \\ CJ + (aId + b^t J)B(x, p) - B(x, ap + b^t Jp) - {}^t JC & 0 \end{pmatrix},$$

where $B_j^i(x, p) = p_k(\Gamma_{l,i}^k J_j^l - \Gamma_{j,l}^k J_i^l)$.

We first notice that $aB_j^i(x, p) - B_j^i(x, ap + b^t Jp) = -bp_k J_s^k(\Gamma_{l,i}^s J_j^l - \Gamma_{j,l}^s J_i^l)$.

Let us compute $D = CJ + (aId + b^t J)B(x, p) - B(x, ap + b^t Jp) - {}^t JC$:

$$D_j^i = bp_k \left[\underbrace{J_j^l \partial x_l J_i^k}_{(1)} + \underbrace{J_i^l \Gamma_{s,l}^k J_j^s}_{(2)} - \underbrace{J_i^l \Gamma_{j,s}^k J_l^s}_{(2)'} - \underbrace{J_s^k \Gamma_{l,i}^s J_j^l}_{(3)} + \underbrace{J_s^k \Gamma_{j,l}^s J_i^l}_{(3)'} - \underbrace{J_i^l \partial x_j J_l^k}_{(1)'} \right].$$

We obtain $(1) + (2) + (3) = J_j^l(\partial x_l J_i^k + J_i^s \Gamma_{l,s}^k - J_s^k \Gamma_{l,i}^s)$ and $(1)' + (2)' + (3)' = J_i^l(\partial x_j J_l^k + J_l^s \Gamma_{j,s}^k - J_s^k \Gamma_{j,l}^s)$. We recognize the coordinates of the tensor A (section 2.3):

$$\partial x_l J_i^k - J_s^k \Gamma_{l,i}^s + J_i^s \Gamma_{l,s}^k = A_{l,i}^k \quad \text{and} \quad \partial x_j J_l^k - J_s^k \Gamma_{j,l}^s + J_l^s \Gamma_{j,s}^k = A_{j,l}^k.$$

Finally $D_j^i = bp_k [J_j^l A_{l,i}^k - J_i^l A_{j,l}^k]$. Then Z is $J^{H, \nabla}$ -holomorphic if and only if $J_j^l A_{l,i}^k = A_{j,l}^k J_i^l$. Since $A_{j,l}^k J_i^l \partial x_k = A(\partial x_j, J \partial x_i)$ and $J_j^l A_{l,i}^k \partial x_k = A(J \partial x_j, \partial x_i)$ this concludes the proof of Theorem 1.3. \square

The almost complex lift \tilde{J} may be characterized generically by the holomorphicity of Z ; more precisely we have :

Corollary 4.2.

- (1) *The multiplication map Z is \tilde{J} -holomorphic and,*
- (2) *Z is $J^{H, \nabla}$ -holomorphic if and only if $\tilde{A}(J, \cdot) = \tilde{A}(\cdot, J)$.*

Proof. Let us prove part (1). Assume ∇ is an almost complex minimal connection on M . We have $\tilde{J} = J^{G, \nabla}$ and by almost complexity of ∇ , A is identically equal to zero. Theorem 1.3 implies the \tilde{J} -holomorphicity of Z .

The part (2) follows from Theorem 1.3 and the equality $J^{H, \nabla} = J^{G, \tilde{\nabla}}$ stated in Corollary 3.2. \square

Remark 4.1. *In case of the tangent bundle TM , the fiberwise multiplication is holomorphic for the complete lift of J if and only if J is integrable. More precisely, “the lack of holomorphicity” of this map is measured by the Nijenhuis tensor (see [3]).*

5. COMPATIBLE LIFTED STRUCTURES AND SYMPLECTIC FORMS

Assume (M, J) is an almost complex manifold. We recall that ω_{st} is the canonical symplectic form on the cotangent bundle on M . From a symplectic point of view, it seems natural to see if the generalized horizontal lifted structure is compatible with ω_{st} . More generally one can search for a lifted structure compatible with ω_{st} .

Proposition 5.1. *There is no almost complex structure J' on the cotangent bundle T^*M , compatible with ω_{st} , such that $\pi : T^*M \rightarrow M$ is (J', J) -holomorphic.*

In particular, the generalized horizontal lift, and so the complete and the horizontal lifts, are not compatible with the canonical symplectic form on T^*M .

