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## Files for Gabriel-Zisman localization

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## FILES FOR GABRIEL-ZISMAN LOCALIZATION

CARLOS SIMPSON

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We present simplified versions of the main files for Gabriel-Zisman localization (the only file which isn't included is `updateA.v` which contains things which should have gone in earlier parts of the set theory and category theory developments). All proofs are taken out, and also all comments have been removed. For proofs as well as the comments in their original form, see the actual Coq files which come with this posting (go to "Other formats" then "Source" on the abstract page). For a general discussion of the historical and little mathematical issues, for the bibliography, and for a more specific discussion of the present proof files, see the companion preprint "Explaining Gabriel-Zisman localization to the computer". The document below was generated by using `coqdoc` to create html files, cutting and pasting those together and editing the resulting text file, then using the "verbatim" environment for each module.

## 1. INSTRUCTIONS

We recall here the basic instructions. Everything should compile under the current version of Coq, version 8.0 patch level 2. See <http://coq.inria.fr/distrib-eng.html> for downloading and installing the Coq proof assistant.

The following instructions are for a Unix-like system. Installation, compilation and utilisation of large multiple-file projects in the Windows version of Coq has, for me at least, been problematic (it would be good if this could be improved in upcoming versions of Coq).

Go to the <http://arxiv.org> website; then to the abstract page for the present preprint. From the abstract page, click on “Other formats” for other download types (rather than “ps” or “pdf”). At the bottom of the resulting page under the rubrique “Source”, click on “Download source”. This yields a tar archive called `0506????.tar.gz` (fill in the correct paper number); save it in the directory you wish to use, then decompact it using the commands

```
gunzip 0506????.tar.gz
tar -xvf 0506????.tar
```

The next steps are reproduced here from the instructions in the Coq reference manual. First, create the makefile (if it isn’t already present).

```
coq_makefile -o Makefile *.v
```

Now create a file called `.depend` and write into it the inter-file dependencies using the commands

```
touch .depend
make depend
```

Subsequently if new files are added to the development, one should redo the command `make depend`.

Now we are ready to compile everything! Type

```
make all
```

and you should get (quickly for the first two, then more slowly for the later files) the following list of compilations:

```
coqc -q -I . axioms
coqc -q -I . tactics
coqc -q -I . set_theory
coqc -q -I . functions
coqc -q -I . notation
coqc -q -I . universe
coqc -q -I . order
coqc -q -I . transfinite
coqc -q -I . ordinal
coqc -q -I . cardinal
coqc -q -I . category
coqc -q -I . functor
coqc -q -I . nat_trans
coqc -q -I . cat_examples
coqc -q -I . limits
coqc -q -I . colimits
coqc -q -I . functor_cat
coqc -q -I . fc_limits
coqc -q -I . little_cat
coqc -q -I . equalizer
```

```

coqc -q -I .  fiproduct
coqc -q -I .  updateA
coqc -q -I .  freecat
coqc -q -I .  qcat
coqc -q -I .  gzdef
coqc -q -I .  left_fractions
coqc -q -I .  gzloc
coqc -q -I .  infinite
coqc -q -I .  lfcx

```

The directory now contains both the `v` files and the `vo` (i.e. compiled) files. The files can be usefully perused using `coqide`. Particularly useful is the `Queries` set of commands in `coqide`, allowing one to view important information from various files in useful chunks.

The files which are new to the Gabriel-Zisman localization are the following:

```

updateA.v
freecat.v
qcat.v
gzdef.v
left_fractions.v
gzloc.v
infinite.v
lfcx.v

```

The other files are identical to the set-theory and category-theory development that was bundled with the previous preprint `math.AG/0410224`.

The remainder of this paper is a listing of everything in the Gabriel-Zisman source files, except all the proofs and the totality of the first file `updateA.v`. This first file contains some things which logically would go with the earlier set-theory and category-theory files; we group them into a new “update” file in order to keep the earlier files identical to the previous version.

For looking at the proofs it is better to use `coqide`, the Coq integrated development environment (which comes with Coq). For example to look at the file `freecat.v`, do:

```
coqide freecat.v
```

(or simply type `coqide` then open the file you want). In `coqide` you get a screen with the source file on the left, and the upper screen on the right is the proof goal window. Thus you can put the cursor in the middle of a proof, click on the “go here” button, and see the state of the proof at that point. Step through proofs with the “next step” button. Using these buttons requires having at least all of the previous files being compiled as per the instructions above—when `coqide` gets to a `Require Export` instruction typically at the start of the file, it looks for a compiled `vo` file.

2. THE FILE `freecat.v`

```
Library freecat
Set Implicit Arguments.
Unset Strict Implicit.
```

```
Require Export updateA.
```

2.1. The module `Uple`.

```
Module Uple.
Export UpdateA.
Export Universe.
```

```
Definition axioms a :=
Function.axioms a &
inc (domain a) nat.
```

```
Definition length a := Bnat (domain a).
```

```
Definition create (f:nat -> E) (l:nat) :=
Function.create (R l) (fun x => f (Bnat x)).
```

```
Lemma domain_create : forall f l, domain (create f l) = R l.
```

```
Lemma length_create : forall f l, length (create f l) = l.
```

```
Definition component (i:nat) a :=
V (R i) a.
```

```
Lemma component_create : forall f l i,
i < l -> component i (create f l) = f i.
```

```
Lemma create_axioms : forall f l, axioms (create f l).
```

```
Lemma eq_create : forall a, axioms a ->
a = create (fun i => component i a) (length a).
```

```
Lemma compare_create : forall f g l m,
l = m -> (forall i, Peano.lt i l -> f i = g i) ->
create f l = create g m.
```

```
Lemma domain_emptyset : domain emptyset = emptyset.
```

```
Lemma empty_axioms : axioms emptyset.
```

```
Lemma length_emptyset : length emptyset = 0.
```

Lemma create\_length\_zero : forall f ,  
create f 0 = emptyset.

Lemma uple\_extensionality : forall a b,  
axioms a -> axioms b ->  
length a = length b ->  
(forall i, i < (length a) -> component i a = component i b) ->  
a=b.

Definition concatenate a b :=  
create (fun i =>  
Y (i < (length a)) (component i a)  
(component (i - (length a)) b))  
((length a) + (length b)).

Lemma length\_concatenate : forall a b,  
length (concatenate a b) = (length a) + (length b).

Lemma concatenate\_axioms : forall a b,  
axioms (concatenate a b).

Lemma component\_concatenate\_first : forall a b i,  
i < (length a) -> component i (concatenate a b) = component i a.

Lemma component\_concatenate\_second : forall a b i,  
i < (plus (length a) (length b)) ->  
(length a) <= i ->  
component i (concatenate a b) = component (i - (length a)) b.

Lemma component\_concatenate\_plus : forall a b i,  
i < (length b) ->  
component ((length a) + i) (concatenate a b) = component i b.

Lemma concatenate\_emptyset\_left : forall a,  
axioms a ->  
concatenate emptyset a = a.

Lemma concatenate\_emptyset\_right : forall a,  
axioms a ->  
concatenate a emptyset = a.

Lemma concatenate\_assoc : forall a b c,  
concatenate a (concatenate b c) =  
concatenate (concatenate a b) c.

Definition uple1 x :=

```
create (fun i:nat => x) 1.
```

```
Lemma uple1_axioms : forall x,
axioms (uple1 x).
```

```
Lemma length_uple1 : forall x, length (uple1 x) = 1.
```

```
Lemma component_uple1 : forall x i,
i < 1 -> component i (uple1 x) = x.
```

```
Lemma eq_uple1 : forall a x,
axioms a -> length a = 1 -> x = component 0 a ->
a = uple1 x.
```

```
Definition utack x a :=
concatenate a (uple1 x).
```

```
Lemma utack_axioms : forall a x,
axioms a -> axioms (utack x a).
```

```
Lemma length_utack : forall a x,
axioms a -> length (utack x a) = (length a) + 1.
```

```
Lemma domain_R_length : forall a,
axioms a -> domain a = R (length a).
```

```
Lemma inc_R_domain : forall a i,
axioms a ->
inc (R i) (domain a) = (i < (length a)).
```

```
Lemma component_utack_old : forall a x i,
axioms a -> i < (length a) ->
component i (utack x a) = component i a.
```

```
Lemma component_utack_new : forall a x i,
axioms a -> i = length a ->
component i (utack x a) = x.
```

```
Definition restrict a i :=
create (fun j => component j a) i.
```

```
Lemma length_restrict : forall a i,
length (restrict a i) = i.
```

```
Lemma component_restrict : forall a i j,
j < i -> component j (restrict a i) = component j a.
```

Lemma restrict\_axioms : forall a i,  
axioms (restrict a i).

Lemma eq\_utack\_restrict : forall a,  
axioms a -> length a > 0 ->  
a = utack (component (length a - 1) a) (restrict a (length a - 1)).

Definition uple\_map (f:E ->E) u :=  
Uple.create (fun i => f (component i u)) (length u).

Lemma length\_uple\_map : forall f u,  
length (uple\_map f u) = length u.

Lemma component\_uple\_map : forall f u i,  
i < length u -> component i (uple\_map f u) = f (component i u).

Lemma axioms\_uple\_map : forall f u, axioms (uple\_map f u).

Lemma uple\_map\_uple1 : forall f u,  
uple\_map f (uple1 u) = uple1 (f u).

Lemma uple\_map\_emptyset : forall f,  
uple\_map f emptyset = emptyset.

Lemma uple\_map\_concatenate : forall f u v,  
uple\_map f (concatenate u v) = concatenate (uple\_map f u)  
(uple\_map f v).

End Uple.

## 2.2. The module Graph.

Module Graph.

Export Uple.

Definition Vertices := R (v\_(r\_(t\_ DOT))).

Definition Edges := R (e\_(d\_(g\_ DOT))).

Definition vertices a := V Vertices a.

Definition edges a := V Edges a.

Definition create v e :=  
denote Vertices v  
(denote Edges e stop).

Definition like a := a = create (vertices a) (edges a).



Lemma vertices\_create : forall v e, vertices (create v e) = v.

Lemma edges\_create : forall v e, edges (create v e) = e.

Lemma create\_like : forall v e, like (create v e).

Lemma like\_extensionality : forall a b,  
like a -> like b -> vertices a = vertices b ->  
edges a = edges b -> a = b.

Definition axioms a :=  
like a &  
(forall u, inc u (edges a) -> Arrow.like u) &  
(forall u, inc u (edges a) -> inc (source u) (vertices a)) &  
(forall u, inc u (edges a) -> inc (target u) (vertices a)).

Lemma axioms\_extensionality : forall a b,  
axioms a -> axioms b -> vertices a = vertices b ->  
edges a = edges b -> a = b.

End Graph.

### 2.3. The module Free\_Category.

Module Free\_Category.

Export Graph.

Definition segment (i:nat) u :=  
component i (arrow u).

Definition seg\_length u :=  
length (arrow u).

Definition arrow\_chain u :=  
Arrow.like u &  
Uple.axioms (arrow u) &  
(0 < seg\_length u -> source (segment 0 u) = source u) &  
(0 < seg\_length u ->  
target (segment (seg\_length u -1) u) = target u) &  
(forall i, i+1 < seg\_length u ->  
source (segment (i+1) u) =  
target (segment i u)) &  
(seg\_length u = 0 -> source u = target u).

Definition mor\_freecat a u :=  
axioms a &  
inc (source u) (vertices a) &

```
inc (target u) (vertices a) &
arrow_chain u &
(forall i, i < seg_length u ->
inc (segment i u) (edges a)).
```

```
Definition vertex_uple u :=
concatenate
(Uple.create (fun i => source (segment i u)) (seg_length u))
(uple1 (target u)).
```

```
Definition vertex_uple' u :=
concatenate (uple1 (source u))
(Uple.create (fun i => target (segment i u)) (seg_length u)).
```

```
Lemma length_vertex_uple : forall u,
length (vertex_uple u) = seg_length u + 1.
```

```
Lemma length_vertex_uple' : forall u,
length (vertex_uple' u) = seg_length u + 1.
```

```
Lemma vertex_uples_same : forall u,
arrow_chain u ->
vertex_uple u = vertex_uple' u.
```

```
Definition freecat_comp u v :=
Arrow.create (source v) (target u)
(concatenate (arrow v) (arrow u)).
```

```
Lemma seg_length_freecat_comp : forall u v,
seg_length (freecat_comp u v) = seg_length u + seg_length v.
```

```
Lemma segment_freecat_comp_first : forall u v i,
i < seg_length v -> segment i (freecat_comp u v)
= segment i v.
```

```
Lemma segment_freecat_comp_second : forall u v i,
seg_length v <= i -> i < seg_length u + seg_length v ->
segment i (freecat_comp u v) =
segment (i - seg_length v) u.
```

```
Definition freecat_id x :=
Arrow.create x x emptyset.
```

```
Lemma seg_length_freecat_id : forall x,
seg_length (freecat_id x) = 0.
```

Lemma arrow\_chain\_extensionality : forall u v,  
 arrow\_chain u -> arrow\_chain v ->  
 seg\_length u = seg\_length v ->  
 source u = source v -> target u = target v ->  
 (forall i, i < seg\_length u -> segment i u = segment i v) ->  
 u = v.

Definition freecat\_edge b :=  
 Arrow.create (source b) (target b) (uple1 b).

Lemma seg\_length\_freecat\_edge : forall b,  
 seg\_length (freecat\_edge b) = 1.

Lemma segment\_freecat\_edge : forall b i,  
 i = 0 -> segment i (freecat\_edge b) = b.

Lemma source\_first\_segment : forall u i,  
 arrow\_chain u -> 0 < seg\_length u -> i = 0 ->  
 source (segment i u) = source u.

Lemma target\_last\_segment : forall u i,  
 arrow\_chain u -> 0 < seg\_length u ->  
 i = seg\_length u - 1 ->  
 target (segment i u) = target u.

Lemma eq\_freecat\_edge : forall u x,  
 arrow\_chain u -> seg\_length u = 1 ->  
 x = (segment 0 u) ->  
 u = freecat\_edge x.

Lemma source\_freecat\_id : forall x,  
 source (freecat\_id x) = x.

Lemma target\_freecat\_id : forall x,  
 target (freecat\_id x) = x.

Lemma source\_freecat\_edge : forall b,  
 source (freecat\_edge b) = source b.

Lemma target\_freecat\_edge : forall b,  
 target (freecat\_edge b) = target b.

Lemma source\_freecat\_comp : forall u v,  
 source (freecat\_comp u v) = source v.

Lemma target\_freecat\_comp : forall u v,  
 target (freecat\_comp u v) = target u.

Lemma arrow\_freecat\_id : forall x,  
 arrow (freecat\_id x) = emptyset.

Lemma arrow\_freecat\_edge : forall b,  
 arrow (freecat\_edge b) = uple1 b.

Lemma arrow\_freecat\_comp : forall u v,  
 arrow (freecat\_comp u v) =  
 concatenate (arrow v) (arrow u).

Lemma arrow\_chain\_freecat\_id : forall x,  
 arrow\_chain (freecat\_id x).

Lemma eq\_freecat\_id : forall u,  
 arrow\_chain u -> seg\_length u = 0 ->  
 u = freecat\_id (source u).

Lemma arrow\_chain\_freecat\_edge : forall b,  
 arrow\_chain (freecat\_edge b).

Lemma arrow\_chain\_freecat\_comp : forall u v,  
 arrow\_chain u -> arrow\_chain v -> source u = target v ->  
 arrow\_chain (freecat\_comp u v).

Lemma freecat\_assoc : forall u v w,  
 freecat\_comp u (freecat\_comp v w) =  
 freecat\_comp (freecat\_comp u v) w.

Lemma freecat\_left\_id' : forall u x,  
 Arrow.like u -> x = target u -> Uple.axioms (arrow u) ->  
 freecat\_comp (freecat\_id x) u = u.

Lemma freecat\_right\_id' : forall u x,  
 Arrow.like u -> x = source u -> Uple.axioms (arrow u) ->  
 freecat\_comp u (freecat\_id x) = u.

Lemma freecat\_left\_id : forall u x,  
 arrow\_chain u -> x = target u ->  
 freecat\_comp (freecat\_id x) u = u.

Lemma freecat\_right\_id : forall u x,  
 arrow\_chain u -> x = source u ->  
 freecat\_comp u (freecat\_id x) = u.

Lemma uples\_contained : forall b u,  
 Uple.axioms u ->

```
(forall i, i < length u -> inc (component i u) b) ->
inc u (powerset (Cartesian.product nat b)).
```

```
Lemma mor_freecat_from_set : forall a u,
mor_freecat a u ->
inc u (Image.create
(product (product (vertices a) (vertices a))
(powerset (product nat (edges a))))
(fun x => Arrow.create (pr1 (pr1 x)) (pr2 (pr1 x)) (pr2 x))).
```

```
Lemma mor_freecat_bounded : forall a,
Bounded.axioms (mor_freecat a).
```

```
Definition freecat_morphisms a :=
Bounded.create (mor_freecat a).
```

```
Lemma inc_freecat_morphisms : forall a u,
inc u (freecat_morphisms a) = mor_freecat a u.
```

```
Definition freecat a :=
Category.Notations.create
(vertices a) (freecat_morphisms a) freecat_comp freecat_id
emptyset.
```

```
Lemma is_mor_freecat : forall a u,
is_mor (freecat a) u = mor_freecat a u.
```

```
Lemma is_ob_freecat : forall a x,
is_ob (freecat a) x = inc x (vertices a).
```

```
Lemma comp_freecat1 : forall a u v,
mor_freecat a u -> mor_freecat a v ->
source u = target v ->
comp (freecat a) u v = freecat_comp u v.
```

```
Lemma id_freecat1 : forall a x,
inc x (vertices a) ->
id (freecat a) x = freecat_id x.
```

```
Lemma structure_freecat : forall a,
structure (freecat a) = emptyset.
```

```
Lemma mor_freecat_id : forall a x,
inc x (vertices a) -> axioms a ->
mor_freecat a (freecat_id x).
```

```
Lemma mor_freecat_comp : forall a u v,
```

```

mor_freecat a u -> mor_freecat a v ->
source u = target v ->
mor_freecat a (freecat_comp u v).

```

```

Lemma mor_freecat_edge : forall a u,
axioms a -> inc u (edges a) ->
mor_freecat a (freecat_edge u).

```

```

Lemma freecat_axioms : forall a,
axioms a -> Category.axioms (freecat a).

```

```

Lemma mor_freecat_rw : forall a u,
Graph.axioms a ->
mor (freecat a) u = mor_freecat a u.

```

```

Lemma ob_freecat_rw : forall a x,
Graph.axioms a ->
ob (freecat a) x = inc x (vertices a).

```

```

Lemma id_freecat : forall a x,
Graph.axioms a -> ob (freecat a) x ->
id (freecat a) x = freecat_id x.

```

```

Lemma comp_freecat : forall a u v,
Graph.axioms a -> mor (freecat a) u ->
mor (freecat a) v -> source u = target v ->
comp (freecat a) u v = freecat_comp u v.

```

```

Definition mor_chain a u :=
arrow_chain u &
Category.axioms a &
ob a (source u) & ob a (target u) &
(forall i, i < (seg_length u) -> mor a (segment i u)).

```

```

Lemma mor_chain_freecat_id : forall a x,
ob a x -> mor_chain a (freecat_id x).

```

```

Lemma mor_chain_freecat_comp : forall a u v,
mor_chain a u -> mor_chain a v ->
source u = target v -> mor_chain a (freecat_comp u v).

```

```

Lemma mor_chain_freecat_edge : forall a u,
mor a u -> mor_chain a (freecat_edge u).

```

```

Definition chain_tack u v :=
freecat_comp (freecat_edge u) v.

```

Lemma arrow\_chain\_chain\_tack : forall u v,  
 arrow\_chain v ->  
 source u = target v -> arrow\_chain (chain\_tack u v).

Lemma source\_chain\_tack : forall u v,  
 source (chain\_tack u v) = source v.

Lemma target\_chain\_tack : forall u v,  
 target (chain\_tack u v) = target u.

Lemma arrow\_chain\_tack : forall u v,  
 arrow (chain\_tack u v) = utack u (arrow v).

Lemma seg\_length\_chain\_tack : forall u v,  
 seg\_length (chain\_tack u v) = seg\_length v + 1.

Lemma segment\_chain\_tack\_old : forall u v i,  
 i < seg\_length v ->  
 segment i (chain\_tack u v) = segment i v.

Lemma segment\_chain\_tack\_new : forall u v i,  
 i = seg\_length v ->  
 segment i (chain\_tack u v) = u.

Lemma mor\_chain\_chain\_tack : forall a u v,  
 mor\_chain a v -> mor a u ->  
 source u = target v -> mor\_chain a (chain\_tack u v).

Definition object\_number i u :=  
 Y (i < seg\_length u) (source (segment i u)) (target u).

Lemma object\_number\_zero : forall u,  
 arrow\_chain u -> object\_number 0 u = source u.

Lemma object\_number\_seg\_length : forall u ,  
 arrow\_chain u -> object\_number (seg\_length u) u = target u.

Lemma source\_segment : forall i u,  
 arrow\_chain u -> i < seg\_length u ->  
 source (segment i u) = object\_number i u.

Lemma target\_segment : forall i u,  
 arrow\_chain u -> i < seg\_length u ->  
 target (segment i u) = object\_number (i+1) u.

Definition chain\_restrict i u :=  
 Arrow.create (source u) (object\_number i u)

(restrict (arrow u) i).

Lemma source\_chain\_restrict : forall i u,  
source (chain\_restrict i u) = source u.

Lemma target\_chain\_restrict : forall i u,  
target (chain\_restrict i u) = object\_number i u.

Lemma seg\_length\_chain\_restrict : forall i u,  
seg\_length (chain\_restrict i u) = i.

Lemma segment\_chain\_restrict : forall i j u,  
j < i ->  
segment j (chain\_restrict i u) = segment j u.

Lemma arrow\_chain\_chain\_restrict : forall i u,  
arrow\_chain u -> i <= seg\_length u ->  
arrow\_chain (chain\_restrict i u).

Lemma chain\_restrict\_refl : forall u,  
arrow\_chain u ->  
chain\_restrict (seg\_length u) u = u.

Lemma eq\_chain\_tack\_restrict : forall u,  
arrow\_chain u -> seg\_length u > 0 ->  
u = chain\_tack (segment (seg\_length u - 1) u)  
(chain\_restrict (seg\_length u - 1) u).

Lemma arrow\_chain\_induction : forall (P:E->Prop),  
(forall x, P (freecat\_id x)) ->  
(forall u v, arrow\_chain v -> source u = target v ->  
P v -> P (chain\_tack u v)) ->  
(forall y, arrow\_chain y -> P y).

Lemma mor\_chain\_chain\_restrict : forall a u i,  
mor\_chain a u -> i <= seg\_length u ->  
mor\_chain a (chain\_restrict i u).

Lemma chain\_restrict\_chain\_tack : forall u v i,  
i = seg\_length v -> arrow\_chain v ->  
source u = target v ->  
chain\_restrict i (chain\_tack u v) = v.

Lemma mor\_chain\_induction : forall a (P:E->Prop),  
(forall x, ob a x -> P (freecat\_id x)) ->  
(forall u v, mor\_chain a v -> mor a u ->  
source u = target v -> P v -> P (chain\_tack u v)) ->



(forall y, mor\_chain a y -> P y).

Definition compose\_chain a u :=  
iterate (seg\_length u) (fun i v => comp a (segment i u) v)  
(id a (source u)).

Lemma compose\_chain\_freecat\_id : forall a x,  
compose\_chain a (freecat\_id x) = id a x.

Lemma compose\_chain\_chain\_tack : forall a u v,  
compose\_chain a (chain\_tack u v) =  
comp a u (compose\_chain a v).

Lemma compose\_chain\_freecat\_edge : forall a u,  
mor a u -> compose\_chain a (freecat\_edge u) = u.

Definition compose\_chain\_facts a u :=  
mor\_chain a u &  
source (compose\_chain a u) = source u &  
target (compose\_chain a u) = target u &  
mor a (compose\_chain a u).

Lemma compose\_chain\_facts\_compose\_chain : forall a u,  
mor\_chain a u -> compose\_chain\_facts a u.

Lemma source\_compose\_chain : forall a u,  
mor\_chain a u -> source (compose\_chain a u) = source u.

Lemma target\_compose\_chain : forall a u,  
mor\_chain a u -> target (compose\_chain a u) = target u.

Lemma mor\_compose\_chain : forall a u, mor\_chain a u ->  
mor a (compose\_chain a u).

Lemma freecat\_comp\_chain\_tack : forall y u v,  
freecat\_comp (chain\_tack y u) v =  
chain\_tack y (freecat\_comp u v).

