The Lamé family of connections on the projective line

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Introduction

A Lamé connection is a rank two connection $(\mathcal{M}, \nabla)$ on the projective line with simple poles at $0, 1, t, \infty$ and local exponents $1/4, -1/4$ at each singular point. The relation with the elliptic curve E, given by the affine equation $w^2 = z(z-1)(z-t)$, is investigated both in an algebraic and a complex analytic context.

For the latter an analytic moduli space for Lamé equations is produced, based on monodromy above $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$ and the fundamental group $\pi_1(E)$. N.J. Hitchin [Hi2] showed that the solutions of an irreducible Lamé connection can be written in terms of the Weierstrass zeta function. This was also noted by F. Beukers (see [Be]) who presented a different proof.

In the algebraic context a universal family, parametrized by a connected, non singular variety $P \to \{t \in \mathbb{C} \,|\, t \neq 0, 1, \infty\}$ of relative dimension 2, of Lamé connections with free vector bundle $\mathcal{M}$ is explicitly computed.

The Riemann-Hilbert correspondence yields an analytic isomorphism between the analytic moduli space and the algebraic one. In particular one observes that P is not the full algebraic moduli space. The missing part is the *special family* of Lamé connections $(\mathcal{M}, \nabla)$ where $\mathcal{M}$ is not free but isomorphic to $O(1) \oplus O(-1)$. This family is computed. For every member of the family the differential Galois group of the Lamé connection is the infinite dihedral group $D_\infty^{\mathrm{SL}_2}$. Here $D_\infty^{\mathrm{SL}_2}$ and $D_n^{\mathrm{SL}_2}$ denote the preimages of the

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infinite dihedral group $D_\infty \subset \mathrm{PSL}_2$ and the finite dihedral group $D_n \subset \mathrm{PSL}_2$ under the map $\mathrm{SL}_2 \to \mathrm{PSL}_2$.

The closed locus $P_{reducible} \subset P$, representing the reducible Lamé systems, is explicitly computed. It turns out to be a non singular divisor. The differential Galois group of a Lamé connection corresponding to $p \in P$ is reducible for $p \in P_{reducible}$ and is an irreducible subgroup of the infinite dihedral group $D_\infty^{\mathrm{SL}_2}$ for $p \notin P_{reducible}$. Let P_N denote the constructible subset of P consisting of the points such that the corresponding Lamé connection has the dihedral group $D_N^{\mathrm{SL}_2}$ as differential Galois group. An algorithm for the computation of the locus P_N, based on division polynomials for the elliptic curve E, the Painlevé VI equation and a transformation of Okamoto, is given. The loci P_N with $N = 2, 3, 4$ are explicitly computed. We note that the locus of the points in P with differential Galois group equal to $D_\infty^{\mathrm{SL}_2}$ is not constructible. The reason is that this linear algebraic group does not satisfy the condition posed in [Sin], (see also [B-vdP]).

A weak point of our presentation of Lamé connections is the separation into two families, the one parametrized by P and the other the special family. In the last section it is shown that one can produce a universal family containing both cases if one considers connection $(\mathcal{M}, \nabla)$ with $\mathcal{M} \cong O(0) \oplus O(-1)$.

The *classical Lamé equation*

$$\mathcal{L}_{n,B}(y) = y'' + \frac{f'}{2f}y' - \frac{n(n+1)z + B}{f}y = 0 \quad \text{with } f = 4(z-e_1)(z-e_2)(z-e_3),$$

distinct e_1, e_2, e_3 and $e_1 + e_2 + e_3 = 0$, has regular singularities at the points e_1, e_2, e_3 with local exponents $0, 1/2$. Further ∞ is a regular singularity with local exponents $-n/2$, $(n+1)/2$. For $n \in \mathbb{Z}$ the difference of the local exponents at ∞ lies in $1/2 + \mathbb{Z}$. It is therefore possible to transform $\mathcal{L}_n$ into a Lamé connection if $n \in \mathbb{Z}$. A suitable transformation yields an isomorphism of $\mathcal{L}_{0,B}$ (with $B \neq 0$) with the special family. This explains in particular the well known result that $D_N^{\mathrm{SL}_2}$ with finite N does not occur as differential Galois group for $\mathcal{L}_{0,B}$.

By an algebraic transformation $\mathcal{L}_{n,B}$ (for $n \geq 1$) is mapped to a 2-dimensional subspace S_n of the universal space P of Lamé connections. The intersection of $S_n \cap P_N$ is equal to the finite set of points in the family $\mathcal{L}_{n,B}$ having differential Galois group $D_N^{\mathrm{SL}_2}$. For $n = 1$ and $N = 2, 3$ our computations agree with the results of [Chi] and [Be-Wa]. In connection with this we observe the following. Any scalar equation for the Lamé connection,

corresponding to a point $p \in P$, has, in general, an apparent singularity, i.e., a singular point different from $0, 1, t, \infty$. This explains that the dimension of P is larger than the dimension of the moduli spaces for the $\mathcal{L}_{n,B}$. Moreover, it is rather exceptional that a point $p \in P_N$ induces a classical Lamé equation with differential Galois group $D_N^{\mathrm{SL}_2}$.

1 Imprimitive differential modules of rank 2

K denotes a differential field. The field of constants C of K is supposed to have characteristic 0 and to be different from K. Let M be a differential module over K of dimension 2 and let B_1, B_2 be a basis of M over K. Any element $m_1 \otimes m_2$ of the second symmetric power $sym^2_K M$ will be written as $m_1 m_2$. In particular B_1^2, B_1B_2, B_2^2 is a basis of $sym^2 M$. Let $L \supset K$ be an extension of degree 2 and let N denote a 1-dimensional differential module over L. We write $\mathrm{Res}_{L/K}(N)$ for N viewed as 2-dimensional differential module over K.

Lemma 1.1 *Let $F = \alpha_1 B_1^2 + \alpha_2 B_1 B_2 + \alpha_3 B_2^2 \neq 0$ generate a differential submodule of $sym^2_K M$. One regards F as element of the polynomial ring $K[B_1, B_2]$. Put $\Delta = \alpha_2^2 - 4\alpha_1\alpha_3$. Then:*
(1) *If F is a K^*-multiple of a square (or equivalently $\Delta = 0$), then we may suppose that $F = B_1^2$. Now KB_1 is a differential submodule of M.*
(2) *If F is a product of two distinct linear factors (or equivalently $\Delta \neq 0$ is a square in K), then we may suppose that $F = B_1B_2$. Now KB_1 and KB_2 are differential submodules of M.*
(3) *Suppose that F is irreducible. We may assume $F = B_2^2 - \Delta B_1^2$. Put $L = K(\delta)$ with $\delta^2 = \Delta$. There exists a 1-dimensional module $N = Le$ over L with $M \cong \mathrm{Res}_{L/K}(N)$ and such that the kernel of the surjective (obvious) morphism of differential modules $sym^2_K M \to \mathrm{Res}_{L/K}(sym^2_L N)$ equals KF. The differential module $N = Le$ with $\partial e = ue$, with the above properties, is unique up to conjugation, i.e., $u \in L$ is replaced by its conjugate $\overline{u}$. Moreover $sym^2_K M$ is the direct sum of $K(B_2^2 - \Delta B_1^2)$ and the differential submodule $K(B_2^2 + \Delta B_1^2) + K(B_1B_2)$.*
(4) *Suppose that C is algebraically closed and that M is irreducible and (for convenience) that the determinant of M (i.e., the exterior product $\Lambda^2 M$) is trivial. Then $sym^2_K M$ contains more than one 1-dimensional submodule, if and only if the differential Galois group of M is $D_2^{\mathrm{SL}_2}$. Moreover, in this case*

there are precisely three 1-dimensional submodules of $sym^2_K M$ and each one defines a different quadratic extension of K.

Proof. (1) and (2) follow by straightforward computation. In case (3) we give M the structure of a vector space over L by prescribing $\delta B_1 = B_2$ and $\delta B_2 = \Delta B_1$. Thus we obtain a vector space $N = LB_1$ over L. Let ∂ on M satisfy $\partial B_1 = aB_1 + bB_2$ with $a, b \in K$. Then ∂ on N is defined by $\partial B_1 = (a + b\delta)B_1$. From the assumption that $\partial(B_2^2 - \Delta B_1^2)$ is a multiple of $(B_2^2 - \Delta B_1^2)$, it follows that $\mathrm{Res}_{L/K}$ is isomorphic to M. The obvious morphism $sym^2_K M \to \mathrm{Res}_{L/K}(N)$ is given by B_1^2, B_1B_2, B_2^2 have images $B_1^2, \delta B_1^2, \delta^2 B_1^2$. The map is surjective and its kernel is generated by $B_2^2 - \Delta B_1^2$. The only freedom one has in giving M a structure of differential module over L is by prescribing $\partial B_1 = (a - b\delta)B_1$. The rest of the statement (3) follows by easy computation.
The first statement of (4) is well known and follows in fact from the classification of the algebraic subgroups of $\mathrm{Sl}_2(C)$. Suppose that the differential Galois group of M is $D_2^{\mathrm{Sl}_2}$. Then the differential Galois group of $sym^2 M$ is the group D_2 (of order 4) and $sym^2 M$ is the direct sum $L_1 \oplus L_2 \oplus L_3$ of submodules. Each L_j corresponds to a different surjective homomorphism $\chi_j : D_2 \to \{\pm 1\}$. The quadratic extension of K corresponding to L_j is the fixed field of the Picard-Vessiot field under the kernel of χ_j.

□

In case (3) of Lemma 1.1, the differential module will be called *imprimitive*.

Remark.
In case (4) of Lemma 1.1, the condition that C is algebraically closed cannot be omitted. We study this in detail.

The assumptions are: M/K irreducible and dimension 2; $\Lambda^2 M$ trivial, i.e., isomorphic to Ke with $\partial e = 0$; $sym^2 M$ has more than one ∂-invariant line. We want to investigate whether the conclusion of part (4) of the lemma is still valid if C is not algebraically closed.

If $\overline{C}K \otimes_K M$ is irreducible, then the differential Galois group of this module over $\overline{C}K$ is $D_2^{\mathrm{SL}_2}$ and there are precisely 3 ∂-invariant lines in $\overline{C}K \otimes sym^2 M$. Two of them are already defined over K and thus the third one is also defined over K.

Suppose that $\overline{C}K \otimes M$ is reducible. This module over $\overline{C}K$ is semi-simple and is in particular the sum of ∂-invariant lines.

Case (i). Assume that $\overline{C}K \otimes M$ has more than two invariant lines. Then the differential Galois group G of this module acts on the solution space V (of dimension 2 over $\overline{C}$) and has more than two invariant lines. Thus G is a subgroup of the group $\overline{C}^*$. Since also $G \subset \mathrm{SL}(V)$, one finds $G \subset \{\pm 1\}$. Since $0 = \overline{C} \otimes ker(\partial, M) = ker(\partial, \overline{C}K \otimes M)$, one concludes that $G = \{\pm 1\}$. Then G acts trivially on $sym^2 V$ and $sym^2(\overline{C}K \otimes M)$ is trivial. This implies that $sym^2 M$ is trivial. The triviality of $\Lambda^2 M$ implies that $M \cong M^*$. Then one finds that $M \otimes M^* \cong M \otimes M \cong sym^2(M) \oplus \Lambda^2 M$ is also a trivial differential module. Amitsur gave a construction of such modules M. For this construction one needs a base field K which is more complicated than a field of the type $C(z)$ with $' = \frac{d}{dz}$. For a modern description, using quaternion fields, and examples we refer to [H-P], section 2.5.

Case (ii) Assume that $\overline{C}K \otimes M$ has precisely two invariant lines. These lines are conjugated under the Galois group of $\overline{C}/C$. Thus there exists a quadratic extension $C(w) \supset C$, with $w^2 \in C$ such that $K(w) \otimes M = K(w)e_1 + K(w)e_2$ and the following properties hold:
(a) Let σ denote the non trivial element of the Galois group of $C(w)/C$. Then we may suppose that $\sigma e_1 = e_2$ and $\sigma e_2 = e_1$.
(b) $\partial e_1 = a e_1$ with $a \in K(w)$ and $\partial e_2 = \sigma(a) e_2$.
(c) $a + \sigma(a) = \frac{f'}{f}$ for some $f \in K^*$.
(d) M is generated by $f_1 = e_1 + e_2$ and $f_2 = w(e_1 - e_2)$.
Further $\partial f_1 = \frac{a+\sigma(a)}{2} f_1 + \frac{a-\sigma(a)}{w} f_2$ and $\partial f_2 = \frac{wa - w\sigma(a)}{2} f_1 + \frac{a+\sigma(a)}{2} f_2$.

We deduce some more properties.
(e) For any $\gamma \in K(w)^*$, the line $K(w)(\gamma e_1 + \sigma(\gamma) e_2)$ is not invariant under ∂. Indeed, this line comes from a line in M. This translates into: for every $\gamma \in \overline{C}K$ with $\gamma \neq 0$ one has $\frac{\gamma'}{\gamma} + a \notin K$.
(f) $sym^2(K(w) \otimes M) = (K(w)e_1^2 + K(w)e_2^2) \oplus K(w)e_1 e_2$ has at least two ∂-invariant $K(w)$-lines which come from $sym^2 M$. The line $K(w)e_1 e_2$ comes from $sym^2 M$. This module over K is trivial since $\partial e_1 e_2 = \frac{f'}{f} e_1 e_2$. Hence there must be a $\gamma \in K(w)^*$ such that $K(w)(\gamma e_1^2 + \sigma(\gamma) e_2^2)$ is invariant under ∂. This translates into $\frac{\gamma'}{\gamma} + 2a \in K$. Thus $a = -\frac{1}{2}\frac{\gamma'}{\gamma} + b$ with $b \in K$.
(g) From (c) it follows that $2b = \frac{f'}{f} + \frac{1}{2}\frac{(N(\gamma))'}{N(\gamma)}$, where $N(\gamma) = \gamma\sigma(\gamma)$ is the norm of γ. Thus $a = -\frac{1}{2}\frac{\gamma'}{\gamma} + \frac{1}{2}\frac{f'}{f} + \frac{1}{4}\frac{N(\gamma)'}{N(\gamma)}$.
(h) $N(\gamma)$ is not a square in K, otherwise the module $sym^2(K(w) \otimes M)$ would be trivial. We conclude the differential Galois group of $\overline{C}K \otimes M$ is the cyclic group of order 4.

Finally, in order to show that there exists an example for the above situation, we consider the case $K = C(z)$ with $' = \frac{d}{dz}$, w as above and we take for the γ in (f) the element $z - w$. With the choice $f = 1$ we find $a = -\frac{1}{4}\frac{1}{z-w} + \frac{1}{4}\frac{1}{z+w}$. One easily verifies this is indeed an example of the required type. □

Corollary 1.2 *Let M be an irreducible differential module of dimension 2 over K. Then M is imprimitive if and only if there exists a quadratic extension $L \supset K$ such that $L \otimes_K M$ has a 1-dimensional submodule.*

Proof. Suppose that M is imprimitive. Then $M = \mathrm{Res}_{L/K}(N)$ for suitable L and N. The kernel of the canonical map $L \otimes_K \mathrm{Res}_{L/K}(N) \to L \otimes_L N = N$ is a 1-dimensional submodule of $L \otimes_K M$.

