



HAL
open science

Geometry of Certain Lie-Frobenius Groups

Jean Michel Dardie, Alberto Medina, Hassène Siby

► **To cite this version:**

Jean Michel Dardie, Alberto Medina, Hassène Siby. Geometry of Certain Lie-Frobenius Groups. 2005.
hal-00005449

HAL Id: hal-00005449

<https://hal.science/hal-00005449>

Preprint submitted on 18 Jun 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Geometry of certain Lie-Frobenius groups

Jean Michel Dardié, Alberto Medina and Hassène Siby

Abstract. Let be $G_{n,p}(\mathbb{K}) = M_{n,p}(\mathbb{K}) \rtimes GL(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , the semi-direct product of the additive group of matrices $M_{n,p}(\mathbb{K})$ by the group $GL(\mathbb{K}^n)$ where the action is done by multiplication of matrices. If $n = kp, p \geq 1$ the Lie group $G_{n,p}(\mathbb{K})$ admits an exact left invariant symplectic form [10]. We study the geometry of this symplectic manifold. If $n > p$ (resp. $n = p$) we prove that $G_{n-p,p}(\mathbb{K})$ (resp $G_{n-1,1}(\mathbb{K})$) is the symplectic reduction of the symplectic orthogonal $(M_{n,p}(\mathbb{K}))^\perp$ of $M_{n,p}(\mathbb{K})$ in $G_{n,p}(\mathbb{K})$ and reciprocally that $G_{n,p}(\mathbb{K})$ is a symplectic double extension, in the sense of [6], of $G_{n-p,p}(\mathbb{K})$ (resp. of $G_{n-1,1}(\mathbb{K})$). Moreover we show that $G_{n,p}(\mathbb{K})$ admits two left transverse Lagrangian foliations (with affine and closed leaves) . Consequently there exists (a left invariant) a canonical torsion free symplectic connection on $G_{n,p}(\mathbb{K})$.

Key words :

Exact symplectic Lie group , Affine Lie group, Symplectic Reduction , Symplectic double extension.

Introduction

The group $G_{n,1}(\mathbb{K}) = Aff(\mathbb{K}^n)$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} admits left invariant symplectic structures which are all exact because $H^2(aff(\mathbb{K}^n), \mathbb{K}) = 0$ ([9]). To give an invariant symplectic form on $G_{n,1}(\mathbb{K})$ is to give a linear form α on the Lie algebra $aff(\mathbb{K}^n)$ such that the 2-cobound $\delta\alpha$ is non degenerate. That is to say α is a point with an open orbit under the coadjoint action of $G_{n,1}(\mathbb{K})$. These points have been characterized in [9] within the framework of study of affine group representations and in [3] where is carried on the study of left invariant symplectic structures of affine group initiated in [4]. In [10], it showed that coadjoint action of Lie group semi-direct product of $M_{n,p}(\mathbb{K})$ and $GL(\mathbb{K}^n)$ by matricial product, where $M_{n,p}(\mathbb{K})$ indicate the additive group of $(n \times p)$ -matrices, admits open orbits if only if $n = kp$. We extend to these groups denoted $G_{n,p}(\mathbb{K})$ the result obtained in [3] for the classical affine group : existence of a unique symplectic structure up isomorphism (Theorem 2.7), existence of transverse symplectic foliations (Theorem 2.6), as well as of a Lagrangian bi-foliation with closed leaves (Theorem 3.2). Such a pair of Lagrangian foliations is important in the quantization procedure (polarization) and implies the existence of a canonical (torsion free) symplectic connection on $G_{n,p}(\mathbb{K})$. The natural left action of $G_{n,p}(\mathbb{K})$ on itself being hamiltonian we can provide (Theorem 2.1) a similar of structure's theorem of simply connected symplectic Lie group from [5].

In the third part we study up the fibrations of this theorem for explicit suggest a construction of $(G_{n,p}(\mathbb{K}), d\alpha_1^+)$ when the generalized double extension of [5] don't apply in this case.

To complete this introduction recall that a symplectic Lie group (G, ω^+) has an affine structure given by the left invariant connection (torsion free and zero curvature) ∇ defined for all $x, y, z \in T_z(G)$ by :

$$\omega^+(\nabla_{x^+}^+ y^+, z^+) = -\omega^+(y^+, [x^+, z^+])$$

where x^+ denotes the left invariant vector field associated to x .

Then one says that the pair (G, ∇^+) is an affine Lie group. In this case the product on \mathcal{G} , $xy = (\nabla_{x^+}^+ y^+)$ is with associator left invariant and verifies the condition $xy - yx = [x, y]$. This structure plays a central rule in this study.