Lemma compose\_chain\_freecat\_comp : forall a u v,  
mor\_chain a u -> mor\_chain a v ->  
source u = target v ->  
compose\_chain a (freecat\_comp u v) =  
comp a (compose\_chain a u) (compose\_chain a v).

Definition chain\_map (fo : E -> E) (fa:E -> E) u :=  
Arrow.create  
(fo (source u)) (fo (target u)) (uple\_map fa (arrow u)).

Lemma source\_chain\_map : forall fo fa u,  
source (chain\_map fo fa u) = fo (source u).

Lemma target\_chain\_map : forall fo fa u,  
target (chain\_map fo fa u) = fo (target u).

Lemma arrow\_chain\_map : forall fo fa u,  
arrow (chain\_map fo fa u) = uple\_map fa (arrow u).

Lemma seg\_length\_chain\_map : forall fo fa u,  
seg\_length (chain\_map fo fa u) = seg\_length u.

Lemma segment\_chain\_map : forall fo fa u i,  
i < seg\_length u -> segment i (chain\_map fo fa u)  
= fa (segment i u).

Lemma arrow\_chain\_chain\_map : forall fo fa u,  
arrow\_chain u ->  
(forall i, i < seg\_length u -> fo (source (segment i u)) =  
source (fa (segment i u))) ->  
(forall i, i < seg\_length u -> fo (target (segment i u)) =  
target (fa (segment i u))) ->  
arrow\_chain (chain\_map fo fa u).

Lemma chain\_map\_freecat\_id : forall fo fm x,  
chain\_map fo fm (freecat\_id x) = freecat\_id (fo x).

Lemma chain\_map\_freecat\_edge : forall fo fm u,  
source (fm u) = fo (source u) ->  
target (fm u) = fo (target u) ->  
chain\_map fo fm (freecat\_edge u) = freecat\_edge (fm u).

Lemma chain\_map\_freecat\_comp : forall fo fm u v,  
chain\_map fo fm (freecat\_comp u v) =  
freecat\_comp (chain\_map fo fm u) (chain\_map fo fm v).

Lemma chain\_map\_chain\_tack : forall fo fm u v,  
target (fm u) = fo (target u) ->  
chain\_map fo fm (chain\_tack u v) =  
chain\_tack (fm u) (chain\_map fo fm v).

Definition free\_functor g a fo fm :=  
Functor.create (freecat g) a (fun u =>  
(compose\_chain a (chain\_map fo fm u))).

Definition free\_functor\_property g a fo fm :=

```

Graph.axioms g &
Category.axioms a &
(forall x, inc x (vertices g) -> ob a (fo x)) &
(forall u, inc u (edges g) -> mor a (fm u)) &
(forall u, inc u (edges g) -> source (fm u) = fo (source u)) &
(forall u, inc u (edges g) -> target (fm u) = fo (target u)).

```

```

Lemma source_free_functor : forall g a fo fm,
source (free_functor g a fo fm) = freecat g.

```

```

Lemma target_free_functor : forall g a fo fm,
target (free_functor g a fo fm) = a.

```

```

Lemma fmor_free_functor : forall g a fo fm u,
Graph.axioms g ->
mor_freecat g u -> fmor (free_functor g a fo fm) u =
(compose_chain a (chain_map fo fm u)).

```

```

Lemma mor_chain_chain_map : forall g a fo fm u,
free_functor_property g a fo fm ->
mor_freecat g u ->
mor_chain a (chain_map fo fm u).

```

```

Lemma fob_free_functor : forall g a fo fm x,
free_functor_property g a fo fm ->
inc x (vertices g) ->
fob (free_functor g a fo fm) x = fo x.

```

```

Lemma free_functor_axioms : forall g a fo fm,
free_functor_property g a fo fm ->
Functor.axioms (free_functor g a fo fm).

```

```

Lemma fmor_ff_freecat_id : forall g a fo fm x,
free_functor_property g a fo fm ->
inc x (vertices g) ->
fmor (free_functor g a fo fm) (freecat_id x) = id a (fo x).

```

```

Lemma fmor_ff_freecat_edge : forall g a fo fm u,
free_functor_property g a fo fm ->
inc u (edges g) ->
fmor (free_functor g a fo fm) (freecat_edge u) = fm u.

```

```

Lemma fmor_comp : forall a f u v,
Functor.axioms f -> source f = a ->
mor a u -> mor a v ->
source u = target v ->
fmor f (comp a u v) = comp (target f) (fmor f u) (fmor f v).

```

```

Lemma fmor_ff_freecat_comp : forall g a fo fm u v,
free_functor_property g a fo fm ->
mor (freecat g) u ->
mor (freecat g) v ->
source u = target v ->
fmor (free_functor g a fo fm) (freecat_comp u v) =
comp a (fmor (free_functor g a fo fm) u)
(fmor (free_functor g a fo fm) v).

```

```

Lemma fmor_ff_chain_tack : forall g a fo fm u v,
free_functor_property g a fo fm ->
inc u (edges g) ->
mor (freecat g) v -> source u = target v ->
fmor (free_functor g a fo fm) (chain_tack u v) =
comp a (fmor (free_functor g a fo fm) v).

```

```

Lemma free_functor_property_fob_fmor : forall g a f,
Functor.axioms f -> Graph.axioms g ->
source f = freecat g -> target f = a ->
free_functor_property g a (fob f)
(fun u => fmor f (freecat_edge u)).

```

```

Definition graph_fmor g f y :=
fmor (free_functor g (target f) (fob f)
(fun u => fmor f (freecat_edge u))) y.

```

```

Lemma graph_fob : forall g f x,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> inc x (vertices g) ->
fob (free_functor g (target f) (fob f)
(fun u => fmor f (freecat_edge u))) x = fob f x.

```

```

Definition graph_fmor_recovers g f y :=
graph_fmor g f y = fmor f y.

```

```

Lemma graph_fmor_recovers_freecat_id : forall g f x,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> inc x (vertices g) ->
graph_fmor_recovers g f (freecat_id x).

```

```

Lemma graph_fmor_recovers_freecat_edge : forall g f u,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> inc u (edges g) ->
graph_fmor_recovers g f (freecat_edge u).

```

```

Lemma graph_fmor_recovers_freecat_comp : forall g f u v,

```

```

Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> mor_freecat g u -> mor_freecat g v ->
source u = target v ->
graph_fmor_recovers g f u -> graph_fmor_recovers g f v ->
graph_fmor_recovers g f (freecat_comp u v).

```

```

Lemma graph_fmor_recovers_chain_tack : forall g f u v,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> inc u (edges g) -> mor_freecat g v ->
source u = target v ->
graph_fmor_recovers g f v ->
graph_fmor_recovers g f (chain_tack u v).

```

```

Lemma inc_object_number_vertices : forall g u i,
Graph.axioms g -> mor_freecat g u ->
i <= seg_length u -> inc (object_number i u) (vertices g).

```

```

Lemma mor_freecat_chain_restrict : forall g u i,
Graph.axioms g -> mor_freecat g u ->
i <= seg_length u -> mor_freecat g (chain_restrict i u).

```

```

Lemma mor_freecat_induction : forall g (P : E -> Prop),
Graph.axioms g ->
(forall x, inc x (vertices g) -> P (freecat_id x)) ->
(forall u v, inc u (edges g) -> mor_freecat g v ->
source u = target v -> P v -> P (chain_tack u v)) ->
(forall u, mor_freecat g u -> P u).

```

```

Lemma graph_fmor_recovers_all : forall g f u,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f -> mor_freecat g u ->
graph_fmor_recovers g f u.

```

```

Lemma eq_free_functor_fob_fmor : forall g f,
Graph.axioms g -> source f = freecat g ->
Functor.axioms f ->
f = free_functor g (target f) (fob f)
(fun u => fmor f (freecat_edge u)).

```

```

Definition free_nt_property g a b (t:E -> E) :=
Graph.axioms g & Functor.axioms a & Functor.axioms b
& source a = freecat g
& source b = freecat g
& target a = target b &
(forall x, inc x (vertices g) -> mor (target a) (t x)) &
(forall x, inc x (vertices g) -> source (t x) = fob a x) &
(forall x, inc x (vertices g) -> target (t x) = fob b x) &

```

```
(forall u, inc u (edges g) ->
comp (target a) (t (target u)) (fmor a (freecat_edge u)) =
comp (target a) (fmor b (freecat_edge u)) (t (source u))).
```

```
Definition free_nt_respects a b (t:E->E) u :=
comp (target a) (t (target u)) (fmor a u) =
comp (target a) (fmor b u) (t (source u)).
```

```
Lemma free_nt_respects_freecat_id : forall g a b t x,
free_nt_property g a b t ->
inc x (vertices g) ->
free_nt_respects a b t (freecat_id x).
```

```
Lemma free_nt_respects_freecat_edge : forall g a b t u,
free_nt_property g a b t ->
inc u (edges g) ->
free_nt_respects a b t (freecat_edge u).
```

```
Lemma free_nt_respects_freecat_comp : forall g a b t u v,
free_nt_property g a b t ->
mor_freecat g u -> mor_freecat g v -> source u = target v ->
free_nt_respects a b t u ->
free_nt_respects a b t v ->
free_nt_respects a b t (freecat_comp u v).
```

```
Lemma free_nt_respects_chain_tack : forall g a b t u v,
free_nt_property g a b t -> inc u (edges g) ->
mor_freecat g v -> source u = target v ->
free_nt_respects a b t v ->
free_nt_respects a b t (chain_tack u v).
```

```
Lemma free_nt_respects_all : forall g a b t u,
free_nt_property g a b t ->
mor_freecat g u ->
free_nt_respects a b t u.
```

```
Lemma free_nt_nat_trans_property : forall g a b t,
free_nt_property g a b t -> Nat_Trans.property a b t.
```

```
Lemma free_nt_prop_eq_nat_trans_prop : forall g a b t,
Graph.axioms g -> source a = freecat g ->
free_nt_property g a b t = Nat_Trans.property a b t.
```

```
Lemma free_nt_property_ntrans : forall g r,
Graph.axioms g -> Nat_Trans.axioms r ->
osource r = freecat g ->
free_nt_property g (source r) (target r) (ntrans r).
```

End Free\_Category.

## 3. THE FILE qcat.v

Require Export updateA.

## 3.1. The module Categorical\_Relation.

Module Categorical\_Relation.

Export UpdateA.

Export Category.

Definition cat\_rel a r :=

Category.axioms a &

is\_relation r &

(forall x y, related r x y -> mor a x) &

(forall x y, related r x y -> mor a y) &

(forall x y, related r x y -> source x = source y) &

(forall x y, related r x y -> target x = target y).

Definition cat\_equiv\_rel a r :=

cat\_rel a r &

is\_equivalence\_relation r &

(forall u, mor a u -> related r u u) &

(forall x y u v, related r x y -> related r u v ->

source x = target u ->

related r (comp a x u) (comp a y v)).

Lemma cat\_rel\_related\_rw : forall a r x y,

cat\_rel a r ->

related r x y =

(related r x y & mor a x & mor a y &

source x = source y & target x = target y).

Lemma cer\_reflexive : forall a r u,

cat\_equiv\_rel a r -> mor a u -> related r u u.

Lemma related\_comp : forall a r x y u v,

cat\_equiv\_rel a r ->

related r x y -> related r u v -> source x = target u ->

related r (comp a x u) (comp a y v).

Lemma cat\_rel\_intersection : forall a z,

nonempty z ->

(forall r, inc r z -> cat\_rel a r) ->

cat\_rel a (intersection z).

Lemma cer\_intersection : forall a z,

nonempty z ->



```
(forall r, inc r z -> cat_equiv_rel a r) ->
cat_equiv_rel a (intersection z).
```

```
Definition coarse a :=
Z (Cartesian.product (morphisms a) (morphisms a))
(fun u => (source (pr1 u) = source (pr2 u) &
target (pr1 u) = target (pr2 u))).
```

```
Lemma related_coarse : forall a x y,
Category.axioms a ->
related (coarse a) x y = (mor a x & mor a y &
source x = source y & target x = target y).
```

```
Lemma cat_rel_coarse : forall a,
Category.axioms a ->
cat_rel a (coarse a).
```

```
Lemma cat_equiv_rel_coarse : forall a,
Category.axioms a ->
cat_equiv_rel a (coarse a).
```

```
Lemma sub_coarse : forall a r,
cat_rel a r ->
sub r (coarse a).
```

```
Definition relations_between r s :=
Z (powerset s) (fun y => sub r y).
```

```
Lemma inc_relations_between : forall r s y,
is_relation s ->
inc y (relations_between r s) =
(sub r y & sub y s & is_relation y).
```

```
Definition cer a r :=
intersection (Z (relations_between r (coarse a))
(fun y=> cat_equiv_rel a y)).
```

```
Lemma cat_equiv_rel_cer : forall a r,
cat_rel a r -> cat_equiv_rel a (cer a r).
```

```
Lemma cat_rel_subset : forall a r,
(exists s, (cat_rel a s & sub r s)) ->
cat_rel a r.
```

```
Lemma sub_cer : forall a r s,
sub r s -> cat_equiv_rel a s ->
sub (cer a r) s.
```

Lemma cer\_contains : forall a r,  
 cat\_rel a r -> sub r (cer a r).

Definition ker f :=  
 Z (coarse (source f)) (fun z => (fmor f (pr1 z) = fmor f (pr2 z))).

Lemma related\_ker : forall f x y,  
 Functor.axioms f ->  
 related (ker f) x y =  
 (mor (source f) x & mor (source f) y &  
 source x = source y & target x = target y & fmor f x = fmor f y).

Lemma is\_relation\_ker : forall f, Functor.axioms f ->  
 is\_relation (ker f).

Lemma cat\_rel\_ker : forall a f,  
 Functor.axioms f -> a = source f ->  
 cat\_rel a (ker f).

Lemma cat\_equiv\_rel\_ker : forall a f,  
 Functor.axioms f -> a = source f ->  
 cat\_equiv\_rel a (ker f).

Definition compatible r f :=  
 Functor.axioms f & cat\_rel (source f) r &  
 sub r (ker f).

Lemma compatible\_related\_ker : forall r f u v,  
 compatible r f -> related r u v -> related (ker f) u v.

Lemma related\_eq\_fmor : forall r f u v,  
 compatible r f -> related r u v -> fmor f u = fmor f v.

Lemma is\_relation\_coarse : forall a,  
 Category.axioms a -> is\_relation (coarse a).

Lemma compatible\_cer : forall a r f,  
 compatible r f -> a = source f ->  
 compatible (cer a r) f.

Lemma compatible\_rw : forall r f,  
 compatible r f =  
 (Functor.axioms f & cat\_rel (source f) r &  
 (forall x y, related r x y -> fmor f x = fmor f y)).

End Categorical\_Relation.

### 3.2. The module Quotient\_Category.

```
Module Quotient_Category.
Export Categorical_Relation.
```

```
Definition arrow_class r u :=
Arrow.create (source u) (target u) (class r u).
```

```
Lemma source_arrow_class : forall r u,
source (arrow_class r u) = source u.
```

```
Lemma target_arrow_class : forall r u,
target (arrow_class r u) = target u.
```

```
Lemma arrow_arrow_class : forall r u,
arrow (arrow_class r u) = class r u.
```

```
Lemma inc_arrow_arrow_class : forall a r u v,
cat_equiv_rel a r -> mor a v ->
inc v (arrow (arrow_class r u)) = related r u v.
```

```
Lemma like_arrow_class : forall r u,
Arrow.like (arrow_class r u).
```

```
Definition is_quotient_arrow a r u :=
cat_equiv_rel a r &
(exists y, mor a y & u = arrow_class r y).
```

```
Lemma is_quotient_arrow_arrow_class : forall a r u,
cat_equiv_rel a r -> mor a u ->
is_quotient_arrow a r (arrow_class r u).
```

```
Definition quotient_morphisms a r :=
Image.create (morphisms a) (arrow_class r).
```

```
Lemma inc_quotient_morphisms : forall a r u,
cat_equiv_rel a r ->
inc u (quotient_morphisms a r) = is_quotient_arrow a r u.
```

```
Lemma related_arrow_class_eq : forall a r u v,
cat_equiv_rel a r ->
mor a u ->
related r u v =
(arrow_class r u = arrow_class r v).
```

```
Definition arrow_rep v := rep (arrow v).
```

Lemma inc\_arrow\_class\_refl : forall a r u,  
 cat\_equiv\_rel a r -> mor a u -> inc u (arrow (arrow\_class r u)).

Lemma nonempty\_arrow : forall a r u,  
 is\_quotient\_arrow a r u ->  
 nonempty (arrow u).

Lemma inc\_arrow\_rep\_arrow : forall a r u,  
 is\_quotient\_arrow a r u ->  
 inc (arrow\_rep u) (arrow u).

Lemma related\_arrow\_rep\_arrow\_class : forall a r u,  
 cat\_equiv\_rel a r -> mor a u ->  
 related r u (arrow\_rep (arrow\_class r u)).

Lemma source\_arrow\_rep : forall a r u,  
 is\_quotient\_arrow a r u -> source (arrow\_rep u) = source u.

Lemma target\_arrow\_rep : forall a r u,  
 is\_quotient\_arrow a r u -> target (arrow\_rep u) = target u.

Lemma arrow\_class\_arrow\_rep : forall a r u,  
 is\_quotient\_arrow a r u ->  
 arrow\_class r (arrow\_rep u) = u.

Lemma inc\_arrow\_facts : forall a r u y,  
 is\_quotient\_arrow a r u -> inc y (arrow u) ->  
 (mor a y & source y = source u & target y = target u).

Lemma mor\_arrow\_rep : forall a r u,  
 is\_quotient\_arrow a r u ->  
 mor a (arrow\_rep u).

Lemma related\_arrow\_rep : forall a r u v,  
 cat\_equiv\_rel a r -> mor a v -> u = arrow\_class r v ->  
 related r v (arrow\_rep u).

Lemma related\_arrow\_rep\_rw : forall a r u v,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u -> mor a v ->  
 related r v (arrow\_rep u) = (u = arrow\_class r v).

Definition quot\_id a r x :=  
 arrow\_class r (id a x).

Definition quot\_comp a r u v :=  
 arrow\_class r (comp a (arrow\_rep u) (arrow\_rep v)).

Lemma source\_quot\_id : forall a r x,  
 ob a x ->  
 source (quot\_id a r x) = x.

Lemma target\_quot\_id : forall a r x,  
 ob a x ->  
 target (quot\_id a r x) = x.

Lemma source\_quot\_comp : forall a r u v,  
 cat\_equiv\_rel a r ->  
 is\_quotient\_arrow a r u ->  
 is\_quotient\_arrow a r v -> source u = target v ->  
 source (quot\_comp a r u v) = source v.

Lemma target\_quot\_comp : forall a r u v,  
 cat\_equiv\_rel a r ->  
 is\_quotient\_arrow a r u ->  
 is\_quotient\_arrow a r v -> source u = target v ->  
 target (quot\_comp a r u v) = target u.

Lemma is\_quotient\_arrow\_quot\_id : forall a r x,  
 cat\_equiv\_rel a r -> ob a x ->  
 is\_quotient\_arrow a r (quot\_id a r x).

Lemma is\_quotient\_arrow\_quot\_comp : forall a r u v,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u ->  
 is\_quotient\_arrow a r v -> source u = target v ->  
 is\_quotient\_arrow a r (quot\_comp a r u v).

Lemma arrow\_class\_eq : forall a r u v,  
 cat\_equiv\_rel a r -> mor a u ->  
 (arrow\_class r u = arrow\_class r v) =  
 (related r u v).

Lemma arrow\_class\_id : forall a r x,  
 arrow\_class r (id a x) = quot\_id a r x.

Lemma quot\_comp\_arrow\_class : forall a r u v,  
 cat\_equiv\_rel a r -> mor a u -> mor a v -> source u = target v ->  
 quot\_comp a r (arrow\_class r u) (arrow\_class r v) =  
 arrow\_class r (comp a u v).

Lemma quot\_id\_left : forall a r a' r' x u,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u ->  
 x = target u -> a' = a -> r' = r ->  
 quot\_comp a r (quot\_id a' r' x) u = u.

Lemma quot\_id\_right : forall a r a' r' x u,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u ->  
 x = source u -> a' = a -> r' = r ->  
 quot\_comp a r u (quot\_id a' r' x) = u.

Lemma quot\_comp\_assoc : forall a r a' r' u v w,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u ->  
 is\_quotient\_arrow a r v ->  
 is\_quotient\_arrow a r w -> source u = target v ->  
 source v = target w -> a' = a -> r' = r ->  
 quot\_comp a r (quot\_comp a' r' u v) w =  
 quot\_comp a r u (quot\_comp a' r' v w).

Definition quotient\_cat a r := Category.Notations.create (objects a)  
 (quotient\_morphisms a r) (quot\_comp a r) (quot\_id a r) (structure a).

Lemma is\_ob\_quotient\_cat : forall a r x,  
 is\_ob (quotient\_cat a r) x = is\_ob a x.

Lemma is\_mor\_quotient\_cat : forall a r u,  
 cat\_equiv\_rel a r ->  
 is\_mor (quotient\_cat a r) u = is\_quotient\_arrow a r u.

Lemma comp\_quotient\_cat : forall a r u v,  
 cat\_equiv\_rel a r -> is\_quotient\_arrow a r u ->  
 is\_quotient\_arrow a r v ->  
 source u = target v ->  
 comp (quotient\_cat a r) u v = quot\_comp a r u v.

Lemma id\_quotient\_cat : forall a r x,  
 cat\_equiv\_rel a r -> ob a x ->  
 id (quotient\_cat a r) x = quot\_id a r x.

Lemma quotient\_cat\_axioms : forall a r,  
 cat\_equiv\_rel a r ->  
 Category.axioms (quotient\_cat a r).

Lemma ob\_quotient\_cat : forall a r x,  
 cat\_equiv\_rel a r ->  
 ob (quotient\_cat a r) x = ob a x.

Lemma mor\_quotient\_cat : forall a r u,  
 cat\_equiv\_rel a r ->  
 mor (quotient\_cat a r) u = is\_quotient\_arrow a r u.

Lemma mor\_quotient\_cat\_ex : forall a r u,

```

cat_equiv_rel a r ->
mor (quotient_cat a r) u =
(exists y, mor a y & u = arrow_class r y).

```

```

Lemma mor_quotient_cat_quot_id : forall a r x,
cat_equiv_rel a r -> ob a x ->
mor (quotient_cat a r) (quot_id a r x).

```

```

Lemma mor_quotient_cat_quot_comp : forall a r u v,
cat_equiv_rel a r ->
mor (quotient_cat a r) u ->
mor (quotient_cat a r) v ->
source u = target v ->
mor (quotient_cat a r) (quot_comp a r u v).

```

```

Lemma mor_quotient_cat_arrow_class : forall a r u,
cat_equiv_rel a r -> mor a u ->
mor (quotient_cat a r) (arrow_class r u).

```

End Quotient\_Category.

### 3.3. The module Quotient\_Functor.

```

Module Quotient_Functor.
Export Quotient_Category.

```

```

Definition qfunctor a b r (fo fm:E-> E) :=
Functor.create a (quotient_cat b r) (fun u => arrow_class r (fm u)).

```

```

Lemma source_qfunctor : forall a b r fo fm,
source (qfunctor a b r fo fm) = a.

```

```

Lemma target_qfunctor : forall a b r fo fm,
target (qfunctor a b r fo fm) = (quotient_cat b r).

```

```

Lemma fmor_qfunctor : forall a b r fo fm u,
mor a u ->
fmor (qfunctor a b r fo fm) u = arrow_class r (fm u).

```

```

Definition qfunctor_property a b r fo fm :=
Category.axioms a &
cat_equiv_rel b r &
(forall x, ob a x -> ob b (fo x)) &
(forall u, mor a u -> mor b (fm u)) &
(forall u, mor a u -> source (fm u) = fo (source u)) &
(forall u, mor a u -> target (fm u) = fo (target u)) &
(forall x, ob a x -> related r (fm (id a x)) (id b (fo x))) &

```

(forall u v, mor a u -> mor a v -> source u = target v ->  
related r (fm (comp a u v)) (comp b (fm u) (fm v))).

Lemma fob\_qfunctor : forall a b r fo fm x,  
qfunctor\_property a b r fo fm -> ob a x ->  
fob (qfunctor a b r fo fm) x = fo x.

Lemma ob\_fob\_qfunctor : forall a b r fo fm x,  
qfunctor\_property a b r fo fm -> ob a x ->  
ob (quotient\_cat b r) (fob (qfunctor a b r fo fm) x).

Lemma mor\_fmor\_qfunctor : forall a b r fo fm u,  
qfunctor\_property a b r fo fm -> mor a u ->  
mor (quotient\_cat b r) (fmor (qfunctor a b r fo fm) u).

