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# RIGIDITY OF AMALGAMATED PRODUCT IN NEGATIVE CURVATURE

GÉRARD BESSON, GILLES COURTOIS, SYLVAIN GALLOT

ABSTRACT. Let  $\Gamma$  be the fundamental group of a compact riemannian manifold  $X$  of sectional curvature  $K \leq -1$  and dimension  $n \geq 3$ . We suppose that  $\Gamma = A *_C B$  is the free product of its subgroups  $A$  and  $B$  over the amalgamated subgroup  $C$ . We prove that the critical exponent  $\delta(C)$  of  $C$  satisfies  $\delta(C) \geq n - 2$ . The equality happens if and only if there exist an embedded compact hypersurface  $Y \subset X$ , totally geodesic, of constant sectional curvature  $-1$ , whose fundamental group is  $C$  and which separates  $X$  in two connected components whose fundamental groups are  $A$  and  $B$ . Similar results hold if  $\Gamma$  is an HNN extension, or more generally if  $\Gamma$  acts on a simplicial tree without fixed point.

## 1. INTRODUCTION

In [15], Y. Shalom proved the following theorem which says that for every lattice  $\Gamma$  in the hyperbolic space and for any decomposition of  $\Gamma$  as an amalgamated product  $\Gamma = A *_C B$ , the group  $C$  has to be “big”. In order to measure how “big”  $C$  is, let us define the critical exponent of a discrete group  $C$  acting on a Cartan Hadamard manifold by

$$\delta(C) = \inf\{s > 0 \mid \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < +\infty\}.$$

**Theorem 1.1.** *Let  $\Gamma$  be a lattice in  $PO(n, 1)$ . Assume that  $\Gamma$  is an amalgamated product of its subgroups  $A$  and  $B$  over  $C$ . Then, the critical exponent  $\delta(C)$  of  $C$  satisfy  $\delta(C) \geq n - 2$ .*

An example is given by any  $n$ -dimensional hyperbolic manifold  $X$  which contains a compact separating connected totally geodesic hypersurface  $Y$ . The Van Kampen theorem then says that the fundamental group  $\Gamma$  of  $X$  is isomorphic to the free product of the fundamental groups of the two halves of  $X - Y$  amalgamated over the fundamental group  $C$  of the incompressible hypersurface  $Y$ . Such examples do exist in dimension 3 thanks to the W.Thurston’s hyperbolization theorem. In any dimension, A. Lubotsky showed that any standard arithmetic lattice of  $PO(n, 1)$  has a finite cover whose fundamental group is an amalgamated product, cf. [11]. In fact, A. Lubotsky proved that any standard arithmetic lattice  $\Gamma$  has a finite index subgroup  $\Gamma_0$  which is mapped onto a nonabelian free group. A nonabelian free group can be written in infinitely many ways as an amalgamated product, so one get infinitely many decomposition of  $\Gamma_0$  as an amalgamated

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product by pulling back the amalgamated decomposition of the nonabelian free group.

In these cases there is equality in theorem 1.1, ie.  $\delta(C) = n - 2$  where  $C$  is the fundamental group of  $Y$ , and  $Y$ . Shalom suggested in [15] that the equality case in the theorem 1.1 happens only in that case.

The aim of this paper is to show that the theorem 1.1 still holds when  $\Gamma$  is the fundamental group of a compact riemannian manifold of variable sectional curvature less than or equal to  $-1$ , and characterize the equality case.

**Theorem 1.2.** *Let  $X$  be an  $n$ -dimensional compact riemannian manifold of sectional curvature  $K \leq -1$ . We assume that the fundamental group  $\Gamma$  of  $X$  is an amalgamated product of its subgroups  $A$  and  $B$  over  $C$  and that neither  $A$  nor  $B$  equals  $\Gamma$ . Then, the critical exponent  $\delta(C)$  of  $C$  satisfy  $\delta(C) \geq n - 2$ . Equality  $\delta(C) = n - 2$  happens if and only if  $C$  cocompactly preserves a totally geodesic isometrically embedded copy  $\mathbb{H}^{n-1}$  of the hyperbolic space of dimension  $n - 1$ . Moreover, in the equality case, the hypersurface  $Y^{n-1} := \mathbb{H}^{n-1}/C$  is embedded in  $X$  and separates  $X$  in two connected components whose fundamental groups are respectively  $A$  and  $B$ .*

