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Geometry of certain Lie-Frobenius groups

Jean Michel Dardié, Alberto Medina and Hassène Siby

Abstract. Let be $G_{n,p}(\mathbb{K}) = M_{n,p}(\mathbb{K}) \rtimes GL(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , the semi-direct product of the additive group of matrices $M_{n,p}(\mathbb{K})$ by the group $GL(\mathbb{K}^n)$ where the action is done by multiplication of matrices. If $n = kp, p \geq 1$ the Lie group $G_{n,p}(\mathbb{K})$ admits an exact left invariant symplectic form [10]. We study the geometry of this symplectic manifold. If $n > p$ (resp. $n = p$) we prove that $G_{n-p,p}(\mathbb{K})$ (resp $G_{n-1,1}(\mathbb{K})$) is the symplectic reduction of the symplectic orthogonal $(M_{n,p}(\mathbb{K}))^\perp$ of $M_{n,p}(\mathbb{K})$ in $G_{n,p}(\mathbb{K})$ and reciprocally that $G_{n,p}(\mathbb{K})$ is a symplectic double extension, in the sense of [6], of $G_{n-p,p}(\mathbb{K})$ (resp. of $G_{n-1,1}(\mathbb{K})$). Moreover we show that $G_{n,p}(\mathbb{K})$ admits two left transverse Lagrangian foliations (with affine and closed leaves) . Consequently there exists (a left invariant) a canonical torsion free symplectic connection on $G_{n,p}(\mathbb{K})$.

Key words :

Exact symplectic Lie group , Affine Lie group, Symplectic Reduction , Symplectic double extension.

Introduction

The group $G_{n,1}(\mathbb{K}) = Aff(\mathbb{K}^n)$ where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} admits left invariant symplectic structures which are all exact because $H^2(aff(\mathbb{K}^n), \mathbb{K}) = 0$ ([9]). To give an invariant symplectic form on $G_{n,1}(\mathbb{K})$ is to give a linear form α on the Lie algebra $aff(\mathbb{K}^n)$ such that the 2-cobound $\delta\alpha$ is non degenerate. That is to say α is a point with an open orbit under the coadjoint action of $G_{n,1}(\mathbb{K})$. These points have been characterized in [9] within the framework of study of affine group representations and in [3] where is carried on the study of left invariant symplectic structures of affine group initiated in [4]. In [10], it showed that coadjoint action of Lie group semi-direct product of $M_{n,p}(\mathbb{K})$ and $GL(\mathbb{K}^n)$ by matricial product, where $M_{n,p}(\mathbb{K})$ indicate the additive group of $(n \times p)$ -matrices, admits open orbits if only if $n = kp$. We extend to these groups denoted $G_{n,p}(\mathbb{K})$ the result obtained in [3] for the classical affine group : existence of a unique symplectic structure up isomorphism (Theorem 2.7), existence of transverse symplectic foliations (Theorem 2.6), as well as of a Lagrangian bi-foliation with closed leaves (Theorem 3.2). Such a pair of Lagrangian foliations is important in the quantization procedure (polarization) and implies the existence of a canonical (torsion free) symplectic connection on $G_{n,p}(\mathbb{K})$. The natural left action of $G_{n,p}(\mathbb{K})$ on itself being hamiltonian we can provide (Theorem 2.1) a similar of structure's theorem of simply connected symplectic Lie group from [5].

In the third part we study up the fibrations of this theorem for explicit suggest a construction of $(G_{n,p}(\mathbb{K}), d\alpha_1^+)$ when the generalized double extension of [5] don't apply in this case.

To complete this introduction recall that a symplectic Lie group (G, ω^+) has an affine structure given by the left invariant connection (torsion free and zero curvature) ∇ defined for all $x, y, z \in T_\varepsilon(G)$ by :

$$\omega^+(\nabla_{x^+}^+ y^+, z^+) = -\omega^+(y^+, [x^+, z^+])$$

where x^+ denotes the left invariant vector field associated to x .

Then one says that the pair (G, ∇^+) is an affine Lie group. In this case the product on \mathcal{G} , $xy = (\nabla_{x^+}^+ y^+)$ is with associator left invariant and verifies the condition $xy - yx = [x, y]$. This structure plays a central rule in this study.

1. Left invariant symplectic structures on the Lie groups $G_{n,p}$

In what follows $G_n(\mathbb{K})$ denotes the classical affine group where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $\mathcal{G}_n(\mathbb{K})$ is its Lie algebra. By analogy, we denote $G_{n,p}(\mathbb{K})$ the group semi-direct product $M_{n,p}(\mathbb{K}) \rtimes Gl(\mathbb{K}^n)$ with $n = kp, p \geq 1$ and $\mathcal{G}_{n,p}(\mathbb{K})$, its Lie algebra. Obviously $G_{n,1}(\mathbb{K})$ is isomorphic to $Aff(\mathbb{K}^n)$. Considering that $H^2(Aff(\mathbb{K}^n), \mathbb{K}) = 0$ (see [3]) any left invariant alternate 2-form is invariantly exact. This result becomes general to $\mathcal{G}_{n,p}(\mathbb{K})$ in the following way :

1.1 Lemma. Every left invariant symplectic form on $G_{n,p}(\mathbb{K})$ is exact.

Proof. If $k = p = 1$; $\mathcal{G}_{1,1}$ is isomorphic to $Aff(\mathbb{K})$ and the result follows. Assume that k (or p) is greater than 1. Let be $\omega \in Z^2(\mathcal{G}_{n,p}; \mathbb{K})$, $\mathcal{G}_{n,p} = M_{p,n}(\mathbb{K}) \rtimes gl(n)$. This means that we have

$$\oint \omega([a, b], c) = 0 \quad (*) \quad \text{for every } a, b, c \text{ in } \mathcal{G}_{n,p}.$$

If we take $a = (x, u)$, $b = (y, v)$ and $c = (0, Id)$

(*) implies $\omega(x, y) = 0$ for every $x, y \in M_{n,p}$, and consequently $\omega(x, b) + \omega(a, y) = \omega(I, uy - vx + [u, v])$.