Lemma source\_fmor\_qfunctor : forall a b r fo fm u,  
qfunctor\_property a b r fo fm -> mor a u ->  
source (fmor (qfunctor a b r fo fm) u) =  
fob (qfunctor a b r fo fm) (source u).

Lemma target\_fmor\_qfunctor : forall a b r fo fm u,  
qfunctor\_property a b r fo fm -> mor a u ->  
target (fmor (qfunctor a b r fo fm) u) =  
fob (qfunctor a b r fo fm) (target u).

Lemma fmor\_qfunctor\_id : forall a b r fo fm x,  
qfunctor\_property a b r fo fm -> ob a x ->  
fmor (qfunctor a b r fo fm) (id a x) =  
id (quotient\_cat b r) (fob (qfunctor a b r fo fm) x).

Lemma fmor\_qfunctor\_comp : forall a b r fo fm u v,  
qfunctor\_property a b r fo fm -> mor a u ->  
mor a v -> source u = target v ->  
fmor (qfunctor a b r fo fm) (comp a u v) =  
comp (quotient\_cat b r) (fmor (qfunctor a b r fo fm) u)  
(fmor (qfunctor a b r fo fm) v).

Lemma qfunctor\_axioms : forall a b r fo fm,  
qfunctor\_property a b r fo fm ->  
Functor.axioms (qfunctor a b r fo fm).

Lemma qfunctor\_extensionality : forall a b r fo fm a' b' r' fo' fm',  
a = a' -> b = b' -> r = r' ->  
qfunctor\_property a b r fo fm ->  
qfunctor\_property a' b' r' fo' fm' ->  
(forall u, mor a u -> related r (fm u) (fm' u)) ->  
qfunctor a b r fo fm = qfunctor a' b' r' fo' fm'.



```

Lemma eq_qfunctor : forall a b r f fo fm,
  Functor.axioms f -> source f = a ->
  target f = (quotient_cat b r) ->
  qfunctor_property a b r fo fm ->
  (forall u, mor a u -> related r (fm u) (arrow_rep (fmor f u))) ->
  f = qfunctor a b r fo fm.

```

```

Lemma qfunctor_property_fob_fmor : forall b r f,
  Functor.axioms f ->
  cat_equiv_rel b r -> target f = (quotient_cat b r) ->
  qfunctor_property (source f) b r
  (fob f) (fun u => arrow_rep (fmor f u)).

```

```

Lemma eq_qfunctor2 : forall b r f,
  Functor.axioms f ->
  cat_equiv_rel b r -> target f = (quotient_cat b r) ->
  f = qfunctor (source f) b r (fob f) (fun u => arrow_rep (fmor f u)).

```

```

Definition qprojection a r :=
  qfunctor a a r (fun (x:E) => x) (fun (u:E)=> u).

```

```

Lemma source_qprojection : forall a r,
  source (qprojection a r) = a.

```

```

Lemma target_qprojection : forall a r,
  target (qprojection a r) = (quotient_cat a r).

```

```

Lemma qprojection_property : forall a r,
  cat_equiv_rel a r ->
  qfunctor_property a a r (fun (x:E) => x) (fun (u:E)=> u).

```

```

Lemma fob_qprojection : forall a r x,
  cat_equiv_rel a r -> ob a x ->
  fob (qprojection a r) x = x.

```

```

Lemma fmor_qprojection : forall a r u,
  mor a u ->
  fmor (qprojection a r) u = (arrow_class r u).

```

```

Lemma fmor_qprojection_arrow_rep : forall a r u,
  is_quotient_arrow a r u ->
  fmor (qprojection a r) (arrow_rep u) = u.

```

```

Lemma qprojection_axioms : forall a r,
  cat_equiv_rel a r ->
  Functor.axioms (qprojection a r).

```

```

Definition qdotted r f :=
  Functor.create (quotient_cat (source f) r) (target f)
  (fun u => fmor f (arrow_rep u)).

```

```

Lemma source_qdotted : forall r f,
  source (qdotted r f) = quotient_cat (source f) r.

```

```

Lemma target_qdotted : forall r f,
  target (qdotted r f) = target f.

```

```

Lemma fmor_qdotted : forall r f u,
  cat_equiv_rel (source f) r
  -> is_quotient_arrow (source f) r u ->
  fmor (qdotted r f) u = fmor f (arrow_rep u).

```

```

Lemma fmor_qdotted_arrow_class : forall r f u,
  cat_equiv_rel (source f) r ->
  compatible r f -> mor (source f) u ->
  fmor (qdotted r f) (arrow_class r u) = fmor f u.

```

```

Lemma fob_qdotted : forall r f x,
  cat_equiv_rel (source f) r ->
  compatible r f -> ob (source f) x ->
  fob (qdotted r f) x = fob f x.

```

```

Lemma fmor_qdotted_quot_id : forall r f x,
  cat_equiv_rel (source f) r ->
  compatible r f -> ob (source f) x ->
  fmor (qdotted r f) (quot_id (source f) r x) = id (target f) (fob f x).

```

```

Lemma fmor_qdotted_quot_comp : forall r f u v,
  cat_equiv_rel (source f) r ->
  compatible r f -> is_quotient_arrow (source f) r u ->
  is_quotient_arrow (source f) r v -> source u = target v ->
  fmor (qdotted r f) (quot_comp (source f) r u v) =
  comp (target f) (fmor (qdotted r f) u) (fmor (qdotted r f) v).

```

```

Lemma qdotted_axioms : forall r f,
  cat_equiv_rel (source f) r ->
  compatible r f -> Functor.axioms (qdotted r f).

```

```

Lemma eq_fcompose_qdotted_qprojection : forall r f,
  cat_equiv_rel (source f) r ->
  compatible r f -> f =
  fcompose (qdotted r f) (qprojection (source f) r).

```

```

Lemma compatible_fcompose_qprojection : forall a r f,
  Functor.axioms f -> cat_equiv_rel a r ->
  source f = quotient_cat a r ->
  compatible r (fcompose f (qprojection a r)).

```

```

Lemma eq_qdotted : forall a r f,
  Functor.axioms f -> cat_equiv_rel a r ->
  source f = quotient_cat a r ->
  f = qdotted r (fcompose f (qprojection a r)).

```

```

End Quotient_Functor.

```

### 3.4. The module Ob\_Iso\_Functor.

```

Module Ob_Iso_Functor.
Export Functor_Cat.
Export Quotient_Functor.

```

```

Definition pull_morphism a f :=
  Functor.create (functor_cat (target f) a)
  (functor_cat (source f) a) (fun u => htrans_right u f).

```

```

Lemma source_pull_morphism : forall f a,
  source (pull_morphism a f) = (functor_cat (target f) a).

```

```

Lemma target_pull_morphism : forall f a,
  target (pull_morphism a f) = (functor_cat (source f) a).

```

```

Lemma fmor_pull_morphism : forall f a u,
  mor (functor_cat (target f) a) u ->
  fmor (pull_morphism a f) u = htrans_right u f.

```

```

Lemma fob_pull_morphism : forall f a y,
  Functor.axioms f -> Category.axioms a ->
  ob (functor_cat (target f) a) y ->
  fob (pull_morphism a f) y = fcompose y f.

```

```

Lemma pull_morphism_axioms : forall f a,
  Functor.axioms f -> Category.axioms a ->
  Functor.axioms (pull_morphism a f).

```

```

Definition faithful f :=
  Functor.axioms f &
  (forall u v, related (ker f) u v -> u = v).

```

```

Definition says f x :=

```

```

Functor.axioms f &
(exists y, (ob (source f) y & fob f y = x)).

```

```

Definition msays f u :=
Functor.axioms f &
(exists y, (mor (source f) y & fmor f y = u)).

```

```

Definition full f :=
Functor.axioms f &
(forall u, mor (target f) u ->
says f (source u) -> says f (target u) -> msays f u).

```

```

Definition ob_inj f :=
Functor.axioms f &
(forall x y, ob (source f) x -> ob (source f) y ->
fob f x = fob f y -> x = y).

```

```

Definition iso_to_full_subcategory f :=
faithful f & full f & ob_inj f.

```

```

Definition ob_image f :=
Z (objects (target f))
(fun x => (exists y, (ob (source f) y & fob f y = x))).

```

```

Definition ob_surj f :=
Functor.axioms f &
(forall x, ob (target f) x -> says f x).

```

```

Definition ob_iso f :=
ob_inj f & ob_surj f.

```

```

Lemma has_finverse_rw : forall f,
has_finverse f = (faithful f & full f & ob_iso f).

```

```

Definition is_full_subcategory a b :=
is_subcategory a b &
(forall u, mor b u -> ob a (source u) -> ob a (target u) ->
mor a u).

```

```

Definition full_subcategory (a:E) (obp : E -> Prop) :=
subcategory a obp (fun u => (obp (source u) & obp (target u))).

```

```

Lemma full_subcategory_property : forall a obp,
Category.axioms a -> subcategory_property a obp
(fun u => (obp (source u) & obp (target u))).

```

```

Lemma full_subcategory_axioms : forall a obp,

```

```

Category.axioms a ->
Category.axioms (full_subcategory a obp).

Lemma is_subcategory_full_subcategory : forall a obp,
Category.axioms a ->
is_subcategory (full_subcategory a obp) a.

Lemma ob_full_subcategory : forall a obp x,
axioms a ->
ob (full_subcategory a obp) x = (ob a x & obp x).

Lemma mor_full_subcategory : forall a obp u,
axioms a ->
mor (full_subcategory a obp) u = (mor a u & obp (source u)
& obp (target u)).

Lemma id_full_subcategory : forall a obp x,
axioms a ->
ob (full_subcategory a obp) x ->
id (full_subcategory a obp) x = id a x.

Lemma comp_full_subcategory : forall a obp u v,
axioms a ->
mor (full_subcategory a obp) u ->
mor (full_subcategory a obp) v ->
source u = target v ->
comp (full_subcategory a obp) u v = comp a u v.

Lemma is_full_subcategory_full_subcategory : forall a obp,
Category.axioms a ->
is_full_subcategory (full_subcategory a obp) a.

Definition subcategory_inclusion a b :=
Functor.create a b (fun u => u).

Lemma source_subcategory_inclusion : forall a b,
source (subcategory_inclusion a b) = a.

Lemma target_subcategory_inclusion : forall a b,
target (subcategory_inclusion a b) = b.

Lemma fmor_subcategory_inclusion : forall a b u,
is_subcategory a b -> mor a u ->
fmor (subcategory_inclusion a b) u = u.

Lemma fob_subcategory_inclusion : forall a b x,
is_subcategory a b -> ob a x ->

```

fob (subcategory\_inclusion a b) x = x.

Lemma subcategory\_inclusion\_axioms : forall a b,  
is\_subcategory a b ->  
Functor.axioms (subcategory\_inclusion a b).

Lemma faithful\_subcategory\_inclusion : forall a b,  
is\_subcategory a b ->  
faithful (subcategory\_inclusion a b).

Lemma ob\_inj\_subcategory\_inclusion : forall a b,  
is\_subcategory a b ->  
ob\_inj (subcategory\_inclusion a b).

Lemma full\_subcategory\_inclusion : forall a b,  
is\_full\_subcategory a b ->  
full (subcategory\_inclusion a b).

Lemma is\_full\_subcategory\_rw : forall a b,  
is\_full\_subcategory a b =  
(is\_subcategory a b & full (subcategory\_inclusion a b)).

Definition ob\_image\_fs f :=  
full\_subcategory (target f) (fun x => inc x (ob\_image f)).

Lemma is\_subcategory\_ob\_image\_fs : forall f,  
Functor.axioms f ->  
is\_subcategory (ob\_image\_fs f) (target f).

Lemma is\_full\_subcategory\_ob\_image\_fs : forall f,  
Functor.axioms f ->  
is\_full\_subcategory (ob\_image\_fs f) (target f).

Lemma inc\_ob\_image : forall f x,  
Functor.axioms f ->  
inc x (ob\_image f) = (exists y, (ob (source f) y & fob f y = x)).

Lemma ob\_ob\_image\_fs : forall f x,  
Functor.axioms f ->  
ob (ob\_image\_fs f) x = inc x (ob\_image f).

Lemma mor\_ob\_image\_fs : forall f u,  
Functor.axioms f ->  
mor (ob\_image\_fs f) u =  
(mor (target f) u & inc (source u) (ob\_image f) &  
inc (target u) (ob\_image f)).

```

Definition ob_image_factor f :=
  Functor.create (source f) (ob_image_fs f)
  (fmor f).

```

```

Lemma source_ob_image_factor : forall f,
  source (ob_image_factor f) = source f.

```

```

Lemma target_ob_image_factor : forall f,
  target (ob_image_factor f) = ob_image_fs f.

```

```

Lemma fmor_ob_image_factor : forall f u,
  Functor.axioms f -> mor (source f) u ->
  fmor (ob_image_factor f) u = fmor f u.

```

```

Lemma fob_ob_image_factor : forall f x,
  Functor.axioms f -> ob (source f) x ->
  fob (ob_image_factor f) x = fob f x.

```

```

Lemma ob_image_factor_axioms : forall f,
  Functor.axioms f ->
  Functor.axioms (ob_image_factor f).

```

```

Lemma ob_image_factorization : forall f,
  Functor.axioms f ->
  f = fcompose
  (subcategory_inclusion (ob_image_fs f) (target f))
  (ob_image_factor f).

```

```

Lemma ob_surj_ob_image_factor : forall f,
  Functor.axioms f ->
  ob_surj (ob_image_factor f).

```

```

Lemma ob_inj_ob_image_factor : forall f,
  Functor.axioms f ->
  ob_inj (ob_image_factor f) = ob_inj f.

```

```

Lemma ob_iso_ob_image_factor : forall f,
  Functor.axioms f ->
  ob_iso (ob_image_factor f) = ob_inj f.

```

```

Lemma full_ob_image_factor : forall f,
  Functor.axioms f ->
  full (ob_image_factor f) = full f.

```

```

Lemma faithful_ob_image_factor : forall f,
  Functor.axioms f ->
  faithful (ob_image_factor f) = faithful f.

```

Lemma iso\_to\_full\_subcategory\_rw : forall f,  
 iso\_to\_full\_subcategory f =  
 (Functor.axioms f & has\_finverse (ob\_image\_factor f)).

Lemma subcategory\_extensionality : forall a b c,  
 is\_subcategory a c -> is\_subcategory b c ->  
 (forall u, mor a u -> mor b u) ->  
 (forall u, mor b u -> mor a u) ->  
 a = b.

Lemma full\_subcategory\_extensionality : forall a b c,  
 is\_full\_subcategory a c -> is\_full\_subcategory b c ->  
 (forall x, ob a x -> ob b x) ->  
 (forall x, ob b x -> ob a x) -> a = b.

Lemma iso\_to\_full\_subcategory\_interp : forall f,  
 iso\_to\_full\_subcategory f =  
 (exists g, (is\_full\_subcategory (target g) (target f) &  
 has\_finverse g &  
 f = fcompose (subcategory\_inclusion (target g) (target f)) g)).

Definition mor\_image f :=  
 Z (morphisms (target f)) (msays f).

Lemma sub\_mor\_image : forall f,  
 sub (mor\_image f) (morphisms (target f)).

Lemma inc\_mor\_image : forall f u,  
 Functor.axioms f ->  
 inc u (mor\_image f) = (exists v, (mor (source f) v &  
 fmor f v = u)).

Definition add\_inverses a s :=  
 Z (morphisms a)  
 (fun y => inc y s \ / (invertible a y & inc (inverse a y) s)).

Lemma sub\_add\_inverses\_refl : forall a s,  
 sub s (morphisms a) -> sub s (add\_inverses a s).

Lemma sub\_add\_inverses\_morphisms : forall a s,  
 sub (add\_inverses a s) (morphisms a).

Lemma inc\_add\_inverses : forall a s y,  
 Category.axioms a -> sub s (morphisms a) ->  
 inc y (add\_inverses a s) =  
 (mor a y & (inc y s \ / (invertible a y & inc (inverse a y) s))).



Definition inverse\_closed a b :=  
 is\_subcategory a b &  
 (forall u, mor a u -> invertible b u -> mor a (inverse b u)).

Lemma sub\_add\_inverses : forall a b s,  
 inverse\_closed a b -> sub s (morphisms a) ->  
 sub (add\_inverses b s) (morphisms a).

Definition generates a s :=  
 Category.axioms a & sub s (morphisms a) &  
 (forall b, is\_subcategory b a -> sub s (morphisms b) -> b = a).

Lemma faithful\_pull\_morphism\_criterion : forall f a,  
 Category.axioms a -> ob\_surj f ->  
 faithful (pull\_morphism a f).

Definition globular t :=  
 (Nat\_Trans.like t) &  
 (Category.axioms (osource t)) &  
 (Category.axioms (otarget t)) &  
 (Functor.axioms (source t)) &  
 (Functor.axioms (target t)) &  
 (source (target t)) = (osource t) &  
 (target (source t)) = (otarget t).

Definition natural\_ob t x :=  
 ob (osource t) x &  
 mor (otarget t) (ntrans t x) &  
 source (ntrans t x) = fob (source t) x &  
 target (ntrans t x) = fob (target t) x.

Definition natural\_mor t u :=  
 mor (osource t) u &  
 natural\_ob t (source u) &  
 natural\_ob t (target u) &  
 comp (otarget t) (ntrans t (target u)) (fmor (source t) u)  
 = comp (otarget t) (fmor (target t) u) (ntrans t (source u)).

Lemma nat\_trans\_axioms\_rw : forall t,  
 Nat\_Trans.axioms t =  
 (globular t &  
 (forall x, ob (osource t) x -> natural\_ob t x) &  
 (forall u, mor (osource t) u -> natural\_mor t u)).

Lemma natural\_mor\_comp : forall t u v,  
 globular t -> natural\_mor t u -> natural\_mor t v ->

source u = target v -> natural\_mor t (comp (osource t) u v).

Lemma natural\_mor\_id : forall t x,  
 globular t -> natural\_ob t x ->  
 natural\_mor t (id (osource t) x).

Lemma natural\_ob\_source : forall t u,  
 globular t -> natural\_mor t u ->  
 natural\_ob t (source u).

Lemma natural\_ob\_target : forall t u,  
 globular t -> natural\_mor t u ->  
 natural\_ob t (target u).

Definition naturality\_subcategory t :=  
 subcategory (osource t) (natural\_ob t) (natural\_mor t).

Lemma is\_subcategory\_naturality\_subcategory : forall t,  
 globular t -> is\_subcategory (naturality\_subcategory t) (osource t).

Lemma ob\_naturality\_subcategory : forall t x,  
 globular t -> ob (naturality\_subcategory t) x = natural\_ob t x.

Lemma mor\_naturality\_subcategory : forall t u,  
 globular t -> mor (naturality\_subcategory t) u = natural\_mor t u.

Lemma invertible\_left\_multiply : forall a u v w,  
 invertible a u -> mor a v -> mor a w ->  
 source u = target v -> source u = target w ->  
 comp a u v = comp a u w -> v = w.

Lemma natural\_mor\_inverse : forall t u,  
 globular t -> invertible (osource t) u ->  
 natural\_mor t u -> natural\_mor t (inverse (osource t) u).

Lemma inverse\_closed\_naturality\_subcategory : forall t,  
 globular t -> inverse\_closed (naturality\_subcategory t) (osource t).

Lemma sub\_add\_inverses\_naturality\_subcategory : forall s t,  
 globular t -> sub s (morphisms (naturality\_subcategory t)) ->  
 sub (add\_inverses (osource t) s)  
 (morphisms (naturality\_subcategory t)).

Definition equalizer\_subcategory f g :=  
 subcategory (source f) (fun x => (fob f x = fob g x))  
 (fun u => (fmor f u = fmor g u)).

```

Lemma equalizer_subcategory_property : forall f g,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  subcategory_property (source f)
  (fun x => (fob f x = fob g x)) (fun u => (fmor f u = fmor g u)).

```

```

Lemma is_subcategory_equalizer_subcategory : forall f g,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  is_subcategory (equalizer_subcategory f g) (source f).

```

```

Lemma ob_equalizer_subcategory : forall f g x,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  ob (equalizer_subcategory f g) x =
  (ob (source f) x & fob f x = fob g x).

```

```

Lemma mor_equalizer_subcategory : forall f g u,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  mor (equalizer_subcategory f g) u =
  (mor (source f) u & fmor f u = fmor g u).

```

```

Lemma mor_equ_subcat_inverse : forall f g u,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  invertible (source f) u ->
  mor (equalizer_subcategory f g) u ->
  mor (equalizer_subcategory f g) (inverse (source f) u).

```

```

Lemma inverse_closed_equalizer_subcategory : forall f g,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  inverse_closed (equalizer_subcategory f g) (source f).

```

```

Lemma mor_equ_subcat_fmor : forall f g h u,
  Functor.axioms f -> Functor.axioms g -> Functor.axioms h ->
  source g = target f -> source h = target f ->
  fcompose g f = fcompose h f ->
  mor (source f) u ->
  mor (equalizer_subcategory g h) (fmor f u).

```

```

Lemma sub_mor_image_equalizer_subcategory : forall f g h,
  Functor.axioms f -> Functor.axioms g -> Functor.axioms h ->
  source g = target f -> source h = target f ->
  fcompose g f = fcompose h f ->
  sub (mor_image f) (morphisms (equalizer_subcategory g h)).

```

```

Lemma sub_add_inverses_equalizer_subcategory : forall f g h,
  Functor.axioms f -> Functor.axioms g -> Functor.axioms h ->
  source g = target f -> source h = target f ->
  fcompose g f = fcompose h f ->
  sub (add_inverses (target f) (mor_image f))
  (morphisms (equalizer_subcategory g h)).

```

```

Lemma equalizer_subcategory_extensionality_criterion :
  forall f g,
  Functor.axioms f -> Functor.axioms g ->
  source f = source g -> target f = target g ->
  equalizer_subcategory f g = source f ->
  f = g.

```

```

Lemma fcompose_eq_shows_eq : forall f g h,
  Functor.axioms f -> Functor.axioms g -> Functor.axioms h ->
  source g = target f -> source h = target f ->
  fcompose g f = fcompose h f ->
  generates (target f) (add_inverses (target f) (mor_image f)) ->
  g = h.

```

```

Lemma ob_inj_pull_morphism_criterion : forall f a,
  Category.axioms a ->
  ob_iso f ->
  generates (target f) (add_inverses (target f) (mor_image f)) ->
  ob_inj (pull_morphism a f).

```

```

Definition full_pull_situation f g h u :=
  Functor.axioms f &
  Functor.axioms g &
  Functor.axioms h &
  Nat_Trans.axioms u &
  source g = target f &
  source h = target f &
  fcompose g f = source u &
  fcompose h f = target u &
  ob_iso f &
  generates (target f) (add_inverses (target f) (mor_image f)).

```

```

Lemma full_pull_additional_facts : forall f g h u,
  full_pull_situation f g h u ->
  (target g = otarget u &
  target h = otarget u &
  source f = osource u &
  source g = source h &
  target g = target h &

```

osource u = source f).

Definition ob\_inverse\_pr f x y :=  
ob (source f) y & fob f y = x.

Definition ob\_inverse f x :=  
choose (ob\_inverse\_pr f x).

Lemma exists\_ob\_inverse\_pr : forall f x,  
ob\_iso f -> ob (target f) x ->  
exists y, ob\_inverse\_pr f x y.

Lemma fob\_ob\_inverse : forall f x,  
ob\_iso f -> ob (target f) x ->  
fob f (ob\_inverse f x) = x.

Lemma ob\_ob\_inverse : forall f x,  
ob\_iso f -> ob (target f) x ->  
ob (source f) (ob\_inverse f x).

Lemma ob\_inverse\_fob : forall f x,  
ob\_iso f -> ob (source f) x ->  
ob\_inverse f (fob f x) = x.

Definition ntdotted f g h u :=  
Nat\_Trans.create g h (fun x => ntrans u (ob\_inverse f x)).

Lemma source\_ntdotted : forall f g h u,  
source (ntdotted f g h u) = g.

Lemma target\_ntdotted : forall f g h u,  
target (ntdotted f g h u) = h.

Lemma osource\_ntdotted : forall f g h u,  
osource (ntdotted f g h u) = source g.

Lemma otarget\_ntdotted : forall f g h u,  
otarget (ntdotted f g h u) = target h.

Lemma globular\_ntdotted : forall f g h u,  
full\_pull\_situation f g h u ->  
globular (ntdotted f g h u).

