Interpolation between logarithmic Sobolev and Poincaré inequalities

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INTERPOLATION BETWEEN LOGARITHMIC SOBOLEV AND POINCARÉ INEQUALITIES

ANTON ARNOLD, JEAN-PHILIPPE BARTIER, AND JEAN DOLBEAULT

Abstract. This note is concerned with intermediate inequalities which interpolate between the logarithmic Sobolev and the Poincaré inequalities. For such *generalized Poincaré inequalities* we improve upon the known constants from the literature.

1. Introduction

In 1989 W. Beckner [B] derived a family of generalized Poincaré inequalities (GPI) for the Gaussian measure that yield a sharp interpolation between the classical Poincaré inequality and the logarithmic Sobolev inequality (LSI) of L. Gross [G]. For any $1 \le p < 2$ these GPIs read

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu\right)^{2/p}\right] \le \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall f \in H^1(d\mu)\,, \tag{1.1}$$

where $\mu(x)$ denotes the normal centered Gaussian distribution on $\mathbb{R}^d$:

$$\mu(x) := (2\pi)^{-d/2}\,e^{-\frac{1}{2}\,|x|^2}\,.$$

For $p = 1$ the GPI (1.1) becomes the Poincaré inequality and in the limit $p \to 2$ it yields the LSI.

Our first result, Theorem 2.3, improves upon (1.1) for functions f that are in the orthogonal of the first eigenspaces of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x\cdot\nabla$.

Generalizations of (1.1) to other probability measures and the quest for "sharpest" constants in such inequalities have attracted lots of interest in the last years ([AD, BCR, LO, W]). In [AMTU] GPIs have been derived for strictly log-concave distribution functions $\nu(x)$:

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall f \in H^1(d\nu)\,, \tag{1.2}$$

where κ is the uniform convexity bound of $-\log\nu(x)$.

Latała and Oleszkiewicz (see [LO]) derived such GPIs under the weaker assumption that $\nu(x)$ satisfies a LSI with constant $0 < \mathcal{C} < \infty$, i.e.,

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\nu}\right) d\nu \le 2\,\mathcal{C}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall\ f \in H^1(d\nu)\,. \tag{1.3}$$

Under the assumption (1.3) they proved for $1 \le p < 2$:

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \mathcal{C}\min\left\{\frac{2}{p}, \frac{1}{2-p}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu\,. \tag{1.4}$$

In the limit $p \to 2$ one recovers again the LSI (1.3). Since this LSI implies a Poincaré inequality (with constant $\mathcal{C}$), the second constant in the above *min* just follows from Hölder's inequality $\left(\int_{\mathbb{R}^d} f\,d\nu\right)^2 \le \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p} = \|f\|^2_{L^p(d\nu)}$ (cf. §3 in [AD]).

Our second result, Theorem 3.1, improves upon the p-dependent constant on the r.h.s. of (1.4).

As a third result we shall derive “refined convex Sobolev inequalities” under the assumption that $\nu(x)$ satisfies a LSI. Such type on inequalities were introduced in [AD] for strictly log-concave distribution functions. They are stronger than Inequality (1.2) in the sense of improving the functional dependance of the l.h.s. of (1.2) on the term $\|f\|_{L^2(d\nu)}/\|f\|_{L^p(d\nu)}$.

2. Generalized Poincaré inequalities for the Gaussian measure

The spectrum of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x\cdot\nabla$ on $L^2(\mathbb{R}^d, d\mu)$ consists of all nonnegative integers $k \in \mathbb{N}$ and the corresponding eigenfunctions are products of one-dimensional Hermite polynomials appropriately normalized. Observing that $\mu\,\mathsf{N}f = -\mathrm{div}(\mu\nabla f)$ we can write for any function $f \in H^1(d\mu)$

$$\int_{\mathbb{R}^d} |\nabla f|^2\, d\mu = \int_{\mathbb{R}^d} f\cdot \mathsf{N}f\, d\mu\ .$$

Extending the proof strategy of Beckner [B], we shall now consider the $L^2(d\mu)$-orthogonal decomposition of f on the eigenspaces of N, i.e.