On the other hand, suppose that $L \otimes_K M$ has a 1-dimensional submodule T_1. Using the conjugation of L/K one finds another 1-dimensional submodule T_2. The 1-dimensional submodule $T_1 \otimes_L T_2$ of $L \otimes_K sym^2_K M$ is invariant under the conjugation of L/K and thus provides a 1-dimensional submodule of $sym^2 M$. □

Suppose that M is an imprimitive differential module over K of dimension 2. For every integer $n \geq 2$, the differential module $sym^n_K M$ has an explicit 2-dimensional factor (direct summand) $\mathrm{Res}_{L/K}(sym^n_L N)$. We use the notation of Lemma 1.1, i.e., $L = K(\delta)$, $\delta^2 = \Delta$, $N = LB_1$ and $\partial B_1 = (a + b\delta)B_1$. Then $sym^n_L N$ has generator $e = B_1^n$ and $\partial e = n(a + b\delta)e$. Then $\mathrm{Res}_{L/K}(sym^n_L N)$ has basis $e, \delta e$ and the matrix of ∂ with respect to this basis reads

$$\begin{pmatrix} na & nb\Delta \\ nb & na + \frac{\Delta'}{2\Delta} \end{pmatrix}.$$

The above method to find a factor of the nth symmetric power can also be given explicitly for scalar equations. Consider a scalar equation $y'' + a_1 y' + a_0 y = 0$ over K with C algebraically closed, such that its differential Galois is an irreducible subgroup of $D^{\mathrm{SL}_2}_\infty$. The (abstract) solution space has a basis y_1, y_2 such that the differential Galois group permutes the two lines Cy_1, Cy_2. Then $S = y_1 y_2$ is a solution of the second symmetric power and $S^2 \in K$. This second symmetric power reads

$$y''' + 3a_1 y'' + (4a_0 + 2a_1^2 + a_1')y' + (2a_0' + 4a_0 a_1)y = 0 \ .$$

The term $\frac{S'}{S}$ can be computed explicitly. As a consequence the expression $\frac{S''}{S} = (\frac{S'}{S})' + (\frac{S'}{S})^2$ is known as element of K. Further $\frac{y_1'}{y_1}$ and $\frac{y_2'}{y_2}$ satisfy the polynomial equation

$$U^2 - \frac{S'}{S}U + (a_0 + a_1\frac{S'}{2S} + \frac{S''}{2S}) = 0 \text{ over } K.$$

From this one derives a differential equation L_n for y_1^n and y_2^n.

$$L_n(y) = y'' + (a_1 + (1-n)\frac{S'}{S})y' + (n^2a_0 + (n^2-n)a_1\frac{S'}{2S} + (n^2-n)\frac{S''}{2S})y.$$

2 Connections with 4 special singular points

Consider an irreducible connection of rank 2 on $\mathbb{P}^1_C$ having 4 singular points $0, 1, t, \infty$ and local exponents $1/4, -1/4$ for each singular point. The generic fibre of this connection is a differential module M of dimension 2 over $C(z)$, the field of rational functions of $\mathbb{P}^1_C$.

Proposition 2.1 *The module M is imprimitive and there is a 1-dimensional submodule of sym^2M such that the corresponding quadratic extension $L \supset C(z)$ has the form $C(z)(w)$ with $w^2 = cz(z-1)(z-t)$ and $c \in C^*$.*

A direct proof by computation will be given later on. First we consider the case where C is the field of complex numbers $\mathbb{C}$ and we present an analytic proof. We start by investigating the monodromy of M. One chooses a base point in $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, t, \infty\}$ and generators $\lambda_1, \ldots, \lambda_4$ of the fundamental group π_1 of this space, consisting of loops around the points $0, 1, t, \infty$ such that $\lambda_1 \cdots \lambda_4 = 1$. The monodromy is a homomorphism $\pi_1 \to \mathrm{Sl}_2(\mathbb{C})$, sending each λ_j to an element B_j of order 4. The monodromy is therefore described by elements $B_1, \ldots, B_4 \in \mathrm{Sl}_2(\mathbb{C})$ of order 4 with $B_1 \cdots B_4 = 1$.

Lemma 2.2 *Let $B_1, \ldots, B_4 \in \mathrm{Sl}_2(\mathbb{C})$ satisfy $B_1 \cdots B_4 = 1$ and $B_j^4 = 1$ for every j. Let G be the group generated by the B_j. There are two possibilities:*
(i) *G is reducible and contained in a Borel subgroup of $\mathrm{Sl}_2(\mathbb{C})$.*
(ii) *G is irreducible and contained in $D_\infty^{\mathrm{Sl}_2}$.*

Proof. We replace each B_j by its image b_j in $\mathrm{PGl}_2(\mathbb{C})$. Each b_j has order 2 and $b_1 \cdots b_4 = 1$. We have to consider several possibilities.

(1) *Suppose that b_1b_2 has precisely two fixed points $p_1, p_2 \in \mathbb{P}^1_{\mathbb{C}}$. Then each b_j interchanges the points p_1, p_2 and the group generated by $b_1, \ldots, b_4$ is an irreducible subgroup of D_∞.*

Proof. b_2b_1 is the inverse of b_1b_2 and has also fixed points p_1, p_2. Now $b_1b_2b_1(p_1) = b_1(p_1)$ and thus $b_1(p_1) \in \{p_1, p_2\}$. If $b_1(p_1) = p_1$, then $b_2(p_1) = b_2b_1(p_1) = p_1$. Also $b_1b_2b_1(p_2) = b_1(p_2)$ and therefore $b_1(p_2) = p_2$. It follows that $b_2(p_2) = p_2$. Since b_1 and b_2 have order 2 one finds the contradiction that $b_1 = b_2$. We conclude that b_1 and b_2 interchanges the points p_1, p_2.

One has $b_1b_2b_3(p_1) = b_3b_2b_1(p_1)$, since $b_1b_2b_3 = (b_1b_2b_3)^{-1} = b_3b_2b_1$. Moreover $b_3b_2b_1(p_1) = b_3(p_1)$. Hence $b_3(p_1) \in \{p_1, p_2\}$. Similarly $b_3(p_2) \in \{p_1, p_2\}$. Suppose that $b_3(p_1) = p_1$ and $b_3(p_2) = p_2$. Then $b_1b_2b_3$ and b_3 have both order two and both p_1, p_2 as fixed points. Then $b_1b_2b_3 = b_3$ and we find the contradiction $b_1b_2 = 1$.

(2) *Suppose that b_1b_2 has precisely one fixed point p. Then p is a fixed point of every b_j and the group generated by the $b_1, \ldots, b_4$ is reducible.*

Proof. $b_2b_1 = (b_1b_2)^{-1}$ has also fixed point p. Now $b_1b_2b_1(p) = b_1(p)$ and thus $b_1(p) = p$. Then also $b_2(p) = p$. Further $b_1b_2b_3(p) = b_3b_2b_1(p) = b_3(p)$ and thus $b_3(p) = p$. Take for convenience $p = \infty$. Then one sees that the group generated by all b_j lies in an upper triangle group of matrices. Hence this group is reducible.

(3) *Suppose that $b_1 = b_2$. Then the group generated by $b_1, \ldots, b_4$ is the same as the group generated by b_1, b_3. If b_1b_3 has precisely two fixed points, then the corresponding group is an irreducible subgroup of D_∞. If b_1b_3 has precisely one fixed point then the corresponding group is reducible. Finally $b_1 = b_3$ also produces a reducible group.*

Proof. One can repeat the arguments in the proofs of (1) and (2) to prove statement (3). □

Second proof. Let E denote the elliptic curve given by the affine equation $w^2 = z(z-1)(z-t)$. Let $p_1, \ldots, p_4 \in E$ denote the points of E with images $0, 1, t, \infty$. Then $E_- := E \setminus \{p_1, \ldots, p_4\} \to \mathbb{P}^1_- := \mathbb{P}^1 \setminus \{0, 1, t, \infty\}$ is a covering of degree 2 and the homomorphism $I : \pi_1(E_-) \to \pi_1(\mathbb{P}^1_-)$ is injective and its image is a subgroup of index 2. Write, as before, $\pi_1(\mathbb{P}^1_-) = < \lambda_1, \ldots, \lambda_4 | \lambda_1 \cdots \lambda_4 = 1 >$. Then the image of I is the kernel of the homomorphism $\pi_1(\mathbb{P}^1_-) \to \{\pm 1\}$ that sends each λ_j to -1. One can give $\pi_1(E_-)$ generators $a, b, \mu_1, \ldots, \mu_4$ such that:
(i) Each μ_j is a loop around p_j and I maps μ_j to a conjugate of λ_j^2.

(ii) There is only one relation, namely $\mu_1\mu_2\mu_3\mu_4 aba^{-1}b^{-1} = 1$.
(iii) The images of a, b in $\pi_1(E)$ under the canonical map $\pi_1(E_-) \to \pi_1(E)$ are generators of $\pi_1(E)$.
(iv) Conjugation with an (or any) element $\alpha \in \pi_1(\mathbb{P}^1_-)$, not contained in $\pi_1(E_-)$, acts as -1 on the abelianized group $(\pi_1(E_-))_{ab}$.
We use the map I to identify $\pi_1(E_-)$ with a subgroup of $\pi_1(\mathbb{P}^1_-)$.

Consider a homomorphism $h : \pi_1(\mathbb{P}^1_-) \to \mathrm{SL}_2(\mathbb{C})$ which sends each λ_j to B_j. Then $h(\mu_j) = -1$ for all j and $h(a)$ commutes with $h(b)$. In particular, $h(\pi_1(E_-))$ is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. Further, h induces a homomorphism $\pi_1(E) \to \mathrm{SL}_2(\mathbb{C})$ which will also be called h. There are the following possibilities:
(a) This abelian subgroup has precisely one invariant line $\mathbb{C}e$ in $\mathbf{C}^2$. Then $\mathbb{C}e$ is also invariant under $h(\pi_1(\mathbb{P}^1_-))$ and the representation h is reducible.
(b) This abelian subgroup has precisely two invariant lines $\mathbb{C}e_1$, $\mathbf{C}e_2$. Then $h : \pi_1(E) \to \mathrm{SL}_2(\mathbb{C})$ has the form $h(c)e_1 = \chi(c)e_1$ and $h(c)e_2 = \chi(c)^{-1}e_2$, for some character χ of $\pi_1(E) \cong \mathbb{Z}^2$ such that $\chi^2 \neq 1$. For any $\alpha \in \pi_1(\mathbb{P}^1_-)$, not contained in $\pi_1(E_-)$, $h(\alpha)$ interchanges the two lines $\mathbb{C}e_1$ and $\mathbb{C}e_2$. This follows from (iv). In particular, the representation h is irreducible and its image is an irreducible subgroup of $D_\infty^{\mathrm{Sl}_2}$.
(c) This abelian subgroup has more than two invariant lines. Then h maps every element of $\pi_1(E_-)$ to ± 1. Then $h(\pi_1(\mathbb{P}^1_-))$ is abelian (actually a cyclic group of order 4) and the representation is reducible. □

Observations 2.3 We supplement the information of Lemma 2.2 with a well known description of the monodromy. The monodromy tuple $(B_1, \ldots, B_4)$, associated to a differential module as above, is only determined up to simultaneous conjugation. The collection $\mathcal{T}$ of all tuples $(B_1, \ldots, B_4) \in \mathrm{Sl}_2^4$ satisfying $B_j^4 = 1$ for every j and $B_1 \cdots B_4 = 1$ is a closed subset of Sl_2^4. The group PSL_2 acts on $\mathcal{T}$ by conjugation. There is a categorical quotient, namely $\mathcal{Q} = \mathrm{Spec}[t_1, t_2, t_3]/(t_1^2+t_2^2+t_3^2+t_1t_2t_3-4)$. The morphism $q : \mathcal{T} \to \mathcal{Q}$ is given by

$$(B_1, \ldots, B_4) \mapsto (t_1 = tr(B_1B_2), t_2 = tr(B_2B_3), t_3 = tr(B_1B_3)).$$

$S = \{(-2,2,2), (2,-2,2), (2,2,-2), (-2,-2,-2)\}$ is the set of the singular points of $\mathcal{Q}$. The preimage of S is the locus in $\mathcal{T}$ describing the reducible groups (this follows easily from the proof of Lemma 2.2). Let $\mathcal{T}_{irred}$ denote the open subset corresponding to the irreducible groups. Then $q : \mathcal{T}_{irred} \to$

$\mathcal{Q} \setminus S$ is a geometric quotient for the action of the group PSL_2 (this follows easily from the second proof of Lemma 2.2). In particular, $\mathcal{T}_{irred}$ is reduced, irreducible, smooth and has dimension 5.

The second proof of Lemma 2.2 leads to a connection between the monodromy of the elliptic curve E and the special monodromy for $\mathbb{P}^1_-$. Let a homomorphism $\chi : \pi_1(E) \to \mathbb{C}^*$, with $\chi^2 \neq 1$ be given. Define $h : \pi_1(E_-) \to \mathrm{SL}_2(\mathbb{C})$ by $h(\mu_j) = -1$ for every j and $h(x)$ is the diagonal matrix with entries $\chi(x), \chi(x)^{-1}$, for $x = a, b$. This homomorphism can be extended to a homomorphism $\phi : \pi_1(\mathbb{P}^1_-) \to \mathrm{SL}_2(\mathbb{C})$ as follows: for any $x \in \pi_1(E_-)$ one defines $\psi(x) = h(x)$ (of course) and $\psi(x\lambda_4) = h(x) \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Using (iv), one easily verifies that ψ is indeed a homomorphism.

We note that two characters $\chi_1, \chi_2 : \pi_1(E) \to \mathbb{C}^*$ with $\chi_1^2 \neq 1 \neq \chi_2^2$ induce isomorphic irreducible representations $\rho_1, \rho_2 : \pi_1(\mathbb{P}^1_-) \to \mathrm{Sl}_2(\mathbb{C})$ if and only if $\chi_2 \in \{\chi_1, \chi_1^{-1}\}$. We see $T_E := \mathrm{Hom}(\pi_1(E), \mathbb{C}^*)$ as an algebraic torus with group of characters $\pi_1(E)$. Let $T_E[2]$ denote the subgroup of the points of order dividing 2. Consider $(T_E \setminus T_E[2])/\sim$, where $\chi_1 \sim \chi_2$ if and only if $\chi_2 \in \{\chi_1, \chi_1^{-1}\}$. The above construction induces an isomorphism of algebraic varieties

$$(T_E \setminus T_E[2])/\sim \longrightarrow \mathcal{Q} \setminus S.$$

This isomorphism extends to an isomorphism $T_E/\sim \longrightarrow \mathcal{Q}$.
The isomorphism can be made explicit as follows. One considers an element $(B_1, \ldots, B_4) \in \mathcal{T}$. Suppose that this tuple generates an irreducible subgroup, then the tuple is an example for case (b) in the second proof of Lemma 2.2. In the conjugacy class of the tuple there is an element of the form

$$\left(\begin{pmatrix} 0 & r \\ -1/r & 0 \end{pmatrix}, \begin{pmatrix} 0 & s \\ -1/s & 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -1/t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),$$

with $s = rt$ because $B_1 \cdots B_4 = 1$. Moreover, $(r, t) \neq (\pm 1, \pm 1)$. The above element is unique up to conjugation with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This operation changes (r, t) into $(1/r, 1/t)$. Thus we find an isomorphism between $\{(\mathbb{C}^*)^2 \setminus \{(\pm 1, \pm 1)\}\}/\sim$ and $\mathcal{Q} \setminus S$, which is essentially the same as the above isomorphism.

