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# Hjelmslev Geometry of Mutually Unbiased Bases

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## Abstract

The basic combinatorial properties of a complete set of mutually unbiased bases (MUBs) of a  $q$ -dimensional Hilbert space  $\mathcal{H}_q$ ,  $q = p^r$  with  $p$  being a prime and  $r$  a positive integer, are shown to be qualitatively mimicked by the configuration of points lying on a proper conic in a projective Hjelmslev plane defined over a Galois ring of characteristic  $p^2$  and rank  $r$ . The  $q$  vectors of a basis of  $\mathcal{H}_q$  correspond to the  $q$  points of a (so-called) neighbour class and the  $q + 1$  MUBs answer to the total number of (pairwise disjoint) neighbour classes on the conic.

**Keywords:** Projective Hjelmslev Planes – Proper Conics – Mutually Unbiased Bases

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As it is well known (see, e.g., [1],[2] and references therein), a complete set of MUBs in  $\mathcal{H}_q$ ,  $q = p^r$  with  $p$  being a prime and  $r$  a positive integer, features altogether  $q^2 + q$  vectors that are partitioned into  $q + 1$  bases of cardinality  $q$ . In our recent paper [3] we have demonstrated that the bases of such a set can be viewed as points of a proper conic (or, more generally, of an oval) in a projective plane of order  $q$ . In this short note we extend and qualitatively finalize this picture by showing that also individual vectors of every such a basis can be represented by points, although these points are of a different nature and require a more general projective setting, that of a projective *Hjelmslev* plane [4]–[7].

To this end in view, we shall first introduce the basics of the Galois ring theory (see, e.g., [8] for the symbols, notation and further details). Let, as above,  $p$  be a prime number and  $r$  a positive integer, and let  $f(x) \in \mathcal{Z}_{p^2}[x]$  be a monic polynomial of degree  $r$  whose image in  $\mathcal{Z}_p[x]$  is irreducible. Then  $GR(p^2, r) \equiv \mathcal{Z}_{p^2}[x]/(f)$  is a ring, called a *Galois ring*, of characteristic  $p^2$  and rank  $r$ , whose maximal ideal is  $pGR(p^2, r)$ . In this ring there exists a non-zero element  $\zeta$  of order  $p^r - 1$  that is a root of  $f(x)$  over  $\mathcal{Z}_{p^2}$ , with  $f(x)$  dividing  $x^{p^r-1} - 1$  in  $\mathcal{Z}_{p^2}[x]$ . Then any element of  $GR(p^2, r)$  can uniquely be written in the form

$$g = a + pb, \tag{1}$$

where both  $a$  and  $b$  belong to the so-called Teichmüller set  $\mathcal{T}_r$ ,

$$\mathcal{T}_r \equiv \{0, 1, \zeta, \zeta^2, \dots, \zeta^{p^r-2}\}, \tag{2}$$

having

$$q = p^r \tag{3}$$

elements. From Eq. (1) it is obvious that  $g$  is a unit (i.e., an invertible element) of  $GR(p^2, r)$  iff  $a \neq 0$  and a zero-divisor iff  $a = 0$ . It then follows that  $GR(p^2, r)$  has  $\#_t = q^2$  elements in total, out of which there are  $\#_z = q$  zero-divisors and  $\#_u = q^2 - q = q(q - 1)$  units. Next, let “ $\bar{\phantom{x}}$ ” denote reduction modulo  $p$ ; then obviously  $\overline{\mathcal{T}_r} = GF(q)$ , the Galois *field* of order  $q$ , and  $\bar{\zeta}$  is a primitive element of  $GF(q)$ . Finally, one notes that any two Galois rings of the same characteristic and rank are isomorphic.

Now we have a sufficient algebraic background to introduce the concept of a projective Hjelmslev plane over  $GR(p^2, r)$ , henceforth referred to as  $PH(2, q)$ .<sup>1</sup>  $PH(2, q)$  is an incidence structure whose points are classes of ordered triples  $(\varrho\check{x}_1, \varrho\check{x}_2, \varrho\check{x}_3)$ , where both  $\varrho$  and at least one  $\check{x}_i$  ( $i=1,2,3$ ) are units, whose lines are classes of ordered triples  $(\sigma\check{l}_1, \sigma\check{l}_2, \sigma\check{l}_3)$ , where both  $\sigma$  and at least one  $\check{l}_i$  ( $i=1,2,3$ ) are units, and the incidence relation is given by

$$\sum_{i=1}^3 \check{l}_i \check{x}_i \equiv \check{l}_1 \check{x}_1 + \check{l}_2 \check{x}_2 + \check{l}_3 \check{x}_3 = 0. \quad (4)$$

From this definition it follows that in  $PH(2, q)$  — as in any *ordinary* projective plane — there is a perfect duality between points and lines. It is also straightforward to see that this plane contains

$$\#_{\text{trip}} = \frac{\#_{\text{t}}^3 - \#_{\text{z}}^3}{\#_{\text{u}}} = \frac{(q^2)^3 - q^3}{q(q-1)} = \frac{q^3(q^3 - 1)}{q(q-1)} = q^2(q^2 + q + 1) \quad (5)$$

points/lines and that the number of points/lines incident with a given line/point is, in light of Eq. (4), equal to the number of non-equivalent couples  $(\varrho\check{x}_1, \varrho\check{x}_2)/(\sigma\check{l}_1, \sigma\check{l}_2)$ , i.e.