$$f = \sum_{k\in\mathbb{N}} f_k,$$

where $\mathsf{N}\, f_k = k\, f_k$. If we denote by π_k the orthogonal projection on the eigenspace of N associated to the eigenvalue $k \in \mathbb{N}$, then $f_k = \pi_k[f]$. Hence,

$$\|f\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} a_k \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu = \sum_{k\in\mathbb{N}} k\, a_k\ ,$$

where $a_k := \|f_k\|^2_{L^2(d\mu)}$. Concerning the evolution equation associated to N

$$u_t = -\mathsf{N}\, u = \Delta u - x\cdot\nabla u$$

with initial data f, one may write

$$u(x,t) = \left(e^{-t\,\mathsf{N}}\, f\right)(x) = \sum_{k\in\mathbb{N}} e^{-k\,t}\, f_k(x)\ ,$$

so that

$$\left\|e^{-t\,\mathsf{N}}\, f\right\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} e^{-2\,k\,t} a_k\ .$$

With these notations, we can now prove the first preliminary result, which generalizes the one stated in [B] in the case $k_0 = 1$.

Lemma 2.1. *Let $f \in H^1(d\mu)$. If $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, then*

$$\int_{\mathbb{R}^d} |f|^2\, d\mu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}} f\right|^2\, d\mu \leq \frac{1-e^{-2k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu\ .$$

Notice that the (constant) component f_0 of f does not contribute to the inequality.

Proof. We use the decomposition on the eigenspaces of N. For any $f_k = \pi_k[f]$, $k \geq k_0$, we have

$$\int_{\mathbb{R}^d} |f_k|^2\, d\mu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}}\, f_k\right|^2\, d\mu = \left(1-e^{-2\,k\,t}\right) a_k\ .$$

For any fixed $t > 0$, the function

$$k \mapsto \frac{1 - e^{-2\,k\,t}}{k}$$

is monotone decreasing: if $k \geq k_0$, then

$$1 - e^{-2\,k\,t} \leq \frac{1 - e^{-2\,k_0\,t}}{k_0}\,k\;.$$

Thus we get

$$\int_{\mathbb{R}^d} |f_k|^2\,d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 d\mu \leq \frac{1 - e^{-2\,k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f_k|^2\,d\mu\;,$$

which proves the result by summation. □

The second preliminary result is Nelson's hypercontractive estimates, see [N]. To make this note selfcontained we include a sketch of the proof given in [G].

Lemma 2.2. *For any $f \in L^p(d\mu)$, $p \in (1,2)$, it holds*

$$\left\| e^{-t\mathsf{N}} f \right\|_{L^2(d\mu)} \leq \|f\|_{L^p(d\mu)} \quad \forall\, t \geq -\frac{1}{2}\log(p-1)\;.$$

Proof. We set

$$F(t) := \left(\int_{\mathbb{R}^d} |u(t)|^{q(t)}\,d\mu \right)^{1/q(t)}$$

with $q(t)$ to be chosen later and $u(x,t) := \left(e^{-t\mathsf{N}} f\right)(x)$. A direct computation gives

$$\frac{F'(t)}{F(t)} = \frac{q'(t)}{q^2(t)} \int_{\mathbb{R}^d} \frac{|u|^q}{F^q} \log\left(\frac{|u|^q}{F^q}\right) d\mu - \frac{4}{F^q}\,\frac{q-1}{q^2} \int_{\mathbb{R}^d} \left| \nabla\left(|u|^{q/2}\right) \right|^2 d\mu\;.$$

We set $v := |u|^{q/2}$, use the LSI (1.3) with $\nu = \mu$ and $\mathcal{C} = 1$, and choose q such that $4\,(q-1) = 2q'$, $q(0) = p$ and $q(t) = 2$. This implies $F'(t) \leq 0$ and ends the proof with $2 = q(t) = 1 + (p-1)\,e^{2t}$. □

We are now ready to state our first main result, which is a straightforward consequence of Lemmata 2.1 and 2.2 for the Gaussian distribution $\mu(x)$.

Theorem 2.3. *Let $f \in H^1(d\mu)$. If $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, then*

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} |f|^2\,d\mu - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu\right)^{2/p}\right] \leq \frac{1-(p-1)^{k_0}}{k_0\,(2-p)} \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu$$

holds for $1 \leq p < 2$.

In the special case $k_0 = 1$ this is exactly the GPI (1.1) due to Beckner, and for $k_0 > 1$ it is a strict improvement for any $p \in [1,2)$.