But this is equivalent to

$$\begin{aligned} \omega(a, b) &= \omega(I, ux - vy) \\ &= -\delta\beta(a, b) \quad \text{with } \beta = \omega(I, \cdot) \quad \blacksquare \end{aligned}$$

N.B In fact for all Lie algebra \mathcal{G} having an element $a \in \mathcal{G}$ such that ad_a is projector, we have $H^2(\mathcal{G}, \mathbb{K}) = 0$.

Remark 1. The mapping

$$\begin{aligned} M_{p,n}(\mathbb{K}) \times gl(n) &\longrightarrow (M_{n,p}(\mathbb{K}) \times gl(n))^* \\ (H, M) &\longmapsto \alpha_{(H,M)} \end{aligned}$$

given by

$$\alpha_{(H,M)}(N, V) = tr(N.H) + tr(M.V)$$

is a linear isomorphism.

In the following, the dual space $\mathcal{G}_{n,p}^*$ of $\mathcal{G}_{n,p}$ is identified with $M_{p,n}(\mathbb{K}) \times gl(n)$. To give an invariant closed 2-form on $G_{n,p}(\mathbb{K})$ it is equivalent to give a linear form α on $\mathcal{G}_{n,p}(\mathbb{K})$. The 2-form $d\alpha^+$, where α^+ denotes the left invariant 1-form defined by α is symplectic if and only if α has an open orbit under the coadjoint representation.

Using the Remark 1 and the natural embedding of $\mathcal{G}_{n,p}$ in $GL(\mathbb{K}^{n+p})$ we can show

1.2 Lemma. The coadjoint representations of $\mathcal{G}_{n,p}$ and $G_{n,p}$ are given by the following formulas :

$$(i) \quad ad_{(0,u)}^*(H, N) = (-H, [u, N])$$

$$(ii) \quad ad_{(x,0)}^*(H, N) = (0, x.H)$$

$$(iii) \quad Ad_{(0,U)}^*(H, N) = (H.U^{-1}, UNU^{-1})$$

$$(iv) \quad Ad_{(X, Id_{\mathbb{K}^n})}^*(H, N) = (H, N + X.H)$$

This implies

$$Ad_{(X,U)}^*(H, N) = (H.U^{-1}, UNU^{-1} + XHU^{-1})$$

Consequently the coadjoint orbit $Ad_{G_{n,p}}^*(\alpha)$ of $\alpha = (H_0, N_0)$ with $H_0 = (0, I_p) \in M_{p,n}$ and

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}$$

is open. Denote by H_0 the (p, n) -matrix whose $p \times p$ blocks are all null except for the last, which is the identity of \mathbb{K}^p and denoted by N_0 the $(n \times n)$ -matrix whose $p \times p$ blocks are all null except the sub-diagonals which are $Id_{\mathbb{K}^p}$. The previous formulas allow us to verify that the orbit of (H_0, N_0) is open since the isotropy subalgebra is trivial.

2. Symplectic Reduction - Left invariant Symplectic foliation on $G_{n,p}(\mathbb{K})$.

Denote by $\omega^+ = d\alpha^+$, $\alpha = (H, N) \in \mathcal{G}_{n,p}^*$ a left invariant symplectic form on $G_{n,p}$.

The action of $G_{n,p}(\mathbb{K})$ on $G_{n,p}(\mathbb{K})$ given by

$$L_G : G_{n,p}(\mathbb{K}) \times G_{n,p}(\mathbb{K}) \longrightarrow G_{n,p}(\mathbb{K}), \quad (\tau, \sigma) \longmapsto \tau\sigma \text{ (product in } G_{n,p} \text{)}$$

is a symplectic action. Moreover L_G is a Hamiltonian action; a momentum mapping for the action is given by

$$\mu : G_{n,p}(\mathbb{K}) \longrightarrow \mathcal{G}_{n,p}(\mathbb{K})^*$$

$$\sigma \longmapsto \mu(\sigma) : x \longmapsto \langle \alpha^+(\sigma), x^-(\sigma) \rangle$$

where $x \in \mathcal{G}_{n,p}$ and x^- is the right invariant vector field associated to x .

