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## Geometry of certain Lie-Frobenius groups

*Jean Michel Dardié, Alberto Medina and Hassène Siby*

**Abstract.** Let be  $G_{n,p}(\mathbb{K}) = M_{n,p}(\mathbb{K}) \rtimes GL(\mathbb{K}^n)$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , the semi-direct product of the additive group of matrices  $M_{n,p}(\mathbb{K})$  by the group  $GL(\mathbb{K}^n)$  where the action is done by multiplication of matrices. If  $n = kp, p \geq 1$  the Lie group  $G_{n,p}(\mathbb{K})$  admits an exact left invariant symplectic form [10]. We study the geometry of this symplectic manifold. If  $n > p$  ( resp.  $n = p$  ) we prove that  $G_{n-p,p}(\mathbb{K})$  (resp  $G_{n-1,1}(\mathbb{K})$  ) is the symplectic reduction of the symplectic orthogonal  $(M_{n,p}(\mathbb{K}))^\perp$  of  $M_{n,p}(\mathbb{K})$  in  $G_{n,p}(\mathbb{K})$  and reciprocally that  $G_{n,p}(\mathbb{K})$  is a symplectic double extension, in the sense of [6], of  $G_{n-p,p}(\mathbb{K})$  ( resp. of  $G_{n-1,1}(\mathbb{K})$  ). Moreover we show that  $G_{n,p}(\mathbb{K})$  admits two left transverse Lagrangian foliations (with affine and closed leaves) . Consequently there exists (a left invariant) a canonical torsion free symplectic connection on  $G_{n,p}(\mathbb{K})$ .

**Key words :**

Exact symplectic Lie group , Affine Lie group, Symplectic Reduction , Symplectic double extension.

### Introduction

The group  $G_{n,1}(\mathbb{K}) = Aff(\mathbb{K}^n)$  where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  admits left invariant symplectic structures which are all exact because  $H^2(aff(\mathbb{K}^n), \mathbb{K}) = 0$  ([9]). To give an invariant symplectic form on  $G_{n,1}(\mathbb{K})$  is to give a linear form  $\alpha$  on the Lie algebra  $aff(\mathbb{K}^n)$  such that the 2-cobound  $\delta\alpha$  is non degenerate. That is to say  $\alpha$  is a point with an open orbit under the coadjoint action of  $G_{n,1}(\mathbb{K})$ . These points have been characterized in [9] within the framework of study of affine group representations and in [3] where is carried on the study of left invariant symplectic structures of affine group initiated in [4]. In [10], it showed that coadjoint action of Lie group semi-direct product of  $M_{n,p}(\mathbb{K})$  and  $GL(\mathbb{K}^n)$  by matricial product, where  $M_{n,p}(\mathbb{K})$  indicate the additive group of  $(n \times p)$ -matrices, admits open orbits if only if  $n = kp$ . We extend to these groups denoted  $G_{n,p}(\mathbb{K})$  the result obtained in [3] for the classical affine group : existence of a unique symplectic structure up isomorphism (Theorem 2.7), existence of transverse symplectic foliations (Theorem 2.6), as well as of a Lagrangian bi-foliation with closed leaves (Theorem 3.2). Such a pair of Lagrangian foliations is important in the quantization procedure (polarization) and implies the existence of a canonical (torsion free) symplectic connection on  $G_{n,p}(\mathbb{K})$ . The natural left action of  $G_{n,p}(\mathbb{K})$  on itself being hamiltonian we can provide (Theorem 2.1) a similar of structure's theorem of simply connected symplectic Lie group from [5].

In the third part we study up the fibrations of this theorem for explicit suggest a construction of  $(G_{n,p}(\mathbb{K}), d\alpha_1^+)$  when the generalized double extension of [5] don't apply in this case.

To complete this introduction recall that a symplectic Lie group  $(G, \omega^+)$  has an affine structure given by the left invariant connection ( torsion free and zero curvature )  $\nabla$  defined for all  $x, y, z \in T_z(G)$  by :

$$\omega^+(\nabla_{x^+}^+ y^+, z^+) = -\omega^+(y^+, [x^+, z^+])$$

where  $x^+$  denotes the left invariant vector field associated to  $x$ .

Then one says that the pair  $(G, \nabla^+)$  is an affine Lie group. In this case the product on  $\mathcal{G}$ ,  $xy = (\nabla_{x^+}^+ y^+)$  is with associator left invariant and verifies the condition  $xy - yx = [x, y]$ . This structure plays a central rule in this study.

### 1. Left invariant symplectic structures on the Lie groups $G_{n,p}$

In what follows  $G_n(\mathbb{K})$  denotes the classical affine group where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and  $\mathcal{G}_n(\mathbb{K})$  is its Lie algebra. By analogy, we denote  $G_{n,p}(\mathbb{K})$  the group semi-direct product  $M_{n,p}(\mathbb{K}) \rtimes Gl(\mathbb{K}^n)$  with  $n = kp, p \geq 1$  and  $\mathcal{G}_{n,p}(\mathbb{K}^n)$ , its Lie algebra. Obviously  $G_{n,1}(\mathbb{K})$  is isomorphic to  $Aff(\mathbb{K}^n)$ . Considering that  $H^2(Aff(\mathbb{K}^n), \mathbb{K}) = 0$  (see [3]) any left invariant alternate 2-form is invariantly exact. This result becomes general to  $\mathcal{G}_{n,p}(\mathbb{K})$  in the following way :

**1.1 Lemma.** Every left invariant symplectic form on  $G_{n,p}(\mathbb{K})$  is exact.

**Proof.** If  $k = p = 1$ ;  $\mathcal{G}_{1,1}$  is isomorphic to  $aff(\mathbb{K})$  and the result follows. Assume that  $k$  (or  $p$ ) is greater than 1. Let be  $\omega \in Z^2(\mathcal{G}_{n,p}; \mathbb{K})$ ,  $\mathcal{G}_{n,p} = M_{p,n}(\mathbb{K}) \rtimes gl(n)$ . This means that we have

$$\oint \omega([a, b], c) = 0 \quad (*) \quad \text{for every } a, b, c \text{ in } \mathcal{G}_{n,p}.$$

If we take  $a = (x, u)$ ,  $b = (y, v)$  and  $c = (0, Id)$

(\*) implies  $\omega(x, y) = 0$  for every  $x, y \in M_{n,p}$ , and consequently  $\omega(x, b) + \omega(a, y) = \omega(I, uy - vx + [u, v])$ .