1. Left invariant symplectic structures on the Lie groups $G_{n,p}$

In what follows $G_n(\mathbb{K})$ denotes the classical affine group where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $\mathcal{G}_n(\mathbb{K})$ is its Lie algebra. By analogy, we denote $G_{n,p}(\mathbb{K})$ the group semi-direct product $M_{n,p}(\mathbb{K}) \rtimes Gl(\mathbb{K}^n)$ with $n = kp, p \geq 1$ and $\mathcal{G}_{n,p}(\mathbb{K}^n)$, its Lie algebra. Obviously $G_{n,1}(\mathbb{K})$ is isomorphic to $Aff(\mathbb{K}^n)$. Considering that $H^2(Aff(\mathbb{K}^n), \mathbb{K}) = 0$ (see [3]) any left invariant alternate 2-form is invariantly exact. This result becomes general to $\mathcal{G}_{n,p}(\mathbb{K})$ in the following way :

1.1 Lemma. Every left invariant symplectic form on $G_{n,p}(\mathbb{K})$ is exact.

Proof. If $k = p = 1$; $\mathcal{G}_{1,1}$ is isomorphic to $aff(\mathbb{K})$ and the result follows. Assume that k (or p) is greater than 1. Let be $\omega \in Z^2(\mathcal{G}_{n,p}; \mathbb{K})$, $\mathcal{G}_{n,p} = M_{p,n}(\mathbb{K}) \rtimes gl(n)$. This means that we have

$$\oint \omega([a, b], c) = 0 \quad (*) \quad \text{for every } a, b, c \text{ in } \mathcal{G}_{n,p}.$$

If we take $a = (x, u)$, $b = (y, v)$ and $c = (0, Id)$

(*) implies $\omega(x, y) = 0$ for every $x, y \in M_{n,p}$, and consequently $\omega(x, b) + \omega(a, y) = \omega(I, uy - vx + [u, v])$.

But this is equivalent to

$$\begin{aligned} \omega(a, b) &= \omega(I, ux - vy) \\ &= -\delta\beta(a, b) \quad \text{with } \beta = \omega(I, \cdot) \quad \blacksquare \end{aligned}$$

N.B In fact for all Lie algebra \mathcal{G} having an element $a \in \mathcal{G}$ such that ad_a is projector, we have $H^2(\mathcal{G}, \mathbb{K}) = 0$.

Remark 1. The mapping

$$\begin{aligned} M_{p,n}(\mathbb{K}) \times gl(n) &\longrightarrow (M_{n,p}(\mathbb{K}) \times gl(n))^* \\ (H, M) &\longmapsto \alpha_{(H,M)} \end{aligned}$$

given by

$$\alpha_{(H,M)}(N, V) = tr(N.H) + tr(M.V)$$

is a linear isomorphism.

In the following, the dual space $\mathcal{G}_{n,p}^*$ of $\mathcal{G}_{n,p}$ is identified with $M_{p,n}(\mathbb{K}) \times gl(n)$. To give an invariant closed 2-form on $G_{n,p}(\mathbb{K})$ it is equivalent to give a linear form α on $\mathcal{G}_{n,p}(\mathbb{K})$. The 2-form $d\alpha^+$, where α^+ denotes the left invariant 1-form defined by α is symplectic if and only if α has an open orbit under the coadjoint representation.

Using the Remark 1 and the natural embedding of $\mathcal{G}_{n,p}$ in $GL(\mathbb{K}^{n+p})$ we can show

1.2 Lemma. The coadjoint representations of $\mathcal{G}_{n,p}$ and $G_{n,p}$ are given by the following formulas :

$$(i) \quad ad_{(0,u)}^*(H, N) = (-H, [u, N])$$

$$(ii) \quad ad_{(x,0)}^*(H, N) = (0, x.H)$$

$$(iii) \quad Ad_{(0,U)}^*(H, N) = (H.U^{-1}, UNU^{-1})$$

$$(iv) \quad Ad_{(X, Id_{\mathbb{K}^n})}^*(H, N) = (H, N + X.H)$$

This implies

$$Ad_{(X,U)}^*(H, N) = (H.U^{-1}, UNU^{-1} + XHU^{-1})$$

Consequently the coadjoint orbit $Ad_{G_{n,p}}^*(\alpha)$ of $\alpha = (H_0, N_0)$ with $H_0 = (0, I_p) \in M_{p,n}$ and

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}$$

is open. Denote by H_0 the (p, n) -matrix whose $p \times p$ blocks are all null except for the last, which is the identity of \mathbb{K}^p and denoted by N_0 the $(n \times n)$ -matrix whose $p \times p$ blocks are all null except the sub-diagonals which are $Id_{\mathbb{K}^p}$. The previous formulas allow us to verify that the orbit of (H_0, N_0) is open since the isotropy subalgebra is trivial.

2. Symplectic Reduction - Left invariant Symplectic foliation on $G_{n,p}(\mathbb{K})$.

Denote by $\omega^+ = d\alpha^+$, $\alpha = (H, N) \in \mathcal{G}_{n,p}^*$ a left invariant symplectic form on $G_{n,p}$.

