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► **To cite this version:**

François Nicoleau. An inverse scattering problem for short-range systems in a time-periodic electric field. *Mathematical Research Letters*, 2005, 12 (16), pp. 885-896. hal-00005485

HAL Id: hal-00005485

<https://hal.science/hal-00005485>

Submitted on 20 Jun 2005

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An inverse scattering problem for short-range systems in a time-periodic electric field.

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Abstract

We consider the time-dependent Hamiltonian $H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x)$ on $L^2(\mathbb{R}^n)$, where the external electric field $E(t)$ and the short-range electric potential $V(t, x)$ are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value E_0 of the external electric field. When $E_0 = 0$, we show that the high energy limit of the scattering operators determines uniquely $V(t, x)$. When $E_0 \neq 0$, the same result holds in dimension $n \geq 3$ for generic short-range potentials. In dimension $n = 2$, one has to assume a stronger decay on the electric potential.

1 Introduction.

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t, x)$ (with the same period T). For the sake of simplicity, we assume that the period $T = 1$.

The corresponding Hamiltonian is given on $L^2(\mathbb{R}^n)$ by :

$$(1.1) \quad H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t, x),$$

where $p = -i\partial_x$. When $E(t) = 0$, the Hamiltonian $H(t)$ describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When $E(t) = \cos(2\pi t) E$ with $E \in \mathbb{R}^n$, the Hamiltonian describes the well-known AC-Stark effect in the E -direction [7].

In this paper, we assume that the external electric field $E(t)$ satisfies :

$$(A_1) \quad t \rightarrow E(t) \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n) \quad , \quad E(t+1) = E(t) \text{ a.e. .}$$

Moreover, we assume that the potential $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$, is time-periodic with period 1, and satisfies the following estimations :

$$(A_2) \quad \forall \alpha \in \mathbb{N}^n, \forall k \in \mathbb{N}, \quad | \partial_t^k \partial_x^\alpha V(t, x) | \leq C_{k,\alpha} \langle x \rangle^{-\delta-|\alpha|}, \text{ with } \delta > 0,$$

where $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$. Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need (A_2) for only k, α with finite order . It is well-known that under assumptions $(A_1) - (A_2)$, $H(t)$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space, [16]. We denote $H(t)$ the self-adjoint realization with domain $D(H(t))$.

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote $H_0(t)$ the free Hamiltonian :

$$(1.2) \quad H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x \quad ,$$

and let $U_0(t, s)$, (resp. $U(t, s)$) be the unitary propagator associated with $H_0(t)$, (resp. $H(t)$) (see section 2 for details).

For short-range potentials, the wave operators are defined for $s \in \mathbb{R}$ and $\Phi \in L^2(\mathbb{R}^n)$ by :

$$(1.3) \quad W^\pm(s) \Phi = \lim_{t \rightarrow \pm\infty} U(s, t) U_0(t, s) \Phi.$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field :

$$(1.4) \quad E_0 = \int_0^1 E(t) dt \quad .$$

• **The case $E_0 = 0$.**

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator U , $\mathcal{H}_c(U)$, $\mathcal{H}_{ac}(U)$, $\mathcal{H}_{sc}(U)$ and $\mathcal{H}_p(U)$ are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of U .

We have the following result ([10], [11], [21]) :

Theorem 1

Assume that hypotheses (A_1) , (A_2) are satisfied with $\delta > 1$ and with $E_0 = 0$.

Then : (i) the wave operators $W^\pm(s)$ exist for all $s \in \mathbb{R}$.

(ii) $W^\pm(s+1) = W^\pm(s)$ and $U(s+1, s) W^\pm(s) = W^\pm(s) U_0(s+1, s)$.

(iii) $\text{Ran}(W^\pm(s)) = \mathcal{H}_{ac}(U(s+1, s))$ and $\mathcal{H}_{sc}(U(s+1, s)) = \emptyset$.

(iv) the purely point spectrum $\sigma_p(U(s+1, s))$ is discrete outside $\{1\}$.

Comments.

1 - The unitary operators $U(s+1, s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems. The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for $n = 3$ and $\delta > 2$, $\mathcal{H}_p(U(s+1, s))$ is finite dimensional.

2 - For general $\delta > 0$, $W^\pm(s)$ do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators $W_{H,J}^\pm$ by solving an Hamilton-Jacobi equation.

