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# Asymptotics of a thermal flow with highly conductive and radiant suspensions

Fadila Bentalha \*, Isabelle Gruais \*\* and Dan Poliřevski \*\*\*

**Abstract.** Radiant spherical suspensions have an  $\varepsilon$ -periodic distribution in a tridimensional incompressible viscous fluid governed by the Stokes-Boussinesq system. We perform the homogenization procedure when the radius of the solid spheres is of order  $\varepsilon^3$  (the critical size of perforations for the Navier-Stokes system) and when the ratio of the fluid/solid conductivities is of order  $\varepsilon^6$ , the order of the total volume of suspensions. Adapting the methods used in the study of small inclusions, we prove that the macroscopic behavior is described by a Brinkman-Boussinesq type law and two coupled heat equations, where certain capacities of the suspensions and of the radiant sources appear.

**Mathematical Subject Classification (2000).** 35B27, 76D07, 76S05.

**Keywords.** Stokes-Boussinesq system, homogenization, non local effects.

## 1 Preliminaries

One main achievement of homogenization theory, introduced by [1] and developed by [2, 3, 4, 5, 6, 7], was the ability to conceptually clarify the relationship between microscopic and macroscopic properties of physical systems, at least as far as the periodic approximation could be acceptable. The major restriction was the technically impossible interplay between different scales: if some quantity varies as the power  $\varepsilon^\alpha$  of the size  $\varepsilon$  of the mesh, then the case where  $\alpha < 0$  leads to blow up at the limit. One major contribution in that direction is the paper by G. Allaire [7] who clearly underlies the role of critical discriminating scales beyond which nothing can be said, but rigidification of elastic systems for instance, and that can however generate a transition state where either 'non local' effects [2, 5] or 'coming from nowhere' terms [1] can emerge.

In this paper, we are interested in the former case which has been thoroughly explored when non local effects concentrate on rod-like one-dimensional submanifolds of the three-dimensional space: see [2] for the Laplacian, [5] for the Elasticity system. This geometry enables the formulation of the limit problem as a rod-like boundary value problem solved by the density of a Radon measure. Our question then was: what happens in other geometries, especially if non local effects are to be supported by a cloud of little particles? The physical opportunity was the example of thermal flows (see [8, 9]) where highly heat conducting spheres are immersed in a Stokes-Boussinesq fluid. It is straightforward

that for some critical size of the particles (eventually  $\varepsilon^3$  when the period of the distribution is  $\varepsilon$ ) the resulting mixture will display a specific behaviour strongly discriminating between a trivial case and a classically homogenized case. Our concern was then to develop new skills to understand how the expected non local effects would be formulated. We found out that the Dirac structure of the masses make the classical formulation in terms of a jump term updated and that it rather generates an additional source coupled with a capacitary term representative of a Brinkman-Boussinesq type law.

More precisely, the physics of the problem may be described as follows. Solid spherical suspensions are  $\varepsilon$ -periodically distributed in a tridimensional bounded domain filled with an incompressible fluid governed by the Stokes-Boussinesq system. We study the homogenization of the convective movement which is generated by highly heterogeneous radiant sources, when the radius of the suspensions is of  $\varepsilon^3$ -order, that is the border case for the Navier-Stokes system (see [7]). Assuming that the conductivity and the radiant source of the fluid have  $\varepsilon^0$ -order, we found that the only regular case in which we have macroscopic effects from both the conductivity and the radiation of the suspensions is when they are of  $\varepsilon^6$ -order. Therefore, we have treated here strictly this case. Nevertheless, the present procedure can be easily adapted to the other cases.

Let  $\Omega \subset \mathbf{R}^3$  be a bounded open set and let

$$Y := \left( -\frac{1}{2}, +\frac{1}{2} \right)^3.$$

$$Y_\varepsilon^k := \varepsilon k + \varepsilon Y, \quad k \in \mathbf{Z}^3.$$

$$\mathbf{Z}_\varepsilon := \{k \in \mathbf{Z}^3, \quad Y_\varepsilon^k \subset \Omega\}$$

The reunion of the suspensions is defined by

$$T_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} B(\varepsilon k, r_\varepsilon),$$

where  $0 < r_\varepsilon \ll \varepsilon$  and  $B(\varepsilon k, r_\varepsilon)$  is the ball of radius  $r_\varepsilon$  centered at  $\varepsilon k$ ,  $k \in \mathbf{Z}_\varepsilon$ .

The fluid domain is given by

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon.$$

Let  $\mathbf{e}^{(3)}$  the last vector of the canonical basis of  $\mathbf{R}^3$ ,  $n$  the normal on  $\partial\Omega_\varepsilon$  in the outward direction and  $[\cdot]_\varepsilon$  the jump across the interface  $\partial T_\varepsilon$ .

For  $a > 0$  (the so-called Rayleigh number),  $b > 0$  ( $b \left(\frac{\varepsilon}{r_\varepsilon}\right)^3$  denoting the ratio of the solid/fluid conductivities),  $f \in C_c(\Omega)$ ,  $g \in C_c(\Omega)$ , where

$$C_c(\Omega) := \{g \in C(\Omega); \quad \text{supp } g \text{ is compact} \},$$

we consider the problem corresponding to the non-dimensional Stokes-Boussinesq system governing the thermal flow of an  $\varepsilon$ -periodic distribution suspension of solid spheres:

To find  $(u^\varepsilon, p^\varepsilon)$ ,  $\theta^\varepsilon, \zeta^\varepsilon$  solution of

$$\operatorname{div} u^\varepsilon = 0, \quad \text{in } \Omega_\varepsilon, \quad (1)$$

$$-\Delta u^\varepsilon + \nabla p^\varepsilon = a\theta^\varepsilon \mathbf{e}^{(3)}, \quad \text{in } \Omega_\varepsilon, \quad (2)$$

$$-\Delta \theta^\varepsilon + u^\varepsilon \nabla \theta^\varepsilon = f, \quad \text{in } \Omega_\varepsilon, \quad (3)$$

$$-\Delta \zeta^\varepsilon = g, \quad \text{in } T_\varepsilon, \quad (4)$$

$$\zeta^\varepsilon = \theta^\varepsilon, \quad \text{on } \partial T_\varepsilon \quad (5)$$

$$\frac{\partial \theta^\varepsilon}{\partial n} = b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \frac{\partial \zeta^\varepsilon}{\partial n}, \quad \text{on } \partial T_\varepsilon \quad (6)$$

$$u^\varepsilon = 0, \quad \text{on } \partial \Omega_\varepsilon, \quad (7)$$

$$\theta^\varepsilon = 0, \quad \text{on } \partial \Omega. \quad (8)$$

Set

$$V_\varepsilon := \{v \in H_0^1(\Omega_\varepsilon; \mathbf{R}^3), \quad \operatorname{div} v = 0\}.$$