Lemma ntrans\_ntdotted : forall f g h u x,  
full\_pull\_situation f g h u ->  
ob (target f) x ->  
ntrans (ntdotted f g h u) x =

`ntrans u (ob_inverse f x).`

Lemma `ntrans_ntdotted_fob` : forall f g h u x,  
`full_pull_situation f g h u ->`  
`ob (source f) x ->`  
`ntrans (ntdotted f g h u) (fob f x)`  
`= ntrans u x.`

Lemma `natural_ob_ntdotted` : forall f g h u x,  
`full_pull_situation f g h u ->`  
`ob (target f) x ->`  
`natural_ob (ntdotted f g h u) x.`

Lemma `natural_mor_ntdotted_fmor` : forall f g h u y,  
`full_pull_situation f g h u ->`  
`mor (source f) y ->`  
`natural_mor (ntdotted f g h u) (fmor f y).`

Lemma `sub_mor_image_naturality_subcategory` : forall f g h u,  
`full_pull_situation f g h u ->`  
`sub (mor_image f)`  
`(morphisms (naturality_subcategory (ntdotted f g h u))).`

Lemma `sub_add_inverses_naturality_subcategory_ntdotted` :  
forall f g h u,  
`full_pull_situation f g h u ->`  
`sub (add_inverses (target f) (mor_image f))`  
`(morphisms (naturality_subcategory (ntdotted f g h u))).`

Lemma `naturality_subcategory_ntdotted_all` : forall f g h u,  
`full_pull_situation f g h u ->`  
`naturality_subcategory (ntdotted f g h u) = target f.`

Lemma `ntdotted_axioms` : forall f g h u,  
`full_pull_situation f g h u ->`  
`Nat_Trans.axioms (ntdotted f g h u).`

Lemma `htrans_right_ntdotted` : forall f g h u,  
`full_pull_situation f g h u ->`  
`htrans_right (ntdotted f g h u) f = u.`

Lemma `full_pull_morphism_criterion` : forall f a,  
`Category.axioms a ->`  
`ob_iso f ->`  
`generates (target f) (add_inverses (target f) (mor_image f)) ->`  
`full (pull_morphism a f).`

```

Lemma iso_to_full_subcategory_pull_morphism_criterion : forall f a,
Category.axioms a ->
ob_iso f ->
generates (target f) (add_inverses (target f) (mor_image f)) ->
iso_to_full_subcategory (pull_morphism a f).

```

```

End Ob_Iso_Functor.

```

### 3.5. The module Associating\_Quotient.

```

Module Associating_Quotient.
Export Quotient_Category.
Export Quotient_Functor.

```

```

Definition rqcat a :=
Category.Notations.like a &
(forall u, is_mor a u -> Arrow.like u) &
(forall x, is_ob a x -> is_mor a (id a x)) &
(forall x, is_ob a x -> source (id a x) = x) &
(forall x, is_ob a x -> target (id a x) = x) &
(forall u, is_mor a u -> is_ob a (source u)) &
(forall u, is_mor a u -> is_ob a (target u)) &
(forall u v, is_mor a u -> is_mor a v -> source u = target v ->
is_mor a (comp a u v)) &
(forall u v, is_mor a u -> is_mor a v -> source u = target v ->
source (comp a u v) = source v) &
(forall u v, is_mor a u -> is_mor a v -> source u = target v ->
target (comp a u v) = target u).

```

```

Lemma rq_is_ob_source : forall a u,
rqcat a -> is_mor a u -> is_ob a (source u).

```

```

Lemma rq_is_ob_target : forall a u,
rqcat a -> is_mor a u -> is_ob a (target u).

```

```

Lemma rq_is_mor_id : forall a x,
rqcat a -> is_ob a x -> is_mor a (id a x).

```

```

Lemma rq_source_id : forall a x,
rqcat a -> is_ob a x -> source (id a x) = x.

```

```

Lemma rq_target_id : forall a x,
rqcat a -> is_ob a x -> target (id a x) = x.

```

```

Lemma rq_is_mor_comp : forall a u v,

```

```
rqcat a -> is_mor a u -> is_mor a v -> source u = target v ->
is_mor a (comp a u v).
```

```
Lemma rq_source_comp : forall a u v,
rqcat a -> is_mor a u -> is_mor a v -> source u = target v ->
source (comp a u v) = source v.
```

```
Lemma rq_target_comp : forall a u v,
rqcat a -> is_mor a u -> is_mor a v -> source u = target v ->
target (comp a u v) = target u.
```

```
Definition left_id_ok a :=
(forall u, is_mor a u -> comp a (id a (target u)) u = u).
```

```
Definition right_id_ok a :=
(forall u, is_mor a u -> comp a u (id a (source u)) = u).
```

```
Definition assoc_ok a :=
(forall u v w, is_mor a u -> is_mor a v -> is_mor a w ->
source u = target v -> source v = target w ->
comp a (comp a u v) w = comp a u (comp a v w)).
```

```
Lemma cat_axioms_rw_rq : forall a,
Category.axioms a =
(rqcat a & left_id_ok a & right_id_ok a & assoc_ok a).
```

```
Definition rqcat_equiv_rel a r :=
rqcat a &
is_equivalence_relation r &
(forall x y, related r x y -> is_mor a x) &
(forall x y, related r x y -> is_mor a y) &
(forall x y, related r x y -> source x = source y) &
(forall x y, related r x y -> target x = target y) &
(forall u, is_mor a u -> related r u u) &
(forall x y u v, related r x y -> related r u v ->
source x = target u ->
related r (comp a x u) (comp a y v)) &
(forall u, is_mor a u ->
related r (comp a (id a (target u)) u) u) &
(forall u, is_mor a u ->
related r (comp a u (id a (source u))) u) &
(forall u v w, is_mor a u -> is_mor a v -> is_mor a w ->
source u = target v -> source v = target w ->
related r (comp a (comp a u v) w) (comp a u (comp a v w))).
```

```
Lemma rq_inc_arrow_arrow_class : forall a r u v,
rqcat_equiv_rel a r -> is_mor a v ->
```



`inc v (arrow (arrow_class r u)) = related r u v.`

`Definition rq_quotient_arrow a r u :=  
 rqcat_equiv_rel a r &  
 (exists y, is_mor a y & u = arrow_class r y).`

`Lemma rq_quotient_arrow_arrow_class : forall a r u,  
 rqcat_equiv_rel a r -> is_mor a u ->  
 rq_quotient_arrow a r (arrow_class r u).`

`Lemma rq_inc_quotient_morphisms : forall a r u,  
 rqcat_equiv_rel a r ->  
 inc u (quotient_morphisms a r) = rq_quotient_arrow a r u.`

`Lemma rq_related_arrow_class_eq : forall a r u v,  
 rqcat_equiv_rel a r ->  
 is_mor a u ->  
 related r u v =  
 (arrow_class r u = arrow_class r v).`

`Lemma rq_inc_arrow_class_refl : forall a r u,  
 rqcat_equiv_rel a r -> is_mor a u -> inc u (arrow (arrow_class r u)).`

`Lemma rq_nonempty_arrow : forall a r u,  
 rq_quotient_arrow a r u ->  
 nonempty (arrow u).`

`Lemma rq_inc_arrow_rep_arrow : forall a r u,  
 rq_quotient_arrow a r u ->  
 inc (arrow_rep u) (arrow u).`

`Lemma rq_related_arrow_rep_arrow_class : forall a r u,  
 rqcat_equiv_rel a r -> is_mor a u ->  
 related r u (arrow_rep (arrow_class r u)).`

`Lemma rq_source_arrow_rep : forall a r u,  
 rq_quotient_arrow a r u -> source (arrow_rep u) = source u.`

`Lemma rq_target_arrow_rep : forall a r u,  
 rq_quotient_arrow a r u -> target (arrow_rep u) = target u.`

`Lemma rq_arrow_class_arrow_rep : forall a r u,  
 rq_quotient_arrow a r u ->  
 arrow_class r (arrow_rep u) = u.`

`Lemma rq_inc_arrow_facts : forall a r u y,  
 rq_quotient_arrow a r u -> inc y (arrow u) ->`

(is\_mor a y & source y = source u & target y = target u).

Lemma rq\_mor\_arrow\_rep : forall a r u,  
 rq\_quotient\_arrow a r u ->  
 is\_mor a (arrow\_rep u).

Lemma rq\_related\_arrow\_rep : forall a r u v,  
 rqcat\_equiv\_rel a r -> is\_mor a v -> u = arrow\_class r v ->  
 related r v (arrow\_rep u).

Lemma rq\_related\_arrow\_rep\_rw : forall a r u v,  
 rqcat\_equiv\_rel a r -> rq\_quotient\_arrow a r u -> is\_mor a v ->  
 related r v (arrow\_rep u) = (u = arrow\_class r v).

Lemma rq\_source\_quot\_comp : forall a r u v,  
 rqcat\_equiv\_rel a r ->  
 rq\_quotient\_arrow a r u ->  
 rq\_quotient\_arrow a r v -> source u = target v ->  
 source (quot\_comp a r u v) = source v.

Lemma rq\_target\_quot\_comp : forall a r u v,  
 rqcat\_equiv\_rel a r ->  
 rq\_quotient\_arrow a r u ->  
 rq\_quotient\_arrow a r v -> source u = target v ->  
 target (quot\_comp a r u v) = target u.

Lemma rq\_quotient\_arrow\_quot\_id : forall a r x,  
 rqcat\_equiv\_rel a r -> is\_ob a x ->  
 rq\_quotient\_arrow a r (quot\_id a r x).

Lemma rq\_quotient\_arrow\_quot\_comp : forall a r u v,  
 rqcat\_equiv\_rel a r -> rq\_quotient\_arrow a r u ->  
 rq\_quotient\_arrow a r v  
 -> source u = target v ->  
 rq\_quotient\_arrow a r (quot\_comp a r u v).

Lemma rq\_arrow\_class\_eq : forall a r u v,  
 rqcat\_equiv\_rel a r -> is\_mor a u ->  
 (arrow\_class r u = arrow\_class r v) =  
 (related r u v).

Lemma rq\_related\_comp : forall a r x y u v,  
 rqcat\_equiv\_rel a r ->  
 related r x y -> related r u v -> source x = target u ->  
 related r (comp a x u) (comp a y v).

Lemma `rq_quot_comp_arrow_class` : forall a r u v,  
`rqcat_equiv_rel a r -> is_mor a u -> is_mor a v ->`  
`source u = target v ->`  
`quot_comp a r (arrow_class r u) (arrow_class r v) =`  
`arrow_class r (comp a u v).`

Lemma `rq_left_id_related` : forall a r u,  
`rqcat_equiv_rel a r ->`  
`is_mor a u ->`  
`related r (comp a (id a (target u )) u) u.`

Lemma `rq_right_id_related` : forall a r u,  
`rqcat_equiv_rel a r ->`  
`is_mor a u ->`  
`related r (comp a u (id a (source u ))) u.`

Lemma `rq_assoc_related` : forall a r u v w,  
`rqcat_equiv_rel a r ->`  
`is_mor a u -> is_mor a v -> is_mor a w ->`  
`source u = target v -> source v = target w ->`  
`related r (comp a (comp a u v) w) (comp a u (comp a v w)).`

Lemma `rq_quot_id_left` : forall a r a' r' x u,  
`rqcat_equiv_rel a r -> rq_quotient_arrow a r u ->`  
`x = target u -> a' = a -> r' = r ->`  
`quot_comp a r (quot_id a' r' x) u = u.`

Lemma `rq_quot_id_right` : forall a r a' r' x u,  
`rqcat_equiv_rel a r -> rq_quotient_arrow a r u ->`  
`x = source u -> a' = a -> r' = r ->`  
`quot_comp a r u (quot_id a' r' x) = u.`

Lemma `rq_quot_comp_assoc` : forall a r a' r' u v w,  
`rqcat_equiv_rel a r -> rq_quotient_arrow a r u ->`  
`rq_quotient_arrow a r v`  
`-> rq_quotient_arrow a r w -> source u = target v ->`  
`source v = target w -> a' = a -> r' = r ->`  
`quot_comp a r (quot_comp a' r' u v) w =`  
`quot_comp a r u (quot_comp a' r' v w).`

Lemma `rq_is_mor_quotient_cat` : forall a r u,  
`rqcat_equiv_rel a r ->`  
`is_mor (quotient_cat a r) u = rq_quotient_arrow a r u.`

Lemma `rq_comp_quotient_cat` : forall a r u v,  
`rqcat_equiv_rel a r -> rq_quotient_arrow a r u ->`  
`rq_quotient_arrow a r v ->`

```
source u = target v ->
comp (quotient_cat a r) u v = quot_comp a r u v.
```

```
Lemma rq_id_quotient_cat : forall a r x,
rqcat_equiv_rel a r -> is_ob a x ->
id (quotient_cat a r) x = quot_id a r x.
```

```
Lemma rq_source_quot_id : forall a r x,
rqcat_equiv_rel a r -> is_ob a x ->
source (quot_id a r x) = x.
```

```
Lemma rq_target_quot_id : forall a r x,
rqcat_equiv_rel a r -> is_ob a x ->
target (quot_id a r x) = x.
```

```
Lemma rqcat_quotient_cat : forall a r,
rqcat_equiv_rel a r ->
rqcat (quotient_cat a r).
```

```
Lemma rq_quotient_cat_axioms : forall a r,
rqcat_equiv_rel a r -> Category.axioms (quotient_cat a r).
```

```
Lemma rq_ob_quotient_cat : forall a r x,
rqcat_equiv_rel a r ->
ob (quotient_cat a r) x = is_ob a x.
```

```
Lemma rq_mor_quotient_cat : forall a r u,
rqcat_equiv_rel a r ->
mor (quotient_cat a r) u = rq_quotient_arrow a r u.
```

```
Lemma rq_mor_quotient_cat_ex : forall a r u,
rqcat_equiv_rel a r ->
mor (quotient_cat a r) u =
(exists y, is_mor a y & u = arrow_class r y).
```

```
Lemma rq_mor_quotient_cat_quot_id : forall a r x,
rqcat_equiv_rel a r -> is_ob a x ->
mor (quotient_cat a r) (quot_id a r x).
```

```
Lemma rq_mor_quotient_cat_quot_comp : forall a r u v,
rqcat_equiv_rel a r ->
mor (quotient_cat a r) u ->
mor (quotient_cat a r) v ->
source u = target v ->
mor (quotient_cat a r) (quot_comp a r u v).
```

```
Lemma rq_mor_quotient_cat_arrow_class : forall a r u,
```

```
rqcat_equiv_rel a r -> is_mor a u ->
mor (quotient_cat a r) (arrow_class r u).
```

```
Definition rqfunctor_property a b r fo fm :=
Category.axioms a &
rqcat_equiv_rel b r &
(forall x, ob a x -> is_ob b (fo x)) &
(forall u, mor a u -> is_mor b (fm u)) &
(forall u, mor a u -> source (fm u) = fo (source u)) &
(forall u, mor a u -> target (fm u) = fo (target u)) &
(forall x, ob a x -> related r (fm (id a x)) (id b (fo x))) &
(forall u v, mor a u -> mor a v -> source u = target v ->
related r (fm (comp a u v)) (comp b (fm u) (fm v))).
```

```
Lemma rq_fob_qfunctor : forall a b r fo fm x,
rqfunctor_property a b r fo fm -> ob a x ->
fob (qfunctor a b r fo fm) x = fo x.
```

```
Lemma rq_ob_fob_qfunctor : forall a b r fo fm x,
rqfunctor_property a b r fo fm -> ob a x ->
ob (quotient_cat b r) (fob (qfunctor a b r fo fm) x).
```

```
Lemma rq_mor_fmor_qfunctor : forall a b r fo fm u,
rqfunctor_property a b r fo fm -> mor a u ->
mor (quotient_cat b r) (fmor (qfunctor a b r fo fm) u).
```

```
Lemma rq_source_fmor_qfunctor : forall a b r fo fm u,
rqfunctor_property a b r fo fm -> mor a u ->
source (fmor (qfunctor a b r fo fm) u) =
fob (qfunctor a b r fo fm) (source u).
```

```
Lemma rq_target_fmor_qfunctor : forall a b r fo fm u,
rqfunctor_property a b r fo fm -> mor a u ->
target (fmor (qfunctor a b r fo fm) u) =
fob (qfunctor a b r fo fm) (target u).
```

```
Lemma rq_fmor_qfunctor_id : forall a b r fo fm x,
rqfunctor_property a b r fo fm -> ob a x ->
fmor (qfunctor a b r fo fm) (id a x) =
id (quotient_cat b r) (fob (qfunctor a b r fo fm) x).
```

```
Lemma rq_fmor_qfunctor_comp : forall a b r fo fm u v,
rqfunctor_property a b r fo fm -> mor a u ->
mor a v -> source u = target v ->
fmor (qfunctor a b r fo fm) (comp a u v) =
comp (quotient_cat b r) (fmor (qfunctor a b r fo fm) u)
```

(fmor (qfunctor a b r fo fm) v).

Lemma qfunctor\_axioms : forall a b r fo fm,  
rqfunctor\_property a b r fo fm ->  
Functor.axioms (qfunctor a b r fo fm).

End Associating\_Quotient.

4. THE FILE `gzdef.v`

```
Require Export freecat.
Require Export qcat.
```

4.1. The module `GZ_Def`.

```
Module GZ_Def.
Export Free_Category.
Export Quotient_Functor.
```

```
Definition Forward := R (f_(o_(r_ DOT))).
Definition Backward := R (b_(k_(d_ DOT))).
```

```
Definition forward_arrow u :=
Arrow.create (source u) (target u) (pair Forward u).
```

```
Definition backward_arrow u :=
Arrow.create (target u) (source u) (pair Backward u).
```

```
Lemma source_forward_arrow : forall u,
source (forward_arrow u) = source u.
```

```
Lemma target_forward_arrow : forall u,
target (forward_arrow u) = target u.
```

```
Lemma source_backward_arrow : forall u,
source (backward_arrow u) = target u.
```

```
Lemma target_backward_arrow : forall u,
target (backward_arrow u) = source u.
```

```
Definition original_arrow u := pr2 (arrow u).
```

```
Lemma original_arrow_forward_arrow : forall u,
original_arrow (forward_arrow u) = u.
```

```
Lemma original_arrow_backward_arrow : forall u,
original_arrow (backward_arrow u) = u.
```

```
Definition direction u := pr1 (arrow u).
```

```
Lemma direction_forward_arrow : forall u,
direction (forward_arrow u) = Forward.
```

```
Lemma direction_backward_arrow : forall u,
direction (backward_arrow u) = Backward.
```

Definition localizing\_system a s :=  
 Category.axioms a &  
 (forall u, inc u s -> mor a u).

Definition loc\_edges a s :=  
 union2 (Image.create (morphisms a) forward\_arrow)  
 (Image.create s backward\_arrow).

Lemma inc\_loc\_edges : forall a s u,  
 localizing\_system a s ->  
 inc u (loc\_edges a s) =  
 ((exists y, (mor a y & u = forward\_arrow y)) \/  
 (exists y, (mor a y & inc y s & u = backward\_arrow y))).

Definition gz\_graph a s := Graph.create (objects a) (loc\_edges a s).

Lemma inc\_vertices\_gz\_graph : forall a s x,  
 localizing\_system a s ->  
 inc x (vertices (gz\_graph a s)) = ob a x.

Lemma inc\_edges\_gz\_graph : forall a s u,  
 localizing\_system a s ->  
 inc u (edges (gz\_graph a s)) = inc u (loc\_edges a s).

Lemma gz\_graph\_axioms : forall a s,  
 localizing\_system a s -> Graph.axioms (gz\_graph a s).

Definition gz\_freecat a s := freecat (gz\_graph a s).

Lemma gz\_freecat\_axioms : forall a s,  
 localizing\_system a s -> Category.axioms (gz\_freecat a s).

Lemma ob\_gz\_freecat : forall a s x,  
 localizing\_system a s ->  
 ob (gz\_freecat a s) x = ob a x.

Lemma mor\_gz\_freecat : forall a s u,  
 localizing\_system a s ->  
 mor (gz\_freecat a s) u = mor\_freecat (gz\_graph a s) u.

Lemma comp\_gz\_freecat : forall a s u v,  
 localizing\_system a s ->  
 mor (gz\_freecat a s) u ->  
 mor (gz\_freecat a s) v ->  
 source u = target v ->  
 comp (gz\_freecat a s) u v = freecat\_comp u v.



Lemma id\_gz\_freecat : forall a s x,  
 localizing\_system a s ->  
 ob a x -> id (gz\_freecat a s) x = freecat\_id x.

Definition forward\_edge u :=  
 freecat\_edge (forward\_arrow u).

Definition backward\_edge u :=  
 freecat\_edge (backward\_arrow u).

Lemma source\_forward\_edge : forall u,  
 source (forward\_edge u) = source u.

Lemma target\_forward\_edge : forall u,  
 target (forward\_edge u) = target u.

Lemma source\_backward\_edge : forall u,  
 source (backward\_edge u) = target u.

Lemma target\_backward\_edge : forall u,  
 target (backward\_edge u) = source u.

Lemma inc\_forward\_arrow\_loc\_edges : forall a s u,  
 localizing\_system a s ->mor a u ->  
 inc (forward\_arrow u) (loc\_edges a s).

Lemma inc\_backward\_arrow\_loc\_edges : forall a s q,  
 localizing\_system a s ->inc q s ->  
 inc (backward\_arrow q) (loc\_edges a s).

Lemma mor\_forward\_edge : forall a s u,  
 localizing\_system a s ->  
 mor a u -> mor (gz\_freecat a s) (forward\_edge u).

Lemma mor\_backward\_edge : forall a s q,  
 localizing\_system a s ->  
 inc q s -> mor (gz\_freecat a s) (backward\_edge q).

Definition gz\_rel a s :=  
 Z (coarse (gz\_freecat a s))  
 (fun z =>  
 ((exists x, (ob a x &  
 z = pair (forward\_edge (id a x)) (freecat\_id x))) \/  
 (exists q, (inc q s & z = pair  
 (freecat\_comp (forward\_edge q) (backward\_edge q))  
 (freecat\_id (target q)))) \/

```
(exists q, (inc q s & z = pair
(freecat_comp (backward_edge q) (forward_edge q))
(freecat_id (source q)))) \ /
(exists u, exists v, (mor a u & mor a v & source u = target v &
z = pair (freecat_comp (forward_edge u) (forward_edge v))
(forward_edge (comp a u v))))).
```

```
Lemma inc_coarse : forall a z,
Category.axioms a ->
inc z (coarse a) =
(exists u, exists v, (mor a u & mor a v & source u = source v
& target u = target v & z=pair u v)).
```

```
Lemma inc_gz_rel : forall a s z,
localizing_system a s ->
inc z (gz_rel a s) =
((exists x, (ob a x &
z = pair (forward_edge (id a x)) (freecat_id x)))\ /
(exists q, (inc q s & z = pair
(freecat_comp (forward_edge q) (backward_edge q))
(freecat_id (target q))))\ /
(exists q, (inc q s & z = pair
(freecat_comp (backward_edge q) (forward_edge q))
(freecat_id (source q))))\ /
(exists u, exists v, (mor a u & mor a v & source u = target v &
z = pair (freecat_comp (forward_edge u) (forward_edge v))
(forward_edge (comp a u v))))).
```

```
Lemma related_gz_rel : forall a s e f,
localizing_system a s ->
related (gz_rel a s) e f =
((exists x, (ob a x &
e = (forward_edge (id a x)) & f = (freecat_id x))) \ /
(exists q, (inc q s & e =
(freecat_comp (forward_edge q) (backward_edge q))
& f = (freecat_id (target q)))) \ /
(exists q, (inc q s & e =
(freecat_comp (backward_edge q) (forward_edge q))
& f = (freecat_id (source q)))) \ /
(exists u, exists v, (mor a u & mor a v & source u = target v &
e = (freecat_comp (forward_edge u) (forward_edge v))
& f = (forward_edge (comp a u v))))).
```

```
Lemma sub_gz_rel_coarse : forall a s,
sub (gz_rel a s) (coarse (gz_freecat a s)).
```

```
Lemma cat_rel_gz_rel : forall a s,
```

```
localizing_system a s ->
cat_rel (gz_freecat a s) (gz_rel a s).
```

```
Definition gz_cer a s := cer (gz_freecat a s) (gz_rel a s).
```

```
Lemma cat_equiv_rel_gz_cer : forall a s,
localizing_system a s ->
cat_equiv_rel (gz_freecat a s) (gz_cer a s).
```

```
Lemma related_gz_cer_first_mor : forall a s e f,
localizing_system a s ->
related (gz_cer a s) e f -> mor (gz_freecat a s) e.
```

```
Lemma related_gz_cer_second_mor : forall a s e f,
localizing_system a s ->
related (gz_cer a s) e f -> mor (gz_freecat a s) f.
```

```
Lemma related_gz_cer_same_source : forall a s e f,
localizing_system a s ->
related (gz_cer a s) e f -> source e = source f.
```

```
Lemma related_gz_cer_same_target : forall a s e f,
localizing_system a s ->
related (gz_cer a s) e f -> target e = target f.
```

```
Lemma related_cer : forall a r u v,
cat_rel a r -> related r u v -> related (cer a r) u v.
```

```
Lemma related_gz_cer_forward_edge_id :
forall a s x,
localizing_system a s ->
ob a x -> related (gz_cer a s) (forward_edge (id a x)) (freecat_id x).
```

```
Lemma related_gz_cer_comp_forward_backward : forall a s q,
localizing_system a s -> inc q s ->
related (gz_cer a s) (freecat_comp (forward_edge q) (backward_edge q))
(freecat_id (target q)).
```

```
Lemma related_gz_cer_comp_backward_forward : forall a s q,
localizing_system a s -> inc q s ->
related (gz_cer a s) (freecat_comp (backward_edge q) (forward_edge q))
(freecat_id (source q)).
```

```
Lemma related_gz_cer_comp_forward : forall a s u v,
localizing_system a s -> mor a u -> mor a v ->
source u = target v ->
related (gz_cer a s) (freecat_comp (forward_edge u) (forward_edge v))
```

(forward\_edge (comp a u v)).