- **The case $E_0 \neq 0$.**

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians) : the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential V satisfying (A_2) has no eigenvalues [2]. The following theorem, due to Moller, is a time-periodic version of these results.

Theorem 2

Assume that hypotheses (A_1) , (A_2) are satisfied with $\delta > \frac{1}{2}$ and with $E_0 \neq 0$.

Then : (i) the Floquet operators $U(s+1, s)$ have purely absolutely continuous spectrum.

(ii) the wave operators $W^\pm(s)$ exist for all $s \in \mathbb{R}$ and are unitary.

(iii) $W^\pm(s+1) = W^\pm(s)$ and $U(s+1, s) W^\pm(s) = W^\pm(s) U_0(s+1, s)$.

The inverse scattering problem.

For $s \in \mathbb{R}$, we define the scattering operators $S(s) = W^{+*}(s) W^-(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s, x)$ from the scattering operators $S(s)$, $s \in [0, 1]$.

In this paper, we prove the following result :

Theorem 3

Assume that $E(t)$ satisfies (A_1) and let V_j , $j = 1, 2$ be potentials satisfying (A_2) . We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{1}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$). Let $S_j(s)$ be the corresponding scattering operators.

Then :

$$\forall s \in [0, 1], S_1(s) = S_2(s) \iff V_1 = V_2 .$$

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder's approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to E_0 . See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t, s)$, $s, t \in \mathbb{R}$, then by virtue of the following relation :

$$(1.5) \quad S(t) = U_0(t, s) S(s) U_0(s, t) ,$$

the potential $V(t, x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2} + 1$ and with $E(t) = 0$ by studying the scattering matrices in a high energy regime.

In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2 Proof of Theorem 3.

2.1 The Avron-Herbst formula.

First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t \in \mathbb{R}}$ be a family of selfadjoint operators on $L^2(\mathbb{R}^n)$ such that $\mathcal{S}(\mathbb{R}^n) \subset D(H(t))$ for all $t \in \mathbb{R}$.

Definition.

We call *propagator* a family of unitary operators on $L^2(\mathbb{R}^n)$, $U(t, s)$, $t, s \in \mathbb{R}$ such that :

- 1 - $U(t, s)$ is a strongly continuous function of $(t, s) \in \mathbb{R}^2$.
- 2 - $U(t, s) U(s, r) = U(t, r)$ for all $t, s, r \in \mathbb{R}$.
- 3 - $U(t, s) (\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$ for all $t, s \in \mathbb{R}$.
- 4 - If $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $U(t, s)\Phi$ is continuously differentiable in t and s and satisfies :

$$i \frac{\partial}{\partial t} U(t, s) \Phi = H(t) U(t, s) \Phi , \quad i \frac{\partial}{\partial s} U(t, s) \Phi = -U(t, s) H(s) \Phi .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3].

In [12], the author gives, for $E_0 \neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all E_0 . We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t) = O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field E_0 , [2]; if we consider the Hamiltonian B_0 on $L^2(\mathbb{R}^n)$,

$$(2.1) \quad B_0 = \frac{1}{2}p^2 - E_0 \cdot x ,$$

we have the following formula :

$$(2.2) \quad e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3} e^{itE_0 \cdot x} e^{-i\frac{t^2}{2}E_0 \cdot p} e^{-it\frac{p^2}{2}} .$$

In the next definition, we give a similar formula for time-dependent electric fields.

Definition.

We consider the family of unitary operators $T(t)$, for $t \in \mathbb{R}$:

$$T(t) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} ,$$

where :

$$(2.3) \quad b(t) = - \int_0^t (E(s) - E_0) ds - \int_0^1 \int_0^t (E(s) - E_0) ds dt .$$

$$(2.4) \quad c(t) = - \int_0^t b(s) ds .$$

$$(2.5) \quad a(t) = \int_0^t \left(\frac{1}{2} b^2(s) - E_0 \cdot c(s) \right) ds .$$

Lemma 4

The family $\{H_0(t)\}_{t \in \mathbb{R}}$ has an unique propagator $U_0(t, s)$ defined by :

$$(2.6) \quad U_0(t, s) = T(t) e^{-i(t-s)B_0} T^*(s) .$$

Proof.