Proof of Proposition 2.1. We first consider the case where C is the field of complex numbers $\mathbb{C}$. According to Lemma 2.2, the monodromy group of M, generated by $B_1, \ldots, B_4$, is an irreducible subgroup of $D_\infty^{\mathrm{Sl}_2}$ and the same

holds for its Zariski closure which is the differential Galois group. It follows that $sym^2 M$ contains a 1-dimensional submodule.

In case the differential Galois group G is $D_\infty^{\mathrm{Sl}_2}$ or $D_n^{\mathrm{Sl}_2}$ with $n > 2$, there is only one 1-dimensional submodule. The corresponding field extension $L \supset \mathbb{C}(z)$ corresponds to the unique surjective homomorphism $h : G \to \{\pm 1\}$. The elements $B_j \in G$ have order 4 and it follows that $h(B_j) = -1$ for every j. This implies that $L \supset \mathbb{C}(z)$ is ramified above each of the points $0, 1, t, \infty$. The extension L is therefore given by $L = \mathbb{C}(z)(w)$ with $w^2 = z(z-1)(z-t)$.

If the differential Galois group G is $D_2^{\mathrm{Sl}_2}$, then there are precisely three 1-dimensional submodules of $sym^2 M$ and there are also precisely three surjective homomorphisms $h : G \to \{\pm 1\}$. As above, it suffices to verify that there exists a homomorphism with $h(B_j) = -1$ for all j. The commutator subgroup of G is identical with the center of G. Consider the images $b_1, \ldots, b_4 \in \mathrm{PSl}_2$ of $B_1, \ldots, B_4$. They generate a commutative group with two generators, say A and B, each with order two. One has $b_j \in \{A, B, AB\}$ for all j. Further the surjective homomorphisms $h : G \to \{\pm 1\}$ are given by $(h(A), h(B)) = (-1, -1), (-1, 1), (1, -1)$. Up to a choice for A and B there are three possibilities:
(i) $(b_1, b_2, b_3, b_4) \mapsto (A, B, A, B)$. The possibilities for $L \supset \mathbb{C}(z)$ are given by the sets of ramified points $\{0, 1, t, \infty\}, \{0, t\}, \{1, \infty\}$.
(ii) $(b_1, b_2, b_3, b_4) \mapsto (A, B, B, A)$. The possibilities for the sets of ramified points are $\{0, 1, t, \infty\}, \{0, \infty\}, \{1, t\}$.
(iii) $(b_1, b_2, b_3, b_4) \mapsto (A, A, B, B)$. The possibilities for the sets of ramified points are $\{0, 1, t, \infty\}, \{0, 1\}, \{t, \infty\}$.
In particular, in each case one of the corresponding fields is $\mathbb{C}(z)(w)$ with $w^2 = z(z-1)(z-t)$. In the algorithmic part of this paper we will verify that each of the cases (i)–(iii) are present in our family.

Now we consider a general case. We may suppose that the algebraic closure $\overline{C}$ of C is a subfield of $\mathbb{C}$. Let M^+ denote the differential module $\overline{C}(z) \otimes_{C(z)} M$. For this differential module the statement of the proposition follows easily from the statement for the case $\mathbb{C}$. Further $sym^2 M^+$ has one or three 1-dimensional submodules. They produce one or three distinct quadratic extensions of $\overline{C}(z)$. It follows that the Galois group of $\overline{C}/C$ preserves these 1-dimensional submodules. Therefore $sym^2 M$ has one or three 1-dimensional submodules. For one of them the field extension is over $\overline{C}(z)$ given by $w^2 = z(z-1)(z-t)$. Thus over the field $C(z)$ this equation reads $w^2 = cz(z-1)(z-t)$ for some $c \in C^*$. □

3 Regular connections on an elliptic curve

In this section the base field will be $\mathbb{C}$, the field of complex numbers. Let M denote a differential module over $\mathbb{C}(z)$ with four singular points $0, 1, t, \infty$ and local exponents $1/4, -1/4$ for each of them. According to Proposition 2.1, M is the restriction to $\mathbb{C}(z)$ of a differential module of rank one over the field $\mathbb{C}(z, w)$ with $w^2 = z(z-1)(z-t)$. After tensoring M with the differential module $(\mathbb{C}(z)e, \partial)$, where $\partial(e) = (\frac{1/4}{z} + \frac{1/4}{z-1} + \frac{1/4}{z-t})e$, the new differential module has local exponents $1/2, 0$ at each singular point. The corresponding rank one differential module over $\mathbb{C}(z, w) = \mathbb{C}(E)$ has no singularities. Here E denotes the elliptic curve given by the equation $w^2 = z(z-1)(z-t)$. We start by describing the regular connections on E.

A regular connection is a pair $(\mathcal{L}, \nabla)$ of a line bundle on E and a connection $\nabla : \mathcal{L} \to \Omega \otimes \mathcal{L}$, where Ω denotes the sheaf of the holomorphic differentials. Its generic stalk is a vector space $\mathbb{C}(E)e$ together with a connection ∇ given by $\nabla e = \omega \otimes e$ where ω of a meromorphic differential on E. The condition that the connection has no singularities translates into ω has only simple poles and all its residues are in $\mathbb{Z}$. Further ω is unique up to the addition of a term $\frac{df}{f}$, with $f \in \mathbb{C}(E)^*$. For any meromorphic differential form ω with at most simple poles, we define $\mathrm{Res}(\omega) := \sum_{p \in E} res_p(\omega)[p]$. This is a divisor of degree 0 on E if all the residues of ω are integers. Let ω define a regular connection and let the line bundle $\mathcal{L}$ correspond to $\mathrm{Res}(\omega)$. Then ω corresponds to a global regular connection $\nabla : \mathcal{L} \to \Omega \otimes \mathcal{L}$ on E.

If $\mathcal{L}$ is trivial (i.e., isomorphic to O_E), then ω has the form $c\frac{dz}{2w}$ for some $c \in \mathbb{C}$.

If $\mathcal{L}$ is not trivial then this line bundle corresponds to a divisor $[q] - [1_E]$, where 1_E is the neutral element of E (as usual taken to be the point $z = \infty$) and $q = (z_0, w_0)$ is some point $(\neq 1_E)$ on the affine curve given by $w^2 = z(z-1)(z-t)$. In this case ω can be written in normalized form as $(c + \frac{w+w_0}{z-z_0})\frac{dz}{2w}$ with $c \in \mathbb{C}$.

Now we make a complex analytic study of the regular connections on E. Consider the exact sequence of sheaves on E

$$0 \to \mathbb{C}^* \to O_E^* \to \Omega \to 0,$$

where $\mathbb{C}^*$ is the constant sheaf, O_E^* is the sheaf of invertible holomorphic functions and Ω is the sheaf of holomorphic differential forms. The morphism

$O_E^* \to \Omega$ is given by $f \mapsto \frac{df}{f}$. This induces an exact sequence for cohomology groups

$$0 \to \mathbb{C}\frac{dz}{2w} \to H^1(E, \mathbb{C}^*) \to H^1(E, O_E^*) \to H^1(E, \Omega) \cdots .$$

$H^1(E, O_E^*)$ is the group of the equivalence classes of line bundles $Pic(E)$. Its subgroup $Pic^0(E)$ of equivalence classes of line bundles of degree 0 is identified with E. Further $H^1(E, \mathbb{C}^*)$ is equal to $\mathrm{Hom}(\pi_1(E), \mathbb{C}^*)$, where $\pi_1(E)$ denotes the fundamental group of E. One easily derives from the above that the following sequence is exact

$$0 \to \mathbb{C}\frac{dz}{2w} \to \mathrm{Hom}(\pi_1(E), \mathbb{C}^*) \to E \to 0.$$

Lemma 3.1 *There is a natural isomorphism of groups*

$$\mathrm{Hom}(\pi_1(E), \mathbb{C}^*) \to \{(\mathcal{L}, \nabla)\}/\sim,$$

where the last group is the group of the equivalence classes of the regular connections on E of rank 1.

Proof. Let $U : \mathbb{C}_u \to E$ denote the uniformization of E. Here u denotes the global parameter of $\mathbb{C}$. The kernel Λ of U is identified with $\pi_1(E)$. Suppose that a homomorphism $\rho : \Lambda \to \mathbb{C}^*$ is given. Let $\mathbb{C}_u \times \mathbb{C}_v$ denote the trivial (geometric) line bundle on $\mathbb{C}_u$ provided with the trivial connection. Here v denotes the global parameter on the linear space $\mathbb{C}$. Let Λ act on $\mathbb{C}_u \times \mathbb{C}_v$ by $\lambda(u, v) = (u + \lambda, \rho(\lambda)v)$. The quotient by the action of Λ is a (geometric) line bundle $(\mathbb{C}_u \times \mathbb{C}_v)/\Lambda \to \mathbb{C}_u/\Lambda = E$ with induced connection. The sheaf of sections of this geometric line bundle is a line bundle $\mathcal{L}$ on E (of degree 0) and is provided with a regular connection.

Let, on the other hand, a regular connection $(\mathcal{L}, \nabla)$ on E be given. Then $U^*(\mathcal{L}, \nabla)$ is a trivial connection on $\mathbb{C}_u$. The group Λ acts on the 1-dimensional solution space of this trivial connection and this produces a homomorphism $\rho : \Lambda \to \mathbb{C}^*$. □

A slightly different way to find the isomorphism of Lemma 3.1 is the following. Consider the exact sequence of sheaves on E:

$$0 \to \mathbb{C}^* \to \mathcal{M}^* \to \Omega_{sp} \to 0,$$

where $\mathcal{M}^*$ denotes the sheaf of the invertible meromorphic functions on E and Ω_{sp} denotes the sheaf of the special meromorphic differentials having at most simple poles and having residues in $\mathbb{Z}$. The morphism $\mathcal{M}^* \to \Omega_{sp}$ is given by $f \mapsto \frac{df}{f}$. The cohomology sequence reads

$$0 \to \mathbb{C}^* \to \mathbb{C}(E)^* \to H^0(E, \Omega_{sp}) \to H^1(E, \mathbb{C}^*) \to 0.$$

Indeed, $H^1(E, \mathcal{M}^*) = 0$ is known. The cokernel of $\mathbb{C}(E)^* \to H^0(E, \Omega_{sp})$ has already been identified with the isomorphism classes of the regular connections on E.

Proposition 3.2 *Let $\pi : E \to \mathbb{P}^1$ denote the morphism, induced by the map $(z, w) \mapsto z$. Let $(\mathcal{L}, \nabla)$ denote a regular connection (of rank one) on E. Then $\pi_*(\mathcal{L}, \nabla)$ is a rank 2 connection on $\mathbb{P}^1$ having 4 regular singular points, namely $0, 1, t, \infty$. At each singular point the local exponents are $0, 1/2$. Further $\pi_* O_E$ is isomorphic to the vector bundle $O(0) \oplus O(-2)$ on $\mathbb{P}^1$ and $\pi_* \mathcal{L}$, with $\mathcal{L} \neq O_E$, is isomorphic to the vector bundle $O(-1) \oplus O(-1)$ on $\mathbb{P}^1$.*

Proof. Let $U : \mathbb{C}_u \to E$ denote again the universal covering. Let $\tilde{\Lambda}$ denote the group of the automorphisms of $\mathbb{C}_u$ generated by the translations over the elements of Λ and the map $u \mapsto -u$. Then $\mathbb{C}_u / \tilde{\Lambda}$ is identified with $\mathbb{P}^1$. The trivial connection on $\mathbb{C}_u$ (provided with a homomorphism $\rho : \Lambda \to \mathbb{C}^*$) yields a connection on $\mathbb{P}^1$ which is clearly regular outside the ramification points. For a point of ramification, say the image in $\mathbb{P}^1$ of the neutral element of E, one easily verifies that the local exponents are $0, 1/2$. Indeed, one knows that the local exponents at a ramification point give, multiplied with the ramification index (in this case 2), the local exponents above. The latter are $\{0, 1\}$.

A line bundle $\mathcal{L}$ on E and its direct image $\pi_* \mathcal{L}$ have cohomology groups of the same dimension. This proves the second statement. □

Remarks 3.3 *From rank one connections on E to Lamé connections.*
Now $\pi_*(\mathcal{L}, \nabla)$ is tensorized with the rank one connection $T := (T, \nabla)$ on $\mathbb{P}^1$ with singularities in $0, 1, t, \infty$, given by $T = O([\infty])e$ (i.e., the sheaf of meromorphic functions having at most a simple pole at ∞) and

$$\nabla e = \left(\frac{-1/4}{z} + \frac{-1/4}{z-1} + \frac{-1/4}{z-t}\right) dz \otimes e = \frac{-dw}{2w} \otimes e.$$

Then $\pi_*(\mathcal{L},\nabla)\otimes T$ has local exponents $1/4,-1/4$ at each singular point. Moreover this rank 2 bundle is free if $\mathcal{L}\neq O_E$ and has the form $O(1)\oplus O(-1)$ in case $\mathcal{L}=O_E$. According to Proposition 2.1 we obtain in this way all the irreducible connections on $\mathbb{P}^1_{\mathbb{C}}$ with 4 singular points $0,1,t,\infty$ and all local exponents $1/4,-1/4$ as push forward of a connection $\nabla e=\omega\otimes e$ on E, where ω is given in standard form by $(c+\frac{w+w_0}{z-z_0})\frac{dz}{2w}-\frac{dw}{2w}$ or $c\frac{dz}{2w}-\frac{dw}{2w}$, with c a constant and $(z_0,w_0)\in E$ (distinct from 1_E). We note that the term $-\frac{dw}{2w}$ takes care of the required shift of the local exponents. The second standard form for ω can be seen as a limit case for the first one where (z_0,w_0) tends to 1_E. This construction produces also four reducible connections on $\mathbb{P}^1_{\mathbb{C}}$, namely for the ω such that $2\omega=\frac{dF}{F}$ for some rational function F on E. The four cases are

$$\omega=\frac{w}{z}\frac{dz}{2w}-\frac{dw}{2w},\quad \frac{w}{z-1}\frac{dz}{2w}-\frac{dw}{2w},\quad \frac{w}{z-t}\frac{dz}{2w}-\frac{dw}{2w},\quad -\frac{dw}{2w}\ .$$

Moreover every irreducible connection on $\mathbb{P}^1_{\mathbb{C}}$ is obtained precisely twice since ω and $-\omega$ produce the same connection. These statements easily follow from Observations 2.3. Indeed, the above construction $(\mathcal{L},\nabla)\mapsto\pi_*(\mathcal{L},\nabla)\otimes T$ translates into the construction, explained in Observations 2.3, which associates to $\chi\in\mathrm{Hom}(\pi_1(E),\mathbb{C}^*)$ a tuple $(B_1,\ldots,B_4)$ (with the properties stated above) modulo the action by conjugation of the group $\mathrm{PSL}_2(\mathbb{C})$.