3. Consequences of the logarithmic Sobolev inequality for general measures

The analysis of the previous section directly generalizes to probability measures with densities with respect to Lebesgue's measure given by

$$\nu(x) := e^{-V(x)}$$

on $\mathbb{R}^d$, that give rise to a LSI (1.3) with a positive constant $\mathcal{C}$. For a LSI to hold, the function $V(x)$ has to grow quadratically at infinity (cf. §7.4 of [ABC..], e.g.). Hence, the operator $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$, considered on $L^2(\mathbb{R}^d, d\nu)$, has again a pure point spectrum. We denote its (nonnegative) eigenvalues by λ_k, $k \in \mathbb{N}$. Notice that $\lambda_0 = 0$ is non-degenerate. The spectral gap λ_1 yields the sharp Poincaré constant $1/\lambda_1$, and it satisfies

$$\frac{1}{\lambda_1} \leq \mathcal{C} \,.$$

This is easily recovered by taking $f = 1 + \varepsilon g$ in (1.3) and letting $\varepsilon \to 0$.

As in the Gaussian case, we make a spectral decomposition of any function $f \in H^1(d\nu)$ and obtain:

$$\|f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} a_k \,, \quad \|\nabla f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} \lambda_k \, a_k \,, \quad \|e^{-t\,\mathsf{N}} f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} e^{-2\,\lambda_k t} a_k$$

where $a_k := \|f_k\|^2_{L^2(d\nu)}$. Our main assumption is therefore that such a decomposition can be done, i.e., that the eigenfunctions of the operator N form a basis of $L^2(d\nu)$ or, equivalently by writing $f\, e^{V/2} =: g$ so that $\int_{\mathbb{R}^d} |f|^2 \, d\nu = \int_{\mathbb{R}^d} |g|^2 \, dx$ and $\int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu = \int_{\mathbb{R}^d} \left(|\nabla g|^2 + (\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V)\, |g|^2 \right) dx$, that the eigenfunctions of the operator $\Delta + (\frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V)$ form a basis of $L^2(dx)$. For consistency, we will therefore require that V is in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ although weaker conditions can also be used. We shall denote this assumption by (H).

Using the analogues of Lemma 2.1 and 2.2 for the distribution $\nu(x)$ we obtain:

$$\int_{\mathbb{R}^d} |f|^2 \, d\nu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f \right|^2 \, d\nu \leq \frac{1 - e^{-2\lambda_{k_0} t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu \tag{3.1}$$

if $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, and

$$\left\| e^{-t\mathsf{N}} f \right\|_{L^2(d\nu)} \leq \|f\|_{L^p(d\nu)} \quad \forall\, t \geq -\frac{\mathcal{C}}{2} \log(p-1) \quad \forall\, p \in (1,2) \,. \tag{3.2}$$

A combination of (3.1) and (3.2), and a summation on all $k \geq k_0$ proves the following result.

Theorem 3.1. *Let ν satisfy the* LSI *(1.3) with the positive constant $\mathcal{C}$ and assume (H). If $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, then*

$$\frac{1}{2-p} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^p \, d\nu \right)^{2/p} \right] \leq C_p \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu \tag{3.3}$$

holds for $1 \leq p < 2$, with

$$C_p := \frac{1 - (p-1)^\alpha}{\lambda_{k_0}\,(2-p)}, \quad \alpha := \lambda_{k_0}\, \mathcal{C} \geq 1 \,.$$