Lemma compatible\_gz\_cer\_criterion : forall a s f,  
 localizing\_system a s ->  
 Functor.axioms f ->  
 source f = gz\_freecat a s ->  
 (forall x, ob a x -> fmor f (forward\_edge (id a x)) =  
 id (target f) (fob f x)) ->  
 (forall q, inc q s ->  
 are\_inverse (target f) (fmor f (forward\_edge q))  
 (fmor f (backward\_edge q))) ->  
 (forall u v, mor a u -> mor a v -> source u = target v ->  
 comp (target f) (fmor f (forward\_edge u)) (fmor f (forward\_edge v)) =  
 fmor f (forward\_edge (comp a u v))) ->  
 compatible (gz\_cer a s) f.

Definition gz\_loc a s :=  
 quotient\_cat (gz\_freecat a s) (gz\_cer a s).

Lemma gz\_loc\_axioms : forall a s,  
 localizing\_system a s ->  
 Category.axioms (gz\_loc a s).

Definition gz\_qprojection a s :=  
 qprojection (gz\_freecat a s) (gz\_cer a s).

Lemma source\_gz\_qprojection : forall a s,  
 localizing\_system a s ->  
 source (gz\_qprojection a s) = gz\_freecat a s.

Lemma target\_gz\_qprojection : forall a s,  
 localizing\_system a s ->  
 target (gz\_qprojection a s) = gz\_loc a s.

Lemma gz\_qprojection\_axioms : forall a s,  
 localizing\_system a s ->  
 Functor.axioms (gz\_qprojection a s).

Definition gz\_proj a s :=  
 qfunctor a (gz\_freecat a s) (gz\_cer a s)  
 (fun x => x) forward\_edge.

Lemma source\_gz\_proj : forall a s,  
 localizing\_system a s ->  
 source (gz\_proj a s) = a.

Lemma target\_gz\_proj : forall a s,

```
localizing_system a s ->
target (gz_proj a s) = gz_loc a s.
```

```
Definition gz_forward a s u :=
arrow_class (gz_cer a s) (forward_edge u).
```

```
Definition gz_backward a s u :=
arrow_class (gz_cer a s) (backward_edge u).
```

```
Lemma source_gz_forward : forall a s u,
localizing_system a s ->
mor a u -> source (gz_forward a s u) = source u.
```

```
Lemma target_gz_forward : forall a s u,
localizing_system a s ->
mor a u -> target (gz_forward a s u) = target u.
```

```
Lemma source_gz_backward : forall a s q,
localizing_system a s ->
inc q s -> source (gz_backward a s q) = target q.
```

```
Lemma target_gz_backward : forall a s q,
localizing_system a s ->
inc q s -> target (gz_backward a s q) = source q.
```

```
Lemma mor_gz_forward : forall a s u,
localizing_system a s ->
mor a u -> mor (gz_loc a s) (gz_forward a s u).
```

```
Lemma mor_gz_backward : forall a s q,
localizing_system a s ->
inc q s -> mor (gz_loc a s) (gz_backward a s q).
```

```
Lemma qfunctor_property_forward_edge :
forall a s, localizing_system a s ->
qfunctor_property a (gz_freecat a s) (gz_cer a s)
(fun x => x) forward_edge.
```

```
Lemma gz_proj_axioms : forall a s,
localizing_system a s ->
Functor.axioms (gz_proj a s).
```

```
Lemma fob_gz_proj : forall a s x,
localizing_system a s ->
ob a x ->
fob (gz_proj a s) x = x.
```

Lemma fmor\_gz\_proj : forall a s u,  
 localizing\_system a s ->  
 mor a u -> fmor (gz\_proj a s) u =  
 gz\_forward a s u.

Lemma gz\_forward\_id : forall a s x,  
 localizing\_system a s ->  
 ob a x -> gz\_forward a s (id a x) = id (gz\_loc a s) x.

Lemma comp\_gz\_forward : forall a s u v,  
 localizing\_system a s -> mor a u -> mor a v ->  
 source u = target v ->  
 comp (gz\_loc a s) (gz\_forward a s u) (gz\_forward a s v) =  
 gz\_forward a s (comp a u v).

Lemma are\_inverse\_gz\_forward\_backward : forall a s q,  
 localizing\_system a s ->  
 inc q s -> are\_inverse (gz\_loc a s) (gz\_forward a s q)  
 (gz\_backward a s q).

Lemma invertible\_gz\_loc\_gz\_forward : forall a s q,  
 localizing\_system a s -> inc q s ->  
 invertible (gz\_loc a s) (gz\_forward a s q).

End GZ\_Def.

#### 4.2. The module GZ\_Thm.

Module GZ\_Thm.  
 Export Ob\_Iso\_Functor.  
 Export GZ\_Def.

Lemma inc\_gz\_forward\_mor\_image : forall a s u,  
 localizing\_system a s ->  
 mor a u -> inc (gz\_forward a s u) (mor\_image (gz\_proj a s)).

Lemma inc\_gz\_backward\_add\_inverses :  
 forall a s q,  
 localizing\_system a s ->  
 inc q s -> inc (gz\_backward a s q)  
 (add\_inverses (gz\_loc a s) (mor\_image (gz\_proj a s))).

Lemma gz\_loc\_induction : forall a s (P:E->Prop),  
 localizing\_system a s ->  
 (forall u, mor a u -> P (gz\_forward a s u)) ->  
 (forall q, inc q s -> P (gz\_backward a s q)) ->

```
(forall x, ob a x -> P (id (gz_loc a s) x)) ->
(forall u v, mor (gz_loc a s) u ->
mor (gz_loc a s) v -> source u = target v ->
P u -> P v -> P (comp (gz_loc a s) u v)) ->
(forall y, mor (gz_loc a s) y -> P y).
```

```
Lemma gz_loc_subcategory_all_criterion : forall a s b,
localizing_system a s -> is_subcategory b (gz_loc a s) ->
(forall u, mor a u -> mor b (gz_forward a s u)) ->
(forall q, inc q s -> mor b (gz_backward a s q)) ->
b = gz_loc a s.
```

```
Lemma add_inverses_mor_image_gz_proj_generates_gz_loc : forall a s,
localizing_system a s ->
generates (gz_loc a s)
(add_inverses (gz_loc a s) (mor_image (gz_proj a s))).
```

```
Lemma ob_gz_loc : forall a s x,
localizing_system a s ->
ob (gz_loc a s) x = ob a x.
```

```
Lemma ob_iso_gz_proj : forall a s,
localizing_system a s -> ob_iso (gz_proj a s).
```

```
Lemma iso_to_subcategory_pull_gz_proj : forall a s b,
localizing_system a s -> Category.axioms b ->
iso_to_full_subcategory (pull_morphism b (gz_proj a s)).
```

```
Lemma gz_proj_epimorphic : forall a s f g,
localizing_system a s -> Functor.axioms f ->
Functor.axioms g -> source f = gz_loc a s ->
source g = gz_loc a s ->
fcompose f (gz_proj a s) = fcompose g (gz_proj a s) ->
f = g.
```

```
Definition loc_compatible a s f :=
localizing_system a s &
Functor.axioms f &
source f = a &
(forall q, inc q s -> invertible (target f) (fmor f q)).
```

```
Definition fr_dotted a s f :=
free_functor (gz_graph a s) (target f)
(fun x => fob f x)
```

```
(fun u => Y (direction u = Forward) (fmor f (original_arrow u))
(inverse (target f) (fmor f (original_arrow u))))).
```

```
Lemma backward_neq_forward : Backward <> Forward.
```

```
Lemma Y_etc_forward_arrow : forall f u,
Y (direction (forward_arrow u) = Forward)
(fmor f (original_arrow (forward_arrow u)))
(inverse (target f)
(fmor f (original_arrow (forward_arrow u)))) = fmor f u.
```

```
Lemma Y_etc_backward_arrow : forall f q,
Y (direction (backward_arrow q) = Forward)
(fmor f (original_arrow (backward_arrow q)))
(inverse (target f)
(fmor f (original_arrow (backward_arrow q)))) =
inverse (target f) (fmor f q).
```

```
Lemma fr_dotted_property : forall a s f,
loc_compatible a s f -> free_functor_property
(gz_graph a s) (target f)
(fun x => fob f x)
(fun u => Y (direction u = Forward) (fmor f (original_arrow u))
(inverse (target f) (fmor f (original_arrow u))))).
```

```
Lemma source_fr_dotted : forall a s f,
source (fr_dotted a s f) = gz_freecat a s.
```

```
Lemma target_fr_dotted : forall a s f,
target (fr_dotted a s f) = target f.
```

```
Lemma fob_fr_dotted : forall a s f x,
loc_compatible a s f -> ob a x ->
fob (fr_dotted a s f) x = fob f x.
```

```
Lemma fmor_fr_dotted_forward_edge : forall a s f u,
loc_compatible a s f -> mor a u ->
fmor (fr_dotted a s f) (forward_edge u) =
fmor f u.
```

```
Lemma fmor_fr_dotted_backward_edge : forall a s f q,
loc_compatible a s f -> inc q s ->
fmor (fr_dotted a s f) (backward_edge q) =
inverse (target f) (fmor f q).
```

```
Lemma fr_dotted_axioms : forall a s f,
loc_compatible a s f ->
```



Functor.axioms (fr\_dotted a s f).

Lemma compatible\_gz\_cer\_fr\_dotted : forall a s f,  
loc\_compatible a s f ->  
compatible (gz\_cer a s) (fr\_dotted a s f).

Definition gz\_dotted a s f :=  
qdotted (gz\_cer a s) (fr\_dotted a s f).

Lemma source\_gz\_dotted : forall a s f,  
source (gz\_dotted a s f) = gz\_loc a s.

Lemma target\_gz\_dotted : forall a s f,  
target (gz\_dotted a s f) = target f.

Lemma fob\_gz\_dotted : forall a s f x,  
loc\_compatible a s f -> ob a x ->  
fob (gz\_dotted a s f) x = fob f x.

Lemma fmor\_gz\_dotted\_gz\_forward : forall a s f u,  
loc\_compatible a s f -> mor a u ->  
fmor (gz\_dotted a s f) (gz\_forward a s u) = fmor f u.

Lemma fmor\_gz\_dotted\_gz\_backward : forall a s f q,  
loc\_compatible a s f -> inc q s ->  
fmor (gz\_dotted a s f) (gz\_backward a s q) =  
inverse (target f) (fmor f q).

Lemma gz\_dotted\_axioms : forall a s f,  
loc\_compatible a s f ->  
Functor.axioms (gz\_dotted a s f).

Lemma fcompose\_gz\_dotted\_gz\_proj : forall a s f,  
loc\_compatible a s f ->  
fcompose (gz\_dotted a s f) (gz\_proj a s) = f.

Lemma loc\_compatible\_fcompose : forall a s g,  
localizing\_system a s ->  
Functor.axioms g ->  
source g = gz\_loc a s ->  
loc\_compatible a s (fcompose g (gz\_proj a s)).

Lemma ob\_image\_pull\_gz\_proj : forall a s b f,  
localizing\_system a s ->

```
Category.axioms b ->  
inc f (ob_image (pull_morphism b (gz_proj a s))) =  
(Functor.axioms f & source f = a & target f = b &  
(forall q, inc q s -> invertible b (fmor f q))).
```

```
End GZ_Thm.
```

5. THE FILE `left_fractions.v`

```
Require Export gzdef.
```

5.1. The module `LeftFractions`.

```
Module LeftFractions.
```

```
Export GZ_Def.
```

```
Definition lf_symbol f t :=
Arrow.create (source f) (source t) (pair f t).
```

```
Definition lf_forward v := pr1 (arrow v).
```

```
Definition lf_backward v := pr2 (arrow v).
```

```
Lemma source_lf_symbol : forall f t, source (lf_symbol f t)
= source f.
```

```
Lemma target_lf_symbol : forall f t, target (lf_symbol f t)
= source t.
```

```
Lemma lf_forward_lf_symbol : forall f t, lf_forward
(lf_symbol f t) = f.
```

```
Lemma lf_backward_lf_symbol : forall f t, lf_backward
(lf_symbol f t) = t.
```

```
Definition lf_symbol_like v :=
v = lf_symbol (lf_forward v) (lf_backward v).
```

```
Lemma lf_symbol_like_lf_symbol : forall f t,
lf_symbol_like (lf_symbol f t).
```

```
Definition lf_symbol_property a s f t :=
localizing_system a s &
mor a f & inc t s & target f = target t.
```

```
Definition is_lf_symbol a s v :=
lf_symbol_like v & lf_symbol_property a s (lf_forward v)
(lf_backward v).
```

```
Lemma is_lf_symbol_lf_symbol : forall a s f t,
lf_symbol_property a s f t ->
is_lf_symbol a s (lf_symbol f t).
```

```
Definition multiplicative_system a s :=
localizing_system a s &
```

(forall y z, inc y s -> inc z s ->  
source y = target z -> inc (comp a y z) s).

Definition has\_left\_fractions a s :=  
multiplicative\_system a s  
&  
(forall x, ob a x -> inc (id a x) s)  
&  
(forall r g, inc r s -> mor a g -> source r = source g ->  
exists u, (is\_lf\_symbol a s u & source u = target r  
& target u = target g & comp a (lf\_backward u) g =  
comp a (lf\_forward u) r))  
&  
(forall v r t, inc v s -> mor a r -> mor a t ->  
source r = target v -> source t = target v ->  
comp a r v = comp a t v ->  
exists w, (inc w s & source w = target r & source w = target t  
& comp a w r = comp a w t)).

Definition lf\_choice a s r g :=  
choose (fun u => (is\_lf\_symbol a s u & source u = target r  
& target u = target g & comp a (lf\_backward u) g =  
comp a (lf\_forward u) r)).

Lemma is\_lf\_symbol\_lf\_choice : forall a s r g,  
has\_left\_fractions a s ->  
inc r s -> mor a g -> source r = source g ->  
is\_lf\_symbol a s (lf\_choice a s r g).

Lemma source\_lf\_choice : forall a s r g,  
has\_left\_fractions a s ->  
inc r s -> mor a g -> source r = source g ->  
source (lf\_choice a s r g) = target r.

Lemma target\_lf\_choice : forall a s r g,  
has\_left\_fractions a s ->  
inc r s -> mor a g -> source r = source g ->  
target (lf\_choice a s r g) = target g.

Lemma comp\_lf\_backward\_lf\_choice : forall a s r g,  
has\_left\_fractions a s ->  
inc r s -> mor a g -> source r = source g ->  
comp a (lf\_backward (lf\_choice a s r g)) g  
= comp a (lf\_forward (lf\_choice a s r g)) r.

Lemma source\_lf\_forward : forall v,  
lf\_symbol\_like v -> source (lf\_forward v) = source v.

Lemma source\_lf\_backward : forall v,  
 lf\_symbol\_like v -> source (lf\_backward v) = target v.

Lemma inc\_ms\_mor : forall a u,  
 (exists s, (localizing\_system a s & inc u s)) ->  
 mor a u.

Lemma inc\_hlf\_mor : forall a u,  
 (exists s, (has\_left\_fractions a s & inc u s)) ->  
 mor a u.

Definition lf\_id\_rep a x :=  
 lf\_symbol (id a x) (id a x).

Lemma source\_lf\_id\_rep : forall a x,  
 ob a x -> source (lf\_id\_rep a x) = x.

Lemma target\_lf\_id\_rep : forall a x,  
 ob a x -> target (lf\_id\_rep a x) = x.

Lemma is\_lf\_symbol\_lf\_id\_rep : forall a s x,  
 has\_left\_fractions a s -> ob a x ->  
 is\_lf\_symbol a s (lf\_id\_rep a x).

Definition lf\_vertex v :=  
 target (lf\_backward v).

Lemma target\_lf\_forward : forall u,  
 (exists a, exists s, (is\_lf\_symbol a s u)) ->  
 target (lf\_forward u) = lf\_vertex u.

Lemma target\_lf\_backward : forall u,  
 target (lf\_backward u) = lf\_vertex u.

Definition lf\_extend a (s:E) p u :=  
 lf\_symbol (comp a p (lf\_forward u)) (comp a p (lf\_backward u)).

Lemma mor\_lf\_forward : forall a u,  
 (exists s, is\_lf\_symbol a s u) ->  
 mor a (lf\_forward u).

Lemma inc\_lf\_backward : forall s u,  
 (exists a, is\_lf\_symbol a s u) ->  
 inc (lf\_backward u) s.

Lemma mor\_lf\_backward : forall a u,

(exists s, is\_lf\_symbol a s u) ->  
mor a (lf\_backward u).

Lemma is\_lf\_symbol\_lf\_extend : forall a s p u,  
has\_left\_fractions a s ->  
is\_lf\_symbol a s u ->  
mor a p -> source p = lf\_vertex u ->  
inc (comp a p (lf\_backward u)) s ->  
is\_lf\_symbol a s (lf\_extend a s p u).

Lemma source\_lf\_extend : forall a s p u,  
is\_lf\_symbol a s u ->  
mor a p -> source p = lf\_vertex u ->  
source (lf\_extend a s p u) = source u.

Lemma target\_lf\_extend : forall a s p u,  
is\_lf\_symbol a s u ->  
mor a p -> source p = lf\_vertex u ->  
target (lf\_extend a s p u) = target u.

Lemma lf\_forward\_lf\_extend : forall a s p u,  
lf\_forward (lf\_extend a s p u) = comp a p (lf\_forward u).

Lemma lf\_backward\_lf\_extend : forall a s p u,  
lf\_backward (lf\_extend a s p u) = comp a p (lf\_backward u).

Lemma lf\_vertex\_lf\_extend : forall a s p u,  
has\_left\_fractions a s ->  
is\_lf\_symbol a s u -> mor a p -> source p = lf\_vertex u ->  
lf\_vertex (lf\_extend a s p u) = target p.

Definition lf\_beyond a s u v :=  
has\_left\_fractions a s &  
is\_lf\_symbol a s u & is\_lf\_symbol a s v &  
source u = source v & target u = target v &  
(exists p, (mor a p & source p = lf\_vertex u &  
lf\_extend a s p u = v)).

Definition lf\_under a s u v :=  
has\_left\_fractions a s &  
is\_lf\_symbol a s u & is\_lf\_symbol a s v &  
source u = source v & target u = target v &  
(exists p, (inc p s & source p = lf\_vertex u &  
lf\_extend a s p u = v)).

Lemma lf\_under\_lf\_beyond : forall a s u v,

`lf_under a s u v -> lf_beyond a s u v.`

`Lemma lf_beyond_lf_extend : forall a s p u,  
has_left_fractions a s ->  
is_lf_symbol a s u -> mor a p ->  
source p = lf_vertex u ->  
inc (comp a p (lf_backward u)) s ->  
lf_beyond a s u (lf_extend a s p u).`

`Lemma inc_comp : forall a s u v,  
multiplicative_system a s ->  
inc u s -> inc v s -> source u = target v ->  
inc (comp a u v) s.`

`Lemma lf_under_lf_extend : forall a s p u,  
has_left_fractions a s ->  
is_lf_symbol a s u -> inc p s ->  
source p = lf_vertex u ->  
lf_under a s u (lf_extend a s p u).`

`Lemma lf_symbol_like_extens : forall u v,  
lf_symbol_like u -> lf_symbol_like v ->  
lf_forward u = lf_forward v -> lf_backward u = lf_backward v ->  
u = v.`

`Lemma lf_symbol_like_lf_extend : forall a s p u,  
lf_symbol_like (lf_extend a s p u).`

`Lemma lf_extend_comp : forall a s u p q,  
is_lf_symbol a s u ->  
mor a p -> mor a q ->  
source q = lf_vertex u ->  
source p = target q ->  
lf_extend a s (comp a p q) u =  
lf_extend a s p (lf_extend a s q u).`

`Lemma lf_beyond_trans : forall a s u v w,  
lf_beyond a s u v -> lf_beyond a s v w ->  
lf_beyond a s u w.`

`Lemma exists_lf_under : forall a s u v,  
lf_beyond a s u v -> exists w,  
(lf_beyond a s v w & lf_under a s u w).`

`Definition lf_equiv a s u v :=  
(exists w, (lf_beyond a s u w & lf_beyond a s v w)).`

Lemma lf\_equiv\_symm : forall a s u v,  
lf\_equiv a s u v -> lf\_equiv a s v u.

Lemma lf\_extend\_id : forall a s u,  
is\_lf\_symbol a s u ->  
lf\_extend a s (id a (lf\_vertex u)) u = u.

Lemma lf\_equiv\_refl : forall a s u,  
has\_left\_fractions a s ->  
is\_lf\_symbol a s u -> lf\_equiv a s u u.

Lemma lf\_beyond\_refl : forall a s u,  
has\_left\_fractions a s ->  
is\_lf\_symbol a s u -> lf\_beyond a s u u.

Lemma lf\_equiv\_beyond\_under : forall a s u v,  
lf\_equiv a s u v -> exists w,  
(lf\_beyond a s u w & lf\_under a s v w).

Lemma exists\_lf\_further : forall a s u v w,  
lf\_beyond a s u v -> lf\_under a s u w ->  
exists z, (lf\_beyond a s v z & lf\_beyond a s w z).

Lemma lf\_equiv\_trans : forall a s u w,  
has\_left\_fractions a s ->  
(exists v, (lf\_equiv a s u v & lf\_equiv a s v w)) ->  
lf\_equiv a s u w.

Definition fills\_in a s u v w :=  
has\_left\_fractions a s &  
is\_lf\_symbol a s u & is\_lf\_symbol a s v & is\_lf\_symbol a s w  
& source u = target v & source w = lf\_vertex v &  
target w = lf\_vertex u &  
comp a (lf\_forward w) (lf\_backward v) =  
comp a (lf\_backward w) (lf\_forward u).

Definition lf\_filler a s u v :=  
lf\_choice a s (lf\_backward v) (lf\_forward u).

Definition lf\_make\_comp a (s:E) u v w :=  
lf\_symbol (comp a (lf\_forward w) (lf\_forward v))  
(comp a (lf\_backward w) (lf\_backward u)).

Definition lf\_comp\_rep a s u v :=  
lf\_make\_comp a s u v (lf\_filler a s u v).



Lemma fills\_in\_lf\_filler : forall a s u v,  
 has\_left\_fractions a s ->  
 is\_lf\_symbol a s u -> is\_lf\_symbol a s v ->  
 source u = target v ->  
 fills\_in a s u v (lf\_filler a s u v).

Lemma source\_lf\_make\_comp : forall a s u v w,  
 fills\_in a s u v w ->  
 source (lf\_make\_comp a s u v w) = source v.

Lemma target\_lf\_make\_comp : forall a s u v w,  
 fills\_in a s u v w ->  
 target (lf\_make\_comp a s u v w) = target u.

Lemma lf\_vertex\_lf\_make\_comp : forall a s u v w,  
 fills\_in a s u v w ->  
 lf\_vertex (lf\_make\_comp a s u v w) =  
 lf\_vertex w.

Lemma is\_lf\_symbol\_lf\_make\_comp : forall a s u v w,  
 fills\_in a s u v w ->  
 is\_lf\_symbol a s (lf\_make\_comp a s u v w).

Lemma lf\_forward\_lf\_make\_comp : forall a s u v w,  
 lf\_forward (lf\_make\_comp a s u v w) =  
 comp a (lf\_forward w) (lf\_forward v).

Lemma lf\_backward\_lf\_make\_comp : forall a s u v w,  
 lf\_backward (lf\_make\_comp a s u v w) =  
 comp a (lf\_backward w) (lf\_backward u).

Lemma lf\_beyond\_fills\_in : forall a s u v w,  
 (exists y, (fills\_in a s u v y &  
 lf\_beyond a s y w)) ->  
 fills\_in a s u v w.

Lemma lf\_beyond\_lf\_make\_comp : forall a s u v w y,  
 fills\_in a s u v w ->  
 lf\_beyond a s w y ->  
 lf\_beyond a s (lf\_make\_comp a s u v w) (lf\_make\_comp a s u v y).

