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► **To cite this version:**

Hubert de Fraysseix, Patrice Ossona de Mendez. Intersection Graphs of Jordan Arcs. Discrete Mathematics and Theoretical Computer Science, 1999, 49, pp.11-28. hal-00005625

HAL Id: hal-00005625

<https://hal.science/hal-00005625>

Submitted on 18 Aug 2005

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Intersection Graphs of Jordan Arcs

P. Ossona de Mendez and H. de Fraysseix

ABSTRACT. A family of Jordan arcs, such that two arcs are nowhere tangent, defines a hypergraph whose vertices are the arcs and whose edges are the intersection points. We shall say that the hypergraph has a *strong intersection representation* and, if each intersection point is interior to at most one arc, we shall say that the hypergraph has a *strong contact representation*.

We first characterize those hypergraphs which have a strong contact representation and deduce some sufficient conditions for a simple planar graph to have a strong intersection representation.

Then, using the Four Color Theorem, we prove that a large class of simple planar graphs have a strong intersection representation.

1. Introduction

A family of Jordan arcs, such that two arcs are nowhere tangent, defines a hypergraph whose vertices are the arcs and whose edges are the intersection points. We shall say that the hypergraph has a *strong intersection representation* and, if each intersection point is interior to at most one arc, we shall say that the hypergraph has a *strong contact representation*.

Such a family classically defines a *string graph*, which is a simple graph, whose vertices are the arcs, two vertices being adjacent if the corresponding arcs intersect at least once [3]. It is not known whether deciding whether a graph is a string graph is algorithmically decidable [11]. However, all planar graphs are obviously string graphs.

If two arcs may only intersect once, we shall say that the intersection graph has an *intersection representation*. Deciding whether a given graph has an intersection representation is known to belong to the NP-complete class [9], even if strong restrictions are requested on the family of arcs [10]. It has been a challenge for long to prove or disprove that all planar graphs have an intersection representation. It is clear that a simple graph has an intersection representation if and only if it has a strong intersection representation.

First, we shall introduce a general framework on strong contact representation.

Then, the introduction of a 4-coloration will lead to prove that a large class of planar graphs have an intersection representation.

Most of the results proved here were first presented in [5] and [7].

This work was partially supported by the **Esprit LTR Project no 20244-ALCOM IT**.

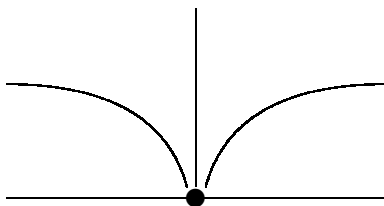
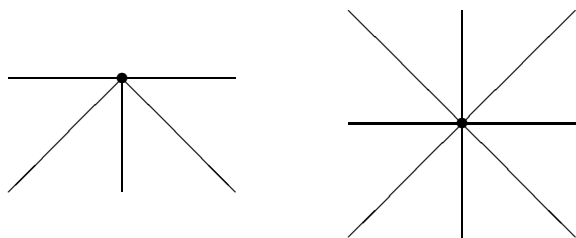


FIGURE 1. Locally one-sided contact

FIGURE 2. A contact and an intersection representations of K_4

GENERAL FRAMEWORK

Arc intersection problems arise topological difficulties that vanish when arcs are only in contact, as the arc-point incidence then defines a plane graph (i.e. a planar graph together with an embedding).

A *contact family* $\mathcal{F} = (\mathcal{V}, \mathcal{P})$ of Jordan arcs is a finite family \mathcal{V} of Jordan arcs in the plane and a finite point set \mathcal{P} on these arcs, such that two arcs share at most one point (called *contact point*), and such that all the contact points belong to \mathcal{P} and are interior to at most one arc. Furthermore, at each point p interior to an arc ν , the arc ν appears twice consecutively in a clockwise traversal around p (i.e. we consider only locally one-sided contacts). This family defines a particular type of intersection graph, called *contact graph* (which may be not planar) and also a colored planar bipartite *arc-point incidence graph* $\text{Incid}(\mathcal{F})$, whose vertex set is the union of the arc set \mathcal{V} (colored white) and the contact point set \mathcal{P} (colored black), and whose incidence is the inclusion relation of a point in an arc.

2. Graphs, Hypergraphs and Frames

2.1. Graphs. Given a graph G , we use the following notations:

- $V(G)$ is the vertex set of G ,
- $E(G)$ is the edge set of G ,
- $N(x)$ is the neighbor set of the vertex x ,
- $N(A) = \bigcup_{x \in A} N(x)$ is the neighbor set of a subset A of vertices,
- $d_G(x)$ is the degree of the vertex x ,
- G_A is the subgraph induced by the subset A of vertices,
- $T(G)$ is the set of bounded triangular faces of G , if G is a plane graph (i.e. a planar graph embedded in the plane).