Thanks to (5), we extend  $\theta^\varepsilon$  on  $T_\varepsilon$  by setting

$$\theta^\varepsilon = \zeta^\varepsilon \quad \text{on } T_\varepsilon.$$

Then, the variational formulation reads:

$$\begin{aligned} \forall (v, q) \in V_\varepsilon \times L^2(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla v \, dx &= a \int_{\Omega_\varepsilon} \theta^\varepsilon v_3 \, dx \\ \int_{\Omega_\varepsilon} q \operatorname{div} u^\varepsilon \, dx &= 0 \end{aligned} \quad (9)$$

$$\begin{aligned} \forall \varphi \in H_0^1(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} \nabla \theta^\varepsilon \nabla \varphi \, dx &+ b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} \nabla \theta^\varepsilon \nabla \varphi \, dx \\ + \int_{\Omega_\varepsilon} u^\varepsilon \varphi \nabla \theta^\varepsilon \, dx &= \int_{\Omega_\varepsilon} f \varphi \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} g \varphi \, dx. \end{aligned} \quad (10)$$

We define  $\mathcal{F}_\varepsilon \in H^{-1}(\Omega)$  by

$$\forall \varphi \in H_0^1(\Omega), \quad \mathcal{F}_\varepsilon(\varphi) := \int_{\Omega_\varepsilon} f \varphi \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} g \varphi \, dx. \quad (11)$$

Then, for  $\alpha > 0$  (we shall choose a suitable value for this parameter later), we can present the variational formulation of the problem (1)–(8):

To find  $(u^\varepsilon, \theta^\varepsilon) \in V_\varepsilon \times H_0^1(\Omega)$  such that

$$\forall (v, \varphi) \in V_\varepsilon \times H_0^1(\Omega), \quad \langle G(u^\varepsilon, \theta^\varepsilon), (v, \varphi) \rangle = \mathcal{F}_\varepsilon(\varphi) \quad (12)$$

where the mapping  $G : V_\varepsilon \times H_0^1(\Omega) \rightarrow V_\varepsilon' \times H^{-1}(\Omega)$  is defined by

$$\begin{aligned} \langle G(u, \theta), (v, \varphi) \rangle &= \alpha \int_{\Omega_\varepsilon} \nabla u \nabla v \, dx - \alpha a \int_{\Omega_\varepsilon} \theta v_3 \, dx \\ &+ \int_{\Omega_\varepsilon} \nabla \theta \nabla \varphi \, dx + \int_{\Omega_\varepsilon} u \varphi \nabla \theta \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} \nabla \theta \nabla \varphi \, dx. \end{aligned}$$

In order to prove the existence theorem for problem (12), we make use of the following result of Gossez.

**Theorem 1.1** *Let  $X$  be a reflexive Banach space and  $G : X \rightarrow X'$  a continuous mapping between the corresponding weak topologies. If*

$$\frac{\langle G\varphi, \varphi \rangle}{|\varphi|_X} \rightarrow \infty \quad \text{as} \quad |\varphi|_X \rightarrow \infty$$

*then  $G$  is a surjection.*

Acting as in the proof of Theorem 5.2.2 [8] Ch 1, Sec. 5, we find that the existence of the weak solutions of problem (12) is assured if  $\alpha$  is chosen sufficiently small.

Moreover, if  $(u^\varepsilon, \theta^\varepsilon)$  is a solution of problem (12), then, by using the weak maximum principle, we obtain that  $\theta^\varepsilon \in L^\infty(\Omega)$ , (see Theorem 3.4 [8] Ch 2, Sec. 3).

**Remark 1.2** For any  $a > 0$ , we have proved the existence of a solution of (12), but we do not have a uniqueness result, except if we assume that  $a > 0$  is small enough.

In the sequel,  $C$  will denote a suitable positive constant independent of  $\varepsilon$  and which may differ from line to line.

## 2 Basic inequalities

Lemma 2.1 and Lemma 2.2 below are set without proof since it is an adaptation of the case  $p = 2$  of Lemma A.3 [2] and Lemma A.4 [2] respectively but with integrals set on spheres.

**Lemma 2.1** *For every  $0 < r_1 < r_2$ , consider:*

$$C(r_1, r_2) := \{x \in \mathbf{R}^3, \quad r_1 < |x| < r_2\}.$$

*Then, if  $u \in H^1(C(r_1, r_2))$ , the following estimate holds true:*

$$|\nabla u|_{C(r_1, r_2)}^2 \geq \frac{4\pi r_1 r_2}{r_2 - r_1} \left| \int_{\mathbf{S}_{r_2}} u \, d\sigma - \int_{\mathbf{S}_{r_1}} u \, d\sigma \right|^2, \quad (13)$$

where

$$\int_{\mathbf{S}_r} \cdot \, d\sigma := \frac{1}{4\pi r^2} \int_{\mathbf{S}_r} \cdot \, d\sigma.$$

**Lemma 2.2** *There exists a positive constant  $C > 0$  such that:  $\forall (R, \alpha) \in \mathbf{R}^+ \times (0, 1)$ ,  $\forall u \in H^1(B(0, R))$ ,*

$$\int_{B(0, R)} |u - \int_{\mathbf{S}_{\alpha R}} u \, d\sigma|^2 \, dx \leq C \frac{R^2}{\alpha} |\nabla u|_{B(0, R)}^2.$$

From now on, we denote by  $R_\varepsilon$  a radius with the property  $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$ , that is :

$$\lim_{\varepsilon \rightarrow 0} \frac{r_\varepsilon}{R_\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{R_\varepsilon}{\varepsilon} = 0. \quad (14)$$

Obviously, its existence is insured by the assumption  $0 < r_\varepsilon \ll \varepsilon$ .

We introduce the measure

$$dm^\varepsilon := \frac{3}{4\pi} \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 1_{T_\varepsilon}(x) dx$$

and denote the norm in  $L^2_{m^\varepsilon}$  by:

$$|\varphi|_{m^\varepsilon}^2 := \int |\varphi|^2 dm^\varepsilon.$$

We denote the domain confined between the spheres of radius  $a$  and  $b$  by

$$\mathcal{C}(a, b) := \{x \in \mathbf{R}^3, a < |x| < b\}$$

and correspondingly

$$\mathcal{C}^k(a, b) := \varepsilon k + \mathcal{C}(a, b),$$

We also use the following notations:

$$\mathcal{C}_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} \mathcal{C}^k(r_\varepsilon, R_\varepsilon).$$

$$S_{r_\varepsilon}^k = \partial B(\varepsilon k, r_\varepsilon), \quad S_{r_\varepsilon} := \cup_{k \in \mathbf{Z}_\varepsilon} S_{r_\varepsilon}^k,$$

$$S_{R_\varepsilon}^k = \partial B(\varepsilon k, R_\varepsilon), \quad S_{R_\varepsilon} := \cup_{k \in \mathbf{Z}_\varepsilon} S_{R_\varepsilon}^k,$$

Consider the piecewise constant functions defined after some  $\theta \in H_0^1(\Omega)$  by

$$\tilde{\tau}^\varepsilon(x) = \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{S_{r_\varepsilon}^k} \theta d\sigma \right) 1_{Y_\varepsilon^k}(x), \quad (15)$$

$$\tilde{\theta}^\varepsilon(x) = \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{S_{R_\varepsilon}^k} \theta d\sigma \right) 1_{Y_\varepsilon^k}(x). \quad (16)$$

**Lemma 2.3** *For every  $\theta \in H_0^1(\Omega)$ , we have*

$$\int_\Omega |\theta - \tilde{\theta}^\varepsilon|^2 dx \leq C \frac{\varepsilon^3}{R_\varepsilon} \int_\Omega |\nabla \theta|^2 dx, \quad (17)$$

$$\int_{T_\varepsilon} |\theta - \tilde{\tau}^\varepsilon|^2 dx \leq C r_\varepsilon^2 \int_{T_\varepsilon} |\nabla \theta|^2 dx \quad (18)$$

$$\int_\Omega |\tilde{\theta}^\varepsilon - \tilde{\tau}^\varepsilon|^2 dx \leq C \frac{\varepsilon^3}{r_\varepsilon} \int_{\mathcal{C}_\varepsilon} |\nabla \theta|^2 dx. \quad (19)$$

where  $\tilde{\theta}^\varepsilon$  and  $\tilde{\tau}^\varepsilon$  are defined by (15) and (16).