But this is equivalent to

$$\begin{aligned} \omega(a, b) &= \omega(I, ux - vy) \\ &= -\delta\beta(a, b) \quad \text{with } \beta = \omega(I, \cdot) \quad \blacksquare \end{aligned}$$

**N.B** In fact for all Lie algebra  $\mathcal{G}$  having an element  $a \in \mathcal{G}$  such that  $ad_a$  is projector, we have  $H^2(\mathcal{G}, \mathbb{K}) = 0$ .

**Remark 1.** The mapping

$$\begin{aligned} M_{p,n}(\mathbb{K}) \times gl(n) &\longrightarrow (M_{n,p}(\mathbb{K}) \times gl(n))^* \\ (H, M) &\longmapsto \alpha_{(H,M)} \end{aligned}$$

given by

$$\alpha_{(H,M)}(N, V) = tr(N.H) + tr(M.V)$$

is a linear isomorphism.

In the following, the dual space  $\mathcal{G}_{n,p}^*$  of  $\mathcal{G}_{n,p}$  is identified with  $M_{p,n}(\mathbb{K}) \times gl(n)$ . To give an invariant closed 2-form on  $G_{n,p}(\mathbb{K})$  it is equivalent to give a linear form  $\alpha$  on  $\mathcal{G}_{n,p}(\mathbb{K})$ . The 2-form  $d\alpha^+$ , where  $\alpha^+$  denotes the left invariant 1-form defined by  $\alpha$  is symplectic if and only if  $\alpha$  has an open orbit under the coadjoint representation.

Using the Remark 1 and the natural embedding of  $\mathcal{G}_{n,p}$  in  $GL(\mathbb{K}^{n+p})$  we can show

**1.2 Lemma.** The coadjoint representations of  $\mathcal{G}_{n,p}$  and  $G_{n,p}$  are given by the following formulas :

$$(i) \quad ad_{(0,u)}^*(H, N) = (-H, [u, N])$$

$$(ii) \quad ad_{(x,0)}^*(H, N) = (0, x.H)$$

$$(iii) \quad Ad_{(0,U)}^*(H, N) = (H.U^{-1}, UNU^{-1})$$

$$(iv) \quad Ad_{(X, Id_{\mathbb{K}^n})}^*(H, N) = (H, N + X.H)$$

This implies

$$Ad_{(X,U)}^*(H, N) = (H.U^{-1}, UNU^{-1} + XHU^{-1})$$

Consequently the coadjoint orbit  $Ad_{G_{n,p}}^*(\alpha)$  of  $\alpha = (H_0, N_0)$  with  $H_0 = (0, I_p) \in M_{p,n}$  and

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}$$

is open. Denote by  $H_0$  the  $(p, n)$ -matrix whose  $p \times p$  blocks are all null except for the last, which is the identity of  $\mathbb{K}^p$  and denoted by  $N_0$  the  $(n \times n)$ -matrix whose  $p \times p$  blocks are all null except the sub-diagonals which are  $Id_{\mathbb{K}^p}$ . The previous formulas allow us to verify that the orbit of  $(H_0, N_0)$  is open since the isotropy subalgebra is trivial.

## 2. Symplectic Reduction - Left invariant Symplectic foliation on $G_{n,p}(\mathbb{K})$ .

Denote by  $\omega^+ = d\alpha^+$ ,  $\alpha = (H, N) \in \mathcal{G}_{n,p}^*$  a left invariant symplectic form on  $G_{n,p}$ .

The action of  $G_{n,p}(\mathbb{K})$  on  $G_{n,p}(\mathbb{K})$  given by

$$L_G : G_{n,p}(\mathbb{K}) \times G_{n,p}(\mathbb{K}) \longrightarrow G_{n,p}(\mathbb{K}), \quad (\tau, \sigma) \longmapsto \tau\sigma \text{ ( product in } G_{n,p} \text{ )}$$

is a symplectic action. Moreover  $L_G$  is a Hamiltonian action; a momentum mapping for the action is given by

$$\mu : G_{n,p}(\mathbb{K}) \longrightarrow \mathcal{G}_{n,p}(\mathbb{K})^*$$

$$\sigma \longmapsto \mu(\sigma) : x \longmapsto \langle \alpha^+(\sigma), x^-(\sigma) \rangle$$

where  $x \in \mathcal{G}_{n,p}$  and  $x^-$  is the right invariant vector field associated to  $x$ .

The subgroup  $\mathcal{H} := M_{n,p}$  is (totally) isotropic for  $\omega^+$  because  $\mathcal{H}$  is an abelian Lie group. Moreover  $L_{\mathcal{H}} : \mathcal{H} \times G_{n,p} \longrightarrow G_{n,p}$  is a hamiltonian action and a momentum mapping for  $L_{\mathcal{H}}$  is given by :

$$m : G_{n,p} \longrightarrow L(\mathcal{H})^*$$

$$\sigma \longmapsto \mu(\sigma)|_{L(\mathcal{H})}$$

where  $L_{\mathcal{H}}$  is the Lie algebra of  $\mathcal{H}$ . We arrive at the following theorem :

**2.1 Theorem .** Let  $(G_{n,p}, d((H, N)^+))$  defined as above. Then

1.  $m^{-1}(H)$  is a closed subgroup of  $G_{n,p}$  and  $\mathcal{H} \subset m^{-1}(H)$

2. The canonical exact sequence of Lie groups

$$(2) \quad \{\epsilon\} \rightarrow \mathcal{H} \rightarrow m^{-1}(H) \rightarrow m^{-1}(H)/\mathcal{H} \rightarrow \{\epsilon\}$$

is split. It is also an exact sequence of affine Lie groups.