The action of $G_{n,p}(\mathbb{K})$ on $G_{n,p}(\mathbb{K})$ given by

$$L_G : G_{n,p}(\mathbb{K}) \times G_{n,p}(\mathbb{K}) \longrightarrow G_{n,p}(\mathbb{K}), \quad (\tau, \sigma) \longmapsto \tau\sigma \text{ (product in } G_{n,p} \text{)}$$

is a symplectic action. Moreover L_G is a Hamiltonian action; a momentum mapping for the action is given by

$$\mu : G_{n,p}(\mathbb{K}) \longrightarrow \mathcal{G}_{n,p}(\mathbb{K})^*$$

$$\sigma \longmapsto \mu(\sigma) : x \longmapsto \langle \alpha^+(\sigma), x^-(\sigma) \rangle$$

where $x \in \mathcal{G}_{n,p}$ and x^- is the right invariant vector field associated to x .

The subgroup $\mathcal{H} := M_{n,p}$ is (totally) isotropic for ω^+ because \mathcal{H} is an abelian Lie group. Moreover $L_{\mathcal{H}} : \mathcal{H} \times G_{n,p} \longrightarrow G_{n,p}$ is a hamiltonian action and a momentum mapping for $L_{\mathcal{H}}$ is given by :

$$m : G_{n,p} \longrightarrow L(\mathcal{H})^*$$

$$\sigma \longmapsto \mu(\sigma)|_{L(\mathcal{H})}$$

where $L_{\mathcal{H}}$ is the Lie algebra of \mathcal{H} . We arrive at the following theorem :

2.1 Theorem . Let $(G_{n,p}, d((H, N)^+))$ defined as above. Then

1. $m^{-1}(H)$ is a closed subgroup of $G_{n,p}$ and $\mathcal{H} \subset m^{-1}(H)$

2. The canonical exact sequence of Lie groups

$$(2) \quad \{\epsilon\} \rightarrow \mathcal{H} \rightarrow m^{-1}(H) \rightarrow m^{-1}(H)/\mathcal{H} \rightarrow \{\epsilon\}$$

is split. It is also an exact sequence of affine Lie groups.

3. The reduced symplectic Lie group $m^{-1}(H)/\mathcal{H}$ is isomorphic to :

$$\begin{array}{ll} G_{n-p,p}(\mathbb{K}) & si \quad n > p \\ G_{p-1,1}(\mathbb{K}) & si \quad n = p > 1 \end{array}$$

4. In the principal bundle

$$(3) \quad m^{-1}(H) \xrightarrow{i} G_{n,p}(\mathbb{K}) \xrightarrow{m} \Theta$$

where Θ is the set of matrices of rank p in $M_{n,p}^* \equiv M_{p,n}$, the fiber is an affine Lie subgroup and m is affine relative to the usual affine structure of $\Theta \subset \mathbb{K}^{np}$.

We need the following lemma for which the proof is obvious.

2.2 Lemma . The mapping m is a surjective submersion.

Furthermore,

$$(4) \quad m((X, T)) = H.T^{-1} \quad \text{where } (X, T) \text{ is in } G_{n,p}(\mathbb{K})$$

Proof. The formula (4) is a direct consequence of the definition of m and the Lemma 1.2. Thus it is clear that m is a surjective submersion on the set Θ of matrices of rank p of $M_{n,p}(\mathbb{K})$.

Proof of the theorem 2.1.

Formula (4) implies that $m^{-1}(H) = \{(X, T) \in G_{n,p}(\mathbb{K})/HT^{-1} = H\}$ is a (closed) subgroup of $G_{n,p}(\mathbb{K})$ which contains \mathcal{H} . Moreover the factor group $m^{-1}(H)/\mathcal{H}$ can be identified with $\{(0, T) \in G_{n,p}/HT^{-1} = H\}$. Thus (2) is an split sequence of Lie groups. On the other hand, since \mathcal{H} is commutative and ω^+ is exact, it follows that $L(\mathcal{H}) \subset L(\mathcal{H})^\perp$ and a straightforward shows that $L(m^{-1}(H)) = L(\mathcal{H})^\perp$.

Moreover from the formula

$$\omega(xy, z) = -\omega(y, [x, z])$$

defining the left symmetric product in $\mathcal{G}_{n,p}(\mathbb{K})$ (see formula (1)) it turns out that $L(\mathcal{H})^\perp$ is a left symmetric subalgebra of $\mathcal{G}_{n,p}(\mathbb{K})$ and that $L(\mathcal{H})$ is a two-side ideal of $L(\mathcal{H})^\perp$. Consequently (2) is an exact sequence of affine Lie groups i.e \mathcal{H} , $m^{-1}(H)$, $m^{-1}(H)/\mathcal{H}$ are affine Lie groups and the applications of formula (2) are affine.