**Remark 1.3.** (i) *By the assumption on  $A$  or  $B$  not being equal to  $\Gamma$  we exclude the trivial decomposition  $\Gamma = \Gamma *_C C$  where  $A = \Gamma$  and  $C = B$  can be an arbitrary subgroup of  $\Gamma$ , for example any cyclic subgroup, in which case the conclusion of theorem 1.2 fails. Also note that because of this assumption on  $A$  and  $B$ , we have  $A \neq C$  and  $B \neq C$ .*

(ii) *Let us recall that standard arithmetic lattices in  $PO(n, 1)$  have finite index subgroup with infinitely many non equivalent decompositions as amalgamated products, cf. [11]. In fact, among these decompositions, all but finitely many of them are such that  $\delta(C) > n - 2$ . Indeed, by theorem 1.2, if  $\delta(C) = n - 2$  then  $C$  is the fundamental group of an embedded totally geodesic hypersurface in  $X$ , but there are only finitely many totally geodesic hypersurfaces by [21].*

When a group is an amalgamated product, it acts on a simplicial tree without fixed point and theorem 1.1 is a particular case of the

**Theorem 1.4.** ([15], theorem 1.6). *Let  $\Gamma \subset SO(n, 1)$ ,  $n \geq 3$ , be a lattice. Suppose  $\Gamma$  acts on a simplicial tree  $T$  without fixed vertex. Then there is an edge of  $T$  whose stabilizer  $C$  satisfies  $\delta(C) \geq n - 2$ .*

In the case  $\Gamma$  is cocompact, the conclusion of theorem 1.4 holds for the stabilizer of any edge which separates the tree  $T$  in two unbounded components, and the proof of this is exactly the same as the proof of theorem 1.2. In particular, when the action of  $\Gamma$  on  $T$  is minimal, (ie. there is no proper subtree of  $T$  invariant by  $\Gamma$ ), the conclusion of theorem 1.4 holds for every edge of  $T$ , in the variable curvature setting, and we are able to handle the equality case.

**Theorem 1.5.** *Let  $\Gamma$  be the fundamental group of an  $n$ -dimensional compact riemannian manifold  $X$  of sectional curvature less than or equal to  $-1$ . Suppose  $\Gamma$  acts minimally on a simplicial tree  $T$  without fixed point. Then, the stabilizer  $C$  of every edge of  $T$  satisfies  $\delta(C) \geq n - 2$ . The equality  $\delta(C) = n - 2$  happens if and only if there exist a compact totally geodesic hypersurface  $Y \subset X$  with fundamental group  $\pi_1(Y) = C$ . Moreover, in that case,  $Y$  with its induced metric has constant sectional curvature  $-1$ .*

Another interesting case contained in theorem 1.5 is the case of *HNN* extension. Let us recall the definition of an *HNN* extension. Let  $A$  and  $C$  be groups and  $f_1 : C \rightarrow A$ ,  $f_2 : C \rightarrow A$  two injective morphisms of  $C$  into  $A$ . The *HNN* extension  $A*_C$  is the group generated by  $A$  and an element  $t$  with the relations  $tf_1(\gamma)t^{-1} = f_2(\gamma)$ . For example, let  $X$  be a compact manifold containing a non separating compact incompressible hypersurface  $Y \subset X$ . Let  $A$  be the fundamental group of the manifold with boundary  $X - Y$  obtained by cutting  $X$  along  $Y$  and let  $C$  be the fundamental group of  $Y$ . The boundary of  $X - Y$  consists in two connected components  $Y_1 \subset X - Y$  and  $Y_2 \subset X - Y$  homeomorphic to  $Y$ . By the incompressibility assumption, these inclusions give rise to two embeddings of  $C$  into  $A$ , and the fundamental group of  $X$  is the associated *HNN* extension  $A*_C$ .