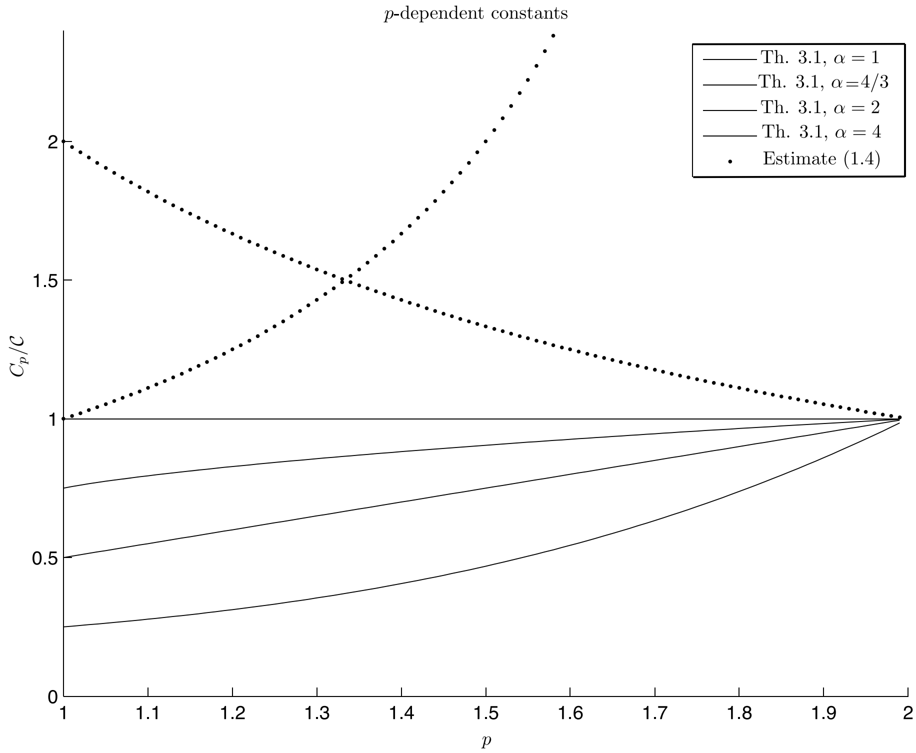

FIG. 3.1. *Comparison of the constants in the GPI for the known estimate (1.4)* $[\cdots]$ *and the new estimates of Theorem 3.1 for various values of* α.

To illustrate how Theorem 3.1 improves upon the estimate (1.4), Fig. 3.1 shows a plot of the p-dependent constant $C_p/\mathcal{C}$ for several values of α.

We shall now comment on the behavior of the p-dependent constant C_p in order to illustrate how Theorem 3.1 improves upon existing results.

Even in the special case $k_0 = 1$, the measure $d\nu$ satisfies in many cases $\alpha = \lambda_1 \mathcal{C} > 1$. On $\mathbb{R}$, e.g., one could consider the example $\nu(x) := c_\varepsilon \exp(-|x| - \varepsilon\, x^2)$ with $\varepsilon \to 0$, which was kindly suggested to us by Michel Ledoux. While the Poincaré constant is here bounded for $\varepsilon \in [0,1]$, the logarithmic Sobolev constant blows up like $\mathcal{O}(1/\varepsilon)$, which can be estimated with Th. 1.1 of [BG] (also see [BR] for a simplified approach and §3 of [L] for a review of applications in geometry). On the other hand, Lemma 2 of [LO] gives a simple sufficient condition in one dimension such that $\alpha = 1$, when $k_0 = 1$. If the logarithmic Sobolev constant takes its minimal value $\mathcal{C} = 1/\lambda_1$ (i.e. $\alpha = 1$), we have $C_p = \mathcal{C}$, for any $p \in [1,2]$ which is the straightforward generalization of (1.1) to the distribution ν. In this case, $C_p = \mathcal{C}$ is moreover optimal. We may indeed consider $f = 1 + \varepsilon g$ in (3.3). By taking the limit $\varepsilon \to 0$, for any $p \in (1,2)$, the best constant in (3.3) satisfies $C_p \geq 1/\lambda_1$, which generalizes the estimate for $p = 2$, $C_2 = \mathcal{C}$. On the opposite, notice that for $k_0 > 1$, $\alpha > 1$ is always true.

For fixed $\alpha \geq 1$, C_p takes the sharp limiting values for the Poincaré inequality ($p = 1$) and the LSI ($p = 2$): $C_1 = 1/\lambda_1$ and $\lim_{p\to 2} C_p = \mathcal{C}$.

For $\alpha > 1$, C_p is monotone increasing in p since it is a difference quotient of the convex function $p \mapsto (p-1)^\alpha$. Hence, $C_p < \mathcal{C}$ for $p < 2$ and $\alpha > 1$, and Theorem 3.1 strictly improves upon the constants of estimate (1.4).

Finally, we consider the situation in which ν only satisfies a Poincaré inequality (with constant $1/\lambda_1$) but *no* LSI: For fixed p and $k_0 = 1$ we have

$$\lim_{\alpha\to\infty} C_p = \frac{1}{\lambda_1(2-p)},$$

which corresponds to the second constant in the *min* of inequality (1.4) (cf. also Theorem 4 in [AD] and §2.2 in [C]).