We can always assume $s = 0$ and we make the following ansatz :

$$(2.7) \quad U_0(t, 0) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} e^{-itB_0} .$$

Since on the Schwartz space, $U_0(t, 0)$ must satisfy :

$$(2.8) \quad i \frac{\partial}{\partial t} U_0(t, 0) = H_0(t) U_0(t, 0) ,$$

the functions $a(t)$, $b(t)$, $c(t)$ solve :

$$(2.9) \quad \dot{b}(t) = -E(t) + E_0, \quad \dot{c}(t) = -b(t), \quad \dot{a}(t) = \frac{1}{2} b^2(t) - E_0 \cdot c(t).$$

We refer to [3] for details and [12] for a different formula. \square

In the same way, in order to define the propagator corresponding to the family $\{H(t)\}$, we consider a Stark Hamiltonian with a time-periodic potential $V_1(t, x)$, (we recall that $c(t)$ is C^1 -periodic function) :

$$(2.10) \quad B(t) = B_0 + V_1(t, x) \quad \text{where} \quad V_1(t, x) = e^{ic(t) \cdot p} V(t, x) e^{-ic(t) \cdot p} = V(t, x + c(t)).$$

Then, $B(t)$ has an unique propagator $R(t, s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t, s)$ for the family $\{H(t)\}$ is defined by :

$$(2.11) \quad U(t, s) = T(t) R(t, s) T^*(s).$$

Comments.

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in \mathbb{R}$:

$$(2.12) \quad U_0(t + 1, s + 1) = U_0(t, s) \quad , \quad U(t + 1, s + 1) = U(t, s) .$$

Thus, the wave operators satisfy $W^\pm(s + 1) = W^\pm(s)$.

2.2 The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder's time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(\mathbb{R}^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.

Before giving the main result of this section, we need some notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where Π_{E_0} is the orthogonal hyperplane to E_0 .
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Phi$, $\Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Psi$.

We have the following high energy asymptotics where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^n)$:

Proposition 5

Under the assumptions of Theorem 3, we have for all $s \in [0, 1]$,

$$\langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \lambda^{-\frac{1}{2}} \left\langle \left(\int_{-\infty}^{+\infty} \partial_x V(s, x + t\omega) dt \right) \Phi, \Psi \right\rangle + o(\lambda^{-\frac{1}{2}}).$$

Comments.

Actually, for the case $n = 2$, $E_0 \neq 0$ and $\delta > \frac{3}{4}$, Proposition 5 is also valid for $\omega \in S^{n-1}$ satisfying $|\omega \cdot E_0| < |E_0|$, (see ([18], [15])).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

Proof of Proposition 5.

For example, let us show Proposition 5 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar. More precisely, see [18] for the case $E_0 = 0$, and for the case $n = 2$, $E_0 \neq 0$, see ([17], Theorem 2.4) and ([15], Theorem 4).

Step 1.

Since $c(t)$ is periodic, $c(t) = O(1)$. Then, $V_1(t, x)$ is a short-range perturbation of B_0 , and we can define the usual wave operators for the pair of Hamiltonians $(B(t), B_0)$:

$$(2.13) \quad \Omega^\pm(s) = s - \lim_{t \rightarrow \pm\infty} R(s, t) e^{-i(t-s)B_0}.$$

Consider also the scattering operators $S_1(s) = \Omega^{+*}(s) \Omega^-(s)$. By virtue of (2.6) and (2.11), it is clear that :

$$(2.14) \quad S(s) = T(s) S_1(s) T^*(s).$$

Using the fact that $e^{-ib(s) \cdot x} p e^{ib(s) \cdot x} = p + b(s)$, we have :

$$(2.15) \quad [S(s), p] = [S(s), p + b(s)] = T(s) [S_1(s), p] T^*(s).$$

Thus,

$$(2.16) \quad \langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \langle [S_1(s), p] T^*(s) \Phi_{\lambda,\omega}, T^*(s) \Psi_{\lambda,\omega} \rangle.$$

In other hand,

$$(2.17) \quad T^*(s) \Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega} e^{ic(s)\cdot(p+\sqrt{\lambda}\omega)} e^{ib(s)\cdot x} e^{ia(s)} \Phi.$$

So, we obtain :

$$(2.18) \quad \langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \langle [S_1(s), p] f_{\lambda,\omega}, g_{\lambda,\omega} \rangle,$$

where

$$(2.19) \quad f = e^{ic(s)\cdot p} e^{ib(s)\cdot x} \Phi \text{ and } g = e^{ic(s)\cdot p} e^{ib(s)\cdot x} \Psi.$$

Clearly, f, g are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.