3. The reduced symplectic Lie group  $m^{-1}(H)/\mathcal{H}$  is isomorphic to :

$$\begin{array}{ll} G_{n-p,p}(\mathbb{K}) & si \quad n > p \\ G_{p-1,1}(\mathbb{K}) & si \quad n = p > 1 \end{array}$$

4. In the principal bundle

$$(3) \quad m^{-1}(H) \xrightarrow{i} G_{n,p}(\mathbb{K}) \xrightarrow{m} \Theta$$

where  $\Theta$  is the set of matrices of rank  $p$  in  $M_{n,p}^* \equiv M_{p,n}$ , the fiber is an affine Lie subgroup and  $m$  is affine relative to the usual affine structure of  $\Theta \subset \mathbb{K}^{np}$ .

We need the following lemma for which the proof is obvious.

**2.2 Lemma .** The mapping  $m$  is a surjective submersion.

Furthermore,

$$(4) \quad m((X, T)) = H.T^{-1} \quad \text{where } (X, T) \text{ is in } G_{n,p}(\mathbb{K})$$

**Proof.** The formula (4) is a direct consequence of the definition of  $m$  and the Lemma 1.2. Thus it is clear that  $m$  is a surjective submersion on the set  $\Theta$  of matrices of rank  $p$  of  $M_{n,p}(\mathbb{K})$ .

**Proof of the theorem 2.1.**

Formula (4) implies that  $m^{-1}(H) = \{(X, T) \in G_{n,p}(\mathbb{K})/HT^{-1} = H\}$  is a (closed) subgroup of  $G_{n,p}(\mathbb{K})$  which contains  $\mathcal{H}$ . Moreover the factor group  $m^{-1}(H)/\mathcal{H}$  can be identified with  $\{(0, T) \in G_{n,p}/HT^{-1} = H\}$ . Thus (2) is an split sequence of Lie groups. On the other hand, since  $\mathcal{H}$  is commutative and  $\omega^+$  is exact, it follows that  $L(\mathcal{H}) \subset L(\mathcal{H})^\perp$  and a straightforward shows that  $L(m^{-1}(H)) = L(\mathcal{H})^\perp$ .

Moreover from the formula

$$\omega(xy, z) = -\omega(y, [x, z])$$

defining the left symmetric product in  $\mathcal{G}_{n,p}(\mathbb{K})$  (see formula (1)) it turns out that  $L(\mathcal{H})^\perp$  is a left symmetric subalgebra of  $\mathcal{G}_{n,p}(\mathbb{K})$  and that  $L(\mathcal{H})$  is a two-side ideal of  $L(\mathcal{H})^\perp$ . Consequently (2) is an exact sequence of affine Lie groups i.e  $\mathcal{H}, m^{-1}(H), m^{-1}(H)/\mathcal{H}$  are affine Lie groups and the applications of formula (2) are affine.

On the other hand a list of the elements of the matrix group  $\{(0, T) \in G_{n,p}(\mathbb{K})/HT^{-1} = H\}$  allows to observe that the group  $m^{-1}(H)/\mathcal{H}$  is isomorphic to the group  $G_{n-p,p}(\mathbb{K})$  if  $n > p$  and isomorphic to  $G_{n-1,1}(\mathbb{K})$  if  $n = p$ . This proves statement 3.

Now , we need to show that the manifold  $\Theta$  is endowed with an affine structure which makes the momentum mapping affine.

Let  $F$  be the subbundle of  $TG_{n,p}(\mathbb{K})$  tangent to the  $L_{\mathcal{H}}$ -orbit and  $F^\perp$  its symplectic orthogonal. Denote by  $\mathcal{F}$  and  $\mathcal{F}^\perp$  the associated foliations respectively. Since  $\mathcal{H}$  is normal in  $G_{n,p}(\mathbb{K})$ , the foliation  $\mathcal{F}^\perp$  is defined either by the left invariant form on  $G_{n,p}(\mathbb{K})$  given by  $\eta'_j := i(e_j^+) \omega^+$  where  $(e_i)$ 's form a

basis of  $L(\mathcal{H})$  and  $i$  denote the interior product, or by the closed forms (thus exact)  $\eta_j := i(e_j^-)\omega^+$ . Obviously the  $\eta_j$  are basic for the fibration. the forms  $\bar{\eta}_j$  which are the projections of  $\eta_j$  by  $m$  define a local parallelism on  $\Theta$ . This parallelism is global and commutative because the  $\bar{\eta}_j$  are exact. Hence  $m$  is affine. ■

Let  $\alpha_1$  and  $\alpha_2$  be the linear forms on  $\mathcal{G}_{n,p}(\mathbb{K})$  defined respectively by  $\alpha_1(x, u) = \text{tr}(H.x)$  and  $\alpha_2(x, u) = \text{tr}(N.u)$ . We get :

**2.3. Theorem.**  $\ker(d\alpha_1)$  and  $\ker(d\alpha_2)$  are supplementary symplectic Lie subalgebras of  $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$ . So they determine two tranverse symplectic foliations left invariant on  $G_{n,p}(\mathbb{K})$

**Proof.** Obviously  $\ker(d\alpha_i)$  is a Lie subalgebra of  $\mathcal{G}_{n,p}(\mathbb{K})$ . In addition the subspaces  $\ker(d\alpha_i)$ ,  $i = 1, 2$  are in direct sum because  $\ker(d\alpha_1) \cap \ker(d\alpha_2) = \{0\}$ .