Moreover:

$$\int_\Omega |\tilde{\theta}^\varepsilon|^2 dx = \int |\tilde{\theta}^\varepsilon|^2 dm_\varepsilon, \quad \int_\Omega |\tilde{\tau}^\varepsilon|^2 dx = \int |\tilde{\tau}^\varepsilon|^2 dm_\varepsilon. \quad (20)$$

**Proof.** Notice that by definition:

$$\int_{\Omega} |\theta - \tilde{\theta}^\varepsilon|^2 dx = \sum_{k \in \mathbf{Z}_\varepsilon} \int_{Y_\varepsilon^k} |\theta - \fint_{\mathbf{S}_{R_\varepsilon^k}^k} \theta d\sigma|^2 dx \leq \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})} |\theta - \fint_{\mathbf{S}_{R_\varepsilon^k}^k} \theta d\sigma|^2 dx$$

where we have used that

$$Y_\varepsilon^k \subset B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})$$

for every  $k \in \mathbf{Z}_\varepsilon$ . We use Lemma 2.2 with

$$R = \frac{\varepsilon\sqrt{3}}{2}, \quad \alpha = \frac{2R_\varepsilon}{\varepsilon\sqrt{3}}$$

to deduce that

$$\begin{aligned} \int_{\Omega} |\theta - \tilde{\theta}^\varepsilon|^2 dx &\leq C \left( \frac{\varepsilon\sqrt{3}}{2} \right)^2 \frac{\varepsilon\sqrt{3}}{2R_\varepsilon} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})} |\nabla\theta|^2 dx \\ &\leq C \frac{\varepsilon^3}{R_\varepsilon} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})} |\nabla\theta|^2 dx \leq C \frac{\varepsilon^3}{R_\varepsilon} \int_{\Omega} |\nabla\theta|^2 dx \end{aligned}$$

which shows (17).

To establish (18), we recall the definition:

$$\int_{T_\varepsilon} |\theta - \tilde{\tau}^\varepsilon|^2 dx = \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} |\theta - \fint_{\mathbf{S}_{r_\varepsilon^k}^k} \theta d\sigma|^2 dx$$

Applying Lemma 2.2 with  $R = r_\varepsilon$  and  $\alpha = 1$ , we get the result

$$\int_{T_\varepsilon} |\theta - \tilde{\tau}^\varepsilon|^2 dx \leq Cr_\varepsilon^2 \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} |\nabla\theta|^2 dx \leq Cr_\varepsilon^2 \int_{T_\varepsilon} |\nabla\theta|^2 dx.$$

We come to (19). Indeed, applying Lemma 2.1 and (14):

$$\begin{aligned} \int_{\Omega} |\tilde{\theta}^\varepsilon - \tilde{\tau}^\varepsilon|^2 dx &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_{Y_\varepsilon^k} \left| \fint_{\mathbf{S}_{R_\varepsilon^k}^k} \theta d\sigma - \fint_{\mathbf{S}_{r_\varepsilon^k}^k} \theta d\sigma \right|^2 dy \\ &\leq \sum_{k \in \mathbf{Z}_\varepsilon} \int_{Y_\varepsilon^k} \frac{(R_\varepsilon - r_\varepsilon)}{4\pi R_\varepsilon r_\varepsilon} dy \int_{C_{r_\varepsilon, R_\varepsilon}^k} |\nabla\theta|^2 dx = \frac{(R_\varepsilon - r_\varepsilon)}{4\pi r_\varepsilon R_\varepsilon} \sum_{k \in \mathbf{Z}_\varepsilon} \varepsilon^3 \int_{C_{r_\varepsilon, R_\varepsilon}^k} |\nabla\theta|^2 dx \\ &= C\varepsilon^3 \frac{(R_\varepsilon - r_\varepsilon)}{4\pi r_\varepsilon R_\varepsilon} \int_{C_\varepsilon} |\nabla\theta|^2 dx \leq C \frac{\varepsilon^3}{r_\varepsilon} \int_{C_\varepsilon} |\nabla\theta|^2 dx. \end{aligned}$$

Finally, a direct computation yields (20). ■

**Proposition 2.4** For any  $\theta \in H_0^1(\Omega)$ , there holds true:

$$\int |\theta|^2 dm_\varepsilon \leq C \max\left(1, \frac{\varepsilon^3}{r_\varepsilon}\right) \int_{\Omega} |\nabla\theta|^2 dx.$$

**Proof.** We have:

$$\begin{aligned}
\int |\theta|^2 dm_\varepsilon &\leq 2 \int |\theta - \tilde{\tau}^\varepsilon|^2 dm_\varepsilon + 2 \int |\tilde{\tau}^\varepsilon|^2 dm_\varepsilon \\
&= 2 \int |\theta - \tilde{\tau}^\varepsilon|^2 dm_\varepsilon + 2 \int_\Omega |\tilde{\tau}^\varepsilon|^2 dx \\
&\leq Cr_\varepsilon^2 \int |\nabla\theta|^2 dm_\varepsilon + 4 \int_\Omega |\tilde{\tau}^\varepsilon - \tilde{\theta}^\varepsilon|^2 dx + 8 \int_\Omega |\tilde{\theta}^\varepsilon - \theta|^2 dx + 8 \int_\Omega |\theta|^2 dx \\
&\leq Cr_\varepsilon^2 \left(\frac{\varepsilon}{r_\varepsilon}\right)^3 \int_{T_\varepsilon} |\nabla\theta|^2 dx + C \frac{\varepsilon^3}{r_\varepsilon} \int_{C_\varepsilon} |\nabla\theta|^2 dx + C \frac{\varepsilon^3}{R_\varepsilon} \int_\Omega |\nabla\theta|^2 dx + C \int_\Omega |\nabla\theta|^2 dx \\
&\leq C \left(\frac{\varepsilon^3}{r_\varepsilon} + \frac{\varepsilon^3}{R_\varepsilon} + 1\right) \int_\Omega |\nabla\theta|^2 dx \leq C \max\left(1, \frac{\varepsilon^3}{r_\varepsilon}\right) \int_\Omega |\nabla\theta|^2 dx
\end{aligned}$$