The subgroup $\mathcal{H} := M_{n,p}$ is (totally) isotropic for ω^+ because \mathcal{H} is an abelian Lie group. Moreover $L_{\mathcal{H}} : \mathcal{H} \times G_{n,p} \longrightarrow G_{n,p}$ is a hamiltonian action and a momentum mapping for $L_{\mathcal{H}}$ is given by :

$$m : G_{n,p} \longrightarrow L(\mathcal{H})^*$$

$$\sigma \longmapsto \mu(\sigma)|_{L(\mathcal{H})}$$

where $L_{\mathcal{H}}$ is the Lie algebra of \mathcal{H} . We arrive at the following theorem :

2.1 Theorem . Let $(G_{n,p}, d((H, N)^+))$ defined as above. Then

1. $m^{-1}(H)$ is a closed subgroup of $G_{n,p}$ and $\mathcal{H} \subset m^{-1}(H)$

2. The canonical exact sequence of Lie groups

$$(2) \quad \{\epsilon\} \rightarrow \mathcal{H} \rightarrow m^{-1}(H) \rightarrow m^{-1}(H)/\mathcal{H} \rightarrow \{\epsilon\}$$

is split. It is also an exact sequence of affine Lie groups.

3. The reduced symplectic Lie group $m^{-1}(H)/\mathcal{H}$ is isomorphic to :

$$\begin{array}{ll} G_{n-p,p}(\mathbb{K}) & si \quad n > p \\ G_{p-1,1}(\mathbb{K}) & si \quad n = p > 1 \end{array}$$

4. In the principal bundle

$$(3) \quad m^{-1}(H) \xrightarrow{i} G_{n,p}(\mathbb{K}) \xrightarrow{m} \Theta$$

where Θ is the set of matrices of rank p in $M_{n,p}^* \equiv M_{p,n}$, the fiber is an affine Lie subgroup and m is affine relative to the usual affine structure of $\Theta \subset \mathbb{K}^{np}$.

We need the following lemma for which the proof is obvious.

2.2 Lemma . The mapping m is a surjective submersion.

Furthermore,

$$(4) \quad m((X, T)) = H.T^{-1} \text{ where } (X, T) \text{ is in } G_{n,p}(\mathbb{K})$$

Proof. The formula (4) is a direct consequence of the definition of m and the Lemma 1.2. Thus it is clear that m is a surjective submersion on the set Θ of matrices of rank p of $M_{n,p}(\mathbb{K})$.

Proof of the theorem 2.1.

Formula (4) implies that $m^{-1}(H) = \{(X, T) \in G_{n,p}(\mathbb{K}) / HT^{-1} = H\}$ is a (closed) subgroup of $G_{n,p}(\mathbb{K})$ which contains \mathcal{H} . Moreover the factor group $m^{-1}(H)/\mathcal{H}$ can be identified with $\{(0, T) \in G_{n,p} / HT^{-1} = H\}$. Thus (2) is an split sequence of Lie groups. On the other hand, since \mathcal{H} is commutative and ω^+ is exact, it follows that $L(\mathcal{H}) \subset L(\mathcal{H})^\perp$ and a straightforward shows that $L(m^{-1}(H)) = L(\mathcal{H})^\perp$.

Moreover from the formula

$$\omega(xy, z) = -\omega(y, [x, z])$$

defining the left symmetric product in $\mathcal{G}_{n,p}(\mathbb{K})$ (see formula (1)) it turns out that $L(\mathcal{H})^\perp$ is a left symmetric subalgebra of $\mathcal{G}_{n,p}(\mathbb{K})$ and that $L(\mathcal{H})$ is a two-side ideal of $L(\mathcal{H})^\perp$. Consequently (2) is an exact sequence of affine Lie groups i.e $\mathcal{H}, m^{-1}(H), m^{-1}(H)/\mathcal{H}$ are affine Lie groups and the applications of formula (2) are affine.

On the other hand a list of the elements of the matrix group $\{(0, T) \in G_{n,p}(\mathbb{K}) / HT^{-1} = H\}$ allows to observe that the group $m^{-1}(H)/\mathcal{H}$ is isomorphic to the group $G_{n-p,p}(\mathbb{K})$ if $n > p$ and isomorphic to $G_{n-1,1}(\mathbb{K})$ if $n = p$. This proves statement 3.

Now , we need to show that the manifold Θ is endowed with an affine structure which makes the momentum mapping affine.

Let F be the subbundle of $TG_{n,p}(\mathbb{K})$ tangent to the $L\mathcal{H}$ -orbit and F^\perp its symplectic orthogonal. Denote by \mathcal{F} and \mathcal{F}^\perp the associated foliations respectively. Since \mathcal{H} is normal in $G_{n,p}(\mathbb{K})$, the foliation \mathcal{F}^\perp is defined either by the left invariant form on $G_{n,p}(\mathbb{K})$ given by $\eta'_j := i(e_j^+) \omega^+$ where (e_i) 's form a

basis of $L(\mathcal{H})$ and i denote the interior product, or by the closed forms (thus exact) $\eta_j := i(e_j^-)\omega^+$. Obviously the η_j are basic for the fibration. the forms $\bar{\eta}_j$ which are the projections of η_j by m define a local parallelism on Θ . This parallelism is global and commutative because the $\bar{\eta}_j$ are exact. Hence m is affine. ■

Let α_1 and α_2 be the linear forms on $\mathcal{G}_{n,p}(\mathbb{K})$ defined respectively by $\alpha_1(x, u) = \text{tr}(H.x)$ and $\alpha_2(x, u) = \text{tr}(N.u)$. We get :

2.3. Theorem. $\ker(d\alpha_1)$ and $\ker(d\alpha_2)$ are supplementary symplectic Lie subalgebras of $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$. So they determine two tranverse symplectic foliations left invariant on $G_{n,p}(\mathbb{K})$

Proof. Obviously $\ker(d\alpha_i)$ is a Lie subalgebra of $\mathcal{G}_{n,p}(\mathbb{K})$. In addition the subspaces $\ker(d\alpha_i)$, $i = 1, 2$ are in direct sum because $\ker(d\alpha_1) \cap \ker(d\alpha_2) = \{0\}$.