If we take  $\alpha = (H_0, N_0)$  we directly observe that we have  $\dim(\ker(\delta\alpha_1)) = p^2(k-1)k$  and  $\dim(\ker(\delta\alpha_2)) = 2p^2k$ . We extends these results about the dimension for all  $\alpha \in \mathcal{G}_{n,p}^*(\mathbb{K})$  with open coadjoint orbit. consequently  $\ker(\delta\alpha_1)$  and  $\ker(\delta\alpha_2)$  are symplectic Lie subalgebras of  $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$ . ■

Let  $C(N_0)$  be the subalgebra of  $gl(\mathbb{K}^n)$  given by :

$$\begin{pmatrix} A_0 & & & & \\ & A_1 & & \ddots & \\ & & \ddots & & \\ & & & \ddots & \\ A_{k-1} & & \cdots & A_1 & A_0 \end{pmatrix}$$

**2.4. Scholie.** The Lie algebra  $\ker(d\alpha_1)$  is isomorphic to the Lie algebra  $\mathcal{G}_{n-p,p}(\mathbb{K})$  if  $n > p$  and  $\mathcal{G}_{p-1,1}(\mathbb{K})$  if  $n = p$  while  $\ker(d\alpha_2)$  is isomorphic to the semi-direct  $M_{n,p}(\mathbb{K}) \rtimes C(N_0)$  if  $n > p$  and to the semi-direct product  $M_{p,p}(\mathbb{K}) \rtimes C(N_0)$  if  $n = p$ .

The following result is the infinitesimal version of Theorem 2.1. It gives a more precise and complete statement of Theorem 2.1.

**2.5 Proposition.** With the notations of Therem 2.1., if  $I = L(\mathcal{H})$ , the canonical sequence of vectoriel spaces

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp/I \rightarrow 0$$

is a split exact sequence of Lie algebras. It is also an exact sequence of left symmetric algebras.

Furthermore the Lie algebra  $\mathcal{G}_{n,p}(\mathbb{K})$  decomposes as a direct sum of Lie subalgebras  $I^\perp$  and  $C(N_0)$  .

**2.6 Theorem.** The symplectic Lie group  $(G_{n,p}, d\alpha^+)$  is endowed with two transversal left invariant symplectic foliations whose leaves are affine submanifolds of  $G_{n,p}(\mathbb{K})$ .

The following assertion specifies the number of open orbits in  $\mathcal{G}_{n,p}^*(\mathbb{K})$  as well as the left invariant symplectic structures on  $G_{n,p}(\mathbb{K})$ .

**2.7 Theorem.**

**a.** There exist two open orbits of the coadjoint representation of  $G_{n,p}(\mathbb{K})$  if  $\mathbb{K} = \mathbb{R}$  and only one if  $\mathbb{K} = \mathbb{C}$

**b.** Up isomorphism there is only one left invariant symplectic structure on  $G_{n,p}$  i.e if  $\omega$  and  $\omega'$  are two left invariant symplectic forms on  $G_{n,p}(\mathbb{K})$ , then there exists an automorphism  $\varphi$  of Lie algebra  $\mathcal{G}_{n,p}(\mathbb{K})$  such that :

$$\omega_\varepsilon(\cdot, \cdot) = \omega'_\varepsilon(\varphi\cdot, \varphi\cdot).$$

The following lemmas set up the main steps of the demonstration of Theorem 2.7. This lemma allow to count the  $Ad_{G_{n,p}}^*$ -orbits

**2.8 Lemma.** If  $Orb_{(H,M)}$  is the coadjoint orbit of  $(H, M) \in \mathcal{G}_{n,p}^* \equiv M_{p,n}(\mathbb{K}) \times gl(n)$  then  $Orb_{(H,M)}$  has an element  $(H'_0, M'_0) \in \mathcal{G}_{n,p}^*$  such that  $H'_0 = (0, \dots, A_p) \in M_{p,n}$ . Moreover, if  $Orb_{(H,M)}$  is open, then  $H'_0$  can be taken as  $H'_0 = (0, \dots, I_p) = H_0$ .

**Proof.** It suffices to remark that there exists  $U \in GL_o(\mathbb{K}^n)$  such that  $HU^{-1} = H_0$ , that is obvious if we look at  $H$  as the matrix of a mapping from  $\mathbb{K}^n$  into  $\mathbb{K}^p$  and  $U^{-1}$  as the matrix of a change of basis in  $\mathbb{K}^n$ . Then the last formula of lemma 1.2 show the first assertion.

Now suppose that  $Orb_{(H,M)} = Orb_{(H'_0, M'_0)}$  is a open orbit. This implies that we have

$$(x, u) \in \mathcal{G}_{n,p}; ad_{(x,u)}^*(\alpha) = 0 \Rightarrow (x, u) = 0$$

In particular

$$ad_{(x,0)}^*(H'_0, M') = 0 \Rightarrow x = 0$$

In other words

$$(**) \quad tr(xH_0u) = 0, u \in gl(n) \Rightarrow x = 0$$

with  $H_0 = (0, A_p)$

But a straight calculation shows that  $(**)$  implies  $A_p$  is invertible.

Finally there exists  $U \in GL_o(\mathbb{K}^n)$  such that  $H'_0U^{-1} = H_0$ , but this relation fix only the last diagonal block of  $U$  to the value  $A^{-1}$ . Therefore if  $k \geq 2$  we able to take an other diagonal block equals to  $\det A^{-1} \cdot Id_{\mathbb{K}^p}$  and completing the diagonale by the 1, to construct a such matrix  $U$  in  $SL(\mathbb{K}^n)$ .