Definition lf\_lean\_to a s e f g h i j :=  
 has\_left\_fractions a s &  
 mor a e & mor a f & mor a g & mor a h & mor a i & mor a j &  
 inc e s & inc h s &  
 source e = source f & source g = target e & source h = target f  
 & target g = target h & source i = target e & source j = target f

& target i = target j &  
 comp a g e = comp a h f & comp a i e = comp a j f.

Lemma show\_lf\_lean\_to : forall a s e f g h i j,  
 has\_left\_fractions a s ->  
 mor a f -> mor a g -> mor a i -> mor a j ->  
 inc e s -> inc h s ->  
 source e = source f -> source g = target e -> source h = target f  
 -> source i = target e -> source j = target f ->  
 comp a g e = comp a h f -> comp a i e = comp a j f->  
 lf\_lean\_to a s e f g h i j.

Definition closes\_lf\_lean\_to a s e f g h i j k l :=  
 lf\_lean\_to a s e f g h i j &  
 mor a k & mor a l & inc l s &  
 source k = target g & source l = target i &  
 target k = target l &  
 comp a k g = comp a l i & comp a k h = comp a l j.

Lemma lf\_lean\_to\_closure : forall a s e f g h i j,  
 lf\_lean\_to a s e f g h i j ->  
 (exists k, exists l, (closes\_lf\_lean\_to a s e f g h i j k l)).

Definition weakly\_reps\_lf\_comp a s u v w :=  
 has\_left\_fractions a s &  
 is\_lf\_symbol a s u & is\_lf\_symbol a s v & is\_lf\_symbol a s w  
 & source u = target v & source w = source v &  
 target w = target u &  
 (exists p, exists q, (mor a p & mor a q &  
 source p = lf\_vertex v & source q = lf\_vertex u &  
 target p = lf\_vertex w & target q = lf\_vertex w &  
 comp a p (lf\_backward v) = comp a q (lf\_forward u) &  
 comp a p (lf\_forward v) = lf\_forward w &  
 comp a q (lf\_backward u) = lf\_backward w)).

Lemma weak\_rep\_lf\_equiv : forall a s u v w y,  
 fills\_in a s u v y ->  
 weakly\_reps\_lf\_comp a s u v w ->  
 lf\_equiv a s w (lf\_make\_comp a s u v y).

Lemma lf\_beyond\_weakly\_reps\_make\_comp : forall a s u v w y,  
 lf\_beyond a s u w -> fills\_in a s w v y ->  
 weakly\_reps\_lf\_comp a s u v (lf\_make\_comp a s w v y).

Lemma weakly\_reps\_make\_comp : forall a s u v y,  
 fills\_in a s u v y ->

`weakly_reps_lf_comp a s u v (lf_make_comp a s u v y).`

Lemma `source_lf_comp_rep` : forall a s u v,  
`has_left_fractions a s ->`  
`is_lf_symbol a s u -> is_lf_symbol a s v ->`  
`source u = target v ->`  
`source (lf_comp_rep a s u v) = source v.`

Lemma `target_lf_comp_rep` : forall a s u v,  
`has_left_fractions a s ->`  
`is_lf_symbol a s u -> is_lf_symbol a s v ->`  
`source u = target v ->`  
`target (lf_comp_rep a s u v) = target u.`

Lemma `is_lf_symbol_lf_comp_rep` : forall a s u v,  
`has_left_fractions a s ->`  
`is_lf_symbol a s u -> is_lf_symbol a s v ->`  
`source u = target v ->`  
`is_lf_symbol a s (lf_comp_rep a s u v).`

Lemma `weakly_reps_lf_comp_lf_comp_rep` : forall a s u v,  
`has_left_fractions a s -> is_lf_symbol a s u ->`  
`is_lf_symbol a s v -> source u = target v ->`  
`weakly_reps_lf_comp a s u v (lf_comp_rep a s u v).`

Lemma `lf_equiv_lf_comp_rep` : forall a s u v w,  
`weakly_reps_lf_comp a s u v w ->`  
`lf_equiv a s w (lf_comp_rep a s u v).`

Lemma `weakly_reps_lf_comp_equiv` : forall a s w1 w2,  
`(exists u, exists v, (weakly_reps_lf_comp a s u v w1 &`  
`weakly_reps_lf_comp a s u v w2)) ->`  
`lf_equiv a s w1 w2.`

Lemma `weakly_reps_make_comp_beyond` : forall a s u v u1 v1 y,  
`has_left_fractions a s -> lf_beyond a s u u1 ->`  
`lf_beyond a s v v1 -> source u = target v ->`  
`fills_in a s u1 v1 y ->`  
`weakly_reps_lf_comp a s u v (lf_make_comp a s u1 v1 y).`

Lemma `weakly_reps_comp_rep_beyond` : forall a s u v u1 v1,  
`has_left_fractions a s -> lf_beyond a s u u1 ->`  
`lf_beyond a s v v1 -> source u = target v ->`  
`weakly_reps_lf_comp a s u v (lf_comp_rep a s u1 v1).`

```

Lemma lf_comp_indep : forall a s u v u1 v1,
has_left_fractions a s -> lf_equiv a s u u1 ->
lf_equiv a s v v1 -> source u = target v ->
lf_equiv a s (lf_comp_rep a s u v) (lf_comp_rep a s u1 v1).

```

```

Lemma lf_equiv_make_comp_comp_rep : forall a s u v y,
fills_in a s u v y ->
lf_equiv a s (lf_make_comp a s u v y) (lf_comp_rep a s u v).

```

```

Definition assoc_board a s u v w x y z :=
fills_in a s u v x &
fills_in a s v w y &
fills_in a s x y z.

```

```

Definition ffb_symbol a (s:E) y z :=
(lf_symbol (comp a (lf_forward z) (lf_forward y)) (lf_backward z)).

```

```

Definition fbb_symbol a (s:E) x z :=
(lf_symbol (lf_forward z) (comp a (lf_backward z) (lf_backward x))).

```

```

Definition ffb_symbol_facts (a s u v w x y z k:E) :=
lf_forward k = comp a (lf_forward z) (lf_forward y) &
lf_backward k = lf_backward z &
source k = lf_vertex w &
target k = lf_vertex x &
target (lf_forward k) = lf_vertex z &
target (lf_backward k) = lf_vertex z &
lf_symbol_like k &
mor a (lf_forward k) &
mor a (lf_backward k) &
inc (lf_backward k) s &
is_lf_symbol a s k &
fills_in a s (lf_make_comp a s u v x) w k.

```

```

Lemma get_ffb_symbol_facts : forall a s u v w x y z,
assoc_board a s u v w x y z ->
ffb_symbol_facts a s u v w x y z (ffb_symbol a s y z).

```

```

Definition fbb_symbol_facts (a s u v w x y z k:E) :=
lf_forward k = lf_forward z &
lf_backward k = comp a (lf_backward z) (lf_backward x) &
source k = lf_vertex y &
target k = lf_vertex u &

```

```

target (lf_forward k) = lf_vertex z &
target (lf_backward k) = lf_vertex z &
lf_symbol_like k &
mor a (lf_forward k) &
mor a (lf_backward k) &
inc (lf_backward k) s &
is_lf_symbol a s k &
fills_in a s u (lf_make_comp a s v w y) k.

```

```

Lemma get_fbb_symbol_facts : forall a s u v w x y z,
assoc_board a s u v w x y z ->
fbb_symbol_facts a s u v w x y z (fbb_symbol a s x z).

```

```

Lemma make_comp_assoc_board : forall a s u v w x y z,
assoc_board a s u v w x y z ->
lf_make_comp a s (lf_make_comp a s u v x) w (fbb_symbol a s y z) =
lf_make_comp a s u (lf_make_comp a s v w y) (fbb_symbol a s x z).

```

```

Lemma lf_comp_rep_assoc : forall a s u v w,
has_left_fractions a s ->
is_lf_symbol a s u -> is_lf_symbol a s v ->
is_lf_symbol a s w ->
source u = target v -> source v = target w ->
lf_equiv a s (lf_comp_rep a s (lf_comp_rep a s u v) w)
(lf_comp_rep a s u (lf_comp_rep a s v w)).

```

```

Definition left_id_filler a (s:E) u :=
lf_symbol (id a (lf_vertex u)) (lf_backward u).

```

```

Definition right_id_filler a (s:E) u :=
lf_symbol (lf_forward u) (id a (lf_vertex u)).

```

```

Lemma ob_lf_vertex : forall a u,
(exists s, (is_lf_symbol a s u)) ->
ob a (lf_vertex u).

```

```

Lemma source_left_id_filler : forall a s u,
is_lf_symbol a s u ->
source (left_id_filler a s u) = lf_vertex u.

```

```

Lemma target_left_id_filler : forall a s u,
is_lf_symbol a s u ->
target (left_id_filler a s u) = target u.

```

```

Lemma source_right_id_filler : forall a s u,
is_lf_symbol a s u ->
source (right_id_filler a s u) = source u.

```

Lemma target\_right\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 target (right\_id\_filler a s u) = lf\_vertex u.

Lemma lf\_forward\_left\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_forward (left\_id\_filler a s u) = id a (lf\_vertex u).

Lemma lf\_backward\_left\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_backward (left\_id\_filler a s u) = lf\_backward u.

Lemma lf\_forward\_right\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_forward (right\_id\_filler a s u) = lf\_forward u.

Lemma lf\_backward\_right\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_backward (right\_id\_filler a s u) = id a (lf\_vertex u).

Lemma lf\_vertex\_left\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_vertex (left\_id\_filler a s u) = lf\_vertex u.

Lemma lf\_vertex\_right\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_vertex (right\_id\_filler a s u) = lf\_vertex u.

Lemma is\_lf\_symbol\_left\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 is\_lf\_symbol a s (left\_id\_filler a s u).

Lemma is\_lf\_symbol\_right\_id\_filler : forall a s u,  
 has\_left\_fractions a s -> is\_lf\_symbol a s u ->  
 is\_lf\_symbol a s (right\_id\_filler a s u).

Lemma lf\_forward\_lf\_id\_rep : forall a x,  
 lf\_forward (lf\_id\_rep a x) = id a x.

Lemma lf\_backward\_lf\_id\_rep : forall a x,  
 lf\_backward (lf\_id\_rep a x) = id a x.

Lemma fills\_in\_left\_id\_filler : forall a s u,  
 has\_left\_fractions a s -> is\_lf\_symbol a s u ->  
 fills\_in a s (lf\_id\_rep a (target u)) u (left\_id\_filler a s u).

Lemma fills\_in\_right\_id\_filler : forall a s u,  
 has\_left\_fractions a s -> is\_lf\_symbol a s u ->  
 fills\_in a s u (lf\_id\_rep a (source u)) (right\_id\_filler a s u).

Lemma lf\_make\_comp\_left\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_make\_comp a s (lf\_id\_rep a (target u)) u  
 (left\_id\_filler a s u) = u.

Lemma lf\_make\_comp\_right\_id\_filler : forall a s u,  
 is\_lf\_symbol a s u ->  
 lf\_make\_comp a s u (lf\_id\_rep a (source u))  
 (right\_id\_filler a s u) = u.

Lemma lf\_left\_id : forall a s u,  
 has\_left\_fractions a s ->  
 is\_lf\_symbol a s u ->  
 lf\_equiv a s (lf\_comp\_rep a s (lf\_id\_rep a (target u)) u) u.

Lemma lf\_right\_id : forall a s u,  
 has\_left\_fractions a s ->  
 is\_lf\_symbol a s u ->  
 lf\_equiv a s (lf\_comp\_rep a s u (lf\_id\_rep a (source u))) u.

Definition taut\_lf\_symbol a y :=  
 lf\_symbol y (id a (target y)).

Lemma source\_taut\_lf\_symbol : forall a y,  
 source (taut\_lf\_symbol a y) = source y.

Lemma target\_taut\_lf\_symbol : forall a y,  
 mor a y ->  
 target (taut\_lf\_symbol a y) = target y.

Lemma lf\_forward\_taut\_lf\_symbol : forall a y,  
 lf\_forward (taut\_lf\_symbol a y) = y.

Lemma lf\_backward\_taut\_lf\_symbol : forall a y,  
 lf\_backward (taut\_lf\_symbol a y) = id a (target y).

Lemma lf\_vertex\_taut\_lf\_symbol : forall a y,  
 mor a y ->  
 lf\_vertex (taut\_lf\_symbol a y) = target y.

Lemma is\_lf\_symbol\_taut\_lf\_symbol : forall a s y,  
 has\_left\_fractions a s -> mor a y ->

`is_lf_symbol a s (taut_lf_symbol a y).`

Lemma `taut_lf_symbol_id` : forall a x,  
ob a x ->  
`taut_lf_symbol a (id a x) = lf_id_rep a x.`

Lemma `fills_in_taut_lf_symbol` : forall a s y z,  
`has_left_fractions a s ->`  
`mor a y -> mor a z -> source y = target z ->`  
`fills_in a s (taut_lf_symbol a y)`  
`(taut_lf_symbol a z) (taut_lf_symbol a y).`

Lemma `comp_taut_lf_symbol` : forall a s y z,  
`has_left_fractions a s ->`  
`mor a y -> mor a z -> source y = target z ->`  
`lf_equiv a s`  
`(lf_comp_rep a s (taut_lf_symbol a y) (taut_lf_symbol a z))`  
`(taut_lf_symbol a (comp a y z)).`

Definition `inverse_lf_symbol a y :=`  
`lf_symbol (id a (target y)) y.`

Lemma `source_inverse_lf_symbol` : forall a y,  
`mor a y ->`  
`source (inverse_lf_symbol a y) = target y.`

Lemma `target_inverse_lf_symbol` : forall a y,  
`mor a y ->`  
`target (inverse_lf_symbol a y) = source y.`

Lemma `lf_forward_inverse_lf_symbol` : forall a y,  
`lf_forward (inverse_lf_symbol a y) = id a (target y).`

Lemma `lf_backward_inverse_lf_symbol` : forall a y,  
`lf_backward (inverse_lf_symbol a y) = y.`

Lemma `lf_vertex_inverse_lf_symbol` : forall a y,  
`lf_vertex (inverse_lf_symbol a y) = target y.`

Lemma `is_lf_symbol_inverse_lf_symbol` : forall a s y,  
`has_left_fractions a s -> inc y s ->`  
`is_lf_symbol a s (inverse_lf_symbol a y).`

Lemma `fills_in_left_lf_inverse` : forall a s y,  
`has_left_fractions a s -> inc y s ->`



```
fills_in a s (inverse_lf_symbol a y) (taut_lf_symbol a y)
(lf_id_rep a (target y)).
```

```
Lemma fills_in_right_lf_inverse : forall a s y,
has_left_fractions a s -> inc y s ->
fills_in a s (taut_lf_symbol a y) (inverse_lf_symbol a y)
(lf_id_rep a (target y)).
```

```
Lemma lf_left_inverse : forall a s y,
has_left_fractions a s -> inc y s ->
lf_equiv a s
(lf_comp_rep a s (inverse_lf_symbol a y) (taut_lf_symbol a y))
(lf_id_rep a (source y)).
```

```
Lemma lf_right_inverse : forall a s y,
has_left_fractions a s -> inc y s ->
lf_equiv a s
(lf_comp_rep a s (taut_lf_symbol a y) (inverse_lf_symbol a y))
(lf_id_rep a (target y)).
```

```
Lemma lf_symbol_equiv : forall a s u,
has_left_fractions a s -> is_lf_symbol a s u ->
lf_equiv a s u
(lf_comp_rep a s
(inverse_lf_symbol a (lf_backward u))
(taut_lf_symbol a (lf_forward u))).
```

End Left\_Fractions.

## 5.2. The module Left\_Fraction\_Category.

```
Module Left_Fraction_Category.
Export Left_Fractions.
Export Associating_Quotient.
```

```
Definition lf_symbol_container a s :=
Image.create (Cartesian.product (morphisms a) s)
(fun z => lf_symbol (pr1 z) (pr2 z)).
```

```
Lemma inc_lf_symbol_container : forall a s u,
is_lf_symbol a s u ->
inc u (lf_symbol_container a s).
```

```
Definition lf_symbol_set a s :=
Z (lf_symbol_container a s) (is_lf_symbol a s).
```

```
Lemma inc_lf_symbol_set : forall a s u,
```

`inc u (lf_symbol_set a s) = is_lf_symbol a s u.`

`Definition lfc_rqcat a s :=  
Category.Notations.create (objects a)  
(lf_symbol_set a s) (lf_comp_rep a s) (lf_id_rep a) (structure a).`

`Lemma is_ob_lfc_rqcat : forall a s x,  
has_left_fractions a s ->  
is_ob (lfc_rqcat a s) x = ob a x.`

`Lemma is_mor_lfc_rqcat : forall a s u,  
has_left_fractions a s ->  
is_mor (lfc_rqcat a s) u = is_lf_symbol a s u.`

`Lemma comp_lfc_rqcat : forall a s u v,  
has_left_fractions a s ->  
is_lf_symbol a s u ->  
is_lf_symbol a s v ->  
source u = target v ->  
comp (lfc_rqcat a s) u v = lf_comp_rep a s u v.`

`Lemma id_lfc_rqcat : forall a s x,  
has_left_fractions a s -> ob a x ->  
id (lfc_rqcat a s) x = lf_id_rep a x.`

`Lemma rqcat_lfc_rqcat : forall a s,  
has_left_fractions a s ->  
rqcat (lfc_rqcat a s).`

`Definition lfer a s :=  
Z (Cartesian.product (lf_symbol_set a s) (lf_symbol_set a s))  
(fun z => lf_equiv a s (pr1 z) (pr2 z)).`

`Lemma related_lfer : forall a s u v,  
has_left_fractions a s ->  
related (lfer a s) u v = lf_equiv a s u v.`

`Lemma related_lfer_rw : forall a s u v,  
has_left_fractions a s ->  
related (lfer a s) u v =  
(is_lf_symbol a s u & is_lf_symbol a s v &  
source u = source v & target u = target v &  
lf_equiv a s u v).`

`Lemma inc_lfer : forall a s x,  
has_left_fractions a s ->  
inc x (lfer a s) =`

(is\_pair x & (related (lfer a s) (pr1 x) (pr2 x))).

Lemma lf\_equiv\_properties : forall a s u v,  
 lf\_equiv a s u v ->  
 (lf\_equiv a s u v &  
 is\_lf\_symbol a s u & is\_lf\_symbol a s v &  
 ob a (source u) & ob a (target u) &  
 ob a (source v) & ob a (target v) &  
 source u = source v & target u = target v).

Lemma lf\_ob\_target : forall a u,  
 (exists s, (is\_lf\_symbol a s u)) ->  
 ob a (target u).

Lemma lf\_ob\_source : forall a u,  
 (exists s, (is\_lf\_symbol a s u)) ->  
 ob a (source u).

Lemma rqcat\_equiv\_rel\_lfer : forall a s,  
 has\_left\_fractions a s ->  
 rqcat\_equiv\_rel (lfc\_rqcat a s) (lfer a s).

Definition left\_frac\_cat a s :=  
 quotient\_cat (lfc\_rqcat a s) (lfer a s).

Lemma left\_frac\_cat\_axioms : forall a s,  
 has\_left\_fractions a s ->  
 Category.axioms (left\_frac\_cat a s).

Definition lf\_class a s u := arrow\_class (lfer a s) u.

Lemma source\_lf\_class : forall a s u,  
 source (lf\_class a s u) = source u.

Lemma target\_lf\_class : forall a s u,  
 target (lf\_class a s u) = target u.

Lemma ob\_left\_frac\_cat : forall a s x,  
 has\_left\_fractions a s ->  
 ob (left\_frac\_cat a s) x = ob a x.

Lemma mor\_left\_frac\_cat : forall a s v,  
 has\_left\_fractions a s ->  
 mor (left\_frac\_cat a s) v =  
 (exists u, (is\_lf\_symbol a s u & v = lf\_class a s u)).

Lemma comp\_left\_frac\_cat\_lf\_class : forall a s u v,

```

has_left_fractions a s ->
is_lf_symbol a s u -> is_lf_symbol a s v ->
source u = target v ->
comp (left_frac_cat a s) (lf_class a s u) (lf_class a s v) =
lf_class a s (lf_comp_rep a s u v).

```

```

Lemma id_left_frac_cat : forall a s x,
has_left_fractions a s -> ob a x ->
id (left_frac_cat a s) x = lf_class a s (lf_id_rep a x).

```

```

Lemma eq_lf_class : forall a s u v,
has_left_fractions a s ->
is_lf_symbol a s u -> is_lf_symbol a s v ->
(lf_class a s u = lf_class a s v) = (lf_equiv a s u v).

```

```

Definition lf_taut a s u :=
lf_class a s (taut_lf_symbol a u).

```

```

Definition lf_inverse a s u :=
lf_class a s (inverse_lf_symbol a u).

```

```

Lemma source_lf_taut : forall a s u,
source (lf_taut a s u) = source u.

```

```

Lemma target_lf_taut : forall a s u,
mor a u ->
target (lf_taut a s u) = target u.

```

```

Lemma source_lf_inverse : forall a s u,
mor a u ->
source (lf_inverse a s u) = target u.

```

```

Lemma target_lf_inverse : forall a s u,
mor a u ->
target (lf_inverse a s u) = source u.

```

```

Lemma mor_lf_taut : forall a s u,
has_left_fractions a s -> mor a u ->
mor (left_frac_cat a s) (lf_taut a s u).

```

```

Lemma mor_lf_inverse : forall a s u,
has_left_fractions a s -> inc u s ->
mor (left_frac_cat a s) (lf_inverse a s u).

```

```

Lemma lf_taut_id : forall a s x,
has_left_fractions a s -> ob a x ->
lf_taut a s (id a x) = id (left_frac_cat a s) x.

```

```

Lemma comp_lf_class : forall a s u v,
has_left_fractions a s ->
is_lf_symbol a s u -> is_lf_symbol a s v ->
source u = target v ->
comp (left_frac_cat a s) (lf_class a s u) (lf_class a s v) =
lf_class a s (lf_comp_rep a s u v).

```

```

Lemma comp_lf_taut : forall a s u v,
has_left_fractions a s ->
mor a u -> mor a v -> source u = target v ->
comp (left_frac_cat a s) (lf_taut a s u) (lf_taut a s v) =
lf_taut a s (comp a u v).

```

```

Lemma are_inverse_lf_taut_lf_inverse : forall a s u,
has_left_fractions a s -> inc u s ->
are_inverse (left_frac_cat a s) (lf_taut a s u) (lf_inverse a s u).

```

```

Lemma is_lf_symbol_arrow_rep : forall a s y,
has_left_fractions a s ->
mor (left_frac_cat a s) y ->
is_lf_symbol a s (arrow_rep y).

```

```

Lemma lf_class_arrow_rep : forall a s y,
has_left_fractions a s ->
mor (left_frac_cat a s) y ->
lf_class a s (arrow_rep y) = y.

```

```

Lemma lf_source_arrow_rep : forall y,
(exists a, exists s, (has_left_fractions a s &
mor (left_frac_cat a s) y)) ->
source (arrow_rep y) = source y.

```

```

Lemma lf_target_arrow_rep : forall y,
(exists a, exists s, (has_left_fractions a s &
mor (left_frac_cat a s) y)) ->
target (arrow_rep y) = target y.

```

```

Lemma comp_etc_arrow_rep : forall a s y,
has_left_fractions a s ->
mor (left_frac_cat a s) y ->
comp (left_frac_cat a s)
(lf_inverse a s (lf_backward (arrow_rep y)))
(lf_taut a s (lf_forward (arrow_rep y))) = y.

```

```

Lemma comp_etc_lf_symbol : forall a s k,
has_left_fractions a s ->

```

```

is_lf_symbol a s k ->
comp (left_frac_cat a s)
(lf_inverse a s (lf_backward k))
(lf_taut a s (lf_forward k)) = lf_class a s k.

```

```

Lemma lfc_mor_expression : forall a s y,
has_left_fractions a s ->
mor (left_frac_cat a s) y ->
(exists u, exists v, (mor a u & inc v s & target u = target v &
y = comp (left_frac_cat a s) (lf_inverse a s v) (lf_taut a s u))).

```

```

Definition lf_proj a s :=
Functor.create a (left_frac_cat a s) (lf_taut a s).

```

```

Lemma source_lf_proj : forall a s,
source (lf_proj a s) = a.

```

```

Lemma target_lf_proj : forall a s,
target (lf_proj a s) = (left_frac_cat a s).

```

```

Lemma lf_proj_property : forall a s,
has_left_fractions a s ->
Functor.property a (left_frac_cat a s) (fun (x:E)=> x)
(lf_taut a s).

```

```

Lemma lf_proj_axioms : forall a s,
has_left_fractions a s ->
Functor.axioms (lf_proj a s).