Proof. Such a lifted structure can be locally written $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, where A, B and C are $(n \times n)$ blocks. Indeed $d\pi$ is locally given by $d\pi = \begin{pmatrix} Id & 0 \end{pmatrix}$ where Id and 0 are $(n \times n)$ blocks and since π is (J', J) -holomorphic we necessarily have this local expression for J' . So if we consider a vector $Y = P_i \partial p_i \neq 0 \in T(T^*M)$ we have $J'Y = (DP)_i \partial p_i$ and then $\omega_{st}(Y, J'Y) = 0$. Hence J' cannot be compatible with ω_{st} . \square

Let $\Gamma \subseteq M$ be a strictly pseudoconvex hypersurface in M . For $x \in \Gamma$, we denote by $N_x^*(\Gamma) = \{p_x \in T_x^*M, (p_x)|_{T_x\Gamma} = 0\}$. The conormal bundle over Γ , defined by the disjoint union $N^*(\Gamma) = \bigcup_{x \in \Gamma} N_x^*(\Gamma)$, is a totally real submanifold of T^*M endowed with the complete lift (see [7] and [2]). Moreover $N^*(\Gamma)$ is Lagrangian for the canonical symplectic form on T^*M . To get a symplectic proof of the total reality of the conormal bundle, it is enough to search for a symplectic form compatible with the complete lift for which $N^*(\Gamma)$ is Lagrangian. More generally we are interested in the compatibility with the generalized horizontal lift. Proposition 1.1 shows that we cannot find such a form.

Let us recall the definitions of Lagrangian and totally real submanifolds :

Definition 5.1.

- (1) *A submanifold N of a symplectic manifold (M', ω') is called Lagrangian for ω' if $\omega'(X, Y) = 0$ for every $X, Y \in \Gamma(TN)$.*
- (2) *A submanifold N of an almost complex manifold (M', J') is totally real if $TN \cap J(TN) = \{0\}$.*

Proof of Proposition 1.1. Let Γ be a strictly pseudoconvex hypersurface of M and $x \in \Gamma$. Since the problem is purely local we can suppose that $M = \mathbb{R}^{2m}$, $J = J_{st} + O(|(x_1, \dots, x_{2m})|)$ and $x = 0$. Since Γ is strictly pseudoconvex we can also suppose that $T_0\Gamma = \{X \in \mathbb{R}^{2m}, X_1 = 0\}$. The two-form ω' is given by $\omega' = \alpha_{i,j} dx^i \wedge dx^j + \beta_{i,j} dp^i \wedge dp^j + \gamma_{i,j} dx^i \wedge dp^j$.

Assume that $\omega'(X, Y) = 0$ for every $X, Y \in TN^*(\Gamma)$. We have $N_0^*(\Gamma) = \{p_0 \in T_0^*\mathbb{R}^{2m}, (p_0)|_{T_0\Gamma} = 0\} = \{(P_1, 0, \dots, 0), P_1 \in \mathbb{R}\}$. Then a vector $Y \in T_{(0,0)}N^*(\Gamma)$ can be

written $Y = X_2\partial x_2 + \cdots + X_{2m}\partial x_{2m} + P_1\partial p_1$. So we have for $2 \leq i < j \leq 2m$:

$$\omega'_{(0;0)}(\partial x_i, \partial x_j) = \alpha_{i,j} = 0.$$

Then $w'_{(0;0)}$ is given by $\omega'_{(0;0)} = \alpha_{1,j}dx_1 \wedge dx^j + \beta_{i,j}dp^i \wedge dp^j + \gamma_{i,j}dx^i \wedge dp^j$.

Since $J_{(0;0)}^{G,\nabla} = \begin{pmatrix} J_{st} & 0 \\ 0 & J_{st} \end{pmatrix}$ we have $J_{(0;0)}^{G,\nabla}Y' = \partial x_{2m}$ for $Y' = \partial x_{2m-1} \neq 0 \in T_{(0;0)}(T^*\Gamma)$.

Thus $\omega'_{(0;0)}(Y', J_{(0;0)}^{G,\nabla}Y') = 0$ and so ω' is not compatible with $J^{G,\nabla}$.

□

Proposition 1.1 is also established for complete and horizontal lifts because $J_{(0;0)}^{G,\nabla} = \tilde{J}_{(0;0)} = J_{(0;0)}^{H,\nabla}$.

Remark 5.1. *Since the conormal bundle of a (strictly pseudoconvex) hypersurface is Lagrangian for the symplectic form ω_{st} on T^*M , Proposition 1.1 shows directly that ω_{st} and $J^{G,\nabla}$ are not compatible.*

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