■

**Lemma 2.5** For  $\varphi \in C_c(\Omega)$  consider the piecewise constant function:

$$\varphi^\varepsilon(x) := \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{Y_\varepsilon^k} \varphi dx \right) 1_{B(\varepsilon k, r_\varepsilon)}(x).$$

Then:

$$\lim_{\varepsilon \rightarrow 0} |\varphi - \varphi^\varepsilon|_{m_\varepsilon} = 0.$$

**Proof.** Notice that

$$|\varphi - \varphi^\varepsilon|_{m_\varepsilon}^2 = \frac{3}{4\pi} \left(\frac{\varepsilon}{r_\varepsilon}\right)^3 \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} |\varphi - \int_{Y_\varepsilon^k} \varphi dy|^2 dx.$$

As we have also

$$|B(\varepsilon k, r_\varepsilon)| = \frac{4\pi}{3} r_\varepsilon^3, \quad \text{card}(\mathbf{Z}_\varepsilon) \simeq \frac{|\Omega|}{\varepsilon^3}$$

then, by the uniform continuity of  $\varphi$  on  $\Omega$ , the result follows. ■

### 3 A priori estimates

In the sequel, we denote

$$\gamma_\varepsilon := \frac{r_\varepsilon}{\varepsilon^3} \tag{21}$$

and we assume that

$$\lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon = \gamma \in ]0, +\infty[. \tag{22}$$

We denote  $\mathcal{F} \in H^{-1}(\Omega)$  by

$$\mathcal{F}(\varphi) := \int_\Omega f \varphi dx + \frac{4\pi b}{3} \int_\Omega g \varphi dx \tag{23}$$



**Proposition 3.1** *We have*

$$\mathcal{F}_\varepsilon \rightharpoonup \mathcal{F} \quad \text{weakly in } H^{-1}(\Omega)$$

**Proof.** For  $\varphi \in H_0^1(\Omega)$  it follows

$$\begin{aligned} |\mathcal{F}_\varepsilon(\varphi)| &\leq |f|_{\Omega_\varepsilon} |\varphi|_{\Omega_\varepsilon} + C \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 |g|_\infty \left| \int_{T_\varepsilon} \varphi \, dx \right| \\ &\leq C |\varphi|_\Omega + C \left| \int \varphi \, dm_\varepsilon \right| \end{aligned} \quad (24)$$

with

$$\left| \int \varphi \, dm_\varepsilon \right| \leq \left( \int dm_\varepsilon \right)^{1/2} \left( \int |\theta^\varepsilon|^2 \, dm_\varepsilon \right)^{1/2} = \sqrt{|\Omega|} \left( \int |\varphi|^2 \, dm_\varepsilon \right)^{1/2}. \quad (25)$$

Notice that due to (22), Proposition 2.4 also reads

$$\int |\varphi|^2 \, dm_\varepsilon \leq C |\nabla \varphi|_\Omega^2. \quad (26)$$

Substituting (25) and (26) into the right-hand side of (24), we get, using Poincaré's inequality,

$$|\mathcal{F}_\varepsilon(\varphi)| \leq C |\nabla \varphi|_\Omega. \quad (27)$$

Now, let  $\varphi \in \mathcal{D}(\Omega)$ . By the Mean Theorem, there exist  $\xi_\varepsilon^k \in B(\varepsilon k, r_\varepsilon)$  such that

$$\begin{aligned} \mathcal{F}_\varepsilon(\varphi) &= \int_{\Omega_\varepsilon} f \varphi \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \sum_{k \in \mathbf{Z}^\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} g(x) \varphi(x) \, dx \\ &= \int_{\Omega_\varepsilon} f \varphi \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \sum_{k \in \mathbf{Z}^\varepsilon} \frac{4\pi}{3} r_\varepsilon^3 g(\xi_\varepsilon^k) \varphi(\xi_\varepsilon^k) \\ &= \int_{\Omega_\varepsilon} f \varphi \, dx + \frac{4\pi b}{3} \sum_{k \in \mathbf{Z}^\varepsilon} |Y_\varepsilon^k| g(\xi_\varepsilon^k) \varphi(\xi_\varepsilon^k). \end{aligned}$$

There follows

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\varphi) = \int_\Omega f \varphi \, dx + \frac{4\pi b}{3} \int_\Omega g \varphi \, dx = \mathcal{F}(\varphi). \quad (28)$$

The proof is completed by (27) and the density of  $\mathcal{D}(\Omega)$  in  $H_0^1(\Omega)$ .  $\blacksquare$

**Proposition 3.2** *If  $(u^\varepsilon, \theta^\varepsilon) \in V_\varepsilon \times H_0^1(\Omega)$  is a solution of the problem (12), and if  $\hat{u}^\varepsilon$  stands for  $u^\varepsilon$  continued with zero to  $\Omega$ , then we have*

$$\hat{u}^\varepsilon \quad \text{and} \quad \theta^\varepsilon \quad \text{are bounded in } H_0^1(\Omega). \quad (29)$$

Moreover,

$$|\nabla \theta^\varepsilon|_{\Omega_\varepsilon}^2 + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 |\nabla \theta^\varepsilon|_{T_\varepsilon}^2 \leq C. \quad (30)$$

**Proof.** Substituting  $v = u^\varepsilon$  in (9) and noticing that

$$\int_{\Omega_\varepsilon} u^\varepsilon \theta^\varepsilon \nabla \theta^\varepsilon dx = \int_{\Omega_\varepsilon} u^\varepsilon \nabla \left( \frac{|\theta^\varepsilon|^2}{2} \right) dx = - \int_{\Omega_\varepsilon} \operatorname{div}(u^\varepsilon) \left( \frac{|\theta^\varepsilon|^2}{2} \right) dx = 0,$$

we get:

$$|\nabla u^\varepsilon|_{\Omega_\varepsilon} \leq a |\theta^\varepsilon|_{\Omega_\varepsilon}, \quad (31)$$

Setting  $\varphi = \theta^\varepsilon$  in (10) and taking into account Proposition 3.1, we find

$$|\nabla \theta^\varepsilon|_{\Omega_\varepsilon}^2 + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 |\nabla \theta^\varepsilon|_{T_\varepsilon}^2 = \mathcal{F}_\varepsilon(\theta^\varepsilon) \leq C |\nabla \theta^\varepsilon|_\Omega \quad (32)$$

Noticing that  $b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \gg 1$ , we deduce from (32):

$$|\nabla \theta^\varepsilon|_\Omega^2 \leq |\nabla \theta^\varepsilon|_{\Omega_\varepsilon}^2 + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 |\nabla \theta^\varepsilon|_{T_\varepsilon}^2 \leq C |\nabla \theta^\varepsilon|_\Omega.$$

Therefore

$$|\nabla \theta^\varepsilon|_\Omega \leq C \quad (33)$$

and thus

$$|\theta^\varepsilon|_\Omega \leq C. \quad (34)$$

Then, (30) follows from (32). Finally, (29) is completed by the estimates (31) and (34).  $\blacksquare$

**Proposition 3.3** *There exist  $u \in H_0^1(\Omega; \mathbf{R}^3)$ ,  $\theta \in H_0^1(\Omega)$  and  $\tau \in L^2(\Omega)$  such that, on some subsequence,*

$$\begin{aligned} \hat{u}^\varepsilon &\rightharpoonup u \quad \text{in } H_0^1(\Omega; \mathbf{R}^3), \\ \theta^\varepsilon &\rightharpoonup \theta \quad \text{in } H_0^1(\Omega), \\ \tilde{\tau}^\varepsilon &\rightharpoonup \tau \quad \text{in } L^2(\Omega), \\ \theta^\varepsilon dm_\varepsilon &\overset{*}{\rightharpoonup} \tau dx \quad \text{in } \mathcal{M}_b(\bar{\Omega}), \end{aligned}$$

where  $\mathcal{M}_b(\bar{\Omega})$  is the set of bounded Radon measures on  $\bar{\Omega}$  and where  $\overset{*}{\rightharpoonup}$  denotes the weak-star convergence in the measures.