```

```

Lemma fob_lf_proj : forall a s x,
has_left_fractions a s -> ob a x ->
fob (lf_proj a s) x = x.

```

```

Lemma fmor_lf_proj : forall a s y,
has_left_fractions a s -> mor a y ->
fmor (lf_proj a s) y = lf_taut a s y.

```

```

Lemma invertible_fmor_lf_proj : forall a s y,
has_left_fractions a s -> inc y s ->
invertible (left_frac_cat a s) (fmor (lf_proj a s) y).

```

```

Lemma inverse_fmor_lf_proj : forall a s y,
has_left_fractions a s -> inc y s ->
inverse (left_frac_cat a s) (fmor (lf_proj a s) y) =
lf_inverse a s y.

```

```

Definition lf_dotted_situation a s f :=
has_left_fractions a s &
Functor.axioms f &
source f = a &
(forall y, inc y s -> invertible (target f) (fmor f y)).

```

```

Definition lf_symb_dot_situation a s f u :=
lf_dotted_situation a s f &
is_lf_symbol a s u.

```

```

Definition lf_symb_dot_facts a s f u:=
lf_symb_dot_situation a s f u &
is_lf_symbol a s u &
lf_dotted_situation a s f &
has_left_fractions a s &
Functor.axioms f &
source f = a &
(forall y, inc y s -> invertible (target f) (fmor f y)) &
mor (source f) (lf_forward u) &
mor (source f) (lf_backward u) &
inc (lf_backward u) s &
mor (target f) (fmor f (lf_forward u)) &
mor (target f) (fmor f (lf_backward u)) &
lf_symbol_like u &
source (fmor f (lf_forward u)) = fob f (source u) &
source (fmor f (lf_backward u)) = fob f (target u) &
target (fmor f (lf_forward u)) = fob f (lf_vertex u) &
target (fmor f (lf_backward u)) = fob f (lf_vertex u) &
invertible (target f) (fmor f (lf_backward u)) &
mor (target f) (inverse (target f) (fmor f (lf_backward u))) &
source (inverse (target f) (fmor f (lf_backward u))) =
fob f (lf_vertex u).

```

```

Lemma lf_symb_dot_situation_rw : forall a s f u,
lf_symb_dot_situation a s f u = lf_symb_dot_facts a s f u.

```

```

Definition lf_symb_dot (a s:E) f u :=
comp (target f)
(inverse (target f) (fmor f (lf_backward u)))
(fmor f (lf_forward u)).

```

```

Lemma source_lf_symb_dot : forall a s f u,
lf_symb_dot_situation a s f u ->
source (lf_symb_dot a s f u) = fob f (source u).

```

```

Lemma target_lf_symb_dot : forall a s f u,

```

```
lf_symb_dot_situation a s f u ->
target (lf_symb_dot a s f u) = fob f (target u).
```

```
Lemma mor_lf_symb_dot : forall a s f u,
lf_symb_dot_situation a s f u ->
mor (target f) (lf_symb_dot a s f u).
```

```
Lemma lf_symb_dot_beyond_invariant : forall a s f u v,
lf_symb_dot_situation a s f u ->
lf_beyond a s u v ->
lf_symb_dot a s f u = lf_symb_dot a s f v.
```

```
Lemma lf_symb_dot_equiv_invariant : forall a s f u v,
lf_symb_dot_situation a s f u ->
lf_equiv a s u v ->
lf_symb_dot a s f u = lf_symb_dot a s f v.
```

```
Lemma lf_symb_dot_make_comp : forall a s f u v w,
lf_symb_dot_situation a s f u ->
lf_symb_dot_situation a s f v ->
fills_in a s u v w ->
lf_symb_dot a s f (lf_make_comp a s u v w) =
comp (target f) (lf_symb_dot a s f u) (lf_symb_dot a s f v).
```

```
Lemma lf_symb_dot_comp_rep : forall a s f u v,
lf_symb_dot_situation a s f u ->
lf_symb_dot_situation a s f v ->
source u = target v ->
lf_symb_dot a s f (lf_comp_rep a s u v) =
comp (target f) (lf_symb_dot a s f u) (lf_symb_dot a s f v).
```

```
Lemma lf_symb_dot_id_rep : forall a s f x,
lf_dotted_situation a s f ->
ob a x ->
lf_symb_dot a s f (lf_id_rep a x) = id (target f) (fob f x).
```

```
Definition lf_dotted a s f :=
Functor.create (left_frac_cat a s) (target f)
(fun u => lf_symb_dot a s f (arrow_rep u)).
```

```
Lemma source_lf_dotted : forall a s f,
source (lf_dotted a s f) = left_frac_cat a s.
```

```
Lemma target_lf_dotted : forall a s f,
target (lf_dotted a s f) = target f.
```

```
Lemma lf_equiv_arrow_rep_lf_class : forall a s u,
```



```

has_left_fractions a s ->
is_lf_symbol a s u ->
lf_equiv a s u (arrow_rep (lf_class a s u)).

```

```

Lemma lf_dotted_property : forall a s f,
lf_dotted_situation a s f ->
Functor.property (left_frac_cat a s) (target f)
(fob f) (fun u => lf_symb_dot a s f (arrow_rep u)).

```

```

Lemma lf_dotted_axioms : forall a s f,
lf_dotted_situation a s f ->
Functor.axioms (lf_dotted a s f).

```

```

Lemma fob_lf_dotted : forall a s f x,
lf_dotted_situation a s f ->
ob a x ->
fob (lf_dotted a s f) x = fob f x.

```

```

Lemma fmor_lf_dotted : forall a s f u,
mor (left_frac_cat a s) u ->
fmor (lf_dotted a s f) u = lf_symb_dot a s f (arrow_rep u).

```

```

Lemma fmor_lf_dotted_lf_class : forall a s f u,
lf_dotted_situation a s f ->
is_lf_symbol a s u ->
fmor (lf_dotted a s f) (lf_class a s u) = lf_symb_dot a s f u.

```

```

Lemma fmor_lf_dotted_lf_taut : forall a s f y,
lf_dotted_situation a s f ->
mor a y ->
fmor (lf_dotted a s f) (lf_taut a s y) = fmor f y.

```

```

Lemma fmor_lf_dotted_lf_inverse : forall a s f y,
lf_dotted_situation a s f ->
inc y s ->
fmor (lf_dotted a s f) (lf_inverse a s y) =
inverse (target f) (fmor f y).

```

```

Lemma fcompose_lf_dotted_lf_proj : forall a s f,
lf_dotted_situation a s f ->
fcompose (lf_dotted a s f) (lf_proj a s) = f.

```

```

Lemma lf_dotted_like_unique : forall a s g h,
has_left_fractions a s ->
Functor.axioms g -> Functor.axioms h ->

```

```
source g = left_frac_cat a s ->  
source h = left_frac_cat a s ->  
(fcompose g (lf_proj a s) = fcompose h (lf_proj a s)) ->  
g = h.
```

```
End Left_Fraction_Category.
```

## 6. THE FILE gzloc.v

Require Export left\_fractions.

## 6.1. The module GZ\_Localization.

Module GZ\_Localization.

Export GZ\_Thm.

Export Left\_Fraction\_Category.

Lemma localizing\_system\_recall : forall a s,  
 localizing\_system a s =  
 (Category.axioms a &  
 (forall u, inc u s -> mor a u)).

Lemma multiplicative\_system\_recall : forall a s,  
 multiplicative\_system a s =  
 (localizing\_system a s &  
 (forall y z, inc y s -> inc z s ->  
 source y = target z -> inc (comp a y z) s)).

Definition localizes a s f :=  
 localizing\_system a s &  
 Functor.axioms f &  
 source f = a &  
 (forall y, inc y s -> invertible (target f) (fmor f y)).

Definition completes\_triangle f g h :=  
 Functor.axioms f & Functor.axioms g & Functor.axioms h &  
 source f = source g & source h = target f &  
 target h = target g & fcompose h f = g.

Definition dotted\_choice f g :=  
 choose (completes\_triangle f g).

Lemma completes\_triangle\_dotted\_choice : forall f g,  
 completes\_triangle f g (dotted\_choice f g) =  
 (exists h, (completes\_triangle f g h)).

Definition is\_localization a s f :=  
 localizes a s f &  
 (forall g, (localizes a s g ->  
 completes\_triangle f g (dotted\_choice f g))) &  
 (forall g h, completes\_triangle f g h ->  
 h = dotted\_choice f g).

Lemma dotted\_choice\_axioms : forall f g,

```
(exists a, exists s, (is_localization a s f & localizes a s g)) ->
Functor.axioms (dotted_choice f g).
```

```
Lemma source_dotted_choice : forall f g,
(exists a, exists s, (is_localization a s f & localizes a s g)) ->
source (dotted_choice f g) = target f.
```

```
Lemma target_dotted_choice : forall f g,
(exists a, exists s, (is_localization a s f & localizes a s g)) ->
target (dotted_choice f g) = target g.
```

```
Lemma fcompose_dotted_choice : forall f g,
(exists a, exists s, (is_localization a s f & localizes a s g)) ->
fcompose (dotted_choice f g) f = g.
```

```
Lemma localizes_fcompose : forall a s f g,
localizes a s f -> Functor.axioms g -> source g = target f ->
localizes a s (fcompose g f).
```

```
Lemma eq_dotted_choice : forall f g h,
completes_triangle f g h ->
(exists a, exists s, (is_localization a s f)) ->
h = dotted_choice f g.
```

```
Lemma fcompose_dotted_choice_dotted_choice : forall f g h,
(exists a, exists s, (is_localization a s f & is_localization a s g &
localizes a s h)) ->
fcompose (dotted_choice g h) (dotted_choice f g) =
dotted_choice f h.
```

```
Lemma dotted_choice_refl : forall f,
(exists a, exists s, (is_localization a s f)) ->
dotted_choice f f = fidentity (target f).
```

```
Lemma are_finverse_dotted_choice : forall f g,
(exists a, exists s,
(is_localization a s f & is_localization a s g)) ->
are_finverse (dotted_choice f g) (dotted_choice g f).
```

```
Lemma has_finverse_dotted_choice : forall f g,
(exists a, exists s,
(is_localization a s f & is_localization a s g)) ->
has_finverse (dotted_choice f g).
```

```
Lemma finverse_dotted_choice : forall f g,
(exists a, exists s,
```

(is\_localization a s f & is\_localization a s g)) ->  
finverse (dotted\_choice f g) = dotted\_choice g f.

Lemma is\_localization\_gz\_proj : forall a s,  
localizing\_system a s ->  
is\_localization a s (gz\_proj a s).

Lemma is\_localization\_lf\_proj : forall a s,  
has\_left\_fractions a s ->  
is\_localization a s (lf\_proj a s).

Lemma are\_finverse\_dotted\_choice\_gz\_proj\_lf\_proj : forall a s,  
has\_left\_fractions a s ->  
are\_finverse (dotted\_choice (gz\_proj a s) (lf\_proj a s))  
(dotted\_choice (lf\_proj a s) (gz\_proj a s)).

Definition oppms s := Image.create s flip.

Lemma inc\_oppms : forall s y,  
inc y (oppms s) = inc (flip y) s.

Lemma oppms\_oppms : forall s, oppms (oppms s) = s.

Lemma localizing\_system\_oppms : forall a s,  
localizing\_system (opp a) (oppms s) = localizing\_system a s.

Lemma multiplicative\_system\_oppms : forall a s,  
multiplicative\_system (opp a) (oppms s) =  
multiplicative\_system a s.

Lemma are\_inverse\_opp : forall a u v,  
are\_inverse (opp a) (flip u) (flip v) =  
are\_inverse a u v.

Lemma invertible\_opp : forall a u,  
invertible (opp a) (flip u) =  
invertible a u.

Lemma localizes\_oppms\_oppf : forall a s f,  
localizes (opp a) (oppms s) (oppf f) =  
localizes a s f.

Lemma completes\_triangle\_oppf : forall f g h,  
completes\_triangle (oppf f) (oppf g) (oppf h) =  
completes\_triangle f g h.

```

Lemma is_localization_oppms_oppf : forall a s f,
is_localization (opp a) (oppms s) (oppf f) =
is_localization a s f.

```

```

Lemma has_left_fractions_rw : forall a s,
has_left_fractions a s = (
multiplicative_system a s
&
(forall x, ob a x -> inc (id a x) s)
&
(forall r g, inc r s -> mor a g -> source r = source g ->
exists p, exists q, (mor a p & inc q s
& target p = target q & source p = target r
& source q = target g & comp a q g =
comp a p r))
&
(forall v r t, inc v s -> mor a r -> mor a t ->
source r = target v -> source t = target v ->
comp a r v = comp a t v ->
exists w, (inc w s & source w = target r & source w = target t
& comp a w r = comp a w t))).

```

```

Definition has_right_fractions a s :=
multiplicative_system a s
&
(forall x, ob a x -> inc (id a x) s)
&
(forall r g, inc r s -> mor a g -> target r = target g ->
exists p, exists q, (mor a q & inc p s &
source p = source q & target p = source g &
target q = source r & comp a g p =
comp a r q))
&
(forall v r t, inc v s -> mor a r -> mor a t ->
source v = target r -> source v = target t ->
comp a v r = comp a v t ->
exists w, (inc w s & source r = target w & source t = target w
& comp a r w = comp a t w)).

```

```

Lemma has_right_fractions_rw : forall a s,
has_right_fractions a s =
has_left_fractions (opp a) (oppms s).

```

```

Definition right_frac_cat a s :=
opp (left_frac_cat (opp a) (oppms s)).

```

Definition rf\_proj a s :=  
 oppf (lf\_proj (opp a) (oppms s)).

Lemma right\_frac\_cat\_axioms : forall a s,  
 has\_right\_fractions a s ->  
 Category.axioms (right\_frac\_cat a s).

Lemma rf\_proj\_axioms : forall a s,  
 has\_right\_fractions a s ->  
 Functor.axioms (rf\_proj a s).

Lemma source\_rf\_proj : forall a s,  
 has\_right\_fractions a s ->  
 source (rf\_proj a s) = a.

Lemma target\_rf\_proj : forall a s,  
 has\_right\_fractions a s ->  
 target (rf\_proj a s) = right\_frac\_cat a s.

Lemma is\_localization\_rf\_proj : forall a s,  
 has\_right\_fractions a s ->  
 is\_localization a s (rf\_proj a s).

Definition lf\_vee a s p q :=  
 localizing\_system a s &  
 mor a p & inc q s & target p = target q.

Definition lf\_vee\_image f p q :=  
 comp (target f) (inverse (target f) (fmor f q))  
 (fmor f p).

Definition lf\_vee\_equivalent a s p q r t :=  
 lf\_vee a s p q & lf\_vee a s r t &  
 source p = source r & source q = source t &  
 (exists y, exists z, (mor a y & mor a z &  
 source y = target p & source z = target r &  
 target y = target z &  
 comp a y p = comp a z r &  
 comp a y q = comp a z t &  
 inc (comp a y q) s)).

Definition left\_fraction\_description a s f :=  
 localizes a s f &  
 ob\_iso f  
 &

```

(forall y, mor (target f) y ->
exists p, exists q, (lf_vee a s p q &
y = lf_vee_image f p q))
&
(forall p q r t, lf_vee a s p q -> lf_vee a s r t ->
(lf_vee_image f p q = lf_vee_image f r t) ->
lf_vee_equivalent a s p q r t).

```

```

Definition rf_wedge a s p q :=
localizing_system a s &
mor a p & inc q s & source p = source q.

```

```

Definition rf_wedge_image f p q :=
comp (target f) (fmor f p) (inverse (target f) (fmor f q)).

```

```

Definition rf_wedge_equivalent a s p q r t :=
rf_wedge a s p q & rf_wedge a s r t &
target p = target r & target q = target t &
(exists y, exists z, (mor a y & mor a z &
target y = source p & target z = source r &
source y = source z &
comp a p y = comp a r z &
comp a q y = comp a t z &
inc (comp a q y) s)).

```

```

Definition right_fraction_description a s f :=
localizes a s f &
ob_iso f
&
(forall y, mor (target f) y ->
exists p, exists q, (rf_wedge a s p q &
y = rf_wedge_image f p q))
&
(forall p q r t, rf_wedge a s p q -> rf_wedge a s r t ->
(rf_wedge_image f p q = rf_wedge_image f r t) ->
rf_wedge_equivalent a s p q r t).

```

```

Lemma lf_vee_image_lf_proj : forall a s p q,
has_left_fractions a s ->
lf_vee a s p q ->
lf_vee_image (lf_proj a s) p q =
lf_class a s (lf_symbol p q).

```

```

Lemma left_fraction_description_lf_proj : forall a s,
has_left_fractions a s ->

```



left\_fraction\_description a s (lf\_proj a s).

Lemma mor\_lf\_vee\_image : forall f p q,  
 (exists a, exists s, (lf\_vee a s p q &  
 localizes a s f)) ->  
 mor (target f) (lf\_vee\_image f p q).

Lemma lf\_vee\_image\_fcompose : forall f g p q,  
 Functor.axioms f -> Functor.axioms g -> source f = target g ->  
 mor (source g) p -> mor (source g) q ->  
 invertible (source f) (fmor g q) ->  
 target p = target q ->  
 lf\_vee\_image (fcompose f g) p q =  
 fmor f (lf\_vee\_image g p q).

Lemma left\_fraction\_description\_invariant : forall a s f g,  
 left\_fraction\_description a s f ->  
 has\_finverse g ->  
 source g = target f ->  
 left\_fraction\_description a s (fcompose g f).

Lemma left\_fraction\_description\_for\_loc : forall a s f,  
 has\_left\_fractions a s ->  
 is\_localization a s f ->  
 left\_fraction\_description a s f.

Lemma left\_fraction\_description\_gz\_proj :  
 forall a s,  
 has\_left\_fractions a s ->  
 left\_fraction\_description a s (gz\_proj a s).

Lemma inverse\_opp : forall a u,  
 invertible a (flip u) ->  
 inverse (opp a) u = flip (inverse a (flip u)).

Lemma right\_fraction\_description\_rw : forall a s f,  
 right\_fraction\_description a s f =  
 left\_fraction\_description (opp a) (oppms s) (oppf f).

Lemma right\_fraction\_description\_for\_loc : forall a s f,  
 has\_right\_fractions a s ->  
 is\_localization a s f ->  
 right\_fraction\_description a s f.

Lemma right\_fraction\_description\_gz\_proj : forall a s,  
 has\_right\_fractions a s ->  
 right\_fraction\_description a s (gz\_proj a s).

End GZ\_Localization.

## 7. THE FILE lfcx.v

Require Export updateA.  
 Require Export gzloc.

## 7.1. The module Coarse\_Cat.

Module Coarse\_Cat.  
 Export GZ\_Localization.

Definition null\_arrow a b := Arrow.create a b emptyset.

Lemma source\_null\_arrow : forall a b, source (null\_arrow a b) = a.

Lemma target\_null\_arrow : forall a b, target (null\_arrow a b) = b.

Lemma arrow\_null\_arrow : forall a b,  
 arrow (null\_arrow a b) = emptyset.

Definition na\_comp u v := null\_arrow (source v) (target u).

Lemma source\_na\_comp : forall u v, source (na\_comp u v) = source v.

Lemma target\_na\_comp : forall u v, target (na\_comp u v) = target u.

Definition is\_null\_arrow u := u = null\_arrow (source u) (target u).

Lemma is\_null\_arrow\_null\_arrow : forall a b,  
 is\_null\_arrow (null\_arrow a b).

Lemma is\_null\_arrow\_na\_comp : forall u v,  
 is\_null\_arrow (na\_comp u v).

Lemma null\_arrow\_extensionality : forall a b,  
 is\_null\_arrow a -> is\_null\_arrow b ->  
 source a = source b -> target a = target b -> a = b.

Lemma na\_comp\_assoc : forall a b c,  
 na\_comp (na\_comp a b) c = na\_comp a (na\_comp b c).

Lemma na\_comp\_null\_arrow : forall a b c d,  
 na\_comp (null\_arrow a b) (null\_arrow c d) =  
 null\_arrow c b.

Lemma na\_left\_id : forall a u,  
 is\_null\_arrow a ->  
 u = null\_arrow (target a) (target a) ->

na\_comp u a = a.

Lemma na\_right\_id : forall a u,  
 is\_null\_arrow a ->  
 u = null\_arrow (source a) (source a) ->  
 na\_comp a u = a.

Definition coarse\_arrows z :=  
 Image.create (product z z)  
 (fun x => null\_arrow (pr1 x) (pr2 x)).

Lemma inc\_coarse\_arrows : forall u z,  
 inc u (coarse\_arrows z) =  
 (is\_null\_arrow u & inc (source u) z & inc (target u) z).

Definition coarse\_cat z :=  
 Category.Notations.create z (coarse\_arrows z)  
 na\_comp (fun x => null\_arrow x x) emptyset.

Lemma is\_ob\_coarse\_cat : forall x z,  
 is\_ob (coarse\_cat z) x = inc x z.

Lemma is\_mor\_coarse\_cat : forall u z,  
 is\_mor (coarse\_cat z) u =  
 (is\_null\_arrow u & inc (source u) z & inc (target u) z).

Lemma comp\_coarse\_cat1 : forall u v z,  
 is\_mor (coarse\_cat z) u -> is\_mor (coarse\_cat z) v ->  
 source u = target v ->  
 comp (coarse\_cat z) u v = na\_comp u v.

Lemma id\_coarse\_cat : forall x z,  
 inc x z -> id (coarse\_cat z) x = null\_arrow x x.

Lemma coarse\_cat\_axioms : forall z,  
 Category.axioms (coarse\_cat z).

Lemma ob\_coarse\_cat : forall x z,  
 ob (coarse\_cat z) x = inc x z.

Lemma mor\_coarse\_cat : forall u z,  
 mor (coarse\_cat z) u =  
 (is\_null\_arrow u & inc (source u) z & inc (target u) z).

Lemma comp\_coarse\_cat : forall u v z,  
 mor (coarse\_cat z) u -> mor (coarse\_cat z) v ->

```

source u = target v ->
comp (coarse_cat z) u v = na_comp u v.

```

```

End Coarse_Cat.

```

## 7.2. The module Lf\_Counterexample.

```

Module Lf_Counterexample.

```

```

Export Coarse_Cat.

```

```

Export Left_Fraction_Category.

```

```

Inductive cx_ob : E -> Prop :=

```

```

is0 : cx_ob (R 0) |

```

```

is1 : cx_ob (R 1) |

```

```

is2 : cx_ob (R 2).

```

```

Ltac is_cx_ob :=

```

```

match goal with

```

```

| |- cx_ob (R 0) => ap is0

```

```

| |- cx_ob (R 1) => ap is1

```

```

| |- cx_ob (R 2) => ap is2

```

```

| _ => fail end.

```

```

Lemma cx_ob_rw : forall x,

```

```

cx_ob x = inc x (R 3).

```

```

Notation na00 := (null_arrow (R 0) (R 0)).

```

```

Notation na11 := (null_arrow (R 1) (R 1)).

```

```

Notation na22 := (null_arrow (R 2) (R 2)).

```

```

Notation na01 := (null_arrow (R 0) (R 1)).

```

```

Notation na02 := (null_arrow (R 0) (R 2)).

```

```

Notation na12 := (null_arrow (R 1) (R 2)).

```

```

Notation na21 := (null_arrow (R 2) (R 1)).

```

```

Inductive cx_mor : E -> Prop :=

```

```

is00 : cx_mor na00 |

```

```

is11 : cx_mor na11 |

```

```

is22 : cx_mor na22 |

```

```

is01 : cx_mor na01 |

```

```

is02 : cx_mor na02 |

```

```

is12 : cx_mor na12 |

```

```

is21 : cx_mor na21 .

```

```

Ltac is_cx_mor :=

```

```

match goal with

```

```

| |- cx_mor na00 => ap is00
| |- cx_mor na11 => ap is11
| |- cx_mor na22 => ap is22
| |- cx_mor na01 => ap is01
| |- cx_mor na02 => ap is02
| |- cx_mor na12 => ap is12
| |- cx_mor na21 => ap is21
| _ => fail end.

```

```

Lemma cx_mor_mor : forall u, cx_mor u ->
mor (coarse_cat (R 3)) u.

```

```

Definition cx := subcategory (coarse_cat (R 3))
cx_ob cx_mor.

```

```

Ltac R_nat_discriminate :=
match goal with
| id1 : R ?X1 = R ?X2 |- _ =>
assert (discr : X1 = X2); [app R_inj | discriminate discr]
| _ => fail
end.

```

```

Lemma cx_property : subcategory_property (coarse_cat (R 3))
cx_ob cx_mor.

```

```

Lemma cx_axioms : Category.axioms cx.

```

```

Lemma ob_cx : forall x, ob cx x = cx_ob x.