• **Step 2 : Modified wave operators.**

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic $U_D(t, s)$ by :

$$(2.20) \quad U_D(t, s) = e^{-i(t-s)B_0} e^{-i \int_0^{t-s} V_1(u+s, up' + \frac{1}{2}u^2 E_0) du},$$

where p' is the projection of p on the orthogonal hyperplane to E_0 .

We define the modified wave operators :

$$(2.21) \quad \Omega_D^\pm(s) = s - \lim_{t \rightarrow \pm\infty} R(s, t) U_D(t, s).$$

It is clear that :

$$(2.22) \quad \Omega_D^\pm(s) = \Omega^\pm(s) e^{-ig^\pm(s, p')},$$

where

$$(2.23) \quad g^\pm(s, p') = \int_0^{\pm\infty} V_1(u + s, up' + \frac{1}{2}u^2 E_0) du.$$

Thus, if we set $S_D(s) = \Omega_D^{+*}(s)\Omega_D^-(s)$, one has :

$$(2.24) \quad S_1(s) = e^{-ig^+(s, p')} S_D(s) e^{ig^-(s, p')}$$

• **Step 3 : High energy estimates.**

Denote $\rho = \min(1, \delta)$. We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

Lemma 6

For $\lambda \gg 1$, we have :

$$(i) \quad \left\| \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^\pm(s, p')} f_{\lambda,\omega} \right\| \leq C (1 + |(t-s)\sqrt{\lambda}|)^{-\frac{1}{2}-\rho}.$$

$$(ii) \quad \left\| \left(R(t, s)\Omega_D^\pm(s) - U_D(t, s) \right) e^{ig^\pm(s, p')} f_{\lambda,\omega} \right\| = O(\lambda^{-\frac{1}{2}}), \text{ uniformly for } t, s \in \mathbb{R}.$$

• **Step 4.**

We denote $F(s, \lambda, \omega) = \langle [S_1(s), p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle$. Using (2.24), we have :

$$\begin{aligned}
F(s, \lambda, \omega) &= \langle [e^{-ig^+(s,p')} S_D(s) e^{ig^-(s,p')}, p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle \\
&= \langle [S_D(s), p] e^{ig^-(s,p')} f_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle \\
&= \langle [S_D(s) - 1, p - \sqrt{\lambda}\omega] e^{ig^-(s,p')} f_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle \\
&= \langle (S_D(s) - 1) e^{ig^-(s,p')} (pf)_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle \\
&\quad - \langle (S_D(s) - 1) e^{ig^-(s,p')} f_{\lambda, \omega}, e^{ig^+(s,p')} (pg)_{\lambda, \omega} \rangle \\
&:= F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega).
\end{aligned}$$

First, let us study $F_1(s, \lambda, \omega)$. Writing $S_D(s) - 1 = (\Omega_D^+(s) - \Omega_D^-(s))^* \Omega_D^-(s)$ and using

$$(2.25) \quad \Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s, t) \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) dt,$$

we obtain :

$$(2.26) \quad S_D(s) - 1 = -i \int_{-\infty}^{+\infty} U_D(t, s)^* \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) R(t, s) \Omega_D^-(s) dt.$$

Thus,

$$\begin{aligned}
F_1(s, \lambda, \omega) &= -i \int_{-\infty}^{+\infty} \langle R(t, s) \Omega_D^-(s) e^{ig^-(s,p')} (pf)_{\lambda, \omega}, \\
&\quad \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s,p')} g_{\lambda, \omega} \rangle dt \\
&= -i \int_{-\infty}^{+\infty} \langle U_D(t, s) e^{ig^-(s,p')} (pf)_{\lambda, \omega}, \\
&\quad \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s,p')} g_{\lambda, \omega} \rangle dt \\
&\quad + R_1(s, \lambda, \omega),
\end{aligned}$$

where :

$$(2.27) \quad R_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} \langle (R(t, s) \Omega_D^-(s) - U_D(t, s)) e^{ig^-(s,p')} (pf)_{\lambda, \omega}, \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s) e^{ig^+(s,p')} g_{\lambda, \omega} \rangle dt.$$