On the other hand a list of the elements of the matrix group $\{(0, T) \in G_{n,p}(\mathbb{K})/HT^{-1} = H\}$ allows to observe that the group $m^{-1}(H)/\mathcal{H}$ is isomorphic to the group $G_{n-p,p}(\mathbb{K})$ if $n > p$ and isomorphic to $G_{n-1,1}(\mathbb{K})$ if $n = p$. This proves statement 3.

Now , we need to show that the manifold Θ is endowed with an affine structure which makes the momentum mapping affine.

Let F be the subbundle of $TG_{n,p}(\mathbb{K})$ tangent to the $L_{\mathcal{H}}$ -orbit and F^\perp its symplectic orthogonal. Denote by \mathcal{F} and \mathcal{F}^\perp the associated foliations respectively. Since \mathcal{H} is normal in $G_{n,p}(\mathbb{K})$, the foliation \mathcal{F}^\perp is defined either by the left invariant form on $G_{n,p}(\mathbb{K})$ given by $\eta'_j := i(e_j^+) \omega^+$ where (e_i) 's form a

basis of $L(\mathcal{H})$ and i denote the interior product, or by the closed forms (thus exact) $\eta_j := i(e_j^-)\omega^+$. Obviously the η_j are basic for the fibration. the forms $\bar{\eta}_j$ which are the projections of η_j by m define a local parallelism on Θ . This parallelism is global and commutative because the $\bar{\eta}_j$ are exact. Hence m is affine. ■

Let α_1 and α_2 be the linear forms on $\mathcal{G}_{n,p}(\mathbb{K})$ defined respectively by $\alpha_1(x, u) = \text{tr}(H.x)$ and $\alpha_2(x, u) = \text{tr}(N.u)$. We get :

2.3. Theorem. $\ker(d\alpha_1)$ and $\ker(d\alpha_2)$ are supplementary symplectic Lie subalgebras of $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$. So they determine two tranverse symplectic foliations left invariant on $G_{n,p}(\mathbb{K})$

Proof. Obviously $\ker(d\alpha_i)$ is a Lie subalgebra of $\mathcal{G}_{n,p}(\mathbb{K})$. In addition the subspaces $\ker(d\alpha_i)$, $i = 1, 2$ are in direct sum because $\ker(d\alpha_1) \cap \ker(d\alpha_2) = \{0\}$.

If we take $\alpha = (H_0, N_0)$ we directly observe that we have $\dim(\ker(\delta\alpha_1)) = p^2(k-1)k$ and $\dim(\ker(\delta\alpha_2)) = 2p^2k$. We extends these results about the dimension for all $\alpha \in \mathcal{G}_{n,p}^*(\mathbb{K})$ with open coadjoint orbit. consequently $\ker(\delta\alpha_1)$ and $\ker(\delta\alpha_2)$ are symplectic Lie subalgebras of $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$. ■

Let $C(N_0)$ be the subalgebra of $gl(\mathbb{K}^n)$ given by :

$$\begin{pmatrix} A_0 & & & & \\ & A_1 & & \ddots & \\ & & \ddots & & \\ & & & \ddots & \\ A_{k-1} & \cdots & A_1 & A_0 & \end{pmatrix}$$

2.4. Scholie. The Lie algebra $\ker(d\alpha_1)$ is isomorphic to the Lie algebra $\mathcal{G}_{n-p,p}(\mathbb{K})$ if $n > p$ and $\mathcal{G}_{p-1,1}(\mathbb{K})$ if $n = p$ while $\ker(d\alpha_2)$ is isomorphic to the semi-direct $M_{n,p}(\mathbb{K}) \rtimes C(N_0)$ if $n > p$ and to the semi-direct product $M_{p,p}(\mathbb{K}) \rtimes C(N_0)$ if $n = p$.

The following result is the infinitesimal version of Theorem 2.1. It gives a more precise and complete statement of Theorem 2.1.

2.5 Proposition. With the notations of Therem 2.1., if $I = L(\mathcal{H})$, the canonical sequence of vectoriel spaces

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp/I \rightarrow 0$$

is a split exact sequence of Lie algebras. It is also an exact sequence of left symmetric algebras.

Furthermore the Lie algebra $\mathcal{G}_{n,p}(\mathbb{K})$ decomposes as a direct sum of Lie subalgebras I^\perp and $C(N_0)$.

2.6 Theorem. The symplectic Lie group $(G_{n,p}, d\alpha^+)$ is endowed with two transversal left invariant symplectic foliations whose leaves are affine submanifolds of $G_{n,p}(\mathbb{K})$.

The following assertion specifies the number of open orbits in $\mathcal{G}_{n,p}^*(\mathbb{K})$ as well as the left invariant symplectic structures on $G_{n,p}(\mathbb{K})$.