4 The analytic universal family

The Legendre family *Legendre* of elliptic curves $w^2=z(z-1)(z-t)$ over $\mathbb{C}$ can be written as algebraic variety

$$\mathrm{Proj}(\mathbb{C}[t,\frac{1}{t(t-1)}][z,w,s]/(sw^2-z(z-s)(z-st)))\to\mathbb{P}^1\setminus\{0,1,\infty\}.$$

If one allows the values $0,1,\infty$ for t, then one obtains an elliptic surface $\mathcal{E}\to\mathbb{P}^1$ having singular fibres above $0,1,\infty$. The uniformization of the Legendre family is equal to $\mathbb{H}\times\mathbb{C}_u$, where $\mathbb{H}$ is the upper half plane and $\mathbb{C}_u$ means $\mathbb{C}$ where we use u as variable. The group acting upon this space is $\mathbb{Z}^2\rtimes\Gamma(2)^*$, where: $\Gamma(2)^*$ is the subgroup of $\mathrm{Sl}_2(\mathbb{Z})$ consisting of the matrices $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ such that $a,d\equiv 1 \mod 4$ and $b,c\equiv 0 \mod 2$. This group is free on two generators and $\Gamma(2)^*\backslash\mathbb{H}$ is isomorphic to $\mathbb{P}^1\setminus\{0,1,\infty\}$.
The action (by conjugation) of $\Gamma(2)^*$ on $\mathbb{Z}^2$ is given by $\gamma\binom{n}{m}\gamma^{-1}=^t\gamma^{-1}\binom{n}{m}$.

The action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\mathbb{H} \times \mathbb{C}_u$ is given by $\gamma(\tau, u) = (\frac{a\tau+b}{c\tau+d}, \frac{u}{c\tau+d})$. The action of $\mathbb{Z}^2$ is given by $\binom{n}{m}(\tau, u) = (\tau, u + n\tau + m)$.

Above the Legendre family we want to construct the universal line bundle with a regular connection. Put $\mathbb{T} = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}^*)$; this is an algebraic torus. Consider the product $\mathbb{T} \times \mathbb{H} \times \mathbb{C}_u \times \mathbb{C}_v$, where the last term means $\mathbb{C}$ with parameter v. This is seen as a geometric line bundle with trivial connection above $\mathbb{T} \times \mathbb{H} \times \mathbb{C}_u$. The group $\mathbb{Z}^2 \rtimes \Gamma(2)^*$ act as follows:
For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)^*$, one defines $\gamma(\rho, \tau, u, v) = (\gamma(\rho), \frac{a\tau+b}{c\tau+d}, \frac{u}{c\tau+d}, v)$, where the action of $\Gamma(2)^*$ on $\mathbb{T}$ is induced by its action on $\mathbb{Z}^2$. Further, for $\binom{n}{m} \in \mathbb{Z}^2$ one defines $\binom{n}{m}(\rho, \tau, u, v) = (\rho, \tau, u + n\tau + m, \rho(\binom{n}{m})v)$.
The quotient $\mathbb{L} := \mathbb{Z}^2 \rtimes \Gamma(2)^* \backslash (\mathbb{T} \times \mathbb{H} \times \mathbb{C}_u \times \mathbb{C}_v)$ is a geometric line bundle with a regular connection above the family of elliptic curves
$\mathbb{Z}^2 \rtimes \Gamma(2)^* \backslash (\mathbb{T} \times \mathbb{H} \times \mathbb{C}_u)$ parametrized by $Par := \Gamma(2)^* \backslash (\mathbb{T} \times \mathbb{H})$. Let $\mathcal{L}$ be the sheaf of sections of $\mathbb{L}$. It has an induced connection and thus we find a universal regular connection $(\mathcal{L}, \nabla)$ above the family of elliptic curves. The parameter space $Par := \Gamma(2)^* \backslash \mathbb{T} \times \mathbb{H}$ is a $\mathbb{T}$-bundle above $\Gamma(2)^* \backslash \mathbb{H} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The family of elliptic curves can be written as $Par \times_{\mathbb{P}^1 \setminus \{0,1,\infty\}} Legendre$

Fix $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\tau \in \mathbb{H}$ with image t and a $\rho \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{C}^*)$. The 'evaluation' of $(\mathcal{L}, \nabla)$ at the point (ρ, t) is a connection on the curve $E_t = E_u/(\mathbb{Z}\tau + \mathbb{Z})$ (with equation $w^2 = z(z-1)(z-t)$) corresponding to the homomorphism $\pi_1(E_t) = \mathbb{Z}\tau + \mathbb{Z} \to \mathbb{C}^*$ given by $n\tau + m \mapsto \rho(\binom{n}{m})$.

One considers the subset $\{(\rho, \tau) | \exists c \in \mathbb{C} \text{ such that } \rho(\binom{n}{m}) = e^{c(n\tau+m)}\}$ of $\mathbb{T} \times \mathbb{H}$. This subset is given by the equation $\frac{\log \rho\binom{1}{0} - \tau \log \rho\binom{0}{1}}{2\pi i} \in \mathbb{Z}\tau + \mathbb{Z}$. (Note that this equation does not depend on the choice of the logarithms). Hence this subset is an analytic divisor on $\mathbb{T} \times \mathbb{H}$. The map $\mathbb{C} \times \mathbb{H} \to \mathbb{T} \times \mathbb{H}$, given by $(c, \tau) \mapsto (\rho, \tau)$ with $\rho\binom{n}{m} = e^{c(n\tau+m)}$, induces an isomorphism of $\mathbb{C} \times \mathbb{H}$ with this divisor. The divisor is invariant under the action of $\Gamma(2)^*$ and yields a divisor Θ in the parameter space Par. The subset of the points in the parameter space where the 'evaluation' of $\mathcal{L}$ is trivial, i.e., isomorphic to O_{E_t}, is precisely Θ. This ends the description of the universal connection of rank one $(\mathcal{L}, \nabla)$ above the space $Par \times_{\mathbb{P}^1 \setminus \{0,1,\infty\}} Legendre$.

The next step is to push this universal connection down to the projective line. One considers the obvious morphism $Legendre \to (\mathbb{P}^1 - \{0, 1, \infty\}) \times \mathbb{P}^1$. This induces a morphism $\pi : Par \times_{\mathbb{P}^1 \setminus \{0,1,\infty\}} Legendre \to Par \times \mathbb{P}^1$. One obtains a connection $\pi_*(\mathcal{L}, \nabla)$ of rank 2 on $Par \times \mathbb{P}^1$. This connection is

tensorized by the rank 1 connection $(O([\infty])e, \nabla)$ given by $\nabla e = -\frac{dw}{2w} \otimes e$. The result is a connection of rank two on the space $Par \times \mathbb{P}^1$ with regular singular sections $Par \times \{0\}, \ldots, Par \times \{\infty\}$ and local exponents $1/4, -1/4$ for each singular section. We delete from Par the closed subset $\Gamma(2)^* \backslash \mathbb{T}[2] \times \mathbb{H}$. On the result Par^* there is a free action of an automorphism of order 2, induced by the map $\mathbb{T} \times \mathbb{H} \to \mathbb{T} \times \mathbb{H}$ given by $(\rho, \tau) \mapsto (\rho^{-1}, \tau)$. The resulting space will de denoted by Par^{**}. This automorphism also acts upon the connection $\pi_*(\mathcal{L}, \nabla) \otimes (O([\infty])e, \nabla)$ restricted to $Par^* \times \mathbb{P}^1$. As a consequence we find a connection $(\mathcal{M}, \nabla)$ on $Par^{**} \times \mathbb{P}^1$. This is finally the universal family of rank two irreducible connections on $\mathbb{P}^1$ with regular singularities at $0, 1, t, \infty$ and local exponents $1/4, -1/4$ at each singular point, that we wanted to construct.

The image Θ^{**} of Θ in Par^{**} is the locus where the vector bundle $\mathcal{M}$ is not free (and actually is isomorphic to $O(1) \oplus O(-1)$). On the complement of the divisor Θ^{**} in Par^{**}, the vector bundle $\mathcal{M}$ is free.

The analytic $\mathbb{T}$-bundle $Par \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is probably not trivial. The same holds for the bundle $Par^{**} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Each fibre is equal to $T \setminus T[2]/\sim$. This space has been identified with $\mathcal{Q} \setminus S$.

The morphism $\Gamma(2)^* \backslash (\mathbb{C} \times \mathbb{H}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is an analytic line bundle and therefore free. It follows that the divisor Θ^{**} on Par^{**} is, as a variety, isomorphic to the space $\mathbb{C}^* \times (\mathbb{P}^1 \setminus \{0, 1, \infty\})$. The restriction of the connection $(\mathcal{M}, \nabla)$ to Θ^{**} will be called the *special family*. In Section 6 we will make this special family explicit. The restriction of $(\mathcal{M}^{**}, \nabla)$ to $Par^{**} \setminus \Theta^{**}$ has the property that the 'evaluation' of $\mathcal{M}^{**}$ at every point of this space is free. The connection $(\mathcal{M}^{**}, \nabla)$ will be studied from an algebraic point of view in Section 5.

5 Algebraic construction of a moduli family

The aim is to construct a 'universal' family of irreducible connections $(\mathcal{M}, \nabla)$ on the projective line with $\mathcal{M}$ free, regular singularities at $0, 1, t, \infty$ and local exponents $1/4, -1/4$ at each singular point.

The parameter t is seen as a point in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus we will work with the projective line over $\mathbb{Q}[t, \frac{1}{t(t-1)}]$. After fixing a basis of $H^0(\mathcal{M})$ and replacing ∇ by $\nabla_{\frac{d}{dz}}$ we obtain a differential operator of the form $\frac{d}{dz} + \sum_{j=1}^{3} \frac{A_j}{z - s_j}$

with $s_1 = 0, s_2 = 1, s_3 = t$ and where the matrices $A_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$ for $j = 1, 2, 3$ satisfy the conditions:

$$a_j^2 + b_j c_j = 1/16 \text{ for } j = 1, 2, 3 \text{ and } (\sum_{j=1}^{3} a_j)^2 + (\sum_{j=1}^{3} b_j)(\sum_{j=1}^{3} c_j) = 1/16.$$

We still have to find the conditions which express that the equation is irreducible and moreover we want to divide by the action (by conjugation) of the group PGl_2.

The structure of the parameter space.
One considers tuples $(A_1, \ldots, A_4)$ of 2×2-matrices such that $\sum A_j = 0$ and each A_j has eigenvalues $1/4, -1/4$. The tuples form an affine algebraic variety Var of dimension 5, having an action of PGl_2, by conjugation. The stabilizer of a tuple $(A_1, \ldots, A_4)$ is not trivial if and only if there are two distinct lines invariant under all A_j. This condition is equivalent to "all A_j commute". In this case the stabilizer is the multiplicative group $\mathbb{G}_m$. The subset Var_1 of Var, consisting of these tuples, is closed; it has dimension 2 and consists of three PGl_2-orbits.

The subset Var_2 of Var, consisting of the tuples such that the $A_1, \ldots, A_4$ have a common eigenvector, is also closed. Let e be the common eigenvector for all A_j. Then $A_j e = \pm 1/4 e$ for all j and the sum of the eigenvalues is 0. This implies that Var_2 has six irreducible components, say $Var_2(i)$ with $i = 1, \ldots, 6$, each of dimension 4. Each $Var_2(i)$ is invariant under the action of PGl_2. Further the quotient of $Var_2(i) \setminus (Var_2(i) \cap Var_1)$ by the action of PGl_2 is seen to be a projective line.

We are interested in the structure of the quotient $\mathrm{PGl}_2 \backslash Var$. There is no geometric quotient. However we will compute the ring R_{inv} of PGl_2-invariant regular functions of Var. First of all, A_4 is normalized to $\left(\begin{smallmatrix} -1/4 & 0 \\ 0 & 1/4 \end{smallmatrix}\right)$. This defines a closed subspace Var' of Var. The stabilizer of A_4, under the action of PGL_2 is its maximal torus, isomorphic to the multiplicative group $\mathbb{G}_m$. Thus R_{inv} can be identified with the ring of the regular functions on Var', invariant under $\mathbb{G}_m$. The ring of regular functions on Var' is $\mathbb{Q}[a_1, a_2, b_1, b_2, c_1, c_2]$ with generating relations: $b_j c_j = 1/16 - a_j^2$ for $j = 1, 2$ and $b_1 c_2 + b_2 c_1 = -2(a_1 - 1/4)(a_2 - 1/4)$. Using these relations one finds a free basis of the above ring over $\mathbb{Q}[a_1, a_2]$ consisting of the monomials:

$$b_1^{>0} b_2^{\geq 0},\ b_2^{>0} c_1^{\geq 0},\ c_1^{\geq 0} c_2^{\geq 0}\ .$$

From this it follows easily that the ring R_{inv} has the form $\mathbb{Q}[a_1, a_2, H]$ with $H = b_2c_1$. There is only one equation, namely

$$H^2 + 2H(a_1 - 1/4)(a_2 - 1/4) + (a_1^2 - 1/16)(a_2^2 - 1/16) = 0 \ .$$

We note that also b_1c_2 satisfies this equation. We could have chosen b_1c_2 for H, however we prefer $H = b_2c_1$ in view of later computations. The singular locus of R_{inv} is equal to

$$(a_1, a_2, H) = (1/4, 1/4, 0), (1/4, -1/4, 0), (-1/4, 1/4, 0) \ .$$

This singular locus is precisely the collection of points where the 4 matrices $A_1, \ldots, A_4$ have a common eigenvector. Hence the regular locus of $\mathrm{Spec}(R_{inv})$ coincides with the variety $\mathrm{PGl}_2\backslash(Var - Var_2)$. This space is not affine and for further calculations we consider the open affine subset given by $(a_1^2 - 1/16)(a_2^2 - 1/16)(a_3^2 - 1/16) \neq 0$ (note that $a_3 = 1/4 - a_1 - a_2$). For the missing points, given by $a_j = \pm 1/4$ with $j = 1, 2, 3$, we make separate calculations. □

The open affine part U of the parameter space P.
The complete parameter space P is given by

$$P := \mathrm{Spec}(\mathbb{Q}[t, \frac{1}{t(1-t)}]) \times (\mathrm{PGL}_2\backslash(Var - Var_2)).$$