**Proof.** From (29), we get, on some subsequence, the following convergences:

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{in } H_0^1(\Omega) \quad (35)$$

$$\theta^\varepsilon \rightarrow \theta \quad \text{in } L^2(\Omega). \quad (36)$$

$$\hat{u}^\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega; \mathbf{R}^3). \quad (37)$$

Moreover, (17) yields

$$|\theta^\varepsilon - \tilde{\theta}^\varepsilon|_\Omega^2 \leq C \frac{r_\varepsilon}{\varepsilon^3} \frac{r_\varepsilon}{R_\varepsilon} |\nabla \theta^\varepsilon|_\Omega^2$$

which obviously yields

$$\lim_{\varepsilon \rightarrow 0} |\theta^\varepsilon - \tilde{\theta}^\varepsilon|_\Omega^2 = 0.$$

Combining with (36), we infer that

$$\tilde{\theta}^\varepsilon \rightarrow \theta \quad \text{in } L^2(\Omega). \quad (38)$$

We set

$$\tau^\varepsilon := \frac{3}{4\pi} \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \theta^\varepsilon 1_{T_\varepsilon}(x), \quad (39)$$

and hence

$$\theta^\varepsilon dm_\varepsilon = \tau^\varepsilon dx.$$

Taking (29) and (26) into account, we obtain

$$\int |\theta^\varepsilon|^2 dm_\varepsilon \leq C.$$

We also remark that for any  $\varphi \in C_c(\Omega)$ , we have

$$\int \varphi dm_\varepsilon \rightarrow \int \varphi dx.$$

Then, using Lemma A-2 of [2], we find that there exists some  $\tau \in L^2(\Omega)$  such that, on some subsequence, the following convergence holds:

$$\theta^\varepsilon dm_\varepsilon \xrightarrow{*} \tau dx, \quad \mathcal{M}_b(\overline{\Omega}). \quad (40)$$

Moreover, recall that from (18) we have, taking into account (30):

$$\int |\theta^\varepsilon - \tilde{\tau}^\varepsilon|^2 dm_\varepsilon \leq Cr_\varepsilon^2 \int |\nabla \theta^\varepsilon|^2 dm_\varepsilon \leq Cr_\varepsilon^2. \quad (41)$$

This implies:

$$(\theta^\varepsilon - \tilde{\tau}^\varepsilon) dm_\varepsilon \xrightarrow{*} h dx, \quad \mathcal{M}_b(\overline{\Omega})$$

for some  $h \in L^2(\Omega)$  and

$$|h|_\Omega^2 \leq \liminf_{\varepsilon \rightarrow 0} \int |\theta^\varepsilon - \tilde{\tau}^\varepsilon|^2 dm_\varepsilon = 0,$$

that is:

$$(\theta^\varepsilon - \tilde{\tau}^\varepsilon) dm_\varepsilon \xrightarrow{*} 0, \quad \mathcal{M}_b(\overline{\Omega}). \quad (42)$$

Notice that from (19):

$$|\tilde{\tau}^\varepsilon|_\Omega^2 \leq 2|\tilde{\tau}^\varepsilon - \tilde{\theta}^\varepsilon|_\Omega^2 + 2|\tilde{\theta}^\varepsilon|_\Omega^2 \leq C \frac{\varepsilon^3}{r_\varepsilon} |\nabla \theta^\varepsilon|_{\mathcal{C}_\varepsilon}^2 + C \leq C, \quad (43)$$

and hence, for some  $\tilde{\tau} \in L^2(\Omega)$ ,

$$\tilde{\tau}^\varepsilon \rightharpoonup \tilde{\tau} \quad \text{in } L^2(\Omega). \quad (44)$$

Combining (40) and (42), we arrive at

$$\tilde{\tau}^\varepsilon dm_\varepsilon \xrightarrow{*} \tau dx, \quad \mathcal{M}_b(\overline{\Omega}).$$

It remains to show that

$$\tilde{\tau} = \tau. \quad (45)$$

To that aim, let  $\varphi \in C_c(\Omega)$  and let

$$\varphi^\varepsilon(x) := \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{Y_\varepsilon^k} \varphi \, dy \right) 1_{B(\varepsilon k, r_\varepsilon)}(x).$$

We have

$$\begin{aligned} & \left| \int_{\Omega} (\tau^\varepsilon - \tilde{\tau}^\varepsilon) \varphi \, dx \right| = \left| \frac{3}{4\pi} \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} \theta^\varepsilon \varphi \, dx - \int_{\Omega} \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{\mathbf{S}_{r_\varepsilon}^k} \theta^\varepsilon \, d\sigma \right) 1_{Y_\varepsilon^k} \varphi \, dx \right| \\ &= \left| \int \theta^\varepsilon \varphi \, dm_\varepsilon - \varepsilon^3 \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{\mathbf{S}_{r_\varepsilon}^k} \theta^\varepsilon \, d\sigma \right) \int_{Y_\varepsilon^k} \varphi \, dx \right| = \left| \int \theta^\varepsilon \varphi \, dm_\varepsilon - \int \tilde{\tau}^\varepsilon \varphi^\varepsilon \, dm_\varepsilon \right| \\ &\leq \left| \int_{\Omega} (\theta^\varepsilon - \tilde{\tau}^\varepsilon) \varphi \, dm_\varepsilon \right| + \left| \int_{\Omega} \tilde{\tau}^\varepsilon (\varphi - \varphi^\varepsilon) \, dm_\varepsilon \right| \\ &\leq |\theta^\varepsilon - \tilde{\tau}^\varepsilon|_{m_\varepsilon} |\varphi|_{m_\varepsilon} + |\tilde{\tau}^\varepsilon|_{m_\varepsilon} |\varphi - \varphi^\varepsilon|_{m_\varepsilon}. \end{aligned} \quad (46)$$

From (20) and (43), we deduce that

$$|\tilde{\tau}^\varepsilon|_{m_\varepsilon} = |\tilde{\tau}^\varepsilon|_{\Omega} \leq C.$$