**2.9 Lemma.** An open orbit  $Orb_{(H_0, M)}$  contains only one element  $(H_0, M')$ , where the block decomposition of  $M'$  can be written :

$$M' = \begin{pmatrix} M_1 & 0 \\ H_1 & 0 \end{pmatrix}$$

with  $(H_1, M_1) \in \mathcal{G}_{n-p,p}^*$ .

**Proof.** Because

$$Ad_{(X, Id)}^*(H_0, M) = (H_0, M + X.H)$$

it is clear that there is only one  $X \in M_{p,n}$  such that  $M' = M + MH_0$ .

**2.10 Lemma.** The linear forms  $(H_0, M')$  and  $(H_o, P')$  on  $\mathcal{G}_{n,p}$  with  $M' = (H_1, M_1)$  and  $P' = (K_1, P_1)$ , defined as in Lemma 2.9, belong to the same  $Ad_{G_{n,p}}^*$ -orbit if and only if  $(H_1, M_1)$  and  $(K_1, P_1)$  are in the same  $Ad_{G_{n-p,p}}^*$ -orbit.

**Proof.**  $(H_0, M')$  and  $(H_o, P')$  are in the same orbit if and only if  $\exists U \in GL_o(\mathbb{K}^n)$  such that  $UM'U^{-1} = P'$  and  $H_0U^{-1} = H_o$ .

The second condition implies

$$U = \begin{pmatrix} U' & 0 \\ X_1 & Id \end{pmatrix}, \quad \text{with} \quad U_1 \in GL_0(\mathbb{K}^{n-p})$$

On other hand  $UM'U^{-1} = P'$  in  $gl(\mathbb{K}^n)$  is equivalent to  $Ad_{(X_1, U_1)}^*(H_1, M_1) = (K_1, P_1)$  in  $\mathcal{G}_{n-p, p}(\mathbb{K})$  ■

**Proof of the theorem 2.7.**

Following the previous Lemmas we have

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{n,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\}$$

On the other hand, using similar arguments as those of the lemmas we can show that

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{1,1}}^* - \text{orbit} \right\} = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{R} \\ 1 & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

If  $\mathbb{K} = \mathbb{C}$  there is only one symplectic structure on  $G_{n,p}$ .

In the case  $\mathbb{K} = \mathbb{R}$ , it follows from the lemmas that every  $Ad_{G_{n,p}}^*$ -open orbit contains an element  $(H_0, N_0)$  where

$$N_0 = \begin{pmatrix} 0 & & & & & \\ A_p & \ddots & & & & \\ & I_p & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & I_p & 0 & \end{pmatrix} \quad \text{with } A_p \text{ invertible}$$

Nevertheless two matrices as  $N_0$  are conjugate by an element  $P$  of  $GL(n)$ .

However the mapping

$$\mathcal{G}_{n,p}^* \longrightarrow \mathcal{G}_{n,p}^*, \quad (g, M) \longmapsto (P^{t-1}g, P^{-1}MP)$$

is dual of the mapping

$$\theta : \mathcal{G}_{n,p} \longrightarrow \mathcal{G}_{n,p} \quad (x, N) \longmapsto (P^{-1}x, P^{-1}NP)$$

and these later is an automorphism of the Lie algebra  $\mathcal{G}_{n,p}$ .

Consequently if  $\omega_1^+$ ,  $\omega_2^+$  are two left invariant symplectic forms on  $G_{n,p}$ , we have :

$$\omega_1^+ = \theta^*(\omega_2^+). \quad \blacksquare$$

**Remark 2.** Notice that the only one left invariant affine structure on  $G_{n,p}$  is given by  $(H_0, N_0)$ .

The following result is an important consequence of the previous study.

**2.11 Proposition.** The identity component of  $G_{n,p}$  is diffeomorphic to an open  $Ad_{G_{n,p}}^*$ -orbit. Consequently  $G_{n,p}$  and  $\{\alpha \in \mathcal{G}_{n,p}^*; \alpha \text{ is Poisson-regular}\}$  are diffeomorphic.

**Proof.** (By induction) We must prove that the orbital mapping in  $(H_0, N_0)$

$$(G_{n,p})_o \longrightarrow \mathcal{G}_{n,p}^*, \quad (X, U) \longmapsto (H_0U^{-1}, UN_0U^{-1} + XH_0U^{-1})$$



has a trivial isotropy.

The result is obvious for  $G_{1,1}$  and this implies that it is also true for  $G_{p,p}$  ( thanks to a sequence double extension ).

Consider the case  $G_{n,p}$  with  $n = kp$ ,  $k \geq 2$ . The equality

$$(H_0U^{-1}, UN_0U^{-1} + XH_0U^{-1}) = (H_0, N_0)$$

implies that  $UN_0U^{-1}$  and  $N_0$  have the same  $p$  block-type ( in particular their last columns are zero). Hence  $X = 0$ .

On the other hand,  $U$  induces an element of  $G_{n-p,p}$  belonging to the  $Ad_{G_{n-p,p}}^*$ -isotropy subgroup in  $N_0$ . Then, if the result is true in  $G_{p,p}$  it is also true in  $G_{2p,p}$  and  $G_{kp,p}$   $p \geq 2$ . ■

### 3. Left invariant Lagrangian foliations and Hess connection .

**3.1.** Let's specify a little bit the Lie algebra isomorphisms indicated by the previous proposition. The isomorphism between  $\text{Rad}(d\alpha_1)$  and  $\mathcal{G}_{n-p,p}(\mathbb{K})$  is determined by the choice of a supplementary subspace of  $\mathcal{K}(H_0) = \{X \in M_{n,p}(\mathbb{K}), H_0 \cdot X = 0\}$  in  $M_{n,p}(\mathbb{K})$ .