4. A refined interpolation inequality

An inequality stronger than (1.2) has been shown by the first and the third author in [AD]. Under the Bakry-Emery condition on the measure $d\nu$, they proved that for all $p \in [1,2)$:

$$(4.1)\quad \frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2\left(\frac{2}{p}-1\right)}\left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{p-1}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$

for any $f \in H^1(d\nu)$, where κ is the uniform convexity bound of $-\log\nu(x)$. The estimate (1.2) is a consequence of this inequality. This can be shown using Hölder's inequality, $\left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p} \le \int_{\mathbb{R}^d} f^2\,d\nu$ and the inequality $\left(1-t^{2-p}\right)/(2-p) \ge 1-t$ for any $t\in[0,1]$, $p\in(1,2)$. With the same notations as in Section 3, we can prove the following result:

THEOREM 4.1. *Let ν satisfy the* LSI *(1.3) with the positive constant $\mathcal{C}$ and assume (H). If $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \ge 1$, then*

$$(4.2)\quad \lambda_{k_0}\max\left\{\frac{\|f\|^2_{L^2(d\nu)}-\|f\|^2_{L^p(d\nu)}}{1-(p-1)^\alpha},\ \frac{\|f\|^2_{L^2(d\nu)}}{\log(p-1)^\alpha}\log\left(\frac{\|f\|^2_{L^p(d\nu)}}{\|f\|^2_{L^2(d\nu)}}\right)\right\} \le \|\nabla f\|^2_{L^2(d\nu)}$$

holds for $1 \le p < 2$, with $\alpha := \lambda_{k_0}\,\mathcal{C} \ge 1$.

Proof. We shall proceed in two steps and derive first for all $\gamma \in (0,2)$ the following inequality, which is inspired by (4.1):

$$(4.3)\quad \frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{\frac{\gamma}{p}}\left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{\frac{2-\gamma}{2}}\right] \le K_p(\gamma)\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu\ ,$$

with

$$K_p(\gamma) := \frac{1-(p-1)^{\alpha\gamma/2}}{\lambda_{k_0}\,(2-p)^2}\ .$$

Step 1: The computations are analogous to the ones of Theorem 3.1. With the same notations as above, the squared bracket of (4.3) can be bounded from above by

$$\mathcal{N} := \|f\|^2_{L^2(d\nu)} - \left\|e^{-t\mathsf{N}}f\right\|^\gamma_{L^2(d\nu)}\|f\|^{2-\gamma}_{L^2(d\nu)} = \sum_{k\ge k_0} a_k - \left(\sum_{k\ge k_0} a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\left(\sum_{k\ge k_0} a_k\right)^{\frac{2-\gamma}{2}}$$

for any $t \ge -\frac{\mathcal{C}}{2}\log(p-1)$ as in (3.2). By Hölder's inequality, we get

$$\sum_{k\ge k_0} a_k\,e^{-\gamma\lambda_k t} = \sum_{k\ge k_0}\left(a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\cdot a_k^{\frac{2-\gamma}{2}} \le \left(\sum_{k\ge k_0} a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\left(\sum_{k\ge k_0} a_k\right)^{\frac{2-\gamma}{2}}.$$

Then

$$\mathcal{N} \le \sum_{k \ge k_0} a_k \left(1 - e^{-\gamma \lambda_k t}\right)$$

can be bounded as in the proof of Theorem 3.1:

$$\mathcal{N} \le \frac{1 - e^{-\gamma \lambda_{k_0} t}}{\lambda_{k_0}} \sum_{k \ge k_0} \lambda_k \, a_k = \frac{1 - e^{-\gamma \lambda_{k_0} t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu$$

using the decay of $k \mapsto \left(1 - e^{-\gamma \lambda_k t}\right) / \lambda_k$. The result then holds using (3.2) with

$$e^{-\gamma \lambda_{k_0} t} = (p-1)^{\gamma \lambda_{k_0} \mathcal{C}/2} \, .$$

Step 2: Next we shall optimize Inequality (4.3) w.r.t. $\gamma \in (0,2)$. After dividing the l.h.s. of (4.3) by $K_p(\gamma)$ we have to find the maximum of the function

$$\gamma \mapsto h(\gamma) := \frac{1 - a^\gamma}{1 - b^\gamma} \, , \qquad \text{with} \quad a = \frac{\|f\|_{L^p(d\nu)}}{\|f\|_{L^2(d\nu)}} \le 1 \, , \quad b = (p-1)^{\alpha/2} \le 1$$

on $\gamma \in [0,2]$. We write $h(\gamma) = g(b^\gamma)$ with

$$g(y) := \frac{1 - y^{\frac{\log a}{\log b}}}{1 - y} \, .$$

For $a < b < 1$ the function $g(y)$ is monotone increasing (since it is a difference quotient of the convex function $y^{\frac{\log a}{\log b}}$). Hence, $h(\gamma)$ is monotone decreasing. Analogously, h is monotone increasing for $b < a < 1$. Hence, the maximum of the function $h(\gamma)$ on $[0,2]$ is either $h(2)$ (if $a > b$) or $\lim_{\gamma \to 0} h(\gamma)$ (in the case $a < b$). This yields the two terms in the *max* of (4.2). □

As in Theorem 3.1, the limiting cases of (4.2) are the sharp Poincaré inequality ($p = 1$) and the LSI ($p = 2$). Inequality (4.1) corresponds to Inequality (4.3) with $\gamma = 2(2-p)$.