**Theorem 1.6.** *Let  $\Gamma$  be the fundamental group of a compact riemannian manifold  $X$  of dimension  $n$  and sectional curvature less than or equal to  $-1$ . Suppose that  $\Gamma = A*_C$  where  $A$  is a proper subgroup of  $\Gamma$ . Then, we have  $\delta(C) \geq n - 2$  and equality  $\delta(C) = n - 2$  if and only if there exist a non separating compact totally geodesic hypersurface  $Y \subset X$  with fundamental group  $\pi_1(Y) = C$ . Moreover, in that case,  $Y$  with its induced metric is of constant sectional curvature  $-1$ , and the *HNN* decomposition arising from  $Y$  is the one we started with.*

Let us summarize the ideas of the proof of theorem 1.2. We work on  $\tilde{X}/C$ . The amalgamation assumption provides an essential hypersurface  $Z$  in  $\tilde{X}/C$ , namely  $Z$  is homologically non trivial in  $\tilde{X}/C$ . The volume of all hypersurfaces homologous to  $Z$  is bounded below by a positive constant because their systole are bounded away from zero. We then construct a smooth map  $F : \tilde{X}/C \rightarrow \tilde{X}/C$ , homotopic to the identity which contracts the volume of all compact hypersurfaces  $Y$  by the factor  $\left(\frac{\delta(C)}{n-2}\right)^{n-1}$ , namely  $vol_{n-1}F(Y) \leq \left(\frac{\delta(C)}{n-2}\right)^{n-1}vol_{n-1}Y$ . This contracting property together with the lower bound of the volume of hypersurfaces in the homology class of  $Z$  gives the inequality  $\delta(C) \geq n - 2$ . This map is different from the map constructed in [3], in particular it can be defined under the single condition that the limit set of  $C$  is not reduced to one point. Moreover, its derivative has an upper bound depending only on the critical exponent of  $C$ .

The equality case goes as follows. When  $\delta(C) = n - 2$ , the map  $F : \tilde{X}/C \rightarrow \tilde{X}/C$  contracts the  $(n - 1)$ -dimensional volumes, ie.  $|Jac_{n-1}F| \leq 1$ .

This contracting property is infinitesimally rigid in the following sense. Let us consider a lift  $\tilde{F}$  of  $F$ . If  $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$  at some point  $x \in \tilde{X}$ , then  $\tilde{F}(x) = x$ , there exists a tangent hyperplane  $E \subset T_x\tilde{X}$  such that  $D\tilde{F}(x)$  is the orthogonal projector of  $T_x\tilde{X}$  onto  $E$  and the limit set  $\Lambda_C$  is contained in the *topological equator*  $E(\infty) \subset \partial\tilde{X}$  associated to  $E$ . By topological equator  $E(\infty) \subset \partial\tilde{X}$  associated to  $E$ , we mean the set of end points of those geodesic rays starting at  $x$  tangentially to  $E$ .

We then prove the existence of a point  $x \in \tilde{X}$  such that

$$(1.1) \quad |\text{Jac}_{n-1}\tilde{F}(x)| = 1.$$

If there would exist a minimizing cycle in the homology class of  $Z$  in  $\tilde{X}/C$ , any point of such a cycle would satisfy (1.1). As no such minimizing cycle a priori exists because of non compactness of  $\tilde{X}/C$ , we prove instead the existence of a  $L^2$  harmonic  $(n-1)$ -form dual to  $Z$ , which is enough to prove existence of a point  $x$  such that (1.1) holds.

At this stage of the proof, there is a big difference between the constant curvature case and the variable curvature case.

In the constant curvature case, any topological equator bounds a totally geodesic hyperbolic hypersurface  $\mathbb{H}^{n-1}$ , and therefore, as the group  $C$  preserves  $\Lambda_C \subset E(\infty) = \partial\mathbb{H}^{n-1}$ , it is not hard to see that  $C$  also preserves  $\mathbb{H}^{n-1}$  and acts cocompactly on it, and the hypersurface of the equality case in theorem 1.2 is  $\mathbb{H}^{n-1}/C$ , [2].