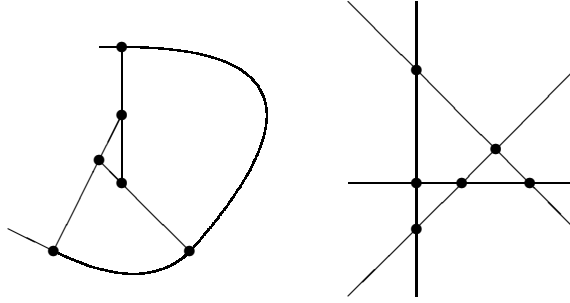


FIGURE 3. A strong contact and a strong intersection representations of K_4

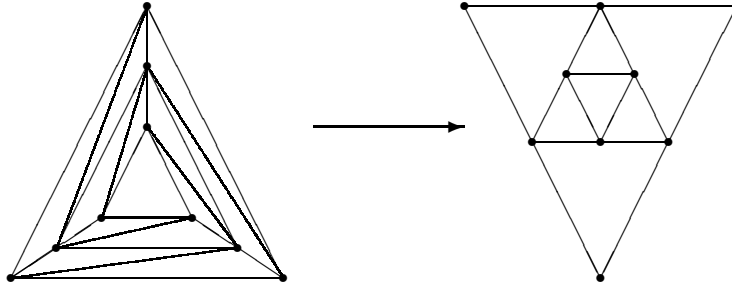


FIGURE 4. A planar graph, which has a contact representation, but no strong contact representation

DEFINITION 2.1. A graph G has a *strong contact representation* if there exists a contact family \mathcal{F} such that the arc-point incidence of \mathcal{F} is the vertex-edge incidence of G

REMARK 2.2. In a strong contact representation, the points represent the edges and thus belong to exactly 2 arcs, unlike the contact points of a contact representation which represent cliques.

2.2. Hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{E})$, where X is a finite set and \mathcal{E} is a family $(E_i, i \in I)$ of subsets of X , such that: $E_i \neq \emptyset (\forall i \in I)$ and $\bigcup_{i \in I} E_i = X$. The elements of X and \mathcal{E} are respectively the *vertices* and the *edges* of the hypergraph. Two vertices $x, y \in X$ are *adjacent* if they both belong to some edge of \mathcal{H} ; two edges E_i, E_j are *adjacent* if their intersection is not empty. A vertex $x \in X$ is *incident* to an edge $E_i \in \mathcal{E}$ if x belongs to E_i (see [1]).

A hypergraph \mathcal{H} is *linear* if any two edges have at most one common element:

$$\forall i \neq j, \quad |E_i \cap E_j| \leq 1$$

The *sub-hypergraph* of \mathcal{H} induced by a subset $Y \subseteq X$ is the hypergraph $\mathcal{H}_Y = (Y, \mathcal{E}_Y)$, where

$$\mathcal{E}_Y = \{E_i \cap Y, \quad E_i \in \mathcal{E}; E_i \cap Y \neq \emptyset\}$$

DEFINITION 2.3. The *equivalent edge number* of \mathcal{H} is the sum

$$\mu(\mathcal{H}) = \sum_{i \in I} (|E_i| - 1)$$

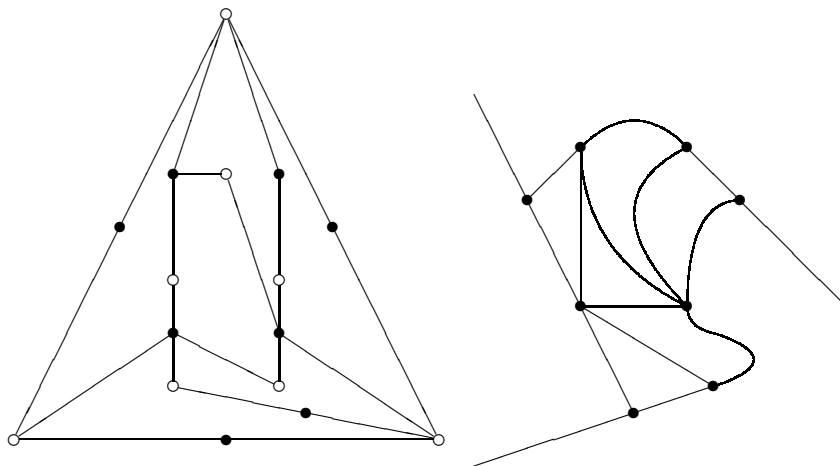


FIGURE 5. The incidence graph $\text{Incid}(\mathcal{H})$ of a planar hypergraph \mathcal{H} and a strong contact representation of \mathcal{H}

DEFINITION 2.4. The *incidence graph* $\text{Incid}(\mathcal{H})$ of \mathcal{H} is the colored bipartite graph on (X, \mathcal{E}) defined by the vertex-edge incidence, a vertex being colored white (resp. black) if it belongs to X (resp. \mathcal{E}).

As a special case, if G is a graph, $\text{Incid}(G)$ is the bicolored vertex-edge incidence graph of G .

DEFINITION 2.5. A hypergraph \mathcal{H} is *planar* if $\text{Incid}(\mathcal{H})$ is planar.

The definition of strong contact representation of graphs extends naturally to hypergraphs:

DEFINITION 2.6. A hypergraph \mathcal{H} has a *strong contact representation* if there exists a contact family \mathcal{F} , whose arc-point incidence is the vertex-edge incidence of \mathcal{H} , so that arcs represent vertices and points represent edges (see Fig. 5).

2.3. Frames.

DEFINITION 2.7. A *frame* is a pair $\Phi = (\Gamma, \Pi)$, where Γ is a plane graph and $\Pi = (F_1, \dots, F_p)$ is a partition of the edge set $E(\Gamma)$, such that each class is either a bounded 2-connected face of Γ or a single edge.

A vertex $x \in V(\Gamma)$ is incident to a class $F_i \in \Pi$ if it is incident to at least one edge in F_i .

A frame Φ defines a hypergraph $\mathcal{H}_\Phi = (V(\Gamma), \mathcal{E})$, where E_i is the set of the vertices incident to the class F_i of Π . We shall say that Φ is a *frame representation* of \mathcal{H}_Φ (see Fig. 6).