2.7 Theorem.

a. There exist two open orbits of the coadjoint representation of $G_{n,p}(\mathbb{K})$ if $\mathbb{K} = \mathbb{R}$ and only one if $\mathbb{K} = \mathbb{C}$

b. Up isomorphism there is only one left invariant symplectic structure on $G_{n,p}$ i.e if ω and ω' are two left invariant symplectic forms on $G_{n,p}(\mathbb{K})$, then there exists an automorphism φ of Lie algebra $\mathcal{G}_{n,p}(\mathbb{K})$ such that :

$$\omega_\varepsilon(\cdot, \cdot) = \omega'_\varepsilon(\varphi \cdot, \varphi \cdot).$$

The following lemmas set up the main steps of the demonstration of Theorem 2.7. This lemma allow to count the $Ad_{G_{n,p}}^*$ -orbits

2.8 Lemma. If $Orb_{(H,M)}$ is the coadjoint orbit of $(H, M) \in \mathcal{G}_{n,p}^* \equiv M_{p,n}(\mathbb{K}) \times gl(n)$ then $Orb_{(H,M)}$ has an element $(H'_0, M'_0) \in \mathcal{G}_{n,p}^*$ such that $H'_0 = (0, \dots, A_p) \in M_{p,n}$. Moreover, if $Orb_{(H,M)}$ is open, then H'_0 can be taken as $H'_0 = (0, \dots, I_p) = H_0$.

Proof. It suffices to remark that there exists $U \in GL_o(\mathbb{K}^n)$ such that $HU^{-1} = H_0$, that is obvious if we look at H as the matrix of a mapping from \mathbb{K}^n into \mathbb{K}^p and U^{-1} as the matrix of a change of basis in \mathbb{K}^n . Then the last formula of lemma 1.2 show the first assertion.

Now suppose that $Orb_{(H,M)} = Orb_{(H'_0, M'_0)}$ is a open orbit. This implies that we have

$$(x, u) \in \mathcal{G}_{n,p}; ad_{(x,u)}^*(\alpha) = 0 \Rightarrow (x, u) = 0$$

In particular

$$ad_{(x,0)}^*(H'_0, M') = 0 \Rightarrow x = 0$$

In other words

$$(**) \quad tr(xH_0u) = 0, u \in gl(n) \Rightarrow x = 0$$

with $H_0 = (0, A_p)$

But a straight calculation shows that $(**)$ implies A_p is invertible.

Finally there exists $U \in GL_o(\mathbb{K}^n)$ such that $H'_0U^{-1} = H_0$, but this relation fix only the last diagonal block of U to the value A^{-1} . Therefore if $k \geq 2$ we able to take an other diagonal block equals to $\det A^{-1} \cdot Id_{\mathbb{K}^p}$ and completing the diagonale by the 1, to construct a such matrix U in $SL(\mathbb{K}^n)$.

2.9 Lemma. An open orbit $Orb_{(H_0, M)}$ contains only one element (H_0, M') , where the block decomposition of M' can be written :

$$M' = \begin{pmatrix} M_1 & 0 \\ H_1 & 0 \end{pmatrix}$$

with $(H_1, M_1) \in \mathcal{G}_{n-p,p}^*$.

Proof. Because

$$Ad_{(X, Id)}^*(H_0, M) = (H_0, M + X.H)$$

it is clear that there is only one $X \in M_{p,n}$ such that $M' = M + MH_0$.

2.10 Lemma. The linear forms (H_0, M') and (H_o, P') on $\mathcal{G}_{n,p}$ with $M' = (H_1, M_1)$ and $P' = (K_1, P_1)$, defined as in Lemma 2.9, belong to the same $Ad_{G_{n,p}}^*$ -orbit if and only if (H_1, M_1) and (K_1, P_1) are in the same $Ad_{G_{n-p,p}}^*$ -orbit.

Proof. (H_0, M') and (H_o, P') are in the same orbit if and only if $\exists U \in GL_o(\mathbb{K}^n)$ such that $UM'U^{-1} = P'$ and $H_0U^{-1} = H_o$.

The second condition implies

$$U = \begin{pmatrix} U' & 0 \\ X_1 & Id \end{pmatrix}, \quad \text{with} \quad U_1 \in GL_0(\mathbb{K}^{n-p})$$

On other hand $UM'U^{-1} = P'$ in $gl(\mathbb{K}^n)$ is equivalent to $Ad_{(X_1, U_1)}^*(H_1, M_1) = (K_1, P_1)$ in $\mathcal{G}_{n-p, p}(\mathbb{K})$ ■

Proof of the theorem 2.7.

Following the previous Lemmas we have

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{n,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\}$$

On the other hand, using similar arguments as those of the lemmas we can show that

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{1,1}}^* - \text{orbit} \right\} = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{R} \\ 1 & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

If $\mathbb{K} = \mathbb{C}$ there is only one symplectic structure on $G_{n,p}$.