In the variable curvature case, we first show the existence of a  $C$ -invariant totally geodesic hypersurface  $\tilde{Z}_\infty \subset \tilde{X}$  whose boundary at infinity coincides with  $\Lambda(C)$ , and then we show that  $\tilde{Z}_\infty$  is isometric to the real hyperbolic space. We then show that  $Y =: \mathbb{H}_{\mathbb{R}}^{n-1}/C$ , which is compact, injects in  $X = \tilde{X}/\Gamma$  and separates  $X$  in two connected components whose fundamental groups are  $A$  and  $B$  respectively.

In order to show the existence of such a totally geodesic hypersurface  $\tilde{Z}_\infty$ , we first prove that  $C$  is a convex cocompact group, ie. the convex hull of the limit set of  $C$  in  $\tilde{X}$  has a compact quotient under the action of  $C$ , and that the limit set of  $C$  is homeomorphic to an  $(n-2)$ -dimensional sphere.

The convex cocompactness property of  $C$  and the fact that the limit set  $\Lambda(C)$  of  $C$  is homeomorphic to an  $(n-2)$ -dimensional topological sphere are the two key points in the equality case.

This compactness property then allows us to prove the existence of a minimizing current in the homology class of the essential hypersurface  $Z \subset \tilde{X}/C$ . By regularity theorem this minimizing current  $\tilde{Z}_\infty$  is a smooth manifold except at a singular set of codimension at least 8. By the contracting properties of our map  $F$ ,  $\tilde{Z}_\infty$  is fixed by  $F$  and the geometric properties of  $F$  at fixed points where the  $(n-1)$ -jacobian of  $F$  equals 1 allows us to prove that  $\tilde{Z}_\infty$  is totally geodesic and isometric to the hyperbolic space.

Let us now briefly describe the proof of the convex cocompactness property of  $C$  in the equality case.

The group  $C$  (or a finite index subgroup of it) actually globally preserves a smooth cocompact hypersurface  $\tilde{Z} \subset \tilde{X}$  which separates  $\tilde{X}$  into two connected components and whose boundary  $\partial\tilde{Z} \subset \partial\tilde{X}$  coincides with  $\Lambda_C \subset E(\infty)$ . In the case where  $C$  wouldn't be convex cocompact, we are able to find an horoball  $HB(\theta_0)$  centered at some point  $\theta_0 \in \Lambda_C$  in the complementary of which lies the hypersurface  $\tilde{Z}$ .

The contradiction then comes from the following.

Consider a sequence of points  $\theta_i \in \partial\tilde{X}$  converging to  $\theta_0$  and geodesic rays  $\alpha_i$  starting from the point  $x \in \tilde{X}$  at which  $|Jac_{n-1}(x)| = 1$  and ending up at  $\theta_i$ . These geodesic rays have to cross  $\tilde{Z}$  at points  $z_i$  which are at bounded distance from the orbit  $Cx$  of  $x$ , therefore the shadows  $\mathcal{O}_i$  of balls centered at these  $z_i$  enlightened from  $x$  have to contain points of  $\Lambda_C$  by the shadow lemma of D. Sullivan. On the other hand, we show that it is possible to choose the sequence  $\theta_i$  in such a way that these shadows  $\mathcal{O}_i$  don't meet  $\Lambda_C$ . This property  $\mathcal{O}_i \cap \Lambda_C = \emptyset$  comes from a choice of  $\theta_i$  such that the distance between  $z_i$  and the set  $H$  of all geodesics rays at  $x$  tangent to  $E \subset T_x\tilde{X}$  tends to  $\infty$ . Intuitively, in order to chose  $z_i$  as far as possible from  $H$ , the points  $\theta_i$  have to be chosen transversally to  $\Lambda_C$ . This transversality condition is not well defined because the limit set  $\Lambda_C$  might be highly non regular. Thus, in order to prove that such a choice is possible, we argue again by contradiction. If for any choice of a sequence  $\theta_i$  converging to  $\theta_0$ , the distance between  $z_i$  and  $H$  stays bounded, then the Gromov distances  $d(\theta_i, \theta_0)$  between  $\theta_i$  and  $\theta_0$  satisfy  $d(\theta_i, \Lambda_C) = o(d(\theta_i, \