PROPOSITION 2.1. *A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is planar if and only if it has a frame representation.*

PROOF. The bijection between the embedded incidence graph of a planar hypergraph and frame representations is shown on Fig. 7:

- From the incidence graph of a planar hypergraph \mathcal{H} , each black vertex (corresponding to some E_i) is split into a C_k (if its degree is $k > 2$), a K_2 (if its degree is 2), or a loop (if its degree is 1), which edges forms a set F_i . Then, the black-white edges are contracted. The resulting graph together with the partition (F_1, \dots, F_m) is a frame representation of \mathcal{H} .

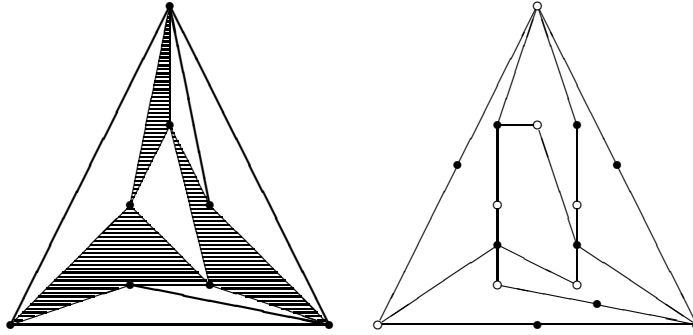


FIGURE 6. A frame representation of a planar hypergraph \mathcal{H} and the incidence graph of \mathcal{H} (the edges of \mathcal{H} are the dashed faces and the thick line segments of the frame)

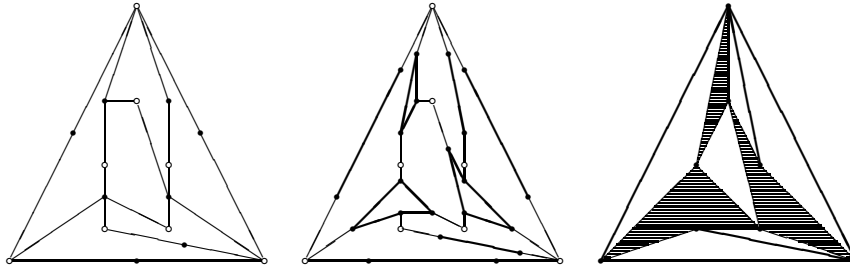


FIGURE 7. Bijection between incidence graphs of planar hypergraphs and frames

- From a frame $\Phi = (\Gamma, \Pi)$, we add a vertex in each face in Π (including the faces defined by a loop) and join it to the vertices of the face, we bisect the non-loop edges that forms a single element class in Π and then delete all the original edges of Γ that have not been bisected.

□

REMARK 2.8. According to the definition of a linear hypergraph, the following conditions are equivalent:

- \mathcal{H} is a linear planar hypergraph,
- \mathcal{H} is a hypergraph which has a C_4 -free frame representation,
- \mathcal{H} is a planar hypergraph and all the frame representations of \mathcal{H} are C_4 -free.

3. Strong Contact Representation

3.1. Strong contact representation of graphs.

THEOREM 3.1. *A graph G has a strong contact representation if and only if it is planar and satisfies:*

$$(3.1) \quad \forall A \subseteq V, \quad |E(G_A)| \leq 2|A|$$

PROOF. Equation (3.1) is equivalent to the existence of a 2-orientation of G , that is an orientation of G such that each vertex has at most 2 incoming edges.

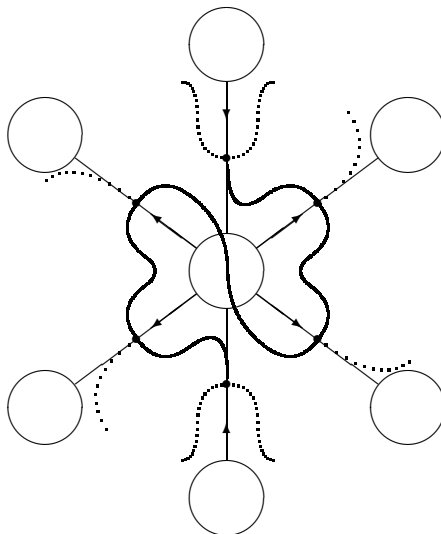


FIGURE 8. A contact representation induced by a 2-orientation

- A strong contact representation of a graph G induces a planar embedding of G and a 2-orientation of it: an edge incident to a vertex v is oriented from v if it corresponds to an internal point of the arc ν representing v ; edges corresponding to point which are internal to no arc are oriented arbitrarily.
- Conversely, an embedding and a 2-orientation of G defines a strong contact representation of G (see Fig. 8).

□

REMARK 3.2. This theorem implies that outer-planar graphs and triangle free planar graphs have a strong contact representation. Actually, bipartite planar graphs have a strong contact representation using segments in two directions [2].