As before we normalize A_4 and identify the right hand side with the regular locus of $\mathrm{Spec}(R_{inv})$. Further we consider the open affine subset U given by $s \neq 0$ where $s = (a_1^2 - 1/16)(a_2^2 - 1/16)(a_3^2 - 1/16)$. Then $H \neq 0$ and thus $b_2c_1 \neq 0$. We normalize further by $c_1 = 1$. The corresponding ring is now $R_U := \mathbb{Q}[t, \frac{1}{t(1-t)}][a_1, a_2, \frac{1}{s}, b_2]/(Rel)$ with $Rel = b_2^2 + 2b_2(a_1 - 1/4)(a_2 - 1/4) + (a_1^2 - 1/16)(a_2^2 - 1/16)$. The other variables are expressed in $a_1, a_2, b_2, 1/s$ by the formulas

$$b_1 = 1/16 - a_1^2; \ c_2 = b_2^{-1}(1/16 - a_2^2); \ a_3 = 1/4 - a_1 - a_2$$

$$b_2^{-1} = -b_2\frac{a_3^2 - 1/16}{s} - 2\frac{(a_1 - 1/4)(a_2 - 1/4)(a_3^2 - 1/16)}{s}$$

$$b_3 = -b_2 - 1/16 + a_1^2; \ c_3 = -1 - b_2^{-1}(1/16 - a_2^2)$$

The locus of the reducible equations.
We search the locus $P_{reducible} \subset P$, consisting of the points in $P(\overline{\mathbb{Q}})$, where

the corresponding differential operator $D := \frac{d}{dz} + \sum_{j=1}^{3} \frac{A_j}{z-s_j}$ is reducible. A point corresponds to prescribed values in $\overline{\mathbb{Q}}$ for $a_1, a_2, a_3, \ldots, t$. Reducibility is equivalent to the existence of a non zero vector $v \in \overline{\mathbb{Q}}[z] \otimes \mathbb{Q}^2$ such that $D(v) = fv$ for some $f \in \overline{\mathbb{Q}}(z)$. The vector v can be normalized such that the g.c.d. in the ring $\overline{\mathbb{Q}}[z]$ of all its coefficients is 1. Thus one considers $v := v_0 + v_1 z + \cdots + v_d z^d$, with all $v_i \in \overline{\mathbb{Q}}^2$, $v_d \neq 0$ and $v_0 + v_1 s + \cdots + v_d s^d \neq 0$ for all $s \in \overline{\mathbb{Q}}$, and satisfying the equation $D(v) = fv$ for some $f \in \overline{\mathbb{Q}}(z)$. One easily verifies that $f = (\sum_{j=1}^{3} \frac{\lambda_j}{z-s_j})$ where the λ_j are eigenvalues of A_j. Expanding the equation $D(v) = fv$ at $z = \infty$ yields that $d = \sum_{j=1}^{4} \lambda_j$, where λ_4 is some eigenvalue of $A_4 = -\sum_{j=1}^{3} A_j$. In this special case there are only two possibilities, namely $d = 0$ and $d = 1$. In first case all A_j have a common eigenvector. This case is excluded by the definition of the parameter space P. The case $d = 1$ is equivalent to the equation

$$(\frac{d}{dz} + \sum_{j=1}^{3} \frac{A_j - 1/4}{z - s_j})(v_0 + v_1 z) = 0 \ .$$

The normalization of A_4 implies $v_1 = e_2 = \binom{0}{1}$, where e_1, e_2 is the standard basis of $\mathbb{Q}^2$. The above equation is equivalent to the system of matrix equations

$$(A_1 - 1/4)v_0 = 0, \ (A_2 - 1/4)(v_0 + e_2) = 0, \ (A_3 - 1/4)(v_0 + te_2) = 0.$$

The existence of a solution v_0 leads to certain relations between the coefficients of the matrices A_1, A_2, A_3, describing in fact $P_{reducible}$.

The direct approach. Let T denote the affine ring of Var'. Recall that $T := \mathbb{Q}[a_1, a_2, b_1, b_2, c_1, c_2]$ with generating relations $b_j c_j = 1/16 - a_j^2$ for $j = 1, 2$ and $b_1 c_2 + b_2 c_1 = -2(a_1 - 1/4)(a_2 - 1/4)$. Consider $T[x_1, x_2]$, where $v_0 = \binom{x_1}{x_2}$. The system of matrix equations defines an ideal $I \subset T[x_1, x_2]$. Using Gröbner basis one calculates $J = I \cap T$. This ideal is invariant under the action by conjugation of $\mathbb{G}_m$. Therefore J is induced by an ideal J_0 of $T^{\mathbb{G}_m} = R_{inv} = \mathbb{Q}[t, \frac{1}{t(t-1)}][a_1, a_2, H]$ (with generating relation $Rel = H^2 + 2H(a_1 - 1/4)(a_2 - 1/4) + (a_1^2 - 1/16)(a_2^2 - 1/16) = 0$) in the sense that $J = J_0 T$. The ideal J_0 turns out to be generated by the two elements

$$f(t) = (a_1 + a_2 - 1/2)^2 t(t-1) + (a_1 + 1/4)^2 t + (a_2 + 1/4)^2 (1 - t) \text{ and}$$

$$2H + (a_1 + a_2 - 1/2)^2 t + (a_2 + 1/4)(a_1 - a_2) \ .$$

One computes that the ideal I also contains the elements $x_1+2b_1t+2b_2t-2b_2$ and $x_2+1/2t-a_1t-a_2t+a_2+1/4$. Thus $T[x_1,x_2]/I = T/J$ and $(T/J)^{\mathbb{G}_m} = R_{inv}/J_0$. The last algebra equals $\mathbb{Q}[t,\frac{1}{t(t-1)}][a_1,a_2]/(f(t))$. One easily verifies that $f(t)$ is irreducible. Therefore $\mathbb{Q}[t,\frac{1}{t(t-1)}][a_1,a_2]/(f(t))$ is a domain of dimension 2. Thus $J_0 \subset R_{inv}$ is a prime ideal of height 1 and defines a closed irreducible subset of $\mathrm{Spec}(R_{inv})$ of codimension 1. The intersection of this closed subset with P is $P_{irreducible}$ and the latter is therefore closed and irreducible of dimension 2. Moreover $P_{irreducible}$ has the following rational parametrization by t,λ

$$a_1 = \frac{t-2\lambda^2}{4t},\ a_2 = \frac{t+1+2\lambda^2+4\lambda}{4(t-1)},\ H = \frac{(\lambda+1)\lambda(\lambda^2-t)}{4t(t-1)}\ ,$$

with $\lambda = ta_1 + ta_2 - t/2 - a_2 - 1/4$.

6 The special family

The 'special family' that we compute here is the family of irreducible connections $(\mathcal{M},\nabla)$ on $\mathbb{P}^1$ with (as before) 4 regular singular points at $0,1,t,\infty$, local exponents $1/4,-1/4$ at each singular point and such that $\mathcal{M}$ is not free. We note that for a suitable shift of the local exponents over integers the connection can be presented with a free vector bundle.

The second exterior power $\Lambda^2(\mathcal{M},\nabla)$ is a rank one connection without singularities on $\mathbb{P}^1$. Hence $\Lambda^2\mathcal{M}$ is the trivial line bundle and the connection on it is also trivial. The irreducibility of $(\mathcal{M},\nabla)$ implies that $\mathcal{M}$ is isomorphic to the sheaf $O(1)\oplus O(-1)$. Using this information one can calculate the a matrix form for the connection. Instead of producing the results of this calculation, we will describe an easier method to obtain the special family.

Proposition 3.2 provides another way to obtain this family. On the elliptic curve E, given by $w^2 = z(z-1)(z-t)$, one considers the connection $\nabla : O_E f_0 \to \Omega\otimes O_E f_0$ given by $\nabla(f_0) = c\frac{dz}{2w}\otimes f_0$ with $c\neq 0$. The direct image, under the morphism $E\to\mathbb{P}^1$ has w.r.t. the basis $f_0, f_1 = wf_0$ the matrix form

$$\frac{d}{dz} + \begin{pmatrix} 0 & c/2 \\ \frac{c}{2w^2} & \frac{w'}{w} \end{pmatrix}\ .$$

One has to shift over $-\frac{w'}{2w}$ in order to obtain the local exponents $1/4,-1/4$. After a conjugation with the constant matrix $\begin{pmatrix} 2c^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ one obtains the

required form (with $d = c^2/4$)

$$\frac{d}{dz} + \begin{pmatrix} -\frac{w'}{2w} & d \\ \frac{1}{w^2} & \frac{w'}{2w} \end{pmatrix} .$$

The above is in fact a family over the base ring $\mathbb{Q}[t, \frac{1}{t(t-1)}, d, d^{-1}]$.

The *classical Lamé equation* is $\mathcal{L}_n := y'' + \frac{f'}{2f}y' - \frac{n(n+1)z+B}{f}y$, with $f = 4(z^3 + az + b)$. The transformation $z \mapsto z - \frac{1+t}{3}$, with suitable t, transforms f into $4z(z-1)(z-t)$ and the equation into

$$y'' + \frac{g'}{2g}y' - \frac{n(n+1)(z-(1+t)/3)+B}{4g}y \text{ with } g = z(z-1)(z-t).$$

The local exponents are $0, 1/2$ for $0, 1, t$ and $-n/2,\ (n+1)/2$ at ∞.
For $n = 0$ one writes $w^2 = g$. The companion matrix equation for the second scalar equation is $\frac{d}{dz} + \begin{bmatrix} 0 & -1 \\ \frac{-B}{4w^2} & \frac{w'}{w} \end{bmatrix}$. The shift yields $\frac{d}{dz} + \begin{bmatrix} -\frac{w'}{2w} & -1 \\ \frac{-B}{4w^2} & \frac{w'}{2w} \end{bmatrix}$.

If $B \neq 0$, then conjugation with a constant matrix yields the equation $\frac{d}{dz} + \begin{bmatrix} -\frac{w'}{2w} & \frac{4}{B} \\ \frac{1}{w^2} & \frac{w'}{2w} \end{bmatrix}$. This is the special family.

7 Explicit formulas and Painlevé VI

7.1 An inverse approach

The family $\{(\mathcal{M}, \nabla)\}$ can be computed directly from the normalized differential form $\omega = (c + \frac{w+w_0}{z-z_0})\frac{dz}{2w}$ of section 3. The differential module above the elliptic curve reads $\partial f_0 = (c + \frac{w+w_0}{z-z_0})\frac{1}{2w}f_0$. With respect to the basis $f_0, f_1 = wf_0$ over the field of rational functions on $\mathbb{P}^1$ one finds the operator

$$\frac{d}{dz} + \begin{bmatrix} \frac{1}{2(z-z_0)} & (c + \frac{w_0}{(z-z_0)})/2 \\ \frac{1}{2w^2}(c + \frac{w_0}{(z-z_0)}) & \frac{w'}{w} + \frac{1}{2(z-z_0)} \end{bmatrix} . \text{ The shift over } -\frac{w'}{2w} \text{ yields}$$

$$L_0 := \frac{d}{dz} + \begin{bmatrix} \frac{1}{2(z-z_0)} - \frac{w'}{2w} & (c + \frac{w_0}{(z-z_0)})/2 \\ \frac{1}{2w^2}(c + \frac{w_0}{(z-z_0)}) & \frac{w'}{2w} + \frac{1}{2(z-z_0)} \end{bmatrix} . \text{ This is replaced by}$$

$L_1 := \begin{pmatrix} 1 & -cz \\ 0 & 1 \end{pmatrix} L_0 \begin{pmatrix} 1 & cz \\ 0 & 1 \end{pmatrix}$ in order to remove the apparent pole of order 2 at ∞.

The formulas for $w_0 \neq 0$ (equivalently $z_0 \neq 0, 1, t$).
z_0 is an apparent singularity of L_1 which can be removed as follows. First calculate the conjugate

$$L_2 := \begin{pmatrix} \frac{1}{2w_0} & -1/2 \\ \frac{1}{2w_0} & 1/2 \end{pmatrix} L_1 \begin{pmatrix} w_0 & w_0 \\ -1 & 1 \end{pmatrix}. \text{ Then again a conjugate}$$

$$L_3 := \begin{pmatrix} 1 & 0 \\ 0 & z - z_0 \end{pmatrix} L_2 \begin{pmatrix} 1 & 0 \\ 0 & (z - z_0)^{-1} \end{pmatrix}.$$

Then L_3 is conjugated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in order to normalize the matrix at infinity to $\begin{pmatrix} -1/4 & 0 \\ 0 & 1/4 \end{pmatrix}$. Finally, a conjugation with a constant matrix of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ is needed to obtain $c_1 = 1$. The final operator $\frac{d}{dz} + U$ belongs to our family, normalized with $c_1 = 1$.

$$U = \frac{1}{\Delta} \begin{bmatrix} N & (cz - cz_0 + w_0)F/t \\ -4zt(-3\,cz_0 + 2\,w_0 + c + ct + c^3)/F + 4t & -N \end{bmatrix}$$

where

$$\Delta = 4z(z-1)(z-t),$$

$$N = z^2 + (2\,c^2 - 2\,z_0)z + 2\,z_0 t - t - 2\,c^2 z_0 - 2\,{z_0}^2 + 2\,z_0$$

$$F = 3\,c{z_0}^2 - 3\,cz_0 t - 3\,cz_0 - w_0 z_0 + c^3 z_0 + 2\,ct - 3\,c^2 w_0 + w_0 t + w_0$$

Remarks 7.1
(1) This formula for U is derived under the assumption that $w_0 \neq 0$. The normalization $c_1 = 1$ introduces the denominator F in the $(2,1)$ entry of U. Thus the formula for U is valid under the assumptions that $w_0 \neq 0$ and $F \neq 0$. The involution of the elliptic curve maps (c, w_0) to $(-c, -w_0)$. One observes that the formula for U is, as it should be, invariant under this map.
(2) Suppose again that $w_0 \neq 0$ and $F \neq 0$. Conjugation of U with the constant matrix $\left(\begin{smallmatrix} t & 0 \\ 0 & F \end{smallmatrix}\right)$ yields

$$U^* = \frac{1}{\Delta} \begin{bmatrix} N & cz - cz_0 + w_0 \\ -4z(-3\,cz_0 + 2\,w_0 + c + ct + c^3) + 4F & -N \end{bmatrix}$$

Thus $\frac{d}{dz} + U$ is equivalent to $\frac{d}{dz} + U^*$. The latter expression is also valid for $F = 0$. We will show that the formula $\frac{d}{dz} + U^*$ remains valid for $w_0 = 0$. Another advantage of U^* is that its $(1,2)$ entry has a zero if $c \neq 0$. We note that U^* is not invariant under the involution of the elliptic curve.

The formulas for $w_0 = 0$ (equivalently $z_0 \in \{0, 1, t\}$).

As before $L_0 := \frac{d}{dz} + \begin{bmatrix} \frac{1}{2(z-z_0)} - \frac{w'}{2w} & c/2 \\ \frac{c}{2w^2} & \frac{w'}{2w} + \frac{1}{2(z-z_0)} \end{bmatrix}$ and L_1 is obtained by conjugating L_0 with $\begin{pmatrix} 1 & c(z-z_0) \\ 0 & 1 \end{pmatrix}$. Further L_2 is obtained from L_1 by conjugation with $\begin{pmatrix} 1 & 0 \\ 0 & (z-z_0)^{-1} \end{pmatrix}$. Finally L_3 is obtained from L_2 by conjugation with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We will not further normalize L_3. One finds the following formulas.

(1) For $z_0 = 0$, $L_3 = \frac{d}{dz} + U_0$ with

$$U_0 = \frac{1}{\Delta} \begin{bmatrix} z^2 - t + 2\,c^2 z & cz \\ 4\,c\,((-1-t-c^2)\,z + 2\,t) & -z^2 + t - 2\,c^2 z \end{bmatrix}$$

(2) For $z_0 = 1$, $L_3 = \frac{d}{dz} + U_1$ with

$$U_1 = \frac{1}{\Delta} \begin{bmatrix} z^2 + 2(c^2-1)z - 2\,c^2 + t & c(z-1) \\ -4z(ct - 2c + c^3) + 4c^3 - 4ct & -z^2 - 2(c^2-1)z + 2\,c^2 - t \end{bmatrix}$$

(3) For $z_0 = t$, $L_3 = \frac{d}{dz} + U_t$ where U_t equals

$$\frac{1}{\Delta} \begin{bmatrix} z^2 + (-2\,t + 2\,c^2)\,z - t\,(-1 + 2\,c^2) & c(z-t) \\ -4z(-2ct + c + c^3) + 4c^3 t - 4ct & -z^2 + (2t - 2c^2)z + t(-1 + 2c^2) \end{bmatrix}$$

These formulas are *specializations* of U^* for $w_0 = 0$ and $z_0 = 0, 1, t$.