Moreover,  $\varphi \in C_c(\Omega)$  yields

$$|\varphi|_{m_\varepsilon} \leq C.$$

Then, (46) becomes

$$\left| \int_{\Omega} (\tau^\varepsilon - \tilde{\tau}^\varepsilon) \varphi \, dx \right| \leq C |\theta^\varepsilon - \tilde{\tau}^\varepsilon|_{m_\varepsilon} + C |\varphi - \varphi^\varepsilon|_{m_\varepsilon}.$$

From (41), we infer that

$$\left| \int_{\Omega} (\tau^\varepsilon - \tilde{\tau}^\varepsilon) \varphi \, dx \right| \leq C r_\varepsilon + C |\varphi - \varphi^\varepsilon|_{m_\varepsilon}. \quad (47)$$

Thus (47) and Lemma 2.5 yield

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\tau^\varepsilon - \tilde{\tau}^\varepsilon) \varphi \, dx = 0.$$

As this holds for every  $\varphi \in C_c(\Omega)$ , the density of  $C_c(\Omega)$  in  $L^2(\Omega)$  together with (44) and (40) imply that  $\tau^\varepsilon \rightharpoonup \tilde{\tau} = \tau$  in  $L^2(\Omega)$ .  $\blacksquare$

## 4 The two macroscopic heat equations

The aim of this section is to pass to the limit as  $\varepsilon \rightarrow 0$  in the variational formulation

$$\begin{aligned} \forall \Phi \in H_0^1(\Omega), \quad & \int_{\Omega_\varepsilon} \nabla \theta^\varepsilon \nabla \Phi \, dx + b \left( \frac{\varepsilon}{r_\varepsilon} \right)^3 \int_{T_\varepsilon} \nabla \theta^\varepsilon \nabla \Phi \, dx + \\ & + \int_{\Omega_\varepsilon} u^\varepsilon \nabla \theta^\varepsilon \Phi \, dx = \mathcal{F}_\varepsilon(\Phi). \end{aligned} \quad (48)$$

Let  $\varphi, \psi \in \mathcal{D}(\Omega)$  and set

$$\varphi^\varepsilon(x) = \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{\mathbf{S}_{r_\varepsilon}^k} \varphi \, d\sigma \right) 1_{Y_\varepsilon^k}(x), \quad (49)$$

$$\psi^\varepsilon(x) = \sum_{k \in \mathbf{Z}_\varepsilon} \left( \int_{\mathbf{S}_{r_\varepsilon}^k} \psi \, d\sigma \right) 1_{Y_\varepsilon^k}(x). \quad (50)$$

Let  $W^\varepsilon$  denote the fundamental solution of the Laplacian, namely

$$\Delta W^\varepsilon = 0 \quad \text{in } \mathcal{C}(r_\varepsilon, R_\varepsilon), \quad (51)$$

$$W^\varepsilon = 1 \quad \text{in } r = r_\varepsilon, \quad (52)$$

$$W^\varepsilon = 0 \quad \text{in } r = R_\varepsilon. \quad (53)$$

The same arguments as in the proof of Lemma A.3 [2] yield

$$W^\varepsilon(r) = \frac{r_\varepsilon}{(R_\varepsilon - r_\varepsilon)} \left( \frac{R_\varepsilon}{r} - 1 \right) \quad \text{if } y \in \mathcal{C}(r_\varepsilon, R_\varepsilon) \quad \text{and} \quad |y| = r. \quad (54)$$

Then, we set

$$w^\varepsilon(x) := \begin{cases} 0 & \text{in } \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon, \\ W^\varepsilon(x - \varepsilon k) & \text{in } \mathcal{C}_\varepsilon^k, \quad \forall k \in \mathbf{Z}_\varepsilon, \\ 1 & \text{in } T_\varepsilon. \end{cases} \quad (55)$$

**Proposition 4.1** *We have*

$$|\nabla w^\varepsilon|_\Omega \leq C \quad (56)$$

**Proof.** Indeed, direct computation shows

$$\begin{aligned} |\nabla w^\varepsilon|_\Omega^2 &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_{C_{r_\varepsilon, R_\varepsilon}^k} |\nabla w^\varepsilon|^2 \, dx \\ &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_0^{2\pi} d\Phi \int_0^\pi \sin \Theta \, d\Theta \int_{r_\varepsilon}^{R_\varepsilon} \frac{dr}{r^2} \left( \frac{r_\varepsilon R_\varepsilon}{R_\varepsilon - r_\varepsilon} \right)^2 \\ &\leq C \frac{|\Omega|}{\varepsilon^3} \left( \frac{1}{r_\varepsilon} - \frac{1}{R_\varepsilon} \right) \left( \frac{r_\varepsilon R_\varepsilon}{R_\varepsilon - r_\varepsilon} \right)^2 \leq C \frac{\gamma_\varepsilon}{(1 - \frac{r_\varepsilon}{R_\varepsilon})}. \end{aligned}$$

The proof is completed by (14) and (22). ■

For  $\varphi, \psi \in \mathcal{D}(\Omega)$ , let us define

$$\Phi^\varepsilon = (1 - w^\varepsilon)\varphi + w^\varepsilon\psi^\varepsilon. \quad (57)$$

**Lemma 4.2** *We have*

$$\lim_{\varepsilon \rightarrow 0} |\Phi^\varepsilon - \varphi|_\Omega = 0.$$

**Proof.** First notice that  $w^\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ . Indeed:

$$|w^\varepsilon|_\Omega = |w^\varepsilon|_{\mathcal{C}_\varepsilon \cup T_\varepsilon} \leq |\mathcal{C}_\varepsilon \cup T_\varepsilon| = \frac{|\Omega|}{\varepsilon^3} \frac{4\pi}{3} R_\varepsilon^3$$

and  $\lim_{\varepsilon \rightarrow 0} \frac{R_\varepsilon}{\varepsilon} = 0$  by assumption (14). As an immediate consequence:

$$(1 - w^\varepsilon)\varphi \rightarrow \varphi \quad \text{in } L^2(\Omega).$$

Moreover, the uniform continuity of  $\psi$  over  $\Omega$  implies that

$$\lim_{\varepsilon \rightarrow 0} |\psi^\varepsilon - \psi|_\infty = 0$$

so that

$$w^\varepsilon \psi^\varepsilon = w^\varepsilon (\psi^\varepsilon - \psi) + w^\varepsilon \psi \rightarrow 0 \quad \text{in } L^2(\Omega).$$

This achieves the proof. ■

**Proposition 4.3** *If  $\theta^\varepsilon$  is solution of (12) and  $\Phi^\varepsilon$  is given by (57) for any  $\psi, \varphi \in \mathcal{D}(\Omega)$ , then we have*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \Phi^\varepsilon + \Phi^\varepsilon u^\varepsilon) \, dx \\ &= \int_{\Omega} \nabla \theta \cdot (\nabla \varphi + \varphi u) \, dx + 4\pi\gamma \int_{\Omega} (\theta - \tau)(\psi - \varphi) \, dx. \end{aligned}$$

**Proof.** First consider

$$\int_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \Phi^\varepsilon + \Phi^\varepsilon u^\varepsilon) \, dx$$

which reduces to

$$\int_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \varphi + \varphi u^\varepsilon) \, dx = \int_{\Omega} \nabla \theta^\varepsilon \cdot (\nabla \varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} + \varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} u^\varepsilon) \, dx.$$