In the case $\mathbb{K} = \mathbb{R}$, it follows from the lemmas that every $Ad_{G_{n,p}}^*$ -open orbit contains an element (H_0, N_0) where

$$N_0 = \begin{pmatrix} 0 & & & & & \\ A_p & \ddots & & & & \\ & I_p & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & I_p & 0 & \end{pmatrix} \quad \text{with } A_p \text{ invertible}$$

Nevertheless two matrices as N_0 are conjugate by an element P of $GL(n)$.

However the mapping

$$\mathcal{G}_{n,p}^* \longrightarrow \mathcal{G}_{n,p}^*, \quad (g, M) \longmapsto (P^{t-1}g, P^{-1}MP)$$

is dual of the mapping

$$\theta : \mathcal{G}_{n,p} \longrightarrow \mathcal{G}_{n,p} \quad (x, N) \longmapsto (P^{-1}x, P^{-1}NP)$$

and these later is an automorphism of the Lie algebra $\mathcal{G}_{n,p}$.

Consequently if ω_1^+ , ω_2^+ are two left invariant symplectic forms on $G_{n,p}$, we have :

$$\omega_1^+ = \theta^*(\omega_2^+). \quad \blacksquare$$

Remark 2. Notice that the only one left invariant affine structure on $G_{n,p}$ is given by (H_0, N_0) .

The following result is an important consequence of the previous study.

2.11 Proposition. The identity component of $G_{n,p}$ is diffeomorphic to an open $Ad_{G_{n,p}}^*$ -orbit. Consequently $G_{n,p}$ and $\{\alpha \in \mathcal{G}_{n,p}^*; \alpha \text{ is Poisson-regular}\}$ are diffeomorphic.

Proof. (By induction) We must prove that the orbital mapping in (H_0, N_0)

$$(G_{n,p})_o \longrightarrow \mathcal{G}_{n,p}^*, \quad (X, U) \longmapsto (H_0U^{-1}, UN_0U^{-1} + XH_0U^{-1})$$

has a trivial isotropy.

The result is obvious for $G_{1,1}$ and this implies that it is also true for $G_{p,p}$ (thanks to a sequence double extension).

Consider the case $G_{n,p}$ with $n = kp$, $k \geq 2$. The equality

$$(H_0U^{-1}, UN_0U^{-1} + XH_0U^{-1}) = (H_0, N_0)$$

implies that UN_0U^{-1} and N_0 have the same p block-type (in particular their last columns are zero). Hence $X = 0$.

On the other hand, U induces an element of $G_{n-p,p}$ belonging to the $Ad_{G_{n-p,p}}^*$ -isotropy subgroup in N_0 . Then, if the result is true in $G_{p,p}$ it is also true in $G_{2p,p}$ and $G_{kp,p}$ $p \geq 2$. ■

3. Left invariant Lagrangian foliations and Hess connection .

3.1. Let's specify a little bit the Lie algebra isomorphisms indicated by the previous proposition. The isomorphism between $\text{Rad}(d\alpha_1)$ and $\mathcal{G}_{n-p,p}(\mathbb{K})$ is determined by the choice of a supplementary subspace of $\mathcal{K}(H_0) = \{X \in M_{n,p}(\mathbb{K}), H_0 \cdot X = 0\}$ in $M_{n,p}(\mathbb{K})$.

Denote by X_0 , the element of $M_{n,p}(\mathbb{K})$ formed by one column of zero-blocks except the last block which is $id_{\mathbb{K}^p}$. Then the mapping $\text{Rad}(d\alpha_1) = \{(0, u) \in \mathcal{G}_{n,p}(\mathbb{K}), H_0 \cdot u = 0\} \longrightarrow \mathcal{G}_{n-p,p}(\mathbb{K})$ such that $(0, u) \longmapsto (U \cdot u_0, \pi_0(u))$ defines such a isomorphism, where $\pi_0(u)$ denotes the matrix of the linear map given by u restricted to $\mathcal{K}(H_0)$. Furthermore the image of the reduced symplectic form is the 2-coboundary associated to $(H_1, N_1) \in \mathcal{G}_{n-p,p}^*(\mathbb{K})$ where H_1 and N_1 are defined by $\text{tr}(N_0u) = \text{tr}(H_1 \cdot u \cdot X_0) + \text{tr}(N_1 \pi_0(u))$. We remark that (H_1, N_1) has the same block type as (H_0, N_0) , so we can repeat the process of symplectic reduction in the same conditions. One obtains a decomposition of the space $\mathcal{G}_{n,p}(\mathbb{K})$ as a direct sum of Lie subalgebras :

$$\mathcal{G}_{n,p}(\mathbb{K}) = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0 \oplus C(N_0) \oplus \cdots \oplus C(N_{p-1})$$