$$\#_{\text{coup}} = \frac{\#_{\text{t}}^2 - \#_{\text{z}}^2}{\#_{\text{u}}} = \frac{(q^2)^2 - q^2}{q(q-1)} = \frac{q^2(q^2 - 1)}{q(q-1)} = q(q+1). \quad (6)$$

These figures should be compared with those characterizing ordinary finite planes of order  $q$ , which read  $\overline{\#}_{\text{trip}} = q^2 + q + 1$  and  $\overline{\#}_{\text{coup}} = q + 1$ , respectively (e.g., [9]).

Any projective Hjelmslev plane,  $PH(2, q)$  in particular, is endowed with a very important, and of crucial relevance when it comes to MUBs, property that has no analogue in an ordinary projective plane — the so-called *neighbour* (or, as occasionally referred to, non-remoteness) relation. Namely (see, e.g., [5]–[7]), we say that two points  $A$  and  $B$  are neighbour, and write  $A \odot B$ , if either  $A = B$ , or  $A \neq B$  and there exist two different lines incident with both; otherwise, they are called nonneighbour, or remote. The same symbol and the dual definition is used for neighbour lines. Let us find the cardinality of the set of neighbours of a given point/line of  $PH(2, q)$ . Algebraically speaking, given a point  $\varrho\check{x}_i$ ,  $i=1,2,3$ , the points that are its neighbours must obviously be of the form  $\varrho(\check{x}_i + p\check{y}_i)$ , with  $\check{y}_i \in \mathcal{T}_r$ . Although there are  $q^3$  different choices for the triple  $(\check{y}_1, \check{y}_2, \check{y}_3)$ , only  $q^3/q = q^2$  of the classes  $\varrho(\check{x}_i + p\check{y}_i)$  represent distinct points because  $\varrho(\check{x}_i + p\check{y}_i)$  and  $\varrho(\check{x}_i + p(\check{y}_i + \kappa\check{x}_i))$  represent one and the same point as  $\kappa$  runs through all the  $q$  elements of  $\mathcal{T}_r$ . Hence, every point/line of  $PH(2, q)$  has  $q^2$  neighbours, the point/line in question inclusive. Following the same line of reasoning, but restricting only to couples of coordinates, we find that given a point  $P$  and a line  $\mathcal{L}$ ,  $P$  incident with  $\mathcal{L}$ , there exist exactly  $(q^2/q =) q$  points on  $\mathcal{L}$  that are neighbour to  $P$  and, dually,  $q$  lines through  $P$  that are neighbour to  $\mathcal{L}$ .

Clearly, as  $A \odot A$  (reflexivity),  $A \odot B$  implies  $B \odot A$  (symmetry) and  $A \odot B$  and  $B \odot C$  implies  $A \odot C$  (transitivity), the neighbour relation is an *equivalence* relation. Given “ $\odot$ ” and a point  $P$ /line  $\mathcal{L}$ , we call the subset of all points  $Q$ /lines  $\mathcal{K}$  of  $PH(2, q)$  satisfying  $P \odot Q/\mathcal{L} \odot \mathcal{K}$  the neighbour class of  $P/\mathcal{L}$ . And since “ $\odot$ ” is an equivalence relation, the aggregate of neighbour classes *partitions* the plane, i.e. the plane consists of a *disjoint union* of neighbour classes of points/lines. The modulo- $p$ -mapping then “induces” a so-called canonical epimorphism of  $PH(2, q)$  into  $PG(2, q)$ , the ordinary projective plane defined over  $GF(q)$ , with the neighbour classes being the cosets of this epimorphism [7]. Loosely rephrased,  $PH(2, q)$  comprises  $q^2 + q + 1$  “clusters” of neighbour points/lines, each of cardinality  $q^2$ , such that its restriction modulo the neighbour relation is the ordinary projective plane  $PG(2, q)$  every single point/line of which encompasses the whole “cluster” of these neighbour points/lines. Analogously, each line of  $PH(2, q)$  consists of  $q + 1$  neighbour classes, each of cardinality  $q$ , such that its “—” image is the ordinary projective line in the  $PG(2, q)$  whose points are exactly these neighbour classes.

<sup>1</sup>This is, of course, a very specific, and rather elementary, kind of projective Hjelmslev plane; its most general axiomatic definition can be found, for example, in [5]–[7].