3.2. Strong contact representation of hypergraphs. We may now generalize this result to hypergraphs. That for, we need first to prove an orientation lemma:

LEMMA 3.3. *The incidence graph $\text{Incid}(\mathcal{H})$ of an hypergraph \mathcal{H} has a 2-1 orientation, that is an orientation such that each white (resp. black) vertex has at most 2 (resp. 1) incoming edges, if and only if it satisfies*

$$(3.2) \quad \forall A \subseteq X, \quad \mu(\mathcal{H}_A) \leq 2|A|$$

PROOF. By classical arguments, $\text{Incid}(\mathcal{H})$ has a 2-1 orientation if and only if it satisfies:

$$(3.3) \quad \forall Y \subseteq X \cup \mathcal{E}, \quad |E(\text{Incid}(\mathcal{H})_Y)| \leq 2|Y \cap X| + |Y \cap \mathcal{E}|$$

That is:

$$(3.4) \quad \forall A \subseteq X, \forall \mathcal{E}' \subseteq \mathcal{E}, \quad |E(\text{Incid}(\mathcal{H})_{A \cup \mathcal{E}'})| \leq 2|A| + |\mathcal{E}'|$$

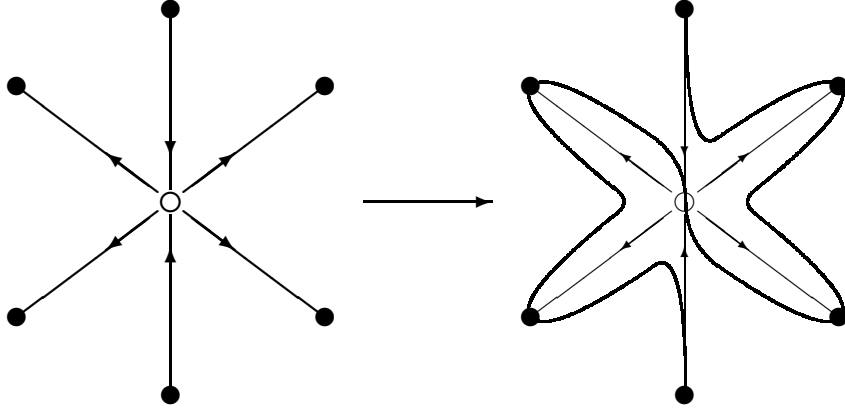


FIGURE 9. Representation of the 2-oriented planar hypergraph from a 2-1 orientation of its incidence graph

It is obviously sufficient to consider pairs (A, \mathcal{E}') where \mathcal{E}' is the neighbor set $N(A)$ of A . Hence, (3.4) may be rewritten:

$$(3.5) \quad \forall A \subseteq X, \quad |E(\text{Incid}(\mathcal{H})_{A \cup N(A)})| \leq 2|A| + |N(A)|$$

Finally,

$$(3.6) \quad |E(\text{Incid}(\mathcal{H})_{A \cup N(A)})| - |N(A)| = \sum_{E \in N(A)} d_{\text{Incid}(\mathcal{H})_{A \cup N(A)}}(E) - |N(A)|$$

$$(3.7) \quad = \sum_{E \in N(A)} (|E \cap A| - 1)$$

$$(3.8) \quad = \mu(\mathcal{H}_A)$$

Hence, the graph $\text{Incid}(\mathcal{H})$ has a 2-1 orientation if and only if (3.2) holds. \square

THEOREM 3.4. *A hypergraph \mathcal{H} has a strong contact representation if and only if \mathcal{H} is planar and satisfies*

$$(3.9) \quad \forall A \subseteq X, \quad \mu(\mathcal{H}_A) \leq 2|A|$$

PROOF. • A strong contact representation of a hypergraph \mathcal{H} induces a planar embedding and a 2-1 orientation of $\text{Incid}(\mathcal{H})$: an edge $\{v, E\}$ is oriented from v to E if the point representing E is interior to the arc representing v .

• Conversely, if \mathcal{H} is a planar hypergraph, an embedding and a 2-1 orientation of $\text{Incid}(\mathcal{H})$ defines a strong contact representation of \mathcal{H} (see Fig. 9). The theorem now follows from Lemma 3.3. \square

3.3. Strong contact representation of frame-represented hypergraphs.

We introduce some definitions and notations related to triangulation:

Let \mathcal{H} be a planar hypergraph and $\Phi = (\Gamma, \Pi)$ be a frame representation of \mathcal{H} .

- Let C be a polygon, then $\text{Triang}(C)$ is the set of the maximal outer-planar graphs which may be obtained by triangulating the inside of C .
- Let E be an edge of \mathcal{H} . Then $\text{Triang}(E) = \{E\}$ if E as cardinality at most two and $\text{Triang}(E) = \text{Triang}(C)$ if C is the polygon of Γ induced by the edges of the class of Π corresponding to the edge E of \mathcal{H} .
- $\text{Triang}(\Phi)$ is the set of the plane multigraphs which may be obtained from Γ by triangulating the faces belonging to Π .

- For any plane graph $G \in \text{Triang}(\Phi)$, $T_\Phi(G)$ denotes the union of the set of all the triangular faces of G which are not faces of Γ and the set of all the triangular faces of Γ which belong to Π .

For any induced plane subgraph H of a plane graph $G \in \text{Triang}(\Phi)$, $T_\Phi(H)$ is the set of all the triangular faces of H which belong to $T_\Phi(G)$. Notice that this set does not depend on the choice of G in $\text{Triang}(\Phi)$.