```

```

Lemma mor_cx : forall u, mor cx u = cx_mor u.

```

```

Inductive in_cx_sys : E -> Prop :=
sys00 : in_cx_sys na00 |
sys11 : in_cx_sys na11 |
sys22 : in_cx_sys na22 |
sys01 : in_cx_sys na01 |
sys02 : in_cx_sys na02.

```

```

Ltac is_cx_sys :=
match goal with
| |- in_cx_sys na00 => ap sys00
| |- in_cx_sys na11 => ap sys11
| |- in_cx_sys na22 => ap sys22
| |- in_cx_sys na01 => ap sys01
| |- in_cx_sys na02 => ap sys02
| _ => fail end.

```

Definition `cx_sys := Z (morphisms cx) in_cx_sys.`

Lemma `inc_cx_sys : forall u,`  
`inc u cx_sys = in_cx_sys u.`

Lemma `source_null_arrow_eq : forall a b c d,`  
`null_arrow a b = null_arrow c d -> a = c.`

Lemma `target_null_arrow_eq : forall a b c d,`  
`null_arrow a b = null_arrow c d -> b = d.`

Lemma `cx_sys_rw : forall u,`  
`inc u cx_sys = (in_cx_sys u &`  
`mor cx u & cx_mor u & ~u=na12 & ~u=na21).`

Lemma `comp_cx : forall u v,`  
`mor cx u -> mor cx v -> source u = target v ->`  
`comp cx u v = na_comp u v.`

Lemma `id_cx : forall x,`  
`ob cx x -> id cx x = null_arrow x x.`

Lemma `localizing_system_cx_sys:`  
`localizing_system cx cx_sys.`

Lemma `multiplicative_system_cx_sys:`  
`multiplicative_system cx cx_sys.`

Lemma `has_left_fractions_cx_sys :`  
`has_left_fractions cx cx_sys.`

Definition `lf110 := lf_symbol na11 na01.`

Definition `lf120 := lf_symbol na12 na02.`

Lemma `lff110 : lf_forward lf110 = na11.`

Lemma `lfb110 : lf_backward lf110 = na01.`

Lemma `lff120 : lf_forward lf120 = na12.`

Lemma `lfb120 : lf_backward lf120 = na02.`

Lemma `is_lf_symbol_lf110 : is_lf_symbol cx cx_sys lf110.`

Lemma is\_lf\_symbol\_lf120 : is\_lf\_symbol cx cx\_sys lf120.

Lemma lfv110 : lf\_vertex lf110 = R 1.

Lemma lfv120 : lf\_vertex lf120 = R 2.

Lemma under\_lf110\_same : forall u,  
lf\_under cx cx\_sys lf110 u -> u = lf110.

Lemma under\_lf120\_same : forall u,  
lf\_under cx cx\_sys lf120 u -> u = lf120.

Lemma lf110\_different\_lf120 : ~lf110 = lf120.

Lemma lf\_beyond\_lf110\_lf120 : lf\_beyond cx cx\_sys lf110 lf120.

Lemma beyond\_not\_under\_counterexample :  
exists a, exists s, exists u, exists v,  
(has\_left\_fractions a s &  
is\_lf\_symbol a s u &  
is\_lf\_symbol a s v &  
lf\_beyond a s u v &  
~(exists w, (lf\_under a s u w & lf\_under a s v w))).

End Lf\_Counterexample.



8. THE FILE `infinite.v`

```
Require Export gzloc.
Require Export cardinal.
```

8.1. The module `Infinite`.

```
Module Infinite.
Export UpdateA.
Export Cardinal.
```

```
Lemma is_infinite_rw1 : forall x,
is_infinite x =
(exists f, Transformation.injective x x f &
~(Transformation.surjective x x f)).
```

```
Lemma is_infinite_rw : forall x,
is_infinite x =
(exists f, exists u, (inc u x & Transformation.injective x x f &
(forall v, inc v x -> ~ (f v = u)))).
```

```
Lemma infinite_sub_infinite : forall x,
(exists y, (is_infinite y & sub y x)) -> is_infinite x.
```

```
Lemma is_finite_natural : forall x, inc x nat ->
is_finite x.
```

```
Lemma is_infinite_nat : is_infinite nat.
```

```
Lemma is_finite_cardinality : forall x,
is_finite (cardinality x) = is_finite x.
```

```
Lemma is_infinite_cardinality : forall x,
is_infinite (cardinality x) = is_infinite x.
```

```
Definition natE : E.
```

```
Lemma is_ordinal_nat : is_ordinal nat.
```

```
Lemma inc_cardinality_nat_or_sub_nat_cardinality : forall x,
(inc (cardinality x) nat) \ / (sub nat (cardinality x)).
```

```
Lemma is_finite_inc_cardinality_nat : forall x,
is_finite x = inc (cardinality x) nat.
```

```
Lemma is_infinite_sub_nat_cardinality : forall x,
```

`is_infinite x = sub nat (cardinality x).`

`Definition plus_one x := tack_on x x.`

`Lemma is_ordinal_plus_one : forall x,  
is_ordinal x -> is_ordinal (plus_one x).`

`Lemma plus_one_R : forall (i:nat),  
plus_one (R i) = R (i+1).`

`Lemma inc_plus_one_nat : forall x,  
inc x nat -> inc (plus_one x) nat.`

`Lemma function_V_tack_on_old : forall f x y z,  
Function.axioms f -> ~inc x (domain f) ->  
inc z (domain f) ->  
V z (tack_on f (pair x y)) = V z f.`

`Lemma function_V_tack_on_new : forall f x y z,  
Function.axioms f -> ~inc x (domain f) ->  
z = x ->  
V z (tack_on f (pair x y)) = y.`

`Lemma are_isomorphic_tack_on : forall x y u v,  
~inc y x -> ~inc v u ->  
are_isomorphic x u ->  
are_isomorphic (tack_on x y) (tack_on u v).`

`Lemma nat_are_isomorphic_eq : forall x y,  
inc x nat -> inc y nat -> are_isomorphic x y ->  
x = y.`

`Lemma nat_cardinality_refl : forall x,  
inc x nat -> cardinality x = x.`

`Lemma cardinality_tack_on : forall x y,  
is_finite x -> ~inc y x ->  
cardinality (tack_on x y) = plus_one (cardinality x).`

`Definition number x := Bnat (cardinality x).`

`Lemma R_number : forall x,  
is_finite x -> R (number x) = cardinality x.`

`Lemma cardinality_emptyset :  
cardinality emptyset = emptyset.`

Lemma number\_emptyset : forall x,  
x = emptyset -> number x = 0.

Lemma number\_tack\_on : forall x y,  
is\_finite x -> ~inc y x ->  
number (tack\_on x y) = (number x) + 1.

Definition take\_out x y :=  
complement x (singleton y).

Lemma inc\_take\_out : forall x y z,  
inc z (take\_out x y) =  
(inc z x & ~z=y).

Lemma tack\_on\_take\_out : forall x y,  
inc y x ->  
tack\_on (take\_out x y) y = x.

Lemma not\_inc\_take\_out : forall x y,  
~inc y (take\_out x y).

Lemma subset\_finite : forall x,  
(exists y, (sub x y & is\_finite y)) -> is\_finite x.

Lemma is\_finite\_take\_out : forall x y,  
is\_finite x -> is\_finite (take\_out x y).

Lemma number\_take\_out : forall x y,  
is\_finite x -> inc y x -> number (take\_out x y) = (number x) -1.

Lemma number\_zero\_emptyset : forall x,  
is\_finite x -> number x = 0 -> x = emptyset.

Lemma number\_gt\_zero\_nonempty : forall x,  
is\_finite x -> number x > 0 ->  
(exists y, inc y x).

Lemma finite\_induction: forall (p:EP) x,  
(p emptyset) ->  
(forall y z, p y -> p (tack\_on y z)) ->  
is\_finite x -> p x.

Lemma image\_create\_tack\_on : forall x y f,  
Image.create (tack\_on x y) f =  
tack\_on (Image.create x f) (f y).

Lemma image\_create\_emptyset : forall f,

Image.create emptyset f = emptyset.

Lemma image\_finite : forall x f,  
is\_finite x -> is\_finite (Image.create x f).

Definition is\_ordinal\_pair x :=  
is\_pair x & is\_ordinal (pr1 x) & is\_ordinal (pr2 x).

Definition op\_lex\_lt x y :=  
is\_ordinal\_pair x &  
is\_ordinal\_pair y &  
(ordinal\_lt (pr1 x) (pr1 y) \/  
(pr1 x = pr1 y & ordinal\_lt (pr2 x) (pr2 y))).

Definition op\_lex\_leq x y :=  
is\_ordinal\_pair x &  
is\_ordinal\_pair y &  
(ordinal\_lt (pr1 x) (pr1 y) \/  
(pr1 x = pr1 y & ordinal\_leq (pr2 x) (pr2 y))).

Lemma ordinal\_leq\_refl : forall x,  
is\_ordinal x -> ordinal\_leq x x.

Lemma ordinal\_lt\_leq : forall x y,  
ordinal\_lt x y -> ordinal\_leq x y.

Lemma ordinal\_leq\_rw : forall x y,  
ordinal\_leq x y =  
(ordinal\_lt x y \/  
(x = y & is\_ordinal x)).

Lemma op\_lex\_lt\_leq : forall x y,  
op\_lex\_lt x y -> op\_lex\_leq x y.

Lemma op\_lex\_leq\_refl : forall x,  
is\_ordinal\_pair x -> op\_lex\_leq x x.

Lemma op\_lex\_leq\_rw : forall x y,  
op\_lex\_leq x y =  
(op\_lex\_lt x y \/  
(x = y & is\_ordinal\_pair x)).

Lemma not\_ordinal\_lt\_refl : forall x,  
~(ordinal\_lt x x).

Lemma not\_op\_lex\_lt\_refl : forall x,  
~ (op\_lex\_lt x x).

Lemma ordinal\_lt\_trans : forall x z,  
 (exists y, (ordinal\_lt x y & ordinal\_lt y z)) ->  
 ordinal\_lt x z.

Lemma op\_lex\_lt\_trans : forall x z,  
 (exists y, (op\_lex\_lt x y & op\_lex\_lt y z)) ->  
 op\_lex\_lt x z.

Lemma op\_lex\_leq\_trans : forall x z,  
 (exists y, (op\_lex\_leq x y & op\_lex\_leq y z)) ->  
 op\_lex\_leq x z.

Lemma not\_ordinal\_lt\_lt : forall x y,  
 ordinal\_lt x y -> ordinal\_lt y x -> False.

Lemma not\_op\_lex\_lt\_lt : forall x y,  
 op\_lex\_lt x y -> op\_lex\_lt y x -> False.

Lemma op\_lex\_leq\_leq\_eq : forall x y,  
 op\_lex\_leq x y -> op\_lex\_leq y x -> x = y.

Lemma op\_lex\_dichotomy : forall x y,  
 is\_ordinal\_pair x -> is\_ordinal\_pair y ->  
 (op\_lex\_lt x y \/ op\_lex\_lt y x \/ x = y).

Definition least\_ordinal z := sow (fun x => inc x z).

Lemma is\_ordinal\_least\_ordinal: forall z,  
 is\_ordinal (least\_ordinal z).

Lemma inc\_least\_ordinal\_ex : forall z,  
 (exists y, (inc y z & is\_ordinal y)) ->  
 inc (least\_ordinal z) z.

Definition set\_of\_ordinals z :=  
 (forall y, inc y z -> is\_ordinal y).

Lemma inc\_least\_ordinal : forall z,  
 set\_of\_ordinals z -> nonempty z ->  
 inc (least\_ordinal z) z.

Lemma ordinal\_leq\_least\_ordinal : forall y z,  
 inc y z -> is\_ordinal y ->  
 ordinal\_leq (least\_ordinal z) y.

Definition set\_of\_ordinal\_pairs z :=  
 (forall y, inc y z -> is\_ordinal\_pair y).

Lemma set\_of\_ordinal\_pairs\_relation : forall z,  
set\_of\_ordinal\_pairs z -> is\_relation z.

Lemma set\_of\_ordinals\_domain : forall z,  
set\_of\_ordinal\_pairs z ->  
set\_of\_ordinals (domain z).

Definition column z x:=  
range (Z z (fun y => pr1 y = x)).

Lemma inc\_column : forall z x y,  
is\_relation z ->  
inc y (column z x) = inc (pair x y) z.

Lemma set\_of\_ordinals\_column : forall z x,  
set\_of\_ordinal\_pairs z -> set\_of\_ordinals (column z x).

Lemma nonempty\_domain : forall z,  
nonempty z -> nonempty (domain z).

Lemma nonempty\_column : forall z x,  
is\_relation z ->  
inc x (domain z) -> nonempty (column z x).

Definition op\_lex\_least z :=  
pair (least\_ordinal (domain z))  
(least\_ordinal (column z (least\_ordinal (domain z)))).

Lemma is\_ordinal\_pair\_op\_lex\_least : forall z,  
is\_ordinal\_pair (op\_lex\_least z).

Lemma inc\_op\_lex\_least : forall z,  
set\_of\_ordinal\_pairs z -> nonempty z ->  
inc (op\_lex\_least z) z.

Lemma pr1\_op\_lex\_least : forall z,  
pr1 (op\_lex\_least z) = least\_ordinal (domain z).

Lemma pr2\_op\_lex\_least : forall z,  
pr2 (op\_lex\_least z) =  
least\_ordinal (column z (least\_ordinal (domain z))).

Lemma op\_lex\_leq\_op\_lex\_least : forall z y,  
set\_of\_ordinal\_pairs z -> inc y z ->  
op\_lex\_leq (op\_lex\_least z) y.

Definition `op_lex z := Order_notation.create z op_lex_leq`.

Lemma `U_op_lex : forall z, U (op_lex z) = z`.

Lemma `leq_op_lex : forall z u v:E, leq (op_lex z) u v = (inc u z & inc v z & op_lex_leq u v)`.

Lemma `op_lex_axioms : forall z:E, set_of_ordinal_pairs z -> Order.Definitions.axioms (op_lex z)`.

Lemma `is_linear_op_lex : forall z:E, set_of_ordinal_pairs z -> is_linear (op_lex z)`.

Lemma `is_well_ordered_op_lex : forall z, set_of_ordinal_pairs z -> is_well_ordered (op_lex z)`.

Definition `prod_reflexive z := are_isomorphic z (Cartesian.product z z)`.

Lemma `prod_reflexive_invariant : forall y z, are_isomorphic y z -> prod_reflexive y -> prod_reflexive z`.

Definition `sub_prod_reflexive z := forall y, sub y z -> is_infinite y -> prod_reflexive y`.

Lemma `sub_prod_reflexive_rw : forall z, sub_prod_reflexive z = (forall y, iso_sub y z -> is_infinite y -> prod_reflexive y)`.

Lemma `sub_prod_reflexive_invariant : forall y z, iso_sub y z -> sub_prod_reflexive z -> sub_prod_reflexive y`.

Lemma `sub_prod_reflexive_prod_reflexive : forall z, is_infinite z -> sub_prod_reflexive z -> prod_reflexive z`.

Lemma `finite_sub_prod_reflexive : forall z, is_finite z -> sub_prod_reflexive z`.

Definition `op_triangle z := Z (Cartesian.product z z) (fun y => (ordinal_leq (pr2 y) (pr1 y)))`.

Lemma `inc_op_triangle : forall y z, is_ordinal z -> inc y (op_triangle z) =`

(is\_ordinal\_pair y & ordinal\_lt (pr1 y) z &  
ordinal\_leq (pr2 y) (pr1 y)).

Lemma set\_of\_ordinal\_pairs\_op\_triangle : forall z,  
is\_ordinal z -> set\_of\_ordinal\_pairs (op\_triangle z).

Lemma iso\_sub\_op\_triangle : forall z,  
is\_ordinal z -> iso\_sub z (op\_triangle z).

Lemma sub\_op\_triangle\_product : forall z,  
sub (op\_triangle z) (product z z).

Lemma same\_cardinality\_op\_triangle : forall z,  
is\_infinite z -> is\_ordinal z -> sub\_prod\_reflexive z ->  
cardinality (op\_triangle z) = cardinality z.

Lemma op\_triangle\_sub : forall y z,  
sub y z -> sub (op\_triangle y) (op\_triangle z).

Definition op\_triangle\_lex z := op\_lex (op\_triangle z).

Lemma U\_op\_triangle\_lex : forall z,  
U (op\_triangle\_lex z) = op\_triangle z.

Lemma leq\_op\_triangle\_lex : forall z u v,  
leq (op\_triangle\_lex z) u v = (inc u (op\_triangle z) &  
inc v (op\_triangle z) & op\_lex\_leq u v).

Lemma is\_well\_ordered\_op\_triangle\_lex : forall z,  
is\_ordinal z -> is\_well\_ordered (op\_triangle\_lex z).

Definition is\_cardinal z :=  
cardinality z = z.

Lemma is\_cardinal\_rw : forall z,  
is\_cardinal z = (is\_ordinal z & (forall y, ordinal\_lt y z ->  
ordinal\_lt (cardinality y) (cardinality z))).

Lemma is\_cardinal\_rw2 : forall z,  
is\_cardinal z = (is\_ordinal z & (forall y, is\_ordinal y ->  
iso\_sub z y -> ordinal\_leq z y)).

Lemma is\_cardinal\_cardinality : forall z,  
is\_cardinal (cardinality z).

Lemma infinite\_cardinality\_tack\_on : forall x y,  
is\_infinite x -> cardinality (tack\_on x y) = cardinality x.



Lemma cardinality\_lt\_ordinal\_lt : forall y z,  
 is\_ordinal y -> is\_ordinal z ->  
 ordinal\_lt (cardinality y) (cardinality z) -> ordinal\_lt y z.

Lemma ord\_card\_leq\_criterion : forall y z,  
 is\_ordinal y -> is\_cardinal z ->  
 (forall x, inc x y -> ordinal\_lt x z) ->  
 ordinal\_leq y z.

Lemma inc\_punctured\_downward\_subset : forall a x y,  
 Order.Definitions.axioms a -> inc x (U a) ->  
 inc y (punctured\_downward\_subset a x) = lt a y x.

Lemma punctured\_downward\_subset\_wo\_avatar : forall a x,  
 is\_well\_ordered a -> inc x (U a) ->  
 Transformation.bijective (punctured\_downward\_subset a x)  
 (wo\_avatar a x) (wo\_avatar a).

Lemma cardinality\_punctured\_downward\_subset : forall a x,  
 is\_well\_ordered a -> inc x (U a) ->  
 (cardinality (punctured\_downward\_subset a x) =  
 cardinality (wo\_avatar a x)).

Lemma card\_leq\_criterion : forall a z,  
 is\_well\_ordered a -> is\_cardinal z ->  
 (forall x, inc x (U a) ->  
 ordinal\_lt (cardinality (punctured\_downward\_subset a x)) z) ->  
 ordinal\_leq (cardinality (U a)) z.

Lemma sub\_punctured\_downward\_cone\_op\_triangle : forall y z,  
 is\_ordinal z -> inc y (op\_triangle z) ->  
 sub (punctured\_downward\_subset (op\_triangle\_lex z) y)  
 (op\_triangle (tack\_on (pr1 y) (pr1 y))).

Definition sub\_prod\_reflexive\_below z :=  
 is\_cardinal z & is\_infinite z &  
 (forall y, ordinal\_lt y z -> sub\_prod\_reflexive y).

Lemma ordinal\_leq\_lt\_trans : forall x y,  
 (exists z, (ordinal\_leq x z & ordinal\_lt z y)) ->  
 ordinal\_lt x y.

Lemma is\_finite\_union2 : forall a b, is\_finite a ->  
 is\_finite b -> is\_finite (union2 a b).

Lemma product\_singleton\_isomorphic : forall a x,  
are\_isomorphic a (product a (singleton x)).

Lemma is\_finite\_product : forall a b,  
is\_finite a -> is\_finite b -> is\_finite (product a b).

Lemma infinite\_tack\_on\_isomorphic: forall z x,  
is\_infinite z -> are\_isomorphic z (tack\_on z x).

Lemma sprb\_op\_triangle\_leq : forall z,  
sub\_prod\_reflexive\_below z ->  
ordinal\_leq (cardinality (op\_triangle z)) z.

Lemma inc\_tack\_on\_infinite\_cardinal : forall z x,  
is\_cardinal z -> is\_infinite z -> inc x z -> inc (tack\_on x x) z.

Lemma tack\_on\_injective : forall a b,  
is\_ordinal a -> is\_ordinal b ->  
tack\_on a a = tack\_on b b -> a = b.

Lemma iso\_sub\_double\_op\_triangle : forall z,  
is\_cardinal z -> is\_infinite z ->  
iso\_sub (product z (R 2)) (op\_triangle z).

Lemma finite\_ordinal\_cardinal : forall x,  
is\_ordinal x -> is\_finite x -> is\_cardinal x.

Lemma ordinal\_lt\_finite\_infinite : forall x y,  
is\_ordinal x -> is\_ordinal y -> is\_finite x  
-> is\_infinite y -> ordinal\_lt x y.

Lemma dumb\_sub\_product : forall z,  
is\_ordinal z -> is\_infinite z ->  
sub (product z (R 2)) (product z z).

Definition lim\_sprb z :=  
sub\_prod\_reflexive\_below (cardinality z) \/  
sub\_prod\_reflexive z .

Lemma cardinality\_double : forall z,  
is\_ordinal z -> is\_infinite z ->  
lim\_sprb z ->  
cardinality (product z (R 2)) = cardinality z.

Lemma cardinal\_leq\_image : forall z f,  
ordinal\_leq (cardinality (Image.create z f)) (cardinality z).

Lemma cardinal\_leq\_iso\_sub : forall a b,  
ordinal\_leq (cardinality a) (cardinality b) ->  
iso\_sub a b.

Lemma ordinal\_leq\_cardinality\_union2\_double : forall a b z,  
ordinal\_leq (cardinality a) (cardinality z) ->  
ordinal\_leq (cardinality b) (cardinality z) ->  
ordinal\_leq (cardinality (union2 a b))  
(cardinality (product z (R 2))).

Lemma lim\_sprb\_cardinal\_union2 : forall a b z,  
lim\_sprb z -> is\_infinite z ->  
ordinal\_leq (cardinality a) (cardinality z) ->  
ordinal\_leq (cardinality b) (cardinality z) ->  
ordinal\_leq (cardinality (union2 a b)) (cardinality z).

Definition reflect\_pair x := pair (pr2 x) (pr1 x).

Definition reflected\_op\_triangle z :=  
Image.create (op\_triangle z) reflect\_pair.

Lemma cardinal\_leq\_reflected\_op\_triangle : forall z,  
ordinal\_leq (cardinality (reflected\_op\_triangle z))  
(cardinality (op\_triangle z)).

Lemma inc\_ordinal\_prod\_itself\_or : forall z x,  
is\_ordinal z ->  
inc x (product z z) =  
(inc x (op\_triangle z) \/ inc x (reflected\_op\_triangle z)).

Lemma ordinal\_prod\_itself\_decomposition : forall z,  
is\_ordinal z ->  
product z z = union2 (op\_triangle z) (reflected\_op\_triangle z).

Lemma lim\_sprb\_cardinal\_product : forall z,  
lim\_sprb z -> is\_infinite z ->  
ordinal\_leq (cardinality (product z z)) (cardinality z).

Lemma lim\_sprb\_sub\_prod\_reflexive : forall z,  
lim\_sprb z -> sub\_prod\_reflexive z.

Lemma ordinal\_lt\_leq\_trans : forall x z,  
(exists y, ordinal\_lt x y & ordinal\_leq y z) ->  
ordinal\_lt x z.

Lemma all\_ordinals\_sub\_prod\_reflexive : forall z,  
is\_ordinal z -> sub\_prod\_reflexive z.

Lemma all\_sub\_prod\_reflexive : forall z,  
sub\_prod\_reflexive z.

Lemma are\_isomorphic\_product : forall z,  
is\_infinite z -> are\_isomorphic z (product z z).

Lemma cardinality\_product\_infinite : forall z,  
is\_infinite z -> cardinality (product z z) = cardinality z.

Lemma cardinality\_union2\_infinite : forall a b z,  
is\_infinite z ->  
ordinal\_leq (cardinality a) (cardinality z) ->  
ordinal\_leq (cardinality b) (cardinality z) ->  
ordinal\_leq (cardinality (union2 a b)) (cardinality z).

Lemma iso\_sub\_union2 : forall a b z,  
is\_infinite z -> iso\_sub a z -> iso\_sub b z ->  
iso\_sub (union2 a b) z.

Lemma russell : forall z, ~(iso\_sub (powerset z) z).

Lemma cardinal\_lt\_powerset : forall z,  
ordinal\_lt (cardinality z) (cardinality (powerset z)).

End Infinite.

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