Using the inequality $1 - x^{\gamma/2} \ge \frac{\gamma}{2}(1 - x)$ with $x = \|f\|^2_{L^p(d\nu)} / \|f\|^2_{L^2(d\nu)} \le 1$ and Hölder's inequality, we obtain for any $\gamma \in (0,2)$,

$$\|f\|^2_{L^2(d\nu)} - \|f\|^\gamma_{L^p(d\nu)} \|f\|^{2-\gamma}_{L^2(d\nu)} \ge \frac{\gamma}{2} \left[\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)} \right] .$$

Inequality (1.2) therefore follows from Inequality (4.3) with $1/\kappa = 2(2-p) \, K_p(\gamma)/\gamma$:

$$\frac{1}{2-p} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^p \, d\nu \right)^{2/p} \right] \le \frac{2(2-p)}{\gamma} \, K_p(\gamma) \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu \, .$$

Notice that $C_p = \frac{1 - (p-1)^\alpha}{\lambda_{k_0} \, (2-p)}$ is always smaller than $\frac{2(2-p)}{\gamma} \, K_p(\gamma) = \frac{2}{\gamma} \, \frac{1 - (p-1)^{\alpha\gamma/2}}{\lambda_{k_0} \, (2-p)}$, as follows again by the above inequality with now $x = (p-1)^\alpha$. See Fig. 4.1 for a comparison of C_p and $K_p(\gamma)$ with $\gamma = 2(2-p)$ in terms of $p \in (1,2)$. However, the inequality

$$\frac{1}{(2-p) \, C_p} \left[\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)} \right] \le \frac{1}{(2-p)^2 \, K_p(\gamma)} \left[\|f\|^2_{L^2(d\nu)} - \|f\|^\gamma_{L^p(d\nu)} \|f\|^{2-\gamma}_{L^2(d\nu)} \right]$$

holds whenever $x = \|f\|^2_{L^p(d\nu)} / \|f\|^2_{L^2(d\nu)} \le (p-1)^\alpha$, by concavity of the map $x \mapsto x^{\gamma/2}$. In such a case, Inequality (4.3) is stronger than Inequality (3.3).

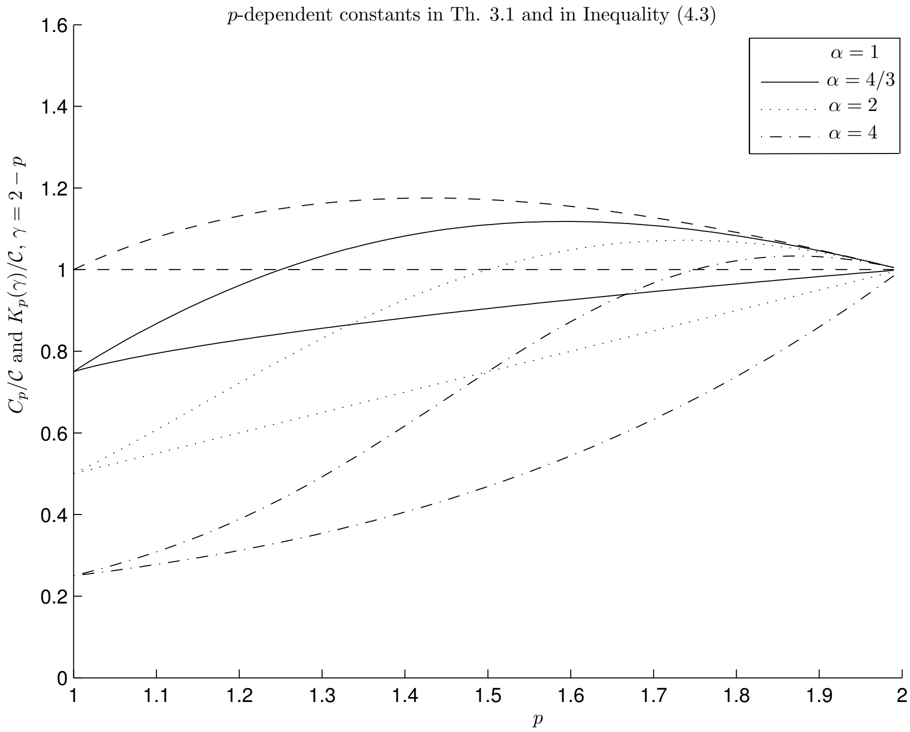

FIG. 4.1. *Comparison of the constants C_p and $K_p(\gamma)$ with $\gamma = 2(2-p)$ in Theorem 3.1 and Inequality (4.3): For fixed α we have $K_p(\gamma) \geq C_p$.*