where \mathcal{K}_i comes from the totally isotropic ideal and $\mathcal{G}_{n-ip,p}(\mathbb{K}) = \{K_i \times \mathcal{G}_{n-(i+1)p,p}(\mathbb{K})\} \oplus C(N_i)$ for $0 \leq i \leq p-1$ (orthogonal direct sum). Using the canonical embedding of $\mathcal{G}_{n,p}(\mathbb{K})$ in $gl(\mathbb{K}^{n+p})$, the subspace $L = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0$ is identified to the subalgebra of strictly upper triangular $(p+1) \times (p+1)$ -matrices by $p \times p$ -blocks and $L' = C(N_0) \oplus \cdots \oplus C(N_{p-1})$ to the subalgebra of lower triangular $(p \times p)$ -matrices by $(p \times p)$ -blocks. As the \mathcal{K}_i and $C(N_i)$ appear as the totally isotropic subspaces at every step of the successive reductions the subalgebras L and L' are lagrangian relative to $d\alpha$ where $\alpha \equiv (H_0, N_0)$.

Denote by Λ and Λ' the connected Lie subgroups of $G_{n,p}(\mathbb{K})$ with Lie algebra L and L' respectively. The natural left actions of Λ and Λ' on $(G_{n,p}(\mathbb{K}), d\alpha^+)$ being hamiltonnians, Theorem 3.1. of ([3]) allows us to assert that Λ and Λ' are closed. So we have proved the following result :

3.2 Theorem. The symplectic Lie group $(G_{n,p}, d\alpha^+)$ is endowed with two transversal left invariant lagrangian foliations with closed and affines leaves

Let $(G_{n,p}(\mathbb{K}), \omega^+)$ be endowed with its affine structure ∇ defined by (1) We recall that a connection on $(G_{n,p}(\mathbb{K}), \omega^+)$ is said to be *symplectic* if and

only if $\nabla\omega = 0$ where $\omega := \omega_\varepsilon^+$, in other words :

$$\nabla_a(\omega(b, c)) = \omega(\nabla_a b, c) + \omega(b, \nabla_a c) \quad \forall a = (x, u), b = (y, v), c = (z, r) \in \mathcal{G}_{n,p}(\mathbb{K})$$

In that follows one identifies an element a of $\mathcal{G}_{n,p}(\mathbb{K})$ with (x, u) , where $x \in M_{n,p}$ and $u \in gl(n)$.

Let \mathcal{L} and \mathcal{L}' be two lagrangian subalgebras of $\mathcal{G}_{n,p}(\mathbb{K})$ such that $\mathcal{G} = \mathcal{L} \oplus \mathcal{L}'$. Then we can write $a = a_1 + a_2$ where $a_1 = (x_1, u_1) \in \mathcal{L}$ and $a_2 = (x_2, u_2) \in \mathcal{L}'$. Then the left symmetric product on $\mathcal{G}_{n,p}(\mathbb{K})$ is given by :

$$(i) \quad (x, 0) \cdot (y, 0) = (l, f)$$

where $l \in M_{n,p}$ is formed from a column of blocks such that the last belongs to $sl(p)$ and $f \in gl(n)$ is an element whose last line of blocks is zero.

$$(ii) \quad (0, u) \cdot (0, v) = (l, -vu)$$

where l is defined as above.

$$(iii) \quad (x, 0) \cdot (0, v) = (l_1, f_1)$$

$$(iv) \quad (0, u) \cdot (y, 0) = (l_2, f_2)$$

where $l_1, l_2 \in M_{n,p}$, $f_1, f_2 \in gl(n)$ are defined as in (i).

The following result is a consequence of the previous discussion

3.3. Corollary. *There exists only one (torsion free) left invariant symplectic connection $\tilde{\nabla}$ (called the Hess'connection) such that*

$$\tilde{\nabla}_a \mathcal{L} \subset \mathcal{L}, \quad \tilde{\nabla}_{a'} \mathcal{L}' \subset \mathcal{L}'.$$

where $a \in \mathcal{L}$ et $a' \in \mathcal{L}'$

This connection is defined by the products :

$$\tilde{\nabla}_{(x,0)^+} (y, 0)^+ = (l, 0)^+ \quad , \quad \tilde{\nabla}_{(0,u)^+} (0, v)^+ = (0, -vu)^+$$

$$\tilde{\nabla}_{(x,0)^+} (0, v)^+ = 0 \quad , \quad \tilde{\nabla}_{(0,u)^+} (y, 0)^+ = (uy, 0)^+$$

Given the kind of results that we have described above, it is natural to ask ourself in what sense the group $G_{n,p}(\mathbb{K})$ is not the symplectic double extension describes in ([6]).