Lebesgue's dominated convergence theorem yields  $\nabla \varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} \rightarrow \nabla \varphi$  in  $L^2(\Omega)$ . Thus, taking (35) into account:

$$\int_{\Omega} \nabla \theta^\varepsilon \cdot \nabla \varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} \, dx \rightarrow \int_{\Omega} \nabla \theta \cdot \nabla \varphi \, dx.$$

Moreover,

$$|1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} u^\varepsilon - u|_\Omega \leq |u^\varepsilon - u|_\Omega + |u|_{\mathcal{C}_\varepsilon \cup T_\varepsilon}$$

and the right-hand side converges to zero because (37) yields

$$u^\varepsilon \rightarrow u \quad \text{in } L^2(\Omega) \tag{58}$$

and we apply Lebesgue's dominated convergence theorem to conclude with the second term. Thus

$$1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} u^\varepsilon \rightarrow u \quad \text{in } L^2(\Omega). \tag{59}$$

Now, as  $\varphi \in C_c(\Omega)$ ,  $\varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} u^\varepsilon \rightarrow \varphi u$  in  $L^2(\Omega)$ . Thus, using (35) again,

$$\int_{\Omega} \nabla \theta^\varepsilon \cdot \varphi 1_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} u^\varepsilon \, dx \rightarrow \int_{\Omega} \nabla \theta \cdot \varphi u \, dx.$$

As a result:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon \setminus \mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \Phi^\varepsilon + \Phi^\varepsilon u^\varepsilon) dx = \int_{\Omega} \nabla \theta \cdot (\nabla \varphi + \varphi u) dx. \quad (60)$$

Now, we come to the remaining part, namely

$$\begin{aligned} & \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \Phi^\varepsilon + \Phi^\varepsilon u^\varepsilon) dx = \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \varphi + \varphi u^\varepsilon) dx \\ & + \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla w^\varepsilon (\psi^\varepsilon - \varphi) + w^\varepsilon (-\nabla \varphi) + w^\varepsilon u^\varepsilon (\psi^\varepsilon - \varphi)) dx \\ & := I_1 + I_2 \end{aligned} \quad (61)$$

We have

$$I_1 = \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot (\nabla \varphi + \varphi u^\varepsilon) dx = \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla \varphi dx + \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \varphi u^\varepsilon dx. \quad (62)$$

In the first term,  $1_{\mathcal{C}_\varepsilon} \nabla \varphi \rightarrow 0$  in  $L^2(\Omega)$  and  $\nabla \theta^\varepsilon \rightharpoonup \nabla \theta$  in  $L^2(\Omega)$  imply

$$\int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla \varphi dx \rightarrow 0. \quad (63)$$

The second term in (62) is handled by using the estimate:

$$|u^\varepsilon|_{\mathcal{C}_\varepsilon} = |1_{\mathcal{C}_\varepsilon} u^\varepsilon|_\Omega \leq |u^\varepsilon - u|_\Omega + |u|_{\mathcal{C}_\varepsilon},$$

where the right hand side tends to zero due to (58). Using  $\nabla \theta^\varepsilon \rightharpoonup \nabla \theta$  in  $L^2(\Omega)$  again, we deduce that

$$\int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \varphi u^\varepsilon dx \rightarrow 0, \quad (64)$$

and hence  $I_1$  tends to zero.

It remains to study the integral  $I_2$  in (61). To that aim, first notice that

$$\begin{aligned} I_2 &= \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla w^\varepsilon (\psi^\varepsilon - \varphi) dx = \\ &= \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla w^\varepsilon (\psi^\varepsilon - \varphi^\varepsilon) dx + \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla w^\varepsilon (\varphi^\varepsilon - \varphi) dx \end{aligned} \quad (65)$$

where  $\varphi^\varepsilon$  has been defined by (49). The second term in the right-hand side of (65) may be estimated by

$$\left| \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla w^\varepsilon (\varphi - \varphi^\varepsilon) dx \right| \leq |\nabla \theta^\varepsilon|_\Omega |\nabla w^\varepsilon|_\Omega |\varphi - \varphi^\varepsilon|_\infty. \quad (66)$$

As  $(w_\varepsilon)$  is bounded in  $H^1(\Omega)$ , (see Proposition 4.1), the right hand side of (66) tends to zero by the uniform continuity of  $\varphi$  over  $\Omega$ .

Going back to the first term in the right hand side of (65), we may write

$$\begin{aligned} & \int_{\mathcal{C}_\varepsilon} \nabla \theta^\varepsilon \cdot \nabla w^\varepsilon (\psi^\varepsilon - \varphi^\varepsilon) dx \\ &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_0^{2\pi} d\Phi \int_0^\pi \sin \Theta d\Theta \int_{r_\varepsilon}^{R_\varepsilon} \frac{\partial \theta^\varepsilon}{\partial r} \Big|_{\mathcal{C}^k(r_\varepsilon, R_\varepsilon)} \frac{d\bar{w}_\varepsilon}{dr} r^2 dr \left( \int_{\mathbf{s}_{r_\varepsilon}^k} \psi d\sigma - \int_{\mathbf{s}_{r_\varepsilon}^k} \varphi d\sigma \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{r_\varepsilon R_\varepsilon}{(R_\varepsilon - r_\varepsilon)} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{\mathbf{S}_1} (\theta^\varepsilon|_{|x-\varepsilon k|=r_\varepsilon} - \theta^\varepsilon|_{|x-\varepsilon k|=R_\varepsilon}) \left( \int_{\mathbf{S}_{r_\varepsilon}^k} \psi \, d\sigma - \int_{\mathbf{S}_{R_\varepsilon}^k} \varphi \, d\sigma \right) d\sigma_1 \\
&= \frac{4\pi r_\varepsilon R_\varepsilon}{\varepsilon^3 (R_\varepsilon - r_\varepsilon)} \int_{\Omega} (\tilde{\tau}^\varepsilon - \tilde{\theta}^\varepsilon)(\psi^\varepsilon - \varphi^\varepsilon) \, dx = \frac{4\pi\gamma_\varepsilon}{\left(1 - \frac{r_\varepsilon}{R_\varepsilon}\right)} \int_{\Omega} (\tilde{\tau}^\varepsilon - \tilde{\theta}^\varepsilon)(\psi^\varepsilon - \varphi^\varepsilon) \, dx
\end{aligned}$$

from which we infer that  $I_2$  is converging to

$$4\pi\gamma \int_{\Omega} (\tau - \theta)(\psi - \varphi) \, dx,$$

and the proof is completed.  $\blacksquare$

We are in the position to state a part of our main result:

**Corollary 4.4** *The limit  $(u, \theta, \tau)$  verifies the following equations:*

$$u \nabla \theta - \Delta \theta + 4\pi\gamma(\theta - \tau) = f \quad \text{in } \Omega, \quad (67)$$

$$\gamma(\tau - \theta) = \frac{b}{3}g \quad \text{in } \Omega. \quad (68)$$

**Proof.** Consider the variational formulation (10) with the test function  $\Phi = \Phi^\varepsilon$  defined by (57) for any  $\varphi, \psi \in \mathcal{D}(\Omega)$ . Then, the left-hand side tends to

$$\int_{\Omega} \nabla \theta \cdot (\nabla \varphi + \varphi u) \, dx + 4\pi\gamma \int_{\Omega} (\tau - \theta)(\psi - \varphi) \, dx. \quad (69)$$

This is a direct consequence of Proposition 4.3 together with the remark that

$$\int_{T_\varepsilon} \nabla \theta^\varepsilon \nabla \Phi^\varepsilon \, dx = 0$$

since  $\Phi^\varepsilon$  is constant on every  $B(\varepsilon k, r_\varepsilon)$ ,  $k \in \mathbf{Z}_\varepsilon$ .