LEMMA 3.5. *\mathcal{H} is a linear hypergraph if and only if all the graphs in $\text{Triang}(\Phi)$ are simple.*

- PROOF. • If \mathcal{H} is not linear, there exists two edges E, E' of \mathcal{H} having at least two vertices x and y in common. Hence, there exists a multigraph in $\text{Triang}(\Phi)$ such that x and y are linked by an edge in $\text{Triang}(E)$ and by an edge induced in $\text{Triang}(E')$.
- If \mathcal{H} is linear, two vertices may simultaneously belong to at most one edge of \mathcal{H} and hence are linked by at most one edge in a graph belonging to $\text{Triang}(\Phi)$. □

LEMMA 3.6. *Let C be a polygon. Then, for any subset $W \subseteq V(C)$, we have:*

$$(3.10) \quad \max_{G \in \text{Triang}(C)} |E(G_W)| - |T(G_W)| = \begin{cases} |W| - 1 & \text{if } W \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

PROOF. The number $|T(G_W)|$ of bounded triangular faces of G_W is one less than the number of faces of G_W . Hence, $|E(G_W)| - |T(G_W)|$ is equal to $|W| - c(G_W)$, where $c(G_W)$ is the number of connected components of G_W . If W is nonempty, this number is at most $|W| - 1$, and this value is achieved by any triangulation G of C , such that G_W is connected. □

LEMMA 3.7. *Let $\Phi = (\Gamma, \Pi)$ be a frame representation of a hypergraph \mathcal{H} . Let $W \subseteq V$ be a subset of the vertex set of Γ . Then,*

$$(3.11) \quad \mu(\mathcal{H}_W) = \max_{G \in \text{Triang}(\Phi)} (|E(G_W)| - |T_\Phi(G_W)|)$$

PROOF. Let Γ_i be the partial graph of Γ induced by (edges of) $F_i \in \Pi$. Let W_i be the subset of W of the vertices incident to F_i . Any graph $G \in \text{Triang}(\Phi)$ is uniquely defined by the list of the graphs $G^{(i)} \in \text{Triang}(\Gamma_i)$.

The number $|E(G_W)|$ of edges of the subgraph of G induced by W is the sum of the numbers $|E(G_{W_i}^{(i)})|$ of edges of the subgraphs of the $G^{(i)}$ induced by W_i .

Similarly, the number $|T_\Phi(G_W)|$ is the sum of the numbers $|T(G_{W_i}^{(i)})|$ of triangles of the subgraphs of the $G^{(i)}$ induced by W_i , as a triangle belongs to $T_\Phi(G_W)$ if and only if its vertices belongs to the same Γ_i .

Hence,

$$(3.12) \quad \mu(\mathcal{H}_W) = \sum_{i, W_i \neq \emptyset} (|W_i| - 1)$$

$$(3.13) \quad = \sum_i \left(\max_{H \in \text{Triang}(\Gamma_i)} (|E(H_{W_i})| - |T(H_{W_i})|) \right)$$

$$(3.14) \quad = \max \left\{ \sum_i (|E(G_{W_i}^{(i)})| - |T(G_{W_i}^{(i)})|), \quad \forall j, G^{(j)} \in \text{Triang}(\Gamma_j) \right\}$$

$$(3.15) \quad = \max_{G \in \text{Triang}(\Phi)} (|E(G_W)| - |T_\Phi(G_W)|)$$

□

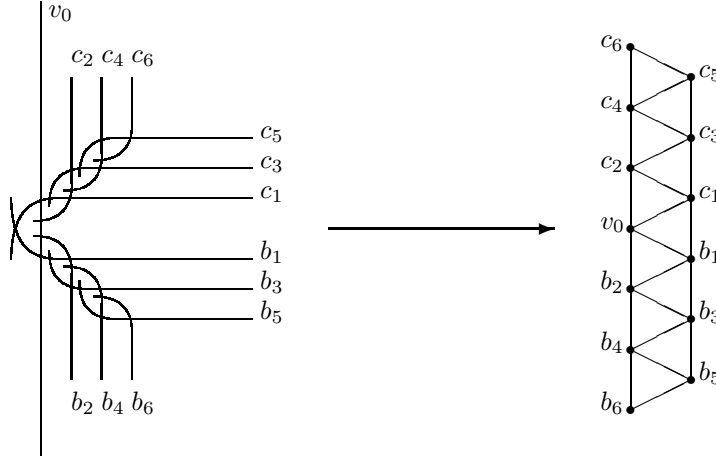


FIGURE 10. The local deformation of a contact point representing a zig-zag graph

THEOREM 3.8. *Let $\Phi = (\Gamma, \Pi)$ be a frame representation of a plane linear hypergraph \mathcal{H} . Then, \mathcal{H} has a strong contact representation if and only if the following inequality holds:*

$$(3.16) \quad \max_{G \in \text{Triang}(\Phi), W \subseteq V(\Gamma)} |E(G_W)| - |T_\Phi(G_W)| - 2|W| \leq 0$$

PROOF. This is a direct consequence of Theorem 3.4 and Lemma 3.7. \square

4. Local Intersections

Let H be a maximal outer-planar graph and let \mathcal{F} be a family of $|V(H)|$ arcs which share a single contact point p .

The graph H is a *local intersection graph* if, for any vertex v of H , there exists a local deformation of \mathcal{F} at p which represents H , and such that

- the arc to which p is interior represents v ,
- the circular order of the arcs is the same as the circular order of the vertices they represent.

A *zig-zag graph* is a maximal outer-planar graph which interior edges defines a path.

LEMMA 4.1. *A zig-zag graph is a local intersection graph.*

PROOF. The construction of a local deformation of a contact point into a zig-zag graph is shown on Fig. 10. \square

REMARK 4.2. One can prove the reverse, that is: a maximal outer-planar graph is a local intersection graph if and only if it is a zig-zag graph.