The interest of inequality (4.1) compared to the result of Theorem 3.1 is that we obtain a nonlinear estimate (see below).

We remark that the first term in the *max* of (4.2) exactly corresponds to Inequality (3.3). Hence, the statement of Theorem 4.1 is always at least as strong as Theorem 3.1. The threshold between the two regimes described by the relative size of the two terms in the *max* of (4.2) is given by

$$\frac{\|f\|_{L^p(d\nu)}}{\|f\|_{L^2(d\nu)}} = (p-1)^{\alpha/2} ,$$

as it is seen from the above proof.

For the special case when $\alpha = 1$ and when κ (the uniform convexity bound on $-\log \nu$) yields the sharp logarithmic Sobolev constant, i.e.

$$\mathcal{C} = 1/\kappa ,$$

we shall now compare Theorem 4.1 to Inequality (4.1). Note that this is a very conservative comparison since (4.1) was derived in [AD] under the Bakry-Emery condition, while the new estimate (4.2) holds under the weaker assumption that $\nu(x)$ satisfies a LSI: For $\|f\|_{L^p(d\nu)}/\|f\|_{L^2(d\nu)}$ "large" (i.e. $a > b$) (4.2) coincides with (1.2), and (4.1) is strictly stronger. On the other hand, for "small" $\|f\|_{L^p(d\nu)}/\|f\|_{L^2(d\nu)}$, the new estimate (4.2) is stronger than (4.1).

In Fig. 4.2 we shall illustrate this comparison (still in the case $\alpha = 1$ and $\mathcal{C} = 1/\kappa$) between the Inequalities (1.2), (4.1), and (4.2). To this end we rewrite them in terms

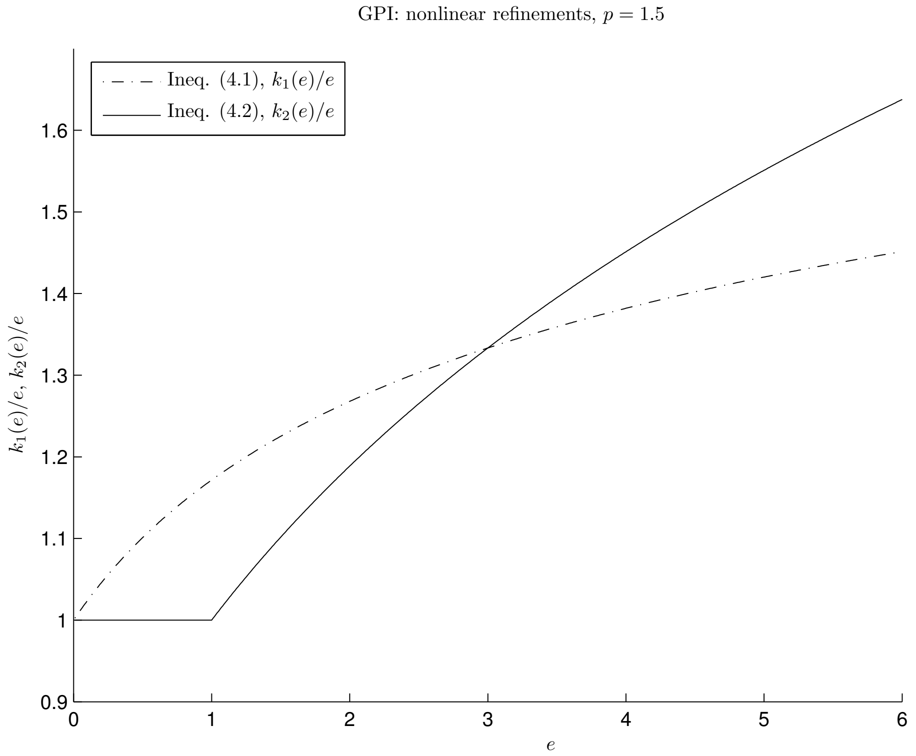

FIG. 4.2. *Comparison of the nonlinear refinements of the GPI (1.2) for $\alpha = 1$ and $p = 1.5$: The estimate (4.1) is known from [AD] and estimate (4.2) is new.*