If we take $\alpha = (H_0, N_0)$ we directly observe that we have $\dim(\ker(\delta\alpha_1)) = p^2(k-1)k$ and $\dim(\ker(\delta\alpha_2)) = 2p^2k$. We extends these results about the dimension for all $\alpha \in \mathcal{G}_{n,p}^*(\mathbb{K})$ with open coadjoint orbit. consequently $\ker(\delta\alpha_1)$ and $\ker(\delta\alpha_2)$ are symplectic Lie subalgebras of $(\mathcal{G}_{n,p}(\mathbb{K}), d\alpha)$. ■

Let $C(N_0)$ be the subalgebra of $gl(\mathbb{K}^n)$ given by :

$$\begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_{k-1} & \cdots & A_1 & A_0 \end{pmatrix}$$

2.4. Scholie. The Lie algebra $\ker(d\alpha_1)$ is isomorphic to the Lie algebra $\mathcal{G}_{n-p,p}(\mathbb{K})$ if $n > p$ and $\mathcal{G}_{p-1,1}(\mathbb{K})$ if $n = p$ while $\ker(d\alpha_2)$ is isomorphic to the semi-direct $M_{n,p}(\mathbb{K}) \rtimes C(N_0)$ if $n > p$ and to the semi-direct product $M_{p,p}(\mathbb{K}) \rtimes C(N_0)$ if $n = p$.

The following result is the infinitesimal version of Theorem 2.1. It gives a more precise and complete statement of Theorem 2.1.

2.5 Proposition. With the notations of Therem 2.1., if $I = L(\mathcal{H})$, the canonical sequence of vectoriel spaces

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp/I \rightarrow 0$$

is a split exact sequence of Lie algebras. It is also an exact sequence of left symmetric algebras.

Furthermore the Lie algebra $\mathcal{G}_{n,p}(\mathbb{K})$ decomposes as a direct sum of Lie subalgebras I^\perp and $C(N_0)$.

2.6 Theorem. The symplectic Lie group $(G_{n,p}, d\alpha^+)$ is endowed with two transversal left invariant symplectic foliations whose leaves are affine submanifolds of $G_{n,p}(\mathbb{K})$.

The following assertion specifies the number of open orbits in $\mathcal{G}_{n,p}^*(\mathbb{K})$ as well as the left invariant symplectic structures on $G_{n,p}(\mathbb{K})$.

2.7 Theorem.

a. There exist two open orbits of the coadjoint representation of $G_{n,p}(\mathbb{K})$ if $\mathbb{K} = \mathbb{R}$ and only one if $\mathbb{K} = \mathbb{C}$

b. Up isomorphism there is only one left invariant symplectic structure on $G_{n,p}$ i.e if ω and ω' are two left invariant symplectic forms on $G_{n,p}(\mathbb{K})$, then there exists an automorphism φ of Lie algebra $\mathcal{G}_{n,p}(\mathbb{K})$ such that :

$$\omega_\varepsilon(.,.) = \omega'_\varepsilon(\varphi., \varphi.).$$

The following lemmas set up the main steps of the demonstration of Theorem 2.7. This lemma allow to count the $Ad_{G_{n,p}}^*$ -orbits

2.8 Lemma. If $Orb_{(H,M)}$ is the coadjoint orbit of $(H, M) \in \mathcal{G}_{n,p}^* \equiv M_{p,n}(\mathbb{K}) \times gl(n)$ then $Orb_{(H,M)}$ has an element $(H'_0, M'_0) \in \mathcal{G}_{n,p}^*$ such that $H'_0 = (0, \dots, A_p) \in M_{p,n}$. Moreover, if $Orb_{(H,M)}$ is open, then H'_0 can be taken as $H'_0 = (0, \dots, I_p) = H_0$.

Proof. It suffices to remark that there exists $U \in GL_o(\mathbb{K}^n)$ such that $HU^{-1} = H_0$, that is obvious if we look at H as the matrix of a mapping from \mathbb{K}^n into \mathbb{K}^p and U^{-1} as the matrix of a change of basis in \mathbb{K}^n . Then the last formula of lemma 1.2 show the first assertion.

Now suppose that $Orb_{(H,M)} = Orb_{(H'_0, M'_0)}$ is a open orbit. This implies that we have

$$(x, u) \in \mathcal{G}_{n,p}; ad_{(x,u)}^*(\alpha) = 0 \Rightarrow (x, u) = 0$$

In particular

$$ad_{(x,0)}^*(H'_0, M') = 0 \Rightarrow x = 0$$

In other words

$$(**) \quad tr(xH_0u) = 0, u \in gl(n) \Rightarrow x = 0$$

with $H_0 = (0, A_p)$

But a straight calculation shows that $(**)$ implies A_p is invertible.