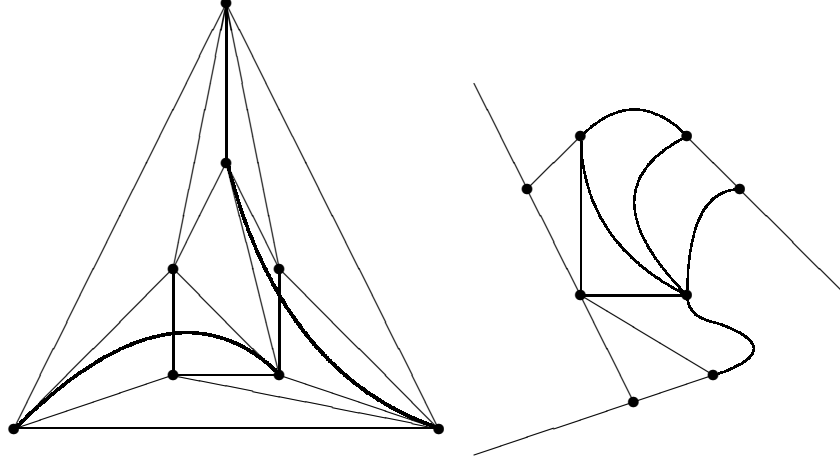
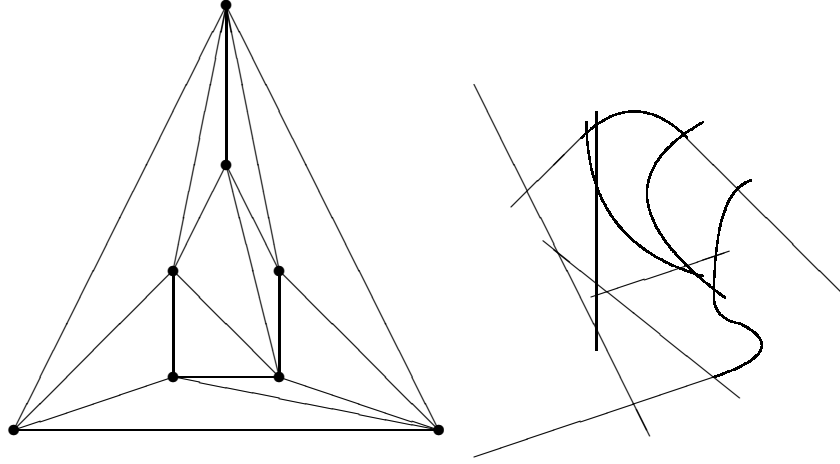
Let $\mathcal{Z}(\Phi)$ denotes the set of all possible graphs obtained from Γ by triangulating the faces in Π using paths, and let $K(\Phi)$ be the graph obtained from Γ by replacing each face in Π by a clique.

THEOREM 4.3. *Let $\Phi = (\Gamma, \Pi)$ be a frame.*

If the following condition holds:

$$(4.1) \quad \max_{G \in \text{Triang}(\Phi), W \subseteq V(\Gamma)} |E(G_W)| - |T_\Phi(G_W)| - 2|W| \leq 0$$

then:

FIGURE 11. A contact representation of $K(\Phi)$ FIGURE 12. A strong intersection representation of a graph $G \in \mathcal{Z}(\Phi)$

- $K(\Phi)$ has a contact representation,
- any graph in $\mathcal{Z}(\Phi)$ has an intersection representation.

PROOF. The conclusion follows from the previous lemma and Theorem 3.8. \square

COROLLARY 4.3.1. Let $\Gamma(V, E)$ be a plane graph and let \mathcal{I} be an independent (i.e. edge-disjoint) set of triangular faces (defining $\Pi = \mathcal{I} \cup \{e, e \in E \setminus (\bigcup_{T \in \mathcal{I}} T)\}$). If the following condition holds:

$$(4.2) \quad \max_{W \subseteq V} |E(\Gamma_W)| - |T(\Gamma_W) \cap \mathcal{I}| - 2|W| \leq 0$$

then Γ has a contact representation.

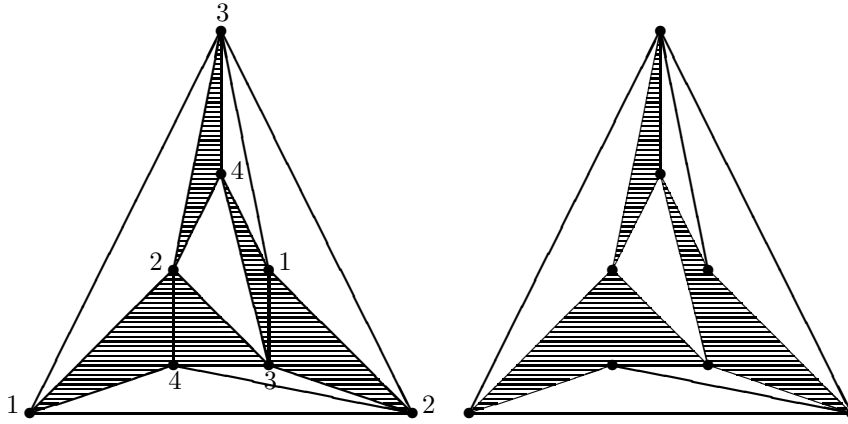


FIGURE 13. The 4-colored plane graph G (the dashed triangles are the members of $T^+(G)$) and the associated frame

REPRESENTATION OF PLANAR GRAPHS

5. The Coloration Method

Let G be a 2-connected 4-colored plane graph. Let $T^+(G)$ be the set of all the bounded triangular faces of G in which two consecutive colors appear clockwise.

First notice that a triangle belonging to $T^+(G)$ is adjacent to at most one other triangle belonging to $T^+(G)$. More precisely, two bounded adjacent triangles both belong to $T^+(G)$ if and only if the vertices of their union are colored $(1, 2, 3, 4)$ in clockwise order.