7.2 A direct approach

On the space $P \setminus P_{reducible}$, the matrix U (or U^*) expresses a_1, a_2, b_2, t in terms of z_0, w_0, c. The aim is to express the rational functions $z_0, \frac{w_0}{c}, c^2$ on this space in terms of a_1, a_2, b_2. One can view the data of U and U^* as polynomials $a_1, a_2, b_2, z_0, w_0, c, t$, generating an ideal in $\mathbb{Q}[a_1, a_2, b_2, t, \frac{1}{t(t-1)}, z_0, w_0, c]$ (b_2 stands for H and there are known relations for a_1, a_2, b_2 and z_0, w_0, c). Elimination for a suitable order produces two elements in this ideal, namely

$$(16t^2a_1^2 - 32a_2^2t + 32a_1a_2t^2 - 32a_1a_2t - 16t^2a_1 + 24a_1t + 16a_2^2 + 16a_2^2t^2 + 8a_2 - 16a_2t^2 + 8a_2t + 1 + 4t^2 - 4t)z_0$$

$$-t\left(-3 + 48\,a_1a_2t + 20\,a_1 - 20\,a_2t + 20\,a_2 - 12\,a_1t - 48\,a_2a_1 + 4\,t + 16\,t{a_1}^2 + 32\,tb_2 - 32\,b_2\right)$$

and

$$((-32+32\,t)\,b_2-(1+4\,a_1)\,(4\,a_1t-4\,a_2t+1+4\,a_2))\,z_0+t\,(1+4\,a_1)^2\ .$$

We recall that the locus $P_{reducible}$ was given by an ideal (Eq_1, Eq_2), given by

$$Eq_1 \;=\; (a_1+a_2-1/2)^2\,t\,(t-1)+(a_1+1/4)^2\,t+(a_2+1/4)^2\,(-t+1)$$

$$Eq_2 \;=\; 2\,b_2+(a_1+a_2-1/2)^2\,t+(a_2+1/4)\,(a_1-a_2)$$

The coefficients P_1, P_2 of z_0 in the above equations satisfy $P_1 = 16 \cdot Eq_1$ and $P_2 = 16(t-1)\cdot Eq_2 - 16\cdot Eq_1$. Thus $(P_1, P_2) = (Eq_1, Eq_2)$. The locus $P \setminus P_{reducible}$ is the union of the two open subsets of P given by $P_1 \neq 0$ and $P_2 \neq 0$. On each one of them, z_0 is a regular expression in a_1, a_2, b_2, t. For $P_2 \neq 0$ this expression is

$$z_0 \;=\; -\frac{t\,(1+4\,a_1)^2}{32\,(t-1)\,b_2-(1+4\,a_1)\,(4\,a_1t-4\,a_2(t-1)+1)}\ .$$

Once z_0 is known, we may choose a solution w_0 of $w_0^2 = z_0(z_0-1)(z_0-t)$. For the zero q of the $(1,2)$ coefficient of U we have two expression, namely $q = z_0 - \frac{w_0}{c}$ and $q = -\frac{t\left(16\,{a_1}^2-1\right)}{16\,(t-1)b_2-t(16\,{a_1}^2-1)}$. This yields a formula for c. We recall that the pair (w_0, c) is unique up to a sign.

7.3 Intermezzo on Painlevé VI

We consider a family of connections, more general than the Lamé connection,

$$\frac{d}{dz}+A=\frac{d}{dz}+\frac{A_1}{z}+\frac{A_2}{z-1}+\frac{A_3}{z-t} \text{ with } A_j=\begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \text{ for } j=1,2,3.$$

Put $A_\infty = -A_1 - A_2 - A_3$. The assumptions are:
A_j has eigenvalues $\pm\theta_j/2$ (equivalently $a_j^2 + b_jc_j = \theta_j^2/4$ for $j = 1,2,3$),
$A_\infty = \begin{pmatrix} -\theta_\infty/2 & 0 \\ 0 & \theta_\infty/2 \end{pmatrix}$ (equivalently $\sum a_j = \theta_\infty/2$, $\sum b_j = 0$, $\sum c_j = 0$).
The $(1,2)$ entry of A has a single zero, namely $q := \frac{b_1 t}{b_1 t + b_2(t-1)}$ (if $b_1t + b_2(t-1) = 0$ then we write $q = \infty$). One considers in this space of connections a family $\frac{d}{dz} + A(t)$ (i.e., all $a_j, b_j, q, \ldots$ are analytic functions of t). Schlesinger's theorem can be formulated as follows (compare [J-M]).

Theorem 7.2 (Schlesinger) *Suppose that $\theta_j \notin \mathbb{Z}$ for $j = 1, 2, 3, \infty$. Consider family $\frac{d}{dz} + A(t)$, holomorphic in t and defined in a neighbourhood of t_0. Suppose that not all $A_j(t_0)$ commute and that $q(t_0) \neq \infty$. Then the family is isomonodromic if and only if $q = q(t)$ satisfies the two equations*

$$q' = 2a_1 \frac{q-1}{t-1} + 2a_2 \frac{q}{t} + (1-\theta_\infty)\frac{q(q-1)}{t(t-1)} \text{ and}$$

$$q'' = 1/2(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t})(q')^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t})q' +$$
$$\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{(\theta_\infty - 1)^2}{2} - \frac{\theta_1^2}{2}\frac{t}{q^2} + \frac{\theta_2^2}{2}\frac{t-1}{(q-1)^2} - \frac{\theta_3^2 - 1}{2}\frac{t(t-1)}{(q-t)^2}\right) .$$

The last equation is the Painlevé VI equation with parameters $(\theta_1, \theta_2, \theta_3, \theta_\infty)$. We note that the first equation for q' is hard to find in the literature. Moreover the Painlevé VI equation is sometimes parametrized in a different way. If q is known, then one recovers all a_j, b_j, c_j (up to the action of $\mathbb{G}_m$ on the b_j and c_j) from the equality $q := \frac{b_1 t}{b_1 t + b_2(t-1)}$ and the equation for q'. The choice $\theta_j = 1/2$ for $j = 1, 2, 3, \infty$ defines the Lamé connection.

A point of order m on the elliptic curve $w^2 = z(z-1)(z-t)$ can be seen as a pair $(z_0(t), w_0(t))$ of algebraic functions in t. It has been shown by É. Picard, [Pic], that the coordinate $z_0(t)$ is a solution of PVI $(0, 0, 0, 1)$ (compare [Maz]).

We will use the *transformation $w_1 w_2 w_1$ of Okamoto* to obtain solutions for PVI $(1/2, 1/2, 1/2, 1/2)$. This transformation is described as follows (see [Ok, p. 356]).
Suppose that q is a solution of PVI for the parameters $(\theta_1, \theta_2, \theta_3, \theta_\infty)$. Then $q + \frac{1}{2p}$ with

$$p = \frac{t(t-1)q'}{2q(q-1)(q-t)} + \frac{\theta_1}{2q} + \frac{\theta_2}{2(q-1)} + \frac{\theta_3 - 1}{2(q-t)}$$

is a solution of PVI for the parameters

$$\theta_1^* = 1/2(\theta_1 - \theta_2 + \theta_3 + \theta_\infty) - 1,\ \theta_2^* = 1/2(-\theta_1 + \theta_2 + \theta_3 + \theta_\infty) - 1,$$

$$\theta_3^* = 1/2(\theta_1 + \theta_2 + \theta_3 - \theta_\infty) + 1,\ \theta_\infty^* = 1/2(\theta_1 + \theta_2 - \theta_3 + \theta_\infty) + 1.$$

In particular, a solution q of PVI $(0,0,0,1)$ yields a solution $q+\frac{1}{2p}$, with $p=\frac{t(t-1)q'}{2q(q-1)(q-t)}-\frac{1}{2(q-t)}$ with $q'=\frac{dq}{dt}$ is a solution of PVI $(-1/2,-1/2,1/2,3/2)$. Accidently this equation is identical with PVI $(1/2,1/2,1/2,1/2)$.

In particular, a point $(z_0(t),w_0(t))$ of order m on the elliptic curve yields the algebraic solution $z_0-\frac{z_0(z_0-1)(z_0-t)}{z_0(z_0-1)-t(t-1)z_0'}$ (with $z_0=z_0(t)$ and $z_0'=\frac{dz_0}{dt}$) for PVI $(1/2,1/2,1/2,1/2)$.

7.4 Points of finite order on the elliptic curve

Each member ∇_* of the Lamé family is induced by a regular equation of rank one on the elliptic curve E, given by $w^2=z(z-1)(z-t)$. Taking the shift into account, ∇_* is the push forward of $\nabla e=\{(c+\frac{w+w_0}{z-z_0})\frac{dz}{2w}-\frac{dw}{2w}\}\otimes e$ or of $\nabla e=\{c\frac{dz}{2w}-\frac{dw}{2w}\}\otimes e$, where c a constant and $(z_0,w_0)\in E$.

For the moment we fix a complex value for t (different from $0,1,\infty$) and take $(z_0,w_0)\in E(\mathbb{C})$. The differential Galois group of ∇_* is equal to $D_n^{\mathrm{SL}_2}$ if and only if the corresponding rank one equation on E has a cyclic differential Galois group of order $2n$. The latter is equivalent to the statement that the equation $\nabla e=(c+\frac{w+w_0}{z-z_0})\frac{dz}{2w}\otimes e$ or $\nabla e=c\frac{dz}{2w}\otimes e$, has a cyclic differential Galois of order m such that $l.c.m.(m,2)=2n$. Thus $m=n$ or $m=2n$ if n is odd and $m=2n$ if n is even. We note that $\nabla e=c\frac{dz}{2w}\otimes e$ has infinite differential Galois group if $c\neq 0$. Thus we may omit this case.

If $\nabla e=(c+\frac{w+w_0}{z-z_0})\frac{dz}{2w}\otimes e$ has a cyclic differential Galois group of order m, then (z_0,w_0) is a point of exact order m. Moreover, for a point (z_0,w_0) of exact order m, there is precisely one value of c such that the differential Galois group is cyclic of order m.

We continue the discussion in section 3 and analyze the exact sequence

$$0\to\mathbb{C}\to\mathrm{Hom}(\pi_1(E),\mathbb{C}^*)\to E\to 0\ .$$

As before, $E=\mathbb{C}/(\mathbb{Z}+\mathbb{C}\tau)$ and we choose as generators a,b of $\pi_1(E)$ the circles $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}\tau$. This identifies $\mathrm{Hom}(\pi_1(E),\mathbb{C}^*)$ with $\mathbb{C}^*\times\mathbb{C}^*$, by $h\mapsto(h(a),h(b))$. The first map of the exact sequence is $d\mapsto(e^d,e^{d\tau})$. Put $S^1=\{z\in\mathbb{C}^*|\ |z|=1\}$. Every element of $\mathbb{C}^*\times\mathbb{C}^*$ can be written uniquely as $(e^d,e^{d\tau})\cdot(s_1,s_2)$ with $d\in\mathbb{C}$ and $s_1,s_2\in S^1$. In particular, the restriction of the second map of the exact sequence to $S^1\times S^1$ is a bijection. Thus for a given point $(z_0,w_0)\in E$ of exact order m, there is precisely one value of c such that $\nabla e=(c+\frac{w+w_0}{z-z_0})\frac{dz}{2w}\otimes e$ has a cyclic differential Galois group of

order m. We note that it is, a priori, difficult to produce a formula for this c. However, F. Beukers has proposed a formula (see [Be]).

Now we consider t as a variable and investigate a family of Lamé connections $\nabla(t)$ with fixed differential Galois group $D_n^{\mathrm{SL}_2}$. The corresponding rank one connection $\nabla e = (c(t) + \frac{w+w_0(t)}{z-z_0(t)})\frac{dz}{2w}$ has the property that $(z_0(t), w_0(t))$ is a point of order m. This point is defined over a suitable algebraic extension of $\mathbb{C}(t)$. The monodromy of this family depends in a continuous way on t and lies in $\mathrm{Hom}(\pi_1(E), \{\zeta \in \mathbb{C}^* \mid \zeta^m = 1\})$. Since this group is finite, the family has constant monodromy.

The corresponding algebraic solution for PVI $(1/2, 1/2, 1/2, 1/2)$ is $z_0 - \frac{w_0}{c}$ according to Theorem 7.1 and the formula for the $(1,2)$ entry of U. A combination of Picard's result and Okamoto's transformation yield the expression $z_0 - \frac{z_0(z_0-1)(z_0-t)}{z_0(z_0-1)-t(t-1)z_0'}$ for this solution. For the uniquely determined $c(t)$ we find therefore the following formula

$$w_0(t)\, c(t) = z_0(t)(z_0(t)-1) - t(t-1)z_0(t)' \ .$$

8 The locus P_N for $D_N^{\mathrm{SL}_2}$ with $N = 2, 3, 4$

8.1 Division polynomials

In this subsection we recall some facts on torsion points on elliptic curves directly related to the locus for $D_n^{\mathrm{SL}_2}$. First we follow [Was] (or [Sil], p. 105) in the description of the division polynomials ψ_m with $m \geq 1$. Suppose that the elliptic curve E is given in the Weierstrass form $w^2 = z^3 + A\,z + B$.

The points $\neq 1_E$ of order dividing m lie on this affine part of the curve E. Their number is $m^2 - 1$ (over an algebraically closed field of characteristic 0). The z-coordinates of these points are the zeros of a polynomial $\psi_m \in \mathbb{Z}[A, B][z, w]$ (with defining relation $w^2 = z^3 + A\,z + B$). For odd m, $\psi_m \in \mathbb{Z}[A, B][z]$ and for m even, $\psi_m \in w\,\mathbb{Z}[A, B][z]$. In the latter case one may replace ψ_m by ψ_m^2 in order to have a polynomial in $\mathbb{Z}[A, B][z]$. These polynomials are given by the following recurrence relations

$$\psi_0 = 0,\ \psi_1 = 1,\ \psi_2 = 2w,\ \psi_3 = 3z^4 + 6Az^2 + 12Bz - A^2 \ ,$$

$$\psi_4 = 4\,w\,(z^6 + 5Az^4 + 20Bz^3 - 5A^2z^2 - 4ABz - 8B^2 - A^3),$$

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \text{ for } m \geq 2 \ ,$$

$$\psi_{2m} = (2w)^{-1} \cdot \psi_m \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \ .$$

The zero set of ψ_m^2 (or its square free part ψ_m^*) is precisely the set of the z-coordinates of the of points with order $\neq 1$ and dividing m.

We specialize this by replacing $\mathbb{Z}[A,B]$ by the function field $F := \mathbb{C}(j)$ of the j-line. Then E is the 'universal curve' above the j-line. The splitting field of F defined by ψ_m^* is the function field of the modular curve $X(m)$. The ramified Galois covering $j : X(m) \to \mathbb{P}^1$ is known to have group $\mathrm{PSL}_2(\mathbb{Z}/m\mathbb{Z})$. It follows that the Galois group Gal_F of $\overline{F}/F$ acts transitively on the points of precise order m. Thus the minimal polynomial for the z-coordinate of a point of precise order m is given by $\prod_{d|m}(\psi_d^*)^{\mu(m/d)}$.