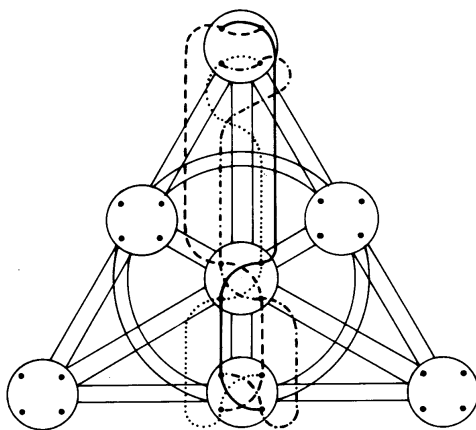


Figure 1: A schematic sketch of the structure of the simplest projective Hjelmslev plane,  $PH(2, 2)$ . Shown are all the 28 of its points (represented by small filled circles), grouped into seven pairwise disjoint sets (neighbour classes) of cardinality four, as well as four of its lines belonging to the same neighbour class (and drawn as solid, dashed, dotted and dot-dashed curves). Also shown is the associated Fano plane,  $PG(2, 2)$ , whose points are represented by seven big circles, six of its lines as pairs of line segments and the remaining line as a pair of concentric circles. Notice the intricate character of pairwise intersections of the four neighbour lines of  $PH(2, 2)$ .

Let us illustrate these remarkable properties on the simplest possible example that is furnished by  $PH(2, q = 2)$ , i.e. the plane defined over  $GR(4, 1)$  whose epimorphic “shadow” is the simplest projective plane  $PG(2, 2)$  — the Fano plane. As partially depicted in Fig. 1, this plane consists of seven classes of quadruples of neighbour points/lines, each point/line featuring three classes of couples of neighbour lines/points. When modulo-two-projected, each quadruple of neighbour points/lines goes into a single point/line of the associated Fano plane.

The most relevant geometrical object for our model [3] is, of course, a *conic*, that is a curve  $\mathcal{Q}$  of  $PH(2, q)$  whose points obey the equation

$$\mathcal{Q} : \sum_{i \leq j} c_{ij} \check{x}_i \check{x}_j \equiv c_{11} \check{x}_1^2 + c_{22} \check{x}_2^2 + c_{33} \check{x}_3^2 + c_{12} \check{x}_1 \check{x}_2 + c_{13} \check{x}_1 \check{x}_3 + c_{23} \check{x}_2 \check{x}_3 = 0, \quad (7)$$

with at least one of the  $c_{ij}$  's being a unit of  $GR(p^2, r)$ . In particular, we are interested in a *proper* conic, which is a conic whose equation cannot be reduced into a form featuring fewer variables whatever coordinate transformation one would employ. It is known (see, e.g., [10]) that the equation of a proper conic of  $PH(2, q)$  can always be brought into a “canonical” form

$$\mathcal{Q}^* : \check{x}_1 \check{x}_3 - \check{x}_2^2 = 0 \quad (8)$$

from which it readily follows that any such conic is endowed, like a line, with  $q^2 + q = q(q+1)$  points;  $q^2$  of them are of the form

$$\varrho \check{x}_i = (1, \sigma, \sigma^2), \quad (9)$$

where the parameter  $\sigma$  runs through all the elements of  $GR(p^2, r)$ , whilst the remaining  $q$  are represented by

$$\varrho \check{x}_i = (0, \delta, 1), \quad (10)$$

with  $\delta$  running through all the zero-divisors of  $GR(p^2, r)$ . And each point of a proper conic, like that of a line, has  $q$  neighbours; for the neighbours of a particular point  $\sigma = \sigma_0$  of (9) are of the form

$$\rho\check{x}_i = (1, \sigma_0 + p\kappa, (\sigma_0 + p\kappa)^2) = (1, \sigma_0 + p\kappa, \sigma_0^2 + p2\kappa) \quad (11)$$

and there are obviously  $q$  of them (the point in question inclusive) as  $\kappa$  runs through  $\mathcal{T}_r$ , and all the  $q$  points of (10) are the neighbours of any of them. All in all, a proper conic, like a line, of  $PH(2, q)$  features  $q + 1$  pairwise disjoint classes of neighbour points, each having  $q$  elements, these classes being the single points of its modular image in  $PG(2, q)$ .

At this point our algebraic geometrical machinery is elaborate enough to generalize and qualitatively complete the geometrical picture of MUBs proposed in [3] where we have argued that a basis of  $\mathcal{H}_q$ ,  $q$  given by (3), can be regarded as a point of an arc in  $PG(2, q)$ , with a complete set of MUBs corresponding to a proper conic (or, in the case of  $p=2$ , to a more general geometrical object called oval). This model, however, lacks a geometrical interpretation of the individual vectors of a basis, which can be achieved in our extended projective setting *à la* Hjelmslev only. Namely, taking any complete, i.e. of cardinality  $q+1$ , set of MUBs, its *bases* are now viewed as the *neighbour classes* of points of a proper conic of  $PH(2, q)$  and the *vectors* of a given basis have their counterpart in the *points* of the corresponding neighbour class. The property of different vectors of a basis being pairwise *orthogonal* is then geometrically embodied in the fact that the corresponding points are all *neighbour*, whilst the property of two different bases being *mutually unbiased* answers to the fact that the points of any two neighbour classes are *remote* from each other. It is left with the reader as an easy exercise to check that “rephrasing these statements modulo  $p$ ” one recovers all the conic-related results of [3].

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