A frame $\Phi = (\Gamma, \Pi)$ is obtained from G and $T^+(G)$:

- The graph Γ is the graph obtained from G by erasing the edges belonging to two triangles belonging to $T^+(G)$.
- The partition Π has the following classes:
 - $\{e\}$, if the edge e belongs to no triangle in $T^+(G)$,
 - T , if the triangle T belongs to $T^+(G)$ and no triangle adjacent to it belongs to $T^+(G)$,
 - $(T_1 \cup T_2) \setminus (T_1 \cap T_2)$, if the triangles T_1 and T_2 are adjacent and both belong to $T^+(G)$.

LEMMA 5.1. *If G has no induced C_4 colored $(1, 2, 3, 4)$ and no separating C_3 with two consecutive colors appearing clockwise, then all the graphs in $\text{Triang}(\Phi)$ have the same property*

PROOF. All the graphs in $\text{Triang}(\Phi)$ may be obtained from G by a sequence of edge-switchings performed on edges not in $E(\Gamma)$.

If such a switching creates a C_3 separator with two consecutive colors appearing clockwise (say v_3, v_1, v_2 , where the edge (v_3, v_1) comes from the switching of an edge (x_2, x_4)), then the original graph has (v_3, x_4, v_1, v_2) as an induced C_4 .

Similarly a C_3 separator with two consecutive colors appearing clockwise may only occur if an induced C_4 colored $(1, 2, 3, 4)$ exists before the edge-switching. \square

LEMMA 5.2. *For any $G \in \text{Triang}(\Phi)$, and for any induced subgraph $H \subseteq G$, we have:*

$$(5.1) \quad T^+(H) = T_\Phi(H)$$

PROOF. This is a direct consequence of the fact that there exists no C_3 separator with two consecutive colors appearing clockwise in G . \square

LEMMA 5.3. *If G has no induced C_4 colored $(1, 2, 3, 4)$ in the clockwise order, we have:*

$$(5.2) \quad |E(G)| - |T^+(G)| - 2|V(G)| \leq -3$$

PROOF. Let us prove the lemma by induction on the size sequence of the faces.

A graph is minimal if each face has length 3 and hence is a maximal planar graph. Then, any triangle of G includes exactly one edge colored $(1, 2)$ or $(3, 4)$ and such an edge belongs to two adjacent triangles. This matching shows that $T^+(G)$ includes either the half of the triangles of G or the half the triangles of G number minus 1 (depending on the coloration of the outer face which never belongs to $T^+(G)$). Hence:

$$\begin{aligned} |E(G)| - |T^+(G)| - 2|V(G)| &\leq (3|V(G)| - 6) - \left(\frac{1}{2}(2|V(G)| - 4) + 1\right) - 2|V(G)| \\ &\leq -3 \end{aligned}$$

Otherwise, we shall construct a 4-colored 2-connected plane graph G' with no face of length 4 colored $(1, 2, 3, 4)$ in the clockwise order that satisfies (together with the associated subset $T^+(G')$ of its face set):

$$(5.3) \quad |E(G')| - |T^+(G')| - 2|V(G')| \geq |E(G)| - |T^+(G)| - 2|V(G)|$$

Let F be a non-triangular face of G of maximal size. We shall consider several cases:

- There exists a color c , such that 4 vertices of F at least are not colored c .
Then G' is obtained from G by adding a vertex x in F colored c , which is linked to all the $k \geq 4$ vertices of F , which are not colored c . We get:

$$(5.4) \quad |V(G')| = |V(G)| + 1$$

$$(5.5) \quad |E(G')| = |E(G)| + k$$

$$(5.6) \quad |T^+(G')| \leq |T(G)| + \left\lceil \frac{2k}{3} \right\rceil$$

Thus,

$$\begin{aligned} |E(G')| - |T^+(G')| - 2|V(G')| &\geq \left\lceil \frac{k}{3} \right\rceil - 2 + |E(G)| - |T^+(G)| - 2|V(G)| \\ &\geq |E(G)| - |T^+(G)| - 2|V(G)| \end{aligned}$$

Remark that no face of length 4 colored $(1, 2, 3, 4)$ in clockwise order may be created that way.

- Otherwise, no color is present twice on F . If $V_i(F)$ denotes the set of the i -colored vertices of F and l denotes the length of F , then:

$$4 \leq l = \left| \bigcup_i V_i(F) \right| = \frac{1}{3} \sum_i \left| \bigcup_{j \neq i} V_j(F) \right| \leq 4$$

Hence, F has length 4.

We triangulate F by adding an edge and, as F is not colored $(1, 2, 3, 4)$ in clockwise order, at most one additional triangle is added to $T^+(G')$. Hence,

$$(5.7) \quad |E(G')| - |T^+(G')| - 2|V(G')| \geq |E(G)| - |T^+(G)| - 2|V(G)|$$

\square

LEMMA 5.4. *If G has no induced C_4 colored $(1, 2, 3, 4)$ in the clockwise order and no separating C_3 with two consecutive colors appearing clockwise, we have:*

$$(5.8) \quad \max_{W \subseteq V(G)} |E(G_W)| - |T^+(G_W)| - 2|V(G_W)| \leq -3$$

PROOF. Let us first prove that the inequality

$$(5.9) \quad |E(G_W)| - |T^+(G_W)| - 2|V(G_W)| \leq -3$$

holds for any induced 2-connected subgraph $H \subseteq G$ in place of G_W .