Finally there exists $U \in GL_o(\mathbb{K}^n)$ such that $H'_0U^{-1} = H_0$, but this relation fix only the last diagonal block of U to the value A^{-1} . Therefore if $k \geq 2$ we able to take an other diagonal block equals to $\det A^{-1} \cdot Id_{\mathbb{K}^p}$ and completing the diagonale by the 1, to construct a such matrix U in $SL(\mathbb{K}^n)$.

2.9 Lemma. An open orbit $Orb_{(H_0, M)}$ contains only one element (H_0, M') , where the block decomposition of M' can be written :

$$M' = \begin{pmatrix} M_1 & 0 \\ H_1 & 0 \end{pmatrix}$$

with $(H_1, M_1) \in \mathcal{G}_{n-p,p}^*$.

Proof. Because

$$Ad_{(X, Id)}^*(H_0, M) = (H_0, M + X.H)$$

it is clear that there is only one $X \in M_{p,n}$ such that $M' = M + MH_0$.

2.10 Lemma. The linear forms (H_0, M') and (H_0, P') on $\mathcal{G}_{n,p}$ with $M' = (H_1, M_1)$ and $P' = (K_1, P_1)$, defined as in Lemma 2.9, belong to the same $Ad_{G_{n,p}}^*$ -orbit if and only if (H_1, M_1) and (K_1, P_1) are in the same $Ad_{G_{n-p,p}}^*$ -orbit.

Proof. (H_0, M') and (H_0, P') are in the same orbit if and only if $\exists U \in GL_o(\mathbb{K}^n)$ such that $UM'U^{-1} = P'$ and $H_0U^{-1} = H_0$.

The second condition implies

$$U = \begin{pmatrix} U' & 0 \\ X_1 & Id \end{pmatrix}, \quad \text{with} \quad U_1 \in GL_0(\mathbb{K}^{n-p})$$

On other hand $UM'U^{-1} = P'$ in $gl(\mathbb{K}^n)$ is equivalent to $Ad_{(X_1, U_1)}^*(H_1, M_1) = (K_1, P_1)$ in $\mathcal{G}_{n-p, p}(\mathbb{K})$ ■

Proof of the theorem 2.7.

Following the previous Lemmas we have

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{n,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\}$$

On the other hand, using similar arguments as those of the lemmas we can show that

$$\text{cardinal} \left\{ \text{open} \quad Ad_{G_{p,p}}^* - \text{orbit} \right\} = \text{cardinal} \left\{ \text{open} \quad Ad_{G_{1,1}}^* - \text{orbit} \right\} = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{R} \\ 1 & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

If $\mathbb{K} = \mathbb{C}$ there is only one symplectic structure on $G_{n,p}$.

In the case $\mathbb{K} = \mathbb{R}$, it follows from the lemmas that every $Ad_{G_{n,p}}^*$ -open orbit contains an element (H_0, N_0) where

$$N_0 = \begin{pmatrix} 0 & & & & \\ A_p & \ddots & & & \\ & I_p & \ddots & & \\ & & \ddots & \ddots & \\ & & & I_p & 0 \end{pmatrix} \quad \text{with } A_p \text{ invertible}$$

Nevertheless two matrices as N_0 are conjugate by an element P of $GL(n)$.

However the mapping

$$\mathcal{G}_{n,p}^* \longrightarrow \mathcal{G}_{n,p}^*, \quad (g, M) \longmapsto (P^{t-1}g, P^{-1}MP)$$

is dual of the mapping

$$\theta : \mathcal{G}_{n,p} \longrightarrow \mathcal{G}_{n,p}, \quad (x, N) \longmapsto (P^{-1}x, P^{-1}NP)$$

and these later is an automorphism of the Lie algebra $\mathcal{G}_{n,p}$.

Consequently if ω_1^+ , ω_2^+ are two left invariant symplectic forms on $G_{n,p}$, we have :

$$\omega_1^+ = \theta^*(\omega_2^+). \quad \blacksquare$$

Remark 2. Notice that the only one left invariant affine structure on $G_{n,p}$ is given by (H_0, N_0) .

The following result is an important consequence of the previous study.

2.11 Proposition. The identity component of $G_{n,p}$ is diffeomorphic to an open $Ad_{G_{n,p}}^*$ -orbit. Consequently $G_{n,p}$ and $\{\alpha \in \mathcal{G}_{n,p}^*; \alpha \text{ is Poisson-regular}\}$ are diffeomorphic.

Proof. (By induction) We must prove that the orbital mapping in (H_0, N_0)

$$(G_{n,p})_o \longrightarrow \mathcal{G}_{n,p}^*, \quad (X, U) \longmapsto (H_0U^{-1}, UN_0U^{-1} + XH_0U^{-1})$$

has a trivial isotropy.

The result is obvious for $G_{1,1}$ and this implies that it is also true for $G_{p,p}$ (thanks to a sequence double extension).

Consider the case $G_{n,p}$ with $n = kp$, $k \geq 2$. The equality

$$(H_0 U^{-1}, U N_0 U^{-1} + X H_0 U^{-1}) = (H_0, N_0)$$

implies that $U N_0 U^{-1}$ and N_0 have the same p block-type (in particular their last columns are zero). Hence $X = 0$.