Denote by  $X_0$ , the element of  $M_{n,p}(\mathbb{K})$  formed by one column of zero-blocks except the last block which is  $id_{\mathbb{K}^p}$ . Then the mapping  $\text{Rad}(d\alpha_1) = \{(0, u) \in \mathcal{G}_{n,p}(\mathbb{K}), H_0 \cdot u = 0\} \longrightarrow \mathcal{G}_{n-p,p}(\mathbb{K})$  such that  $(0, u) \longmapsto (U \cdot u_0, \pi_0(u))$  defines such a isomorphism, where  $\pi_0(u)$  denotes the matrix of the linear map given by  $u$  restricted to  $\mathcal{K}(H_0)$ . Furthermore the image of the reduced symplectic form is the 2-coboundary associated to  $(H_1, N_1) \in \mathcal{G}_{n-p,p}^*(\mathbb{K})$  where  $H_1$  and  $N_1$  are defined by  $\text{tr}(N_0u) = \text{tr}(H_1 \cdot u \cdot X_0) + \text{tr}(N_1 \pi_0(u))$ . We remark that  $(H_1, N_1)$  has the same block type as  $(H_0, N_0)$ , so we can repeat the process of symplectic reduction in the same conditions. One obtains a decomposition of the space  $\mathcal{G}_{n,p}(\mathbb{K})$  as a direct sum of Lie subalgebras :

$$\mathcal{G}_{n,p}(\mathbb{K}) = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0 \oplus C(N_0) \oplus \cdots \oplus C(N_{p-1})$$

where  $\mathcal{K}_i$  comes from the totally isotropic ideal and  $\mathcal{G}_{n-ip,p}(\mathbb{K}) = \{K_i \times \mathcal{G}_{n-(i+1)p,p}(\mathbb{K})\} \oplus C(N_i)$  for  $0 \leq i \leq p-1$  (orthogonal direct sum). Using the canonical embedding of  $\mathcal{G}_{n,p}(\mathbb{K})$  in  $gl(\mathbb{K}^{n+p})$ , the subspace  $L = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0$  is identified to the subalgebra of strictly upper triangular  $(p+1) \times (p+1)$ -matrices by  $p \times p$ -blocks and  $L' = C(N_0) \oplus \cdots \oplus C(N_{p-1})$  to the subalgebra of lower triangular  $(p \times p)$ -matrices by  $(p \times p)$ -blocks. As the  $\mathcal{K}_i$  and  $C(N_i)$  appear as the totally isotropic subspaces at every step of the successive reductions the subalgebras  $L$  and  $L'$  are lagrangian relative to  $d\alpha$  where  $\alpha \equiv (H_0, N_0)$ .

Denote by  $\Lambda$  and  $\Lambda'$  the connected Lie subgroups of  $G_{n,p}(\mathbb{K})$  with Lie algebra  $L$  and  $L'$  respectively. The natural left actions of  $\Lambda$  and  $\Lambda'$  on  $(G_{n,p}(\mathbb{K}), d\alpha^+)$  being hamiltonnians, Theorem 3.1. of ([3]) allows us to assert that  $\Lambda$  and  $\Lambda'$  are closed. So we have proved the following result :

**3.2 Theorem.** The symplectic Lie group  $(G_{n,p}, d\alpha^+)$  is endowed with two transversal left invariant lagrangian foliations with closed and affines leaves

Let  $(G_{n,p}(\mathbb{K}), \omega^+)$  be endowed with its affine structure  $\nabla$  defined by (1) We recall that a connection on  $(G_{n,p}(\mathbb{K}), \omega^+)$  is said to be *symplectic* if and

only if  $\nabla\omega = 0$  where  $\omega := \omega_\varepsilon^+$ , in other words :

$$\nabla_a(\omega(b, c)) = \omega(\nabla_a b, c) + \omega(b, \nabla_a c) \quad \forall a = (x, u), b = (y, v), c = (z, r) \in \mathcal{G}_{n,p}(\mathbb{K})$$

In that follows one identifies an element  $a$  of  $\mathcal{G}_{n,p}(\mathbb{K})$  with  $(x, u)$ , where  $x \in M_{n,p}$  and  $u \in gl(n)$ .

Let  $\mathcal{L}$  and  $\mathcal{L}'$  be two lagrangian subalgebras of  $\mathcal{G}_{n,p}(\mathbb{K})$  such that  $\mathcal{G} = \mathcal{L} \oplus \mathcal{L}'$ . Then we can write  $a = a_1 + a_2$  where  $a_1 = (x_1, u_1) \in \mathcal{L}$  and  $a_2 = (x_2, u_2) \in \mathcal{L}'$ . Then the left symmetric product on  $\mathcal{G}_{n,p}(\mathbb{K})$  is given by :

$$(i) \quad (x, 0) \cdot (y, 0) = (l, f)$$

where  $l \in M_{n,p}$  is formed from a column of blocks such that the last belongs to  $sl(p)$  and  $f \in gl(n)$  is an element whose last line of blocks is zero.

$$(ii) \quad (0, u) \cdot (0, v) = (l, -vu)$$

where  $l$  is defined as above.

$$(iii) \quad (x, 0) \cdot (0, v) = (l_1, f_1)$$

$$(iv) \quad (0, u) \cdot (y, 0) = (l_2, f_2)$$

where  $l_1, l_2 \in M_{n,p}$ ,  $f_1, f_2 \in gl(n)$  are defined as in (i).