The answer is clearly no if $k > 1$; in fact to have $G_{n,p}(\mathbb{K})$ be the symplectic double extension of $G_{n-p,p}(\mathbb{K})$ in the sense of [6] it is necessary that I^\perp be a Lie ideal of $\mathcal{G}_{n,p}(\mathbb{K})$. Considering the proposition 2.3 involve the existence of a Lie ideal of $gl(\mathbb{K}^n)$ isomorphic to $\mathcal{G}_{n-p,p}(\mathbb{K})$.

4. $G_{n,p}(\mathbb{K})$ as symplectic double extension of $G_{n-p,p}(\mathbb{K})$

We have shown in the previous paragraph that the techniquess of symplectic double extension developed in ([5]) do not apply to $G_{n,p}(\mathbb{K})$ for $k > 1$.

We reconsider the study of the canonical fibrations (2) and (3) to try to understand how work this example. We have observed in 2.5. that for the symplectic Lie algebra $(\mathcal{G}_{n,p}, d\alpha)$ where $\alpha \cong (H_0, N_0)$, a section of the canonical exact sequence

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp / I \rightarrow 0$$

is determined by the choice of an element X_0 in $M_{n,p}(\mathbb{K}) \setminus \mathcal{K}(H_0)$ where $\mathcal{K}(H_0)$ is the kernel of H_0 .

4.4 Proposition. The manifold $G_{n,p}(\mathbb{K})$ is diffeomorphic to $\coprod V_\gamma \times m^{-1}(H_0) / \sim$ where $(\alpha, \sigma) \sim (\beta, \tau)$ if and only if $\alpha = \beta$ and $\sigma = \Gamma_{\gamma_1 \gamma_2}(\tau)$ for (α, σ) in $V_{\gamma_1} \times m^{-1}(H_0)$ and (β, τ) in $V_{\gamma_2} \times m^{-1}(H_0)$. The cocycle $\Gamma_{\gamma_1 \gamma_2}$ is defined by **(9)**.

Acknowledgements. The third author was partially supported by NSF grant DMS-0204100 and The Department of Mathematics of Penn state University. He also wishes to thank The Department of Romances Languages and the Department of Mathematics at The University of North Carolina at Chapel Hill for their kind welcome during his stay. Thanks to Jim Stasheff for editing this paper and thanks to Patrick Eberlein for his helpful discussions and suggestions.

JeanMichel DARDIE
Université de Montpellier2
Département de Mathématiques
34095 Montpellier cedex5
UMR 5149 du CNRS

Alberto MEDINA
Université de Montpellier2
Département de Mathématiques
34095 Montpellier cedex5
UMR 5149 du CNRS

dardie@darboux.math.univ – montp2.fr medina@darboux.math.univ – montp2.fr

Hassene SIBY
Université de Montpellier2
Département de Mathématiques
34095 Montpellier cedex5
UMR 5149 du CNRS
siby@darboux.math.univ – montp2.fr

RÉFÉRENCES

- [1] R. ABRAHAM J.E. MARSDEN, *Fundation of Mechanics*, Benjamin/Cummings Pub., Princeton, N.J, 1978.
- [2] Y. AGOAKA , Uniqueness of Left Invariant Symplectic Structures on the Affine Lie Group, *Proc.Amer.Math.Soc.* **Vol.129**,N°9 (2001), pp 2753-2762.
- [3] M. BORDEMANN , A. MEDINA, A. OUADFEL, Le groupe affine comme variété symplectique, *Tohuku Math. J.* **45**,N°3 (1993), pp 423-436.
- [4] J.M. DARDIÉ , Groupes de Lie à structure symplectique ou kählérienne et double extension. *Thèse de Doctorat, Univ.Montpellier II*,1993.
- [5] J.M. DARDIÉ, A. MEDINA, Groupes de Lie à structure symplectique invariante, in "Séminaire Gaston Darboux", Montpellier **t.328, série I**,(1990-1991),p.77-85.
- [6] J.M. DARDIÉ, A. MEDINA, Double extension symplectique d'un groupe de Lie symplectique, *Adv. Math.* **117** (1996), pp 208-227.
- [7] J.M. DARDIÉ, A. MEDINA, Algèbres de Lie kählériennes et double extension , *J. Algebra.* **185** (1996), pp 774-795.
- [8] H.HESS, Connxions on Symplectic Manifolds and Geometric Quantization , Differential geometrical methods in mathematical physics, (*Proc. Conf., Aix-en-Provence/Salamanca*) (1979), pp 153-166.
- [9] A. MEDINA, PH. REVOY Groupes de Lie à structure symplectique invariante , *Groupoids and Integrable Systems*, M.R.S.I.Pub. Berkeley (1991), pp 247-266.
- [10] M. RAIS, La représentation coadjointe du groupe affine, *Ann. Inst. Fourier* **28** (1978), pp 207-237.
- [11] A. WEINSTEIN, Symplectic Manifolds and Their Lagrangian submanifolds, *Adv. Math.* **6** (1971), pp 329-346.