Actually, let $T^+(H)$ be the set of all the bounded triangles of H in which two consecutive colors appear clockwise. As G has no separating C_3 with two consecutive colors appearing clockwise, $T^+(H) = T(H) \cap T^+(G)$. Moreover, H has no face of length 4 colored $(1, 2, 3, 4)$ in clockwise order as it would be an induced 4-cycle of G .

Hence, by the preceding lemma, any 2-connected induced subgraph H of G satisfies:

$$(5.10) \quad |E(H)| - |T^+(H)| - 2|V(H)| \leq -3$$

From this inequality, we deduce that the same holds for any induced subgraph H of G (remark that the weaker inequality $|E(H)| - |T^+(H)| - 2|V(H)| \leq 0$ could not be extended to not 2-connected graphs). \square

THEOREM 5.5. *Let G be a 4-colored 2-connected plane graph with no induced 4-cycle colored $(1, 2, 3, 4)$ in the clockwise order and no separating C_3 with two consecutive colors appearing clockwise.*

Then, G has an intersection representation.

PROOF. From the previous lemmas, we get

$$(5.11) \quad \max_{G \in \text{Triang}(\Phi), W \subseteq V(\Gamma)} |E(G_W)| - |T_\Phi(G_W)| - 2|W| \leq -3$$

and, as G belongs to $\mathcal{Z}(\Phi) = \text{Triang}(\Phi)$, the results follows. \square

COROLLARY 5.5.1. *Let G be an internally 5-connected planar graph. Then, G has an intersection representation.*

THEOREM 5.6. *Let G be a 3-colored 2-connected plane graph with no separating C_3 having two consecutive colors appearing clockwise,*

Then, G has a contact representation.

PROOF. As previously, we get

$$(5.12) \quad \max_{G \in \text{Triang}(\Phi), W \subseteq V(\Gamma)} |E(G_W)| - |T_\Phi(G_W)| - 2|W| \leq -3$$

and, as G is equal to $K(\Phi)$, the results follows. \square

COROLLARY 5.6.1. *Let G be a 4-connected 3-colorable planar graph. Then, G has a contact representation.*

6. Cut and Paste

By cutting C_3 -separated components and pasting back their representation we obtain the following extension of Theorem 5.5:

THEOREM 6.1. *Let G be a 4-colored plane graph with no induced 4-cycle colored $(1, 2, 3, 4)$ in the clockwise order.*

Then, G has an intersection representation.

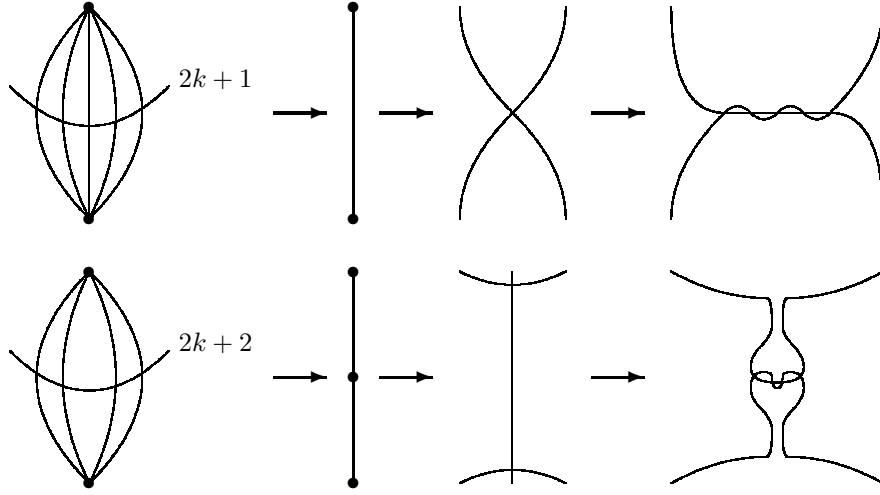


FIGURE 14. The existence of a strong intersection representation for all simple planar graphs is equivalent to the existence of a strong intersection representation for all planar multigraphs

7. Conclusion

Here we consider general graphs and hypergraphs (with possible loops and multiple edges).

DEFINITION 7.1. A hypergraph \mathcal{H} (or a multigraph G) has a *strong intersection representation* if there exists a family \mathcal{F} of Jordan arcs, whose arc-point incidence is the vertex-edge incidence of \mathcal{H} (or G): arcs represent vertices and points represent edges.

PROBLEM 1. Has any planar simple graph an intersection representation?

PROBLEM 2. Has any planar multigraph a strong intersection representation?

PROPOSITION 7.1. *The problems 1 and 2 are equivalent:*

Any planar simple graph has an intersection representation if and only if any planar multigraph has a strong intersection representation.

PROOF. If any planar multigraph has a strong intersection representation, any planar simple graph has indeed an intersection representation.

Conversely, if any planar simple graph has an intersection representation, any planar simple graph has also a strong intersection representation obtained by a trivial local deformation of the arcs at the intersection points. If a planar multigraph G has multiple edges, we consider the planar simple graph G' obtained from G by replacing any edge set E_i linking two vertices x and y by:

- a single edge $\{x, y\}$ if $|E_i|$ is odd,
- a bisected edge $\{x, z\}, \{z, y\}$ if $|E_i|$ is even.

Then, a strong intersection representation of G' induces a strong intersection representation of G (see Fig. 14). \square

PROBLEM 3. Has any planar hypergraph a strong intersection representation?

ACKNOWLEDGMENT 1. The authors would like to thank R. Šámal for his helpful comments and remarks.

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