The following result is a consequence of the previous discussion

**3.3. Corollary.** *There exists only one (torsion free) left invariant symplectic connection  $\tilde{\nabla}$  ( called the Hess'connection ) such that*

$$\tilde{\nabla}_a \mathcal{L} \subset \mathcal{L}, \quad \tilde{\nabla}_{a'} \mathcal{L}' \subset \mathcal{L}'.$$

where  $a \in \mathcal{L}$  et  $a' \in \mathcal{L}'$

*This connection is defined by the products :*

$$\tilde{\nabla}_{(x,0)^+} (y, 0)^+ = (l, 0)^+ \quad , \quad \tilde{\nabla}_{(0,u)^+} (0, v)^+ = (0, -vu)^+$$

$$\tilde{\nabla}_{(x,0)^+} (0, v)^+ = 0 \quad , \quad \tilde{\nabla}_{(0,u)^+} (y, 0)^+ = (uy, 0)^+$$

Given the kind of results that we have described above, it is natural to ask ourself in what sense the group  $G_{n,p}(\mathbb{K})$  is not the symplectic double extension describes in ([6]).

The answer is clearly no if  $k > 1$ ; in fact to have  $G_{n,p}(\mathbb{K})$  be the symplectic double extension of  $G_{n-p,p}(\mathbb{K})$  in the sense of [6] it is necessary that  $I^\perp$  be a Lie ideal of  $\mathcal{G}_{n,p}(\mathbb{K})$ . Considering the proposition 2.3 involve the existence of a Lie ideal of  $gl(\mathbb{K}^n)$  isomorphic to  $\mathcal{G}_{n-p,p}(\mathbb{K})$ .

#### 4. $G_{n,p}(\mathbb{K})$ as symplectic double extension of $G_{n-p,p}(\mathbb{K})$

We have shown in the previous paragraph that the techniquess of symplectic double extension developed in ([5]) do not apply to  $G_{n,p}(\mathbb{K})$  for  $k > 1$ .

We reconsider the study of the canonical fibrations (2) and (3) to try to understand how work this example. We have observed in 2.5. that for the symplectic Lie algebra  $(\mathcal{G}_{n,p}, d\alpha)$  where  $\alpha \cong (H_0, N_0)$ , a section of the canonical exact sequence

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp/I \rightarrow 0$$

is determined by the choice of an element  $X_0$  in  $M_{n,p}(\mathbb{K}) \setminus \mathcal{K}(H_0)$  where  $\mathcal{K}(H_0)$  is the kernel of  $H_0$ .

Conversely, we consider the reduced algebra  $(\mathcal{G}_{n-p,p}, d\alpha')$  with  $\alpha' \equiv (H_1, N_1)$  where  $H_1 = (0, \dots, 0, I_p)$  and

$$N_1 = \begin{pmatrix} 0 & & & & & \\ I_p & \ddots & & & & \\ & I_p & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & I_p & 0 \end{pmatrix}$$

Let  $i$  be the canonical inclusion from  $M_{n-p,p}$  to  $M_{n,p}$  obtained by putting zero in the  $n-p+1, \dots, n$  rows, and let  $Z \in M_{n,p} \setminus M_{n-p,p}$  have rank  $p$  (e.g.  $Z = H_0$ ).

Denote by  $r : M_{n-p,p} \longrightarrow M_{n,p}$  the linear mapping given by

$$r \circ i = id_{M_{n-p,p}} \text{ and } r(Z) = 0$$

and let  $H \in M_{n,p}^*$  satisfy

$$(5) \quad H \circ i = 0 \text{ and } H.Z = Id_p$$

We consider the regular representation :

$$(6) \quad \eta : \mathcal{G}_{n-p,p} \rightarrow gl(n), (x, u) \mapsto (i \circ u \circ r + H.i(x))$$

From  $\eta$  we deduce a representation of Lie groups :

$$(7) \quad \rho : G_{n-p,p} \rightarrow GL(n), (X, U) \mapsto (i \circ U \circ r + i(X).H)$$

verifying,

$$\rho_{*,\epsilon} = \eta$$

Moreover consider the inclusion :

$$R : M_{n,p}^* \times gl(n-p) \rightarrow gl(n)$$

given by

$$(8) \quad R(H, N) = i \circ N \circ r + Z.(H \circ r)$$

Then we can state the following result.

**4.1 Theorem.** Consider the symplectic Lie group  $(G_{n-p,p}, d(H_1, N_1)^+)$ . If  $H$  is the linear form given by (4), and  $N = R(H_1, N_1)$  with  $R$  given by (7), then  $(G_{n,p}, d(H, N)^+)$  is a symplectic Lie group such that  $(G_{n-p,p}, d(H_1, N_1)^+)$  is the reduced symplectic Lie group as described in theorem 2.1.

**Proof.** By definition of  $H$  we have

$$m^{-1}(H) = M_{n,p} \times \rho(G_{n-p,p})$$

It remains to prove that  $d(H_1, N_1)^+$  is the reduced symplectic form of  $d(H, N)^+|_{m^{-1}(H)}$ . Consider the decomposition  $M_{n,p}(\mathbb{K}) = i(M_{n-p,p}(\mathbb{K})) \otimes \mathbb{K}.Z$

On the other hand a straight verification proves that  $tr(H_1x) + tr(N_1u) = tr(R(H_1, N_1) \cdot (\eta(x, u)))$  for all  $(x, u)$  of  $\mathcal{G}_{n-p,p}(\mathbb{K})$ . Finally as  $N = R(H_1, N_1)$  and  $d((H, 0)^+)$  vanish on  $m^{-1}(H)$  it follows that  $d((H_1, N_1)^+)$  is the reduced form. This ends the proof. ■

**Remark 4.** A similar argument proves that for  $p \geq 2$  :  $G_{p,1} = Aff(\mathbb{K}^p)$  is a symplectic reduction of  $G_{p,p}$  and  $G_{p,p}$  is a symplectic double extension of  $G_{p,1}$

#### 4.2 Study of the fibration :

$$(3) \quad m^{-1}(H) \xrightarrow{i} GA_{n,p}(\mathbb{K}) \xrightarrow{m} \Theta$$

According to Theorem 1.2. we can consider the case where  $\alpha \equiv (H_0, N_0)$ . Let  $C = C(N_0) \cap GL(\mathbb{K}^n)$  or, if we prefer  $C$  is the Lie subgroup of  $GL(\mathbb{K}^n)$  formed by matrices of type

$$\begin{pmatrix} A_0 & & & & \\ & A_1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & A_1 & A_0 \end{pmatrix}$$

with  $A_0$  is invertible.