of the non-negative functional

$$\mathsf{e}_p[f] := \frac{\|f\|^2_{L^2(d\nu)}}{\|f\|^2_{L^p(d\nu)}} - 1 \,,$$

which can be interpreted as an *entropy* (cf. [AD] for details), and the corresponding *entropy production*

$$\mathsf{I}_p[f] := \frac{2(2-p)}{\|f\|^2_{L^p(d\nu)}} \, \|\nabla f\|^2_{L^2(d\nu)} \,.$$

The GPI (1.2) is then the following *linear* lower bound for $\mathsf{I}_p[f]$:

$$\mathsf{e}_p[f] \le \frac{1}{2\kappa} \, \mathsf{I}_p[f] \,,$$

while (4.1) and (4.2) are *nonlinear refinements*:

$$k_1(\mathsf{e}_p[f]) \le \frac{1}{2\kappa} \, \mathsf{I}_p[f], \quad k_1(e) := \frac{1}{2-p}[e + 1 - (e+1)^{p-1}] \ge e \,,$$

and, respectively,

$$k_2(\mathsf{e}_p[f]) \le \frac{1}{2\kappa} \, \mathsf{I}_p[f], \quad k_2(e) := \max \left\{ e, \, \frac{2-p}{|\log(p-1)|} \, (e+1) \log(e+1) \right\} \ge e \,.$$

We remark that for the logarithmic entropy similar nonlinear estimates are discussed in §§1.3, 4.3 of [L].

In the above notation, Inequality (4.3) corresponds to

$$k(\gamma, \mathsf{e}_p[f]) \le \frac{1}{2\kappa}\, \mathsf{I}_p[f], \quad k(\gamma,e) := (2-p)(e+1)\frac{1-(e+1)^{-\gamma/2}}{1-(p-1)^{\gamma/2}} \text{ for any } \gamma \in (0,2]\ .$$

For $0 < e \le (2-p)/(p-1)$, we have $k(0,e) := \frac{2-p}{|\log(p-1)|}\,(e+1)\log(e+1) \le k(\gamma,e) \le k(2,e) = e = k_2(e)$ for any $\gamma \in (0,2)$, and $e = k(2,e) \le k(\gamma,e) \le k_2(e)$ if $e \ge (2-p)/(p-1)$. This explains the discontinuity of $k_2'(e)$ at $e = (2-p)/(p-1)$, see e.g. Fig. 4.2.

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REFERENCES

[ABC..] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Société Mathématique de France, 2000.

[AD] A. Arnold, J. Dolbeault, *Refined convex Sobolev Inequalities*, to appear in J. Functional Analysis, 2005.

[AMTU] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, *On logarithmic Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations, 26 no. 1-2 (2001), pp. 43–100.

[BCR] F. Barthe, P. Cattiaux, C. Roberto, *Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry*, preprint, 2005.

[BR] F. Barthe and C. Roberto, *Sobolev inequalities for probability measures on the real line*, Studia Math., 159 (2003), pp. 481–497.

[B] W. Beckner, *A generalized Poincaré inequality for Gaussian measures*, Proc. Amer. Math. Soc., 105 (1989), pp. 397–400.

[BG] S.G. Bobkov, F. Götze, *Exponential Integrability and Transportation Cost Related to Logarithmic Sobolev Inequalities*, J. Functional Analysis, 163 (1999), pp. 1–28.

[C] D. Chafaï, *Entropies, Convexity, and Functional Inequalities*, J. Math. Kyoto Univ., 44-2 (2004), pp. 325–363.

[G] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. of Math., 97 (1975), pp. 1061–1083.

[LO] R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré*, in Geometric aspects of functional analysis, vol. 1745 of Lecture Notes in Math., Springer, Berlin, 2000, pp. 147–168.

[L] M. Ledoux, *The geometry of Markov diffusion generators*, Ann. Fac. Sci. Toulouse Math., 6 (2000), pp. 305–366.

[N] E. Nelson, *The free Markoff field*, J. Functional Analysis, 12 (1973), pp. 211–227.

[W] F.Y. Wang, *A Generalization of Poincaré and Log-Sobolev Inequalities*, Potential Analysis, 22 (2005), pp. 1–15.