On the other hand, U induces an element of $G_{n-p,p}$ belonging to the $Ad_{G_{n-p,p}}^*$ -isotropy subgroup in N_0 . Then, if the result is true in $G_{p,p}$ it is also true in $G_{2p,p}$ and $G_{kp,p}$ $p \geq 2$. ■

3. Left invariant Lagrangian foliations and Hess connection .

3.1. Let's specify a little bit the Lie algebra isomorphisms indicated by the previous proposition. The isomorphism between $\text{Rad}(d\alpha_1)$ and $\mathcal{G}_{n-p,p}(\mathbb{K})$ is determined by the choice of a supplementary subspace of $\mathcal{K}(H_0) = \{X \in M_{n,p}(\mathbb{K}), H_0.X = 0\}$ in $M_{n,p}(\mathbb{K})$.

Denote by X_0 , the element of $M_{n,p}(\mathbb{K})$ formed by one column of zero-blocks except the last block which is $id_{\mathbb{K}^p}$. Then the mapping $\text{Rad}(d\alpha_1) = \{(0, u) \in \mathcal{G}_{n,p}(\mathbb{K}), H_0.u = 0\} \longrightarrow \mathcal{G}_{n-p,p}(\mathbb{K})$ such that $(0, u) \longmapsto (U.u_0, \pi_0(u))$ defines such a isomorphism, where $\pi_0(u)$ denotes the matrix of the linear map given by u restricted to $\mathcal{K}(H_0)$. Furthermore the image of the reduced symplectic form is the 2-coboundary associated to $(H_1, N_1) \in \mathcal{G}_{n-p,p}^*(\mathbb{K})$ where H_1 and N_1 are defined by $\text{tr}(N_0 u) = \text{tr}(H_1.u.X_0) + \text{tr}(N_1 \pi_0(u))$. We remark that (H_1, N_1) has the same block type as (H_0, N_0) , so we can repeat the process of symplectic reduction in the same conditions. One obtains a decomposition of the space $\mathcal{G}_{n,p}(\mathbb{K})$ as a direct sum of Lie subalgebras :

$$\mathcal{G}_{n,p}(\mathbb{K}) = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0 \oplus C(N_0) \oplus \cdots \oplus C(N_{p-1})$$

where \mathcal{K}_i comes from the totally isotropic ideal and $\mathcal{G}_{n-ip,p}(\mathbb{K}) = \{K_i \times \mathcal{G}_{n-(i+1)p,p}(\mathbb{K})\} \oplus C(N_i)$ for $0 \leq i \leq p-1$ (orthogonal direct sum). Using the canonical embedding of $\mathcal{G}_{n,p}(\mathbb{K})$ in $gl(\mathbb{K}^{n+p})$, the subspace $L = \mathcal{K}_{p-1} \oplus \cdots \oplus \mathcal{K}_0$ is identified to the subalgebra of strictly upper triangular $(p+1) \times (p+1)$ -matrices by $p \times p$ -blocks and $L' = C(N_0) \oplus \cdots \oplus C(N_{p-1})$ to the subalgebra of lower triangular $(p \times p)$ -matrices by $(p \times p)$ -blocks. As the \mathcal{K}_i and $C(N_i)$ appear as the totally isotropic subspaces at every step of the successive reductions the subalgebras L and L' are lagrangian relative to $d\alpha$ where $\alpha \equiv (H_0, N_0)$.

Denote by Λ and Λ' the connected Lie subgroups of $G_{n,p}(\mathbb{K})$ with Lie algebra L and L' respectively. The natural left actions of Λ and Λ' on $(G_{n,p}(\mathbb{K}), d\alpha^+)$ being hamiltonnians, Theorem 3.1. of ([3]) allows us to assert that Λ and Λ' are closed. So we have proved the following result :

3.2 Theorem. The symplectic Lie group $(G_{n,p}, d\alpha^+)$ is endowed with two transversal left invariant lagrangian foliations with closed and affines leaves .

Let $(G_{n,p}(\mathbb{K}), \omega^+)$ be endowed with its affine structure ∇ defined by (1) We recall that a connection on $(G_{n,p}(\mathbb{K}), \omega^+)$ is said to be *symplectic* if and

only if $\nabla\omega = 0$ where $\omega := \omega_\varepsilon^+$, in other words :

$$\nabla_a(\omega(b, c)) = \omega(\nabla_a b, c) + \omega(b, \nabla_a c) \quad \forall a = (x, u), b = (y, v), c = (z, r) \in \mathcal{G}_{n,p}(\mathbb{K})$$

In that follows one identifies an element a of $\mathcal{G}_{n,p}(\mathbb{K})$ with (x, u) , where $x \in M_{n,p}$ and $u \in gl(n)$.

Let \mathcal{L} and \mathcal{L}' be two lagrangian subalgebras of $\mathcal{G}_{n,p}(\mathbb{K})$ such that $\mathcal{G} = \mathcal{L} \oplus \mathcal{L}'$. Then we can write $a = a_1 + a_2$ where $a_1 = (x_1, u_1) \in \mathcal{L}$ and $a_2 = (x_2, u_2) \in \mathcal{L}'$. Then the left symmetric product on $\mathcal{G}_{n,p}(\mathbb{K})$ is given by :

$$(i) \quad (x, 0) \cdot (y, 0) = (l, f)$$

where $l \in M_{n,p}$ is formed from a column of blocks such that the last belongs to $sl(p)$ and $f \in gl(n)$ is an element whose last line of blocks is zero.

$$(ii) \quad (0, u) \cdot (0, v) = (l, -vu)$$

where l is defined as above.

$$(iii) \quad (x, 0) \cdot (0, v) = (l_1, f_1)$$

$$(iv) \quad (0, u) \cdot (y, 0) = (l_2, f_2)$$

where $l_1, l_2 \in M_{n,p}$, $f_1, f_2 \in gl(n)$ are defined as in (i).