The convergence of the right-hand side is obtained by using the uniform continuity of  $\psi$  and by Proposition 3.1. Thus we find the variational formulation of (67)-(68) and the proof is completed.  $\blacksquare$

## 5 The homogenized problem

Proposition 3.3 yields the existence of some  $u \in H_0^1(\Omega; \mathbf{R}^3)$  with  $\operatorname{div}(u) = 0$  and for which the following convergence holds on some subsequence

$$\hat{u}^\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega; \mathbf{R}^3).$$

From [7], we find that there exists an extension of the pressure (denoted by  $\hat{p}^\varepsilon$ ) and some  $p \in L^2(\Omega)$  such that

$$\hat{p}^\varepsilon \rightharpoonup p \quad \text{in } L^2(\Omega)/\mathbf{R}.$$

We denote by  $(w_\varepsilon^k, q_\varepsilon^k) \in H^1(\mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2})) \times L_0^2(\mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}))$  the only solution of the following Stokes problem

$$\operatorname{div} w_\varepsilon^k = 0 \quad \text{in } \mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}),$$



$$\begin{aligned}
-\Delta w_\varepsilon^k + \nabla q_\varepsilon^k &= 0 \quad \text{in } \mathcal{C}(r_\varepsilon, \frac{\varepsilon}{2}), \\
w_\varepsilon^k &= 0 \quad \text{if } r = r_\varepsilon, \\
w_\varepsilon^k &= \mathbf{e}^{(k)} \quad \text{if } r = \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently, we define

$$v_\varepsilon^k(x) = \begin{cases} 0 & \text{if } x \in T_\varepsilon, \\ w_\varepsilon^k(x - \varepsilon i) & \text{if } x \in \mathcal{C}^i(r_\varepsilon, \frac{\varepsilon}{2}), \quad i \in \mathbf{Z}_\varepsilon, \\ \mathbf{e}^{(k)} & \text{if } x \in \Omega_\varepsilon \setminus \cup_{i \in \mathbf{Z}_\varepsilon} \mathcal{C}^i(r_\varepsilon, \frac{\varepsilon}{2}). \end{cases}$$

For  $\varphi \in \mathcal{D}(\Omega)$ , we set  $v = \varphi v_\varepsilon^k$  in (9) and then using the energy method like in [1] we find the equation that the velocity field satisfies in  $H^{-1}(\Omega)$ :

$$-\Delta u + 6\pi\gamma u = -\nabla p + a\theta \mathbf{e}^{(3)} \quad \text{in } \Omega. \quad (70)$$

Finally, we summarize the results of Proposition 3.3, Corollary 4.4 together with the relation (70) into our main theorem.

**Theorem 5.1** *If  $(u^\varepsilon, p^\varepsilon)$  is a solution of problem (12), then the following convergences hold on some subsequence*

$$\begin{aligned}
\hat{u}^\varepsilon &\rightharpoonup u \quad \text{in } H_0^1(\Omega; \mathbf{R}^3), \\
\theta^\varepsilon &\rightharpoonup \theta \quad \text{in } H_0^1(\Omega), \\
\theta^\varepsilon dm_\varepsilon &\xrightarrow{*} \tau dx \quad \text{in } \mathcal{M}_b(\overline{\Omega}),
\end{aligned}$$

where  $(u, \theta) \in H_0^1(\Omega; \mathbf{R}^3) \times H_0^1(\Omega)$ , which stand for the macroscopic velocity and temperature of the fluid, and  $\tau \in L^2(\Omega)$ , which stands for the macroscopic temperature of the vanished suspensions, form a solution of the following system:

$$\begin{aligned}
\operatorname{div} u &= 0 \quad \text{in } \Omega, \\
-\Delta u + 6\pi\gamma u &= -\nabla p + a\theta \mathbf{e}^{(3)} \quad \text{in } \Omega, \\
u \nabla \theta - \Delta \theta + 4\pi\gamma(\theta - \tau) &= f \quad \text{in } \Omega, \\
\gamma(\tau - \theta) &= \frac{b}{3}g \quad \text{in } \Omega.
\end{aligned}$$

**Remark 5.2** In the present case, with suspensions of critical size, the Brinkman-Boussinesq equation was an expected result; nevertheless, our proof is different from that of [7], which treated the homogenization of the Navier-Stokes equations for perforated domains in a similar case.

**Remark 5.3** Our two-temperature model, with  $\gamma$  as transfer coefficient, is the macroscopic effect of the assumption on the the ratio of the fluid/solid conductivities.

**Remark 5.4** The appearance of the source term  $\frac{4\pi b}{3}g$  in the second macroscopic heat equation is strictly the consequence of the assumption on the microscopic radiation.

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## References

- [1] D. CIORANESCU AND F. MURAT, Un terme étrange venu d'ailleurs, I and II. *Nonlinear Partial Differential Equations and their Applications. College de France Seminar*, **II** and **III**, Paris (1979/1980) and (1980/1981) ; *Research Notes in Mathematics*, **60** and **70**, Pitman, London (1982) and (1983), 98–138 and 154–178
- [2] M. BELLIEUD AND G. BOUCHITTÉ, Homogenization of elliptic problems in a fiber reinforced structure. Non local effects. *Ann. Scuola Norm. Sup. Pis Cl. Sci.* **26** (1998), 407–436.
- [3] J. CASADO-DIAZ, Two-scale convergence for nonlinear Dirichlet problems in perforated domains. *Proceedings of the Royal Society of Edinburgh* **130 A** (2000), 249–276.
- [4] M. BRIANE AND N. TCHOU, Fibered microstructure for some non-local Dirichlet forms. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **30** (2001), 681–712.
- [5] M. BELLIEUD AND I. GRUAIS, Homogenization of an elastic material reinforced by very stiff or heavy fibers. Non local effects. Memory effects. *J. Math. Pures Appl.*, to appear (2004).
- [6] U. MOSCO, Composite media and asymptotic Dirichlets forms. *J. Functional Anal.* **123** (1994), 368–421.
- [7] G. ALLAIRE, Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational. Mech. Anal.* **113** (1991), 209–259.
- [8] H. ENE AND D. POLIŠEVSKI, *Thermal Flow in Porous Media*. D. Reidel Pub. Co., Dordrecht (1987).
- [9] D. POLIŠEVSKI, Thermal flow through a porous radiant of low conductivity. *J. Appl. Math. Phys. (ZAMP)* **53** (2002), 12–19.

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