Here we are interested in the division polynomials over the field $\mathbb{C}(t)$ which is a Galois extension of $\mathbb{C}(j)$ with Galois group S_3. The substitution $z \to z + \frac{t+1}{3}$ brings $z(z-1)(z-t)$ into the Weierstrass form $z^3 + A(t)z + B(t)$. Put $\phi_m = \psi_m(A(t), B(t), z - \frac{t+1}{3})$. This is the division polynomial for the Legendre family. Let ϕ_m^* denote the square free part of ϕ_m. The z-coordinates of the points of precise order m are the zeros of the polynomial $\prod_{d|m}(\phi_d^*)^{\mu(m/d)}$.

For odd m this polynomial is irreducible. For even m this polynomial has three irreducible factors, which are permuted by the Galois group S_3 of $\mathbb{C}(t)/\mathbb{C}(j)$. These statements can be deduced from the Galois action of $\mathrm{Gal}_{\mathbb{C}(t)}$ on the group $(\mathbb{Z}/m\mathbb{Z})^2$ of all points of order dividing m.

We note that $\sigma \in S_3$ permutes in fact the three points of ramification and permutes the three finite singular points of the Lamé connections.

8.2 Points of order 2

The points of precise order two on the elliptic curve yield reducible Lamé connections with cyclic differential Galois groups of order 4. The formulas for these connections are obtained from the formula $\frac{d}{dz} + U^*$ of subsection 7.1, specialized with $w_0 = 0,\ c = 0$ and $z_0 = 0, 1, t$. Put $\Delta = 4z(z-1)(z-t)$. One finds

(1) For $z_0 = 0$, $\frac{d}{dz} + \frac{1}{\Delta}\begin{bmatrix} z^2 - t & 0 \\ 0 & -z^2 + t \end{bmatrix}$.

(2) For $z_0 = 1$, $\frac{d}{dz} + \frac{1}{\Delta}\begin{bmatrix} z^2 - 2z + t & 0 \\ 0 & -z^2 + 2z - t \end{bmatrix}$.

(3) For $z_0 = t$, $\frac{d}{dz} + \frac{1}{\Delta}\begin{bmatrix} z^2 - 2tz + t & 0 \\ 0 & -z^2 + 2tz - t \end{bmatrix}$.

These equations do not correspond to points of the parameter space P. They correspond in fact to the three singular points of $\mathrm{Spec}(R_{inv})$.

8.3 The locus P_2 for $D_2^{\mathrm{SL}_2}$ and points of order 4

The equation for the z-coordinates of the points of precise order 4 is

$$\frac{\phi_4(z)}{\phi_2(z)} = (z^2 - t)(z^2 - 2\,z + t)(z^2 - 2\,t\,z + t) \in C(t)[z]\ .$$

The three irreducible factors produce the components of P_2. Let z_0 denote a zero of the polynomial. The formulas $w_0^2 = z_0(z_0 - 1)(z_0 - t)$ and $w_0\,c = z_0(z_0 - 1) - t(t-1)\frac{dz_0}{dt}$ and the universal family of subsection 7.1 yield explicit formulas for the connection above the three components of P_2. Each component is an open subset of the projective line, parametrized by $\lambda := z_0$. This open subset is determined by the condition $t \neq 0, 1, \infty$.

1. The first factor is parametrized by $z_0 = \lambda,\ t = \lambda^2$. One has further $2\,w_0\,c = 2\,t - z_0 - z_0 t$. Put $\Delta = 8\,(z - \lambda^2)\,(z-1)\,z$. After conjugation of the system by $\begin{pmatrix} 1 & 0 \\ 0 & 8\lambda \end{pmatrix}$, one obtains

$$\frac{1}{\Delta}\begin{bmatrix} 2\,z^2 - (\lambda+1)^2\,z + \lambda\,(\lambda^2+1) & (\lambda-1)^2\,(\lambda+1)^2\,(-\lambda - z) \\ \lambda - 3\,z & -2\,z^2 + (\lambda+1)^2\,z - \lambda\,(\lambda^2+1) \end{bmatrix}$$

2. The second factor is parametrized by $z_0 = \lambda,\ t = -\lambda^2 + 2\,\lambda$. One has $2w_0c = -t - z_0 t + 2\,z_0$. Put $\Delta = 8\,(z + \lambda^2 - 2\,\lambda)\,(z-1)\,z$. After conjugation of the system by $\begin{pmatrix} \lambda+2 & 0 \\ 0 & 8 \end{pmatrix}$, one obtains

$$\frac{1}{\Delta}\begin{bmatrix} 2\,z^2 + \lambda\,(-4+\lambda)\,z - \lambda^2\,(\lambda-2) & \lambda\,(\lambda-2)\,(\lambda-2+z) \\ (\lambda-2)\,(-\lambda-2+3\,z)\,\lambda & -2\,z^2 - \lambda(-4+\lambda)z + \lambda^2(\lambda-2) \end{bmatrix}.$$

3. The third factor is parametrized by $z_0 = \lambda,\ t = \frac{\lambda^2}{2\lambda-1}$. One has $2w_0c = 2\,tz_0 - t - z_0$. Put $\Delta = 8\,(2\,\lambda-1)\,z\,(z-1)\left(z - \frac{\lambda^2}{2\,\lambda-1}\right)$. One finds

$$\frac{1}{\Delta}\begin{bmatrix} (2\,\lambda-1)\,(2\,z^2 + (-2\,\lambda-1)\,z + \lambda) & 1/8\,\frac{(4\,\lambda-1)((-2\,\lambda+1)z+\lambda)}{\lambda} \\ 8\,\frac{\lambda\,((-6\,\lambda+3)z+\lambda\,(4\,\lambda-1))}{4\,\lambda-1} & (2\lambda-1)(-2z^2 + (2\lambda+1)z - \lambda) \end{bmatrix}.$$

After conjugation with $\begin{pmatrix} 4\lambda-1 & 0 \\ 0 & 8\lambda \end{pmatrix}$ one obtains the system

$$\frac{1}{\Delta}\begin{bmatrix} (2\,\lambda-1)\,(2\,z^2+(-2\,\lambda-1)\,z+\lambda) & (-2\,\lambda+1)\,z+\lambda \\ (-6\,\lambda+3)\,z+\lambda\,(4\,\lambda-1) & (2\lambda-1)(-2z^2+(2\lambda+1)z-\lambda) \end{bmatrix}$$

8.4 The locus P_3 for $D_3^{\mathrm{SL}_2}$ and points of order 3 and 6

The locus P_3 is derived from the points of order 3 and 6. The z-coordinates of the points of precise order 3 are given by the irreducible polynomial

$$\phi_3(z) = -3\,z^4+4\,z^3t+4\,z^3-6\,z^2t+t^2$$

For the points of precise order 6 this is the polynomial $F_6 := \frac{\phi_6(z)}{\phi_3(z)\phi_2(z)}$. The latter has three irreducible factors

$$F_{6,a} = -z^4+4\,z^3-6\,z^2t+4\,zt^2-t^2\ ,$$
$$F_{6,b} = z^4-4\,z^3t+6\,z^2t-4\,zt+t^2\ ,$$
$$F_{6,c} = z^4-6\,z^2t+4\,zt^2+4\,zt-3\,t^2\ .$$

These define the four connected components of P_3. Each one has a rational parametrization. For a zero z_0 of ϕ_6 one has $w_0^2 = z_0(z_0-1)(z_0-t)$ and $w_0c = z_0(z_0-1)-t(t-1)\frac{dz_0}{dt}$. We note that the solutions q of PVI attached to this examples happen to have poles that do not correspond to $t \in \{0,1,\infty\}$. Using section 7.1, one derives a formula for the connection above each of these components.

1. $\phi_3(z_0,t)=0$ is parametrized by $z_0 = \frac{\lambda^2}{2\,\lambda-1}$, $t = -\frac{(\lambda-2)\lambda^3}{2\,\lambda-1}$. Moreover $6w_0c = t-2\,z_0t-2\,z_0+3\,{z_0}^2$. After conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & 24\lambda \end{pmatrix}$ the system is

$$\begin{bmatrix} \frac{G_1}{(2\lambda-1)\Delta} & \frac{2^5\,(\lambda-2)\,(\lambda+1)^2\left(\left(\lambda^2-\lambda+1\right)z+\lambda^2\,(\lambda-2)\right)}{\Delta} \\ \frac{9(\lambda-2)\left((2\lambda-1)\,z-\lambda^2\right)}{(2\lambda-1)\Delta} & \frac{-G_1}{(2\lambda-1)\Delta} \end{bmatrix}$$

where

$$\Delta = 6^3\,z\,(z-1)\left(z+\frac{(\lambda-2)\,\lambda^3}{2\,\lambda-1}\right)$$

$G_1 \;=\; (-54+108\,\lambda)\,z^2+12\,\left(\lambda^2-4\,\lambda+1\right)(\lambda+1)^2\,z-6\,\lambda^2\,(\lambda-2)\left(4\,\lambda^2-\lambda+4\right)$

2. The parametrization of $F_{6,a}(z_0,t)=0$ given by $z_0 \;=\; \lambda^2, \quad t \;=\; -\frac{(\lambda-2)\lambda^3}{2\,\lambda-1}$. Conjugation with $\begin{pmatrix} \lambda^3+6\,\lambda-2 & 0 \\ 0 & 18\lambda \end{pmatrix}$ yields the system $\begin{bmatrix} \frac{H_1}{\Delta} & \frac{Q}{\Delta} \\ \frac{H_2}{\Delta} & \frac{-H_1}{\Delta} \end{bmatrix}$,

$$\text{where } \Delta \;=\; 36\;(2\,\lambda-1)\,z\,(z-1)\left(z+\frac{(\lambda-2)\,\lambda^3}{2\,\lambda-1}\right)\;,$$

$$Q \;=\; (2\,\lambda-4)\,(\lambda+1)^2\left((2\,\lambda-1)\,z+\lambda^2\,(\lambda-2)\right)\;,$$

$$H_1 = (18\lambda-9)z^2+2(2\lambda-1)(2\lambda^3-6\lambda^2-1)z-\lambda^2(\lambda-2)(2\,\lambda^3+3\lambda-4),$$

$$H_2 \;=\; (2\,\lambda-4)((2\,\lambda-1)\left(5\,\lambda^2-2\,\lambda+2\right)z-\lambda^2\left(\lambda^3+6\,\lambda-2\right))\;.$$

3. The parametrization of $F_{6,b}(z_0,t) \;=\; 0$ is $\quad z_0 \;=\; -\frac{\lambda\,(\lambda-2)}{2\,\lambda-1}, \;\; t \;=\; -\frac{(\lambda-2)\lambda^3}{2\,\lambda-1}$. After conjugation by $\begin{pmatrix} 2\,\lambda^3-6\,\lambda^2-1 & 0 \\ 0 & \lambda^2 \end{pmatrix}$ one obtains the system

$$\begin{bmatrix} \frac{H_1}{\Delta} & \dfrac{1/9\;(2\,\lambda-1)\,(\lambda-2)\,(\lambda+1)^2\,(\lambda-z)}{\Delta} \\ \dfrac{H_2}{\Delta} & \frac{-H_1}{\Delta} \end{bmatrix}$$

where

$$\Delta \;=\; 36\;(2\,\lambda-1)\,z\,(z-1)\left(z+\frac{(\lambda-2)\,\lambda^3}{2\,\lambda-1}\right)\;,$$

$$H_1 \;=\; (18\,\lambda-9)\,z^2+2\;(\lambda-2)\left(\lambda^3+6\,\lambda-2\right)z+\lambda\;(\lambda-2)\left(4\,\lambda^3-3\,\lambda^2-2\right)\;,$$

$$H_2 \;=\; (36\,\lambda-72)(-(2\,\lambda-1)\left(2\,\lambda^2-2\,\lambda+5\right)z-\lambda\;\left(2\,\lambda^3-6\,\lambda^2-1\right))\;.$$

4. The parametrization of $F_{6,c}(z_0,t)=0$ is $\;z_0 \;=\; -\lambda\;(\lambda-2)\,,\; t=-\frac{(\lambda-2)\lambda^3}{2\,\lambda-1}$. After conjugation by $\begin{pmatrix} (\lambda^2-4\,\lambda+1)\,(\lambda^2-\lambda+1) & 0 \\ 0 & 18\lambda^2 \end{pmatrix}$ one obtains the system

$$\begin{bmatrix} \frac{H_1}{\Delta} & \dfrac{2\;(\lambda+1)^2\,(\lambda-2)\,((-2\,\lambda+1)\,z+\lambda\;(\lambda^2-\lambda+1))}{\Delta} \\ \frac{H_2}{\Delta} & \frac{-H_1}{\Delta} \end{bmatrix}$$

where

$$\Delta \;=\; 36\;(2\,\lambda-1)\,(z-1)\left(z+\frac{(\lambda-2)\,\lambda^3}{2\,\lambda-1}\right)z\;,$$

$$Q \;=\; (2\,\lambda-4)\,(\lambda+1)^2\left((-2\,\lambda+1)\,z+\lambda\,\left(\lambda^2-\lambda+1\right)\right)\;,$$

$$\begin{aligned} H_1 \;&=\; (-9+18\,\lambda)\,z^2+4\;(2\,\lambda-1)\,(\lambda-2)\,(\lambda+1)^2\,z \\ &\quad +\lambda\;(\lambda-2)\left(2\,\lambda^4-4\,\lambda^3-3\,\lambda^2-4\,\lambda+2\right)\;, \end{aligned}$$

$$H_2 = -(2\,\lambda-4)((2\,\lambda-1)(5\,\lambda^2-8\lambda+5)z+\lambda(\lambda^2-4\lambda+1)(\lambda^2-\lambda+1)).$$

8.5 The locus P_4 and points of order 8

The polynomial $\frac{\phi_8(z)}{\phi_4(z)}$ has three irreducible factors. One of them is $F_8(z) =$

$$z^8-20tz^6+32(t+t^2)z^5-(16t+58t^2+16t^3)z^4+(2t^2+32t^3)z^3-20t^3z^2+t^4.$$

This factor describes one of the three connected components of P_4. A rational parametrization of $F_8(z_0,t)=0$ is given by

$$z_0 \;=\; -4\,\frac{(\lambda-1)^3\,\lambda}{(2\,\lambda-1)^3},\quad t \;=\; 16\,\frac{\lambda^4\,(\lambda-1)^4}{(2\,\lambda-1)^4}$$