The following result is a consequence of the previous discussion

3.3. Corollary. *There exists only one (torsion free) left invariant symplectic connection $\tilde{\nabla}$ (called the Hess'connection) such that*

$$\tilde{\nabla}_a \mathcal{L} \subset \mathcal{L}, \quad \tilde{\nabla}_{a'} \mathcal{L}' \subset \mathcal{L}'.$$

where $a \in \mathcal{L}$ et $a' \in \mathcal{L}'$

This connection is defined by the products :

$$\tilde{\nabla}_{(x,0)^+}(y, 0)^+ = (l, 0)^+ \quad , \quad \tilde{\nabla}_{(0,u)^+}(0, v)^+ = (0, -vu)^+$$

$$\tilde{\nabla}_{(x,0)^+}(0, v)^+ = 0 \quad , \quad \tilde{\nabla}_{(0,u)^+}(y, 0)^+ = (uy, 0)^+$$

Given the kind of results that we have described above, it is natural to ask ourself in what sense the group $G_{n,p}(\mathbb{K})$ is not the symplectic double extension describes in ([6]).

The answer is clearly no if $k > 1$; in fact to have $G_{n,p}(\mathbb{K})$ be the symplectic double extension of $G_{n-p,p}(\mathbb{K})$ in the sense of [6] it is necessary that I^\perp be a Lie ideal of $\mathcal{G}_{n,p}(\mathbb{K})$. Considering the proposition 2.3 involve the existence of a Lie ideal of $gl(\mathbb{K}^n)$ isomorphic to $\mathcal{G}_{n-p,p}(\mathbb{K})$.

4. $G_{n,p}(\mathbb{K})$ as symplectic double extension of $G_{n-p,p}(\mathbb{K})$

We have shown in the previous paragraph that the techniquess of symplectic double extension developed in ([5]) do not apply to $G_{n,p}(\mathbb{K})$ for $k > 1$.

We reconsider the study of the canonical fibrations (2) and (3) to try to understand how work this example. We have observed in 2.5. that for the symplectic Lie algebra $(\mathcal{G}_{n,p}, d\alpha)$ where $\alpha \cong (H_0, N_0)$, a section of the canonical exact sequence

$$0 \rightarrow I \rightarrow I^\perp \rightarrow I^\perp/I \rightarrow 0$$

is determined by the choice of an element X_0 in $M_{n,p}(\mathbb{K}) \setminus \mathcal{K}(H_0)$ where $\mathcal{K}(H_0)$ is the kernel of H_0 .

Conversely, we consider the reduced algebra $(\mathcal{G}_{n-p,p}, d\alpha')$ with $\alpha' \equiv (H_1, N_1)$ where $H_1 = (0, \dots, 0, I_p)$ and

$$N_1 = \begin{pmatrix} 0 & & & & \\ I_p & \ddots & & & \\ & I_p & \ddots & & \\ & & \ddots & \ddots & \\ & & & I_p & 0 \end{pmatrix}$$

Let i be the canonical inclusion from $M_{n-p,p}$ to $M_{n,p}$ obtained by putting zero in the $n-p+1, \dots, n$ rows, and let $Z \in M_{n,p} \setminus M_{n-p,p}$ have rank p (e.g. $Z = H_0$).

Denote by $r : M_{n-p,p} \longrightarrow M_{n,p}$ the linear mapping given by

$$r \circ i = id_{M_{n-p,p}} \text{ and } r(Z) = 0$$

and let $H \in M_{n,p}^*$ satisfy

$$(5) \quad H \circ i = 0 \text{ and } H.Z = Id_p$$

We consider the regular representation :

$$(6) \quad \eta : \mathcal{G}_{n-p,p} \rightarrow gl(n), (x, u) \mapsto (i \circ u \circ r + H.i(x))$$

From η we deduce a representation of Lie groups :

$$(7) \quad \rho : G_{n-p,p} \rightarrow GL(n), (X, U) \mapsto (i \circ U \circ r + i(X).H)$$

verifying,

$$\rho_{*,\epsilon} = \eta$$

Moreover consider the inclusion :

$$R : M_{n,p}^* \times gl(n-p) \rightarrow gl(n)$$

given by

$$(8) \quad R(H, N) = i \circ N \circ r + Z.(H \circ r)$$

Then we can state the following result.

4.1 Theorem. Consider the symplectic Lie group $(G_{n-p,p}, d(H_1, N_1)^+)$. If H is the linear form given by (4), and $N = R(H_1, N_1)$ with R given by (7), then $(G_{n,p}, d(H, N)^+)$ is a symplectic Lie group such that $(G_{n-p,p}, d(H_1, N_1)^+)$ is the reduced symplectic Lie group as described in theorem 2.1.