According to Lemma 2.2.  $m(C) = \{(A_{k-1}, \dots, A_1, A_0) : A_0 \text{ invertible}\}$ , denote by  $V_0$  this open set of  $\Theta$ .

Let be  $V_\gamma = V_0 \cdot \sigma_\gamma$  where  $\gamma = (i_1, \dots, i_p)$  and  $\sigma_\gamma$  is the element of  $GL(\mathbb{K}^n)$  which realizes the permutation of the  $k$  last columns with the columns indexed by  $\gamma$  in  $M_{p,n}(\mathbb{K})$ . The sets  $V_\gamma$  are clearly the open set of  $\times$  and  $m$  defines a diffeomorphism of  $C \cdot \sigma_\gamma$  on  $V_\gamma$ . Denote  $S_\gamma$  the embedding of  $V_\gamma$  in  $GL(\mathbb{K}^n)$  such that  $S_\gamma(V_\gamma) = C \cdot \sigma_\gamma$  and  $m \circ S_\gamma = id_{V_\gamma}$  for all multi-indices  $\gamma$ . We have the following result :

**4.3 Lemma.** The  $V_\gamma$  as defined above , with  $\gamma = (i_1, \dots, i_p)$  for  $1 \leq i_1 < i_2 < \dots < i_p \leq n$ , form an open trivializing cover for the fibration (2).

#### Proof.

The elements  $V_\gamma$  recover  $\Theta$  since  $\Theta$  is formed by matrices of rank  $p$  in  $M_{p,n}(\mathbb{K})$  whose first  $p$  columns are independent in  $\mathbb{K}^p$ . By contrast  $V_0$  is formed of matrices for which the last  $p$  columns are independent.

To show that the  $V_\gamma$  are trivialising for the fibration (3) amounts to proving that we have  $m^{-1}(V_\gamma) = S_\gamma(V_\gamma) \cdot m^{-1}(H_0)$ . For all  $\sigma$  of  $G_{n,p}(\mathbb{K})$  we have the identity  $m^{-1}(m(\sigma)) = \sigma \cdot m^{-1}(H_0)$ . Indeed if  $\sigma' \in m^{-1}(m(\sigma))$  we have by definition of  $m$  the formula,

$$\langle Ad_\sigma^*(H, 0); (X, 0) \rangle = \langle Ad_{\sigma'}^*(H, 0); (X, 0) \rangle \text{ for all } X \text{ in } M_{n,p}(\mathbb{K})$$

or what amounts to the same thing

$$\langle Ad_{\sigma^{-1}\sigma'}^*(H, 0); (X, 0) \rangle = \langle (H, 0); (X, 0) \rangle \text{ for all } X \text{ in } M_{n,p}(\mathbb{K})$$

This last relation means that  $\sigma^{-1}\sigma' \in m^{-1}(H_0)$  by the identity below and the fact that the  $V_\gamma$  are open and trivialising. The trivialisations are then given by the map  $\phi_\gamma : m^{-1}(V_\gamma) \rightarrow V_\gamma \times m^{-1}(H_0)$ ;  $S_\gamma(\alpha) \cdot \sigma \mapsto (\alpha, \sigma)$  for all multi-indices  $\gamma = (i_1, \dots, i_p)$  with  $1 \leq i_1 < i_2 < \dots < i_p \leq n$ . The cocycle defining the fibration (2) is given by

$$(9) \quad \Gamma_{\gamma_1 \gamma_2} : V_{\gamma_1} \cap V_{\gamma_2} \rightarrow m^{-1}(H_0); \alpha \mapsto S_{\gamma_1}^{-1}(\alpha) \cdot S_{\gamma_2}(\alpha)$$

Thus we have proved the following result.

**4.4 Proposition.** The manifold  $G_{n,p}(\mathbb{K})$  is diffeomorphic to  $\coprod V_\gamma \times m^{-1}(H_0) / \sim$  where  $(\alpha, \sigma) \sim (\beta, \tau)$  if and only if  $\alpha = \beta$  and  $\sigma = \Gamma_{\gamma_1 \gamma_2}(\tau)$  for  $(\alpha, \sigma)$  in  $V_{\gamma_1} \times m^{-1}(H_0)$  and  $(\beta, \tau)$  in  $V_{\gamma_2} \times m^{-1}(H_0)$ . The cocycle  $\Gamma_{\gamma_1 \gamma_2}$  is defined by **(9)**.

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*JeanMichel DARDIE*  
*Université de Montpellier2*  
*Département de Mathématiques*  
*34095 Montpellier cedex5*  
*UMR 5149 du CNRS*

*Alberto MEDINA*  
*Université de Montpellier2*  
*Département de Mathématiques*  
*34095 Montpellier cedex5*  
*UMR 5149 du CNRS*

*dardie@darboux.math.univ – montp2.fr   medina@darboux.math.univ – montp2.fr*

*Hassene SIBY*  
*Université de Montpellier2*  
*Département de Mathématiques*  
*34095 Montpellier cedex5*  
*UMR 5149 du CNRS*  
*siby@darboux.math.univ – montp2.fr*

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