After conjugation with

$$\begin{bmatrix} (36\,\lambda^4-32\,\lambda^3+4\,\lambda^2+1)\,(6\,\lambda^2-4\,\lambda+1) & 0 \\ 0 & 32\;(\lambda-1)\,\lambda^3\,(2\,\lambda-1)^3 \end{bmatrix}$$

one obtains the system

$$\begin{bmatrix} \frac{H_1}{\Delta} & \frac{Q}{\Delta} \\ \frac{H_2}{\Delta} & \frac{-H_1}{\Delta} \end{bmatrix}$$

where

$$\Delta \;=\; 32\;(2\,\lambda-1)^7\,z\,(z-1)\left(z-16\,\frac{\lambda^4\,(\lambda-1)^4}{(2\,\lambda-1)^4}\right)$$

$$\begin{aligned} Q \;&=\; 1/4\;\left(2\,\lambda^2-1\right)^2\left(2\,\lambda^2-4\,\lambda+1\right)^2 \\ &\quad \left(-4\,\lambda\;\left(6\,\lambda^2-4\,\lambda+1\right)(\lambda-1)^3+\left(2\,\lambda^2-4\,\lambda+3\right)(2\,\lambda-1)^3\,z\right) \end{aligned}$$

$$H_1 =$$
$$-4\lambda(144\lambda^8 - 448\lambda^7 + 608\lambda^6 - 416\lambda^5 + 152\lambda^4 - 48\lambda^3 + 24\lambda^2 - 8\lambda + 1)(\lambda - 1)^3$$
$$-(9 - 16\lambda + 4\lambda^2 + 4\lambda^4)(2\lambda^2 - 4\lambda + 1)^2(2\,\lambda - 1)^3 z + 8(2\,\lambda - 1)^7 z^2$$

$$\begin{aligned} H_2 \;=\; & 16\,\lambda\,\left(36\,\lambda^4 - 32\,\lambda^3 + 4\,\lambda^2 + 1\right)\left(6\,\lambda^2 - 4\,\lambda + 1\right)(\lambda - 1)^3 \\ & +12\,\left(40\,\lambda^6 - 80\,\lambda^5 + 100\,\lambda^4 - 112\,\lambda^3 + 86\,\lambda^2 - 36\,\lambda + 7\right)(2\,\lambda - 1)^3\, z \end{aligned}$$

Remarks 8.1 We note that the formula for $L_n(y)$, given at the end of section 1, provides an efficient method to verify that the differential Galois group of a given equation is contained in $D_n^{SL_2}$. It suffices to test if $L_n(y) = 0$ (for n even) or $L_{2n}(y) = 0$ (for n odd) has a rational solution. In the formula for $L_n(y)$ the exponential solution S of the second symmetric power of $L(y)$ is needed. If $L(y)$ is obtained from the system $\frac{d}{dz} + U$ using the cyclic vector $(1, 0)$, then

$$S = \frac{z - z_0}{\sqrt{z(z-1)(z-t)}}.$$

8.6 Decidable

Let a differential operator $\frac{d}{dz} + \sum_{j=1}^{3} \frac{A_i}{z - s_j}$ be given where the A_j satisfy the conditions of section 5. We suppose that the equation is irreducible and that the field K, generated by all entries (including t), is a finite extension of $\mathbb{Q}(t)$. We claim that there is an algorithm deciding whether the differential Galois group is $D_n^{\mathrm{Sl}_2}$ for some integer $n \geq 2$ or equals $D_\infty^{\mathrm{Sl}_2}$. This claim is equivalent to the assertion that there is an algorithm deciding whether a point on the elliptic curve $w^2 = cz(z-1)(z-t)$ has finite order or not. This problem has been studied by B. Work, F. Baldassarri and B.M. Trager et al.(see [Tra]).

If K is a number field, then according to a theorem of L. Merel [Mer], there is an effective function N of $[K : \mathbb{Q}]$ such that any elliptic curve, defined over K, has, over K, no point of order $> N$. Hence there is an algorithm in this situation.

Suppose that t is transcendent. Then there is also an effective bound on the order n of points on E with values in K. This bound comes from the fact that the genera of the modular curves $X_1(n)$ tend to infinity. If the genus of $X_1(n)$ is larger than the genus of K (viewed as a curve) then the elliptic curve has no points of order n with values in K. Again there is an algorithm in this case.

9 Variations on the Lamé connection

The aim is to describe the family of connections (M, ∇) on P^1 with vector bundle $M \cong O(0) \oplus O(-1)$, connection ∇ with regular singular points $0, 1, t, \infty$ and local exponents $0, 1/2$ for the first three and $0, -1/2$ for ∞.

We present M as the sub-bundle $Oe_1 \oplus O(-[\infty])e_2$ of the free bundle $Oe_1 \oplus Oe_2$. The calculations will be given in terms of this basis e_1, e_2. The group of the automorphisms Aut of M equals

$$\begin{pmatrix} a & c+dz \\ 0 & b \end{pmatrix}, \text{ with } a, b \in C^* \text{ and } c, d \in C \ .$$

$\nabla : M \to \Omega([0] + [1] + [t] + [\infty]) \otimes M$, and thus

$$\nabla e_1 = (\sum_{j=1,2,3} \frac{a_j}{z - s_j})dz \otimes e_1 + (\sum_{j=1,2,3} \frac{b_j}{z - s_j})dz \otimes e_2 \text{ with } \sum b_j = 0 \ ,$$

$$\nabla e_2 = (c_0 + \sum_{j=1,2,3} \frac{c_j}{z - s_j})dz \otimes e_1 + (\sum_{j=1,2,3} \frac{d_j}{z - s_j})dz \otimes e_2 \ ,$$

where $s_1, s_2, s_3 = 0, 1, t$. One finds that $L = \nabla_{\frac{d}{dz}}$ equals

$$L = \frac{d}{dz} + \sum_{j=1,2,3} \frac{1}{z - s_j} \begin{pmatrix} a_j & c_j \\ b_j & d_j \end{pmatrix} + \begin{pmatrix} 0 & c_0 \\ 0 & 0 \end{pmatrix} \ ,$$

with relations $a_j + d_j = 1/2$, $a_j d_j = b_j c_j$ for $j = 1, 2, 3$ and $\sum b_j = 0$. Now we compute the matrix of ∇, locally at $z = \infty$. We use the notations $s = 1/z$ and $f_1 = e_1, \ f_2 = se_2$.

$$\nabla f_1 = -(\sum \frac{a_j}{1 - s_j s})\frac{ds}{s} \otimes f_1 - (s^{-1} \sum \frac{b_j}{1 - s_j s})\frac{ds}{s} \otimes f_2 \ ,$$

$$\nabla f_2 = -(c_0 + \sum \frac{c_j}{1 - s_j s})\frac{ds}{s} \otimes f_1 + (1 - \sum \frac{d_j}{1 - s_j s})\frac{ds}{s} \otimes f_2 \ .$$

Thus the "residue matrix" of ∇ at ∞ is equal to

$$\begin{pmatrix} -\sum a_j & -c_0 - \sum c_j \\ -\sum b_j s_j & 1 - \sum d_j \end{pmatrix} \ .$$

This matrix has eigenvalues $0, -1/2$ and this gives the relations

$$-\sum a_j + 1 - \sum d_j = -1/2 \text{ and}$$

$$(-\sum a_j)(1 - \sum d_j) = (c_0 + \sum c_j) \cdot \sum b_j s_j \ .$$

The first relation is superfluous. In total we have 13 variables and 8 (nice) relations; this yields a space S of dimension 5. After division by the automorphism group of the bundle M we find a space of dimension 2 (as it should be). We make this precise by computing, step by step, S/Aut. The expression

$$\tilde{L} := \begin{pmatrix} 1 & \alpha z \\ 0 & 1 \end{pmatrix} \cdot L \cdot \begin{pmatrix} 1 & -\alpha z \\ 0 & 1 \end{pmatrix}, \text{ for a constant } \alpha,$$

is again a connection of the required type. Let its coefficients be denoted by $\tilde{a}_1, \ldots, \tilde{c}_0$. One calculates that

$$\tilde{c}_0 = c_0 + \alpha(-1 - \sum a_j + \sum d_j) - \alpha^2 \sum s_j b_j \ ,$$

and $\tilde{b}_j = b_j$ for all j.

Suppose that the coefficient $\sum b_j s_j$ *of* α^2 *is zero.*
(This is a rather special). Then $(-\sum a_j)(1 - \sum d_j) = 0$ and the coefficient of α is not zero. In this case there is a unique α such that $\tilde{c}_0 = 0$.
After performing this transformation, we may suppose that $c_0 = 0$. Now we consider the transformation

$$\tilde{L} := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \cdot L \cdot \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}, \text{ for a constant } \beta \ .$$

The $(1,2)$ coefficient of the new matrix is

$$-\beta \sum \frac{a_j}{z - s_j} - \beta^2 \sum \frac{b_j}{z - s_j} + \beta \sum \frac{d_j}{z - s_j} + \sum \frac{c_j}{z - s_j} \ .$$

One develops this coefficient at $z = \infty$ as series in s, this yields

$$s\{\beta(\sum d_j - \sum a_j) + \sum c_j\} + s^2 * \ ,$$

and there is a unique β such that the coefficient of s is 0. After performing this transformation, one has that $\sum c_j = 0$ and the 'residue matrix' of ∇ at $z = \infty$ is

$$\begin{pmatrix} -\sum a_j & 0 \\ 0 & 1 - \sum d_j \end{pmatrix} .$$

There are two possibilities for this matrix, namely $\sum a_j = 0,\ 1-\sum d_j = -1/2$ or $\sum a_j = -1/2,\ 1-\sum d_j = 0$. We continue with the first case. One performs the transformation

$$\tilde{L} := \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot L \cdot \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \text{ for a non zero constant } \gamma .$$

If $\sum \frac{b_j}{z-s_j} = 0$ or $\sum \frac{c_j}{z-s_j} = 0$, then ∇ is 'very reducible' in the sense that Oe_1 or $O(-[\infty])e_2$ is invariant. *We remove this very reducible case from the parameter space.*
Suppose that $\sum \frac{b_j}{z-s_j} \neq 0$, then is follows that $(b_1, b_2, b_3) \sim (t-1, -t, 1)$. One may normalize, using γ, such that $(b_1, b_2, b_3) = (t-1, -t, 1)$.

For the second possibility, $\sum a_j = -1/2$ and $1 - \sum d_j = 0$, one can perform the same steps. We note that this case is not equivalent under the action of Aut to the case $\sum a_j = 0$.

Suppose that the coefficient $\sum b_j s_j$ of α^2 is not zero.
(This is the general situation). There are two values for α which transform c_0 into 0. At least one of them allows a value of β which transforms, in the next step, $\sum c_j$ into 0. Thus we may already suppose that $c_0 = 0,\ \sum c_j = 0$. Now we consider again the transformation

$$\tilde{L} := \begin{pmatrix} 1 & \beta + \alpha z \\ 0 & 1 \end{pmatrix} \cdot L \cdot \begin{pmatrix} 1 & -\beta - \alpha z \\ 0 & 1 \end{pmatrix}, \text{ for constants } \alpha, \beta,$$

and suppose that the property $c_0 = 0,\ \sum c_j = 0$ is preserved. We observe that $(-\sum a_j)(1 - \sum d_j) = 0$. Suppose that $\sum a_j = 0$. Then one computes the existence of a unique non zero element $\beta + \alpha z$, that preserves $c_0 = 0,\ \sum c_j = 0$. This element transforms $\sum a_j = 0$ into $\sum a_j = 1/2$. If one starts with $\sum d_j = 1$ (or equivalently $\sum a_j = 1/2$), there is again a unique non zero element $\beta + \alpha z$, that preserves $c_0 = 0,\ \sum c_j = 0$. This element transforms $\sum a_j = 1/2$ into $\sum a_j = 0$. Thus we can eliminate the action of the group of transformation $\{\begin{pmatrix} 1 & \beta + \alpha z \\ 0 & 1 \end{pmatrix} \mid \alpha, \beta \in C\}$ by requiring

$c_0 = 0,\ \sum c_j = 0,\ \sum a_j = 0$. Then we are left with the action of the group $\{\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \mid \gamma \in C^*\}$. This action can be eliminated by requiring that, say, $\sum b_j s_j = 1$.

A space of connections

We restrict ourselves to connections with $c_0 = 0, \sum c_j = 0$. Of the two possibilities for $\sum a_j$ we choose $\sum a_j = 0$. Here we have omitted the rather mysterious special situation $\sum s_j b_j = 0$ and $\sum a_j = 1/2$. As before, the connection is called 'very reducible' if either Oe_1 or $O(-[\infty])e_2$ is invariant under the connection. Consider the space X of connections given by:

$$c_0 = 0, \sum a_j = 0, \sum b_j = 0, \sum c_j = 0, a_j(1/2 - a_j) = b_j c_j \ (j = 1, 2, 3),$$

$(b_1, b_2, b_3) \neq (0, 0, 0)$ and $(c_1, c_2, c_3) \neq (0, 0, 0)$. Thus we have excluded the 'very reducible' connections. We still have to divide X by the action of $\mathbb{G}_m$. Any $g \in \mathbb{G}_m = C^*$ multiplies the b_j with g, the c_j with g^{-1} and leaves the a_j invariant.

Consider the $\mathbb{G}_m$-invariant morphism $X \to \mathbb{A}^2 \times \mathbb{P}^1$, which maps a point of X to $((a_1, a_2, a_3), [b_1, b_2, b_3])$. For convenience we have identified $\mathbb{A}^2$ with $\{(a_1, a_2, a_3) \in \mathbf{A}^3 \mid \sum a_j = 0\}$ and similarly we write $\mathbb{P}^1$ for $\{[b_1, b_2, b_3] \in \mathbb{P}^2 \mid \sum b_j = 0\}$. The image lies in the subset Y of $(\mathbb{A}^2 \setminus \{(0, 0, 0)\}) \times \mathbb{P}^1$ defined by the equation $b_2 b_3 a_1(1/2 - a_1) + b_1 b_3 a_2(1/2 - a_2) + b_1 b_2 a_3(1/2 - a_3) = 0$. A small calculation shows that Y is in fact the geometric quotient of X for the action of $\mathbb{G}_m$. One can verify that Y is non singular. In fact, the fibres of $Y \to \mathbb{P}^1$ are non singular conics in $\mathbb{A}^2$. The fibres are connected except for the points $[1, -1, 0],\ [1, 0, -1],\ [0, 1, -1]$. For each of these points the fibre consists of two parallel lines.

We note that $\mathbb{A}^2, \mathbb{P}^1$ should be seen as affine and projective variety over the affine scheme $\mathrm{Spec}(C[t, \frac{1}{t(t-1)}])$. The same holds for X and Y.

The special family

On the elliptic curve one considers the family of connections $\nabla e = c\frac{dz}{w} \otimes e$ (with $c \neq 0$) on the trivial line bundle $O_E e$. This is pushed down to the projective line as the family

$$\frac{d}{dz} + \begin{pmatrix} 0 & c \\ \frac{c}{w^2} & \frac{w'}{w} \end{pmatrix} \ ;$$

this is already a connection on M with the prescribed local exponents. Further $[b_1, b_2, b_3] = [t - 1, -t, 1] \in \mathbb{P}^1$ and $\sum b_j s_j = 0$. Conjugation with the

matrix $\begin{pmatrix} 1 & \beta+\alpha z \\ 0 & 1 \end{pmatrix}$, with $\beta = 2(1+t)/3 + 8c^3/3$ and $\alpha = -2c$ transforms these connections into the points of Y with

$$a_1 = \frac{2c^2(1+t)/3 + 8c^4/3}{t},\ a_2 = \frac{2c^2(1+t)/3 + 8c^4/3 - 2c^2}{1-t} \text{ and}$$

$$a_3 = \frac{2c^2(1+t)/3 + 8c^4/3 - 2c^2 t}{t^2 - t},\ [b_1, b_2, b_3] = [t-1, -t, 1]\ .$$

As before, the locus of the reducible equations can be computed. The same holds for the universal family of connections, expressed in terms of the differential $(c + \frac{w+w_0}{z-z_0})\frac{dz}{2w}$.

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