Proof. By definition of H we have

$$m^{-1}(H) = M_{n,p} \times \rho(G_{n-p,p})$$

It remains to prove that $d(H_1, N_1)^+$ is the reduced symplectic form of $d(H, N)^+|_{m^{-1}(H)}$. Consider the decomposition $M_{n,p}(\mathbb{K}) = i(M_{n-p,p}(\mathbb{K})) \otimes \mathbb{K}.Z$

On the other hand a straight verification proves that $tr(H_1 x) + tr(N_1 u) = tr(R(H_1, N_1) \cdot (\eta(x, u)))$ for all (x, u) of $\mathcal{G}_{n-p,p}(\mathbb{K})$. Finally as $N = R(H_1, N_1)$ and $d((H, 0)^+)$ vanish on $m^{-1}(H)$ it follows that $d((H_1, N_1)^+)$ is the reduced form. This ends the proof. ■

Remark 4. A similar argument proves that for $p \geq 2$: $G_{p,1} = Aff(\mathbb{K}^p)$ is a symplectic reduction of $G_{p,p}$ and $G_{p,p}$ is a symplectic double extension of $G_{p,1}$

4.2 Study of the fibration :

$$(3) \quad m^{-1}(H) \xrightarrow{i} GA_{n,p}(\mathbb{K}) \xrightarrow{m} \Theta$$

According to Theorem 1.2. we can consider the case where $\alpha \equiv (H_0, N_0)$. Let $C = C(N_0) \cap GL(\mathbb{K}^n)$ or, if we prefer C is the Lie subgroup of $GL(\mathbb{K}^n)$ formed by matrices of type

$$\begin{pmatrix} A_0 & & & \\ A_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ A_{k-1} & \cdots & A_1 & A_0 \end{pmatrix}$$

with A_0 is invertible.

According to Lemma 2.2. $m(C) = \{(A_{k-1}, \dots, A_1, A_0) : A_0 \text{ invertible}\}$, denote by V_0 this open set of Θ .

Let be $V_\gamma = V_0 \cdot \sigma_\gamma$ where $\gamma = (i_1, \dots, i_p)$ and σ_γ is the element of $GL(\mathbb{K}^n)$ which realizes the permutation of the k last columns with the columns indexed by γ in $M_{p,n}(\mathbb{K})$. The sets V_γ are clearly the open set of \times and m defines a diffeomorphism of $C \cdot \sigma_\gamma$ on V_γ . Denote S_γ the embedding of V_γ in $GL(\mathbb{K}^n)$ such that $S_\gamma(V_\gamma) = C \cdot \sigma_\gamma$ and $m \circ S_\gamma = id_{V_\gamma}$ for all multi-indices γ . We have the following result :

4.3 Lemma. The V_γ as defined above , with $\gamma = (i_1, \dots, i_p)$ for $1 \leq i_1 < i_2 < \dots < i_p \leq n$, form an open trivializing cover for the fibration (2).

Proof.

The elements V_γ recover Θ since Θ is formed by matrices of rank p in $M_{p,n}(\mathbb{K})$ whose first p columns are independent in \mathbb{K}^p . By contrast V_0 is formed of matrices for which the last p columns are independent.

To show that the V_γ are trivialising for the fibration (3) amounts to proving that we have $m^{-1}(V_\gamma) = S_\gamma(V_\gamma) \cdot m^{-1}(H_0)$. For all σ of $G_{n,p}(\mathbb{K})$ we have the identity $m^{-1}(m(\sigma)) = \sigma \cdot m^{-1}(H_0)$. Indeed if $\sigma' \in m^{-1}(m(\sigma))$ we have by definition of m the formula,

$$\langle Ad_\sigma^*(H, 0); (X, 0) \rangle = \langle Ad_{\sigma'}^*(H, 0); (X, 0) \rangle \text{ for all } X \text{ in } M_{n,p}(\mathbb{K})$$

or what amounts to the same thing

$$\langle Ad_{\sigma^{-1}\sigma'}^*(H, 0); (X, 0) \rangle = \langle (H, 0); (X, 0) \rangle \text{ for all } X \text{ in } M_{n,p}(\mathbb{K})$$

This last relation means that $\sigma^{-1}\sigma' \in m^{-1}(H_0)$ by the identity below and the fact that the V_γ are open and trivialising. The trivialisations are then given by the map $\phi_\gamma : m^{-1}(V_\gamma) \rightarrow V_\gamma \times m^{-1}(H_0); S_\gamma(\alpha) \cdot \sigma \mapsto (\alpha, \sigma)$ for all multi-indices $\gamma = (i_1, \dots, i_p)$ with $1 \leq i_1 < i_2 < \dots < i_p \leq n$. The cocycle defining the fibration (2) is given by

$$(9) \quad \Gamma_{\gamma_1 \gamma_2} : V_{\gamma_1} \cap V_{\gamma_2} \rightarrow m^{-1}(H_0); \alpha \mapsto S_{\gamma_1}^{-1}(\alpha) \cdot S_{\gamma_2}(\alpha)$$

Thus we have proved the following result.

4.4 Proposition. The manifold $G_{n,p}(\mathbb{K})$ is diffeomorphic to $\coprod V_\gamma \times m^{-1}(H_0) / \sim$ where $(\alpha, \sigma) \sim (\beta, \tau)$ if and only if $\alpha = \beta$ and $\sigma = \Gamma_{\gamma_1 \gamma_2}(\tau)$ for (α, σ) in $V_{\gamma_1} \times m^{-1}(H_0)$ and (β, τ) in $V_{\gamma_2} \times m^{-1}(H_0)$. The cocycle $\Gamma_{\gamma_1 \gamma_2}$ is defined by (9).

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