By Lemma 6, it is clear that $R_1(s, \lambda, \omega) = O(\lambda^{-1})$. Thus, writing $t = \frac{\tau}{\sqrt{\lambda}} + s$, we obtain :

$$(2.28) \quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \langle U_D(\frac{\tau}{\sqrt{\lambda}} + s, s) e^{ig^-(s,p')} (pf)_{\lambda,\omega} , \\ \left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda}E_0) \right) \\ U_D(\frac{\tau}{\sqrt{\lambda}} + s, s) e^{ig^+(s,p')} g_{\lambda,\omega} \rangle d\tau + O(\lambda^{-1}) .$$

Denote by $f_1(\tau, s, \lambda, \omega)$ the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

$$(2.29) \quad |f_1(\tau, s, \lambda, \omega)| \leq C (1 + |\tau|)^{-\frac{1}{2}-\rho} .$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(s, \lambda, \omega)$, it suffices to determine $\lim_{\lambda \rightarrow +\infty} f_1(\tau, s, \lambda, \omega)$.

Let us denote :

$$(2.30) \quad U^\pm(t, s, p') = e^{i \int_t^{\pm\infty} V_1(u+s, up' + \frac{1}{2}u^2 E_0) du} .$$

We have :

$$(2.31) \quad f_1(\tau, s, \lambda, \omega) = \langle e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p') (pf)_{\lambda,\omega} , \\ \left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda} E_0) \right) e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p') g_{\lambda,\omega} \rangle .$$

Using the Avron-Herbst formula (2.2), we deduce that :

$$(2.32) \quad f_1(\tau, s, \lambda, \omega) = \langle e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p') (pf)_{\lambda,\omega} , \\ \left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}p' + \frac{\tau^2}{2\lambda} E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p') g_{\lambda,\omega} \rangle .$$

Then, we obtain :

$$(2.33) \quad f_1(\tau, s, \lambda, \omega) = \langle e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} U^-(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) pf , \\ \left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} E_0) \right) \\ e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} U^+(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) g \rangle .$$

Since

$$(2.34) \quad e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} = e^{-i\frac{\tau\sqrt{\lambda}}{2}} e^{-i\tau\omega.p} e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} ,$$

we have

$$(2.35) \quad f_1(\tau, s, \lambda, \omega) = \langle e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^-\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) pf, \left(V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x + \tau\omega + \frac{\tau^2}{2\lambda} E_0\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} E_0\right) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} U^+\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) g \rangle .$$

Since $|V_1(u + s, u(p' + \sqrt{\lambda}\omega) + \frac{1}{2}u^2 E_0)| \leq C(u^2 + 1)^{-\delta} \in L^1(\mathbb{R}^+, du)$, it is easy to show (using Lebesgue's theorem again) that :

$$(2.38) \quad s - \lim_{\lambda \rightarrow +\infty} U^\pm\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) = 1 .$$

Then,

$$(2.39) \quad \lim_{\lambda \rightarrow +\infty} f_1(\tau, s, \lambda, \omega) = \langle pf, (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) g \rangle .$$

So, we have obtained :

$$(2.40) \quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \langle pf, \left(\int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) g \rangle + o\left(\frac{1}{\sqrt{\lambda}}\right).$$

In the same way, we obtain

$$(2.41) \quad F_2(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \langle f, \left(\int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) pg \rangle + o\left(\frac{1}{\sqrt{\lambda}}\right),$$

so

$$(2.42) \quad F(s, \lambda, \omega) = F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega)$$

$$(2.43) \quad = \frac{1}{\sqrt{\lambda}} \langle f, \left(\int_{-\infty}^{+\infty} \partial_x V_1(s, x + \tau\omega) d\tau \right) g \rangle + o\left(\frac{1}{\sqrt{\lambda}}\right) .$$

Using (2.19) and $\partial_x V(s, x + \tau\omega) = e^{-ic(s)p} \partial_x V_1(s, x + \tau\omega) e^{ic(s)p}$, we obtain :

$$(2.44) \quad F(s, \lambda, \omega) = \frac{1}{\sqrt{\lambda}} \langle \Phi, \left(\int_{-\infty}^{+\infty} \partial_x V(s, x + \tau\omega) d\tau \right) \Psi \rangle + o\left(\frac{1}{\sqrt{\lambda}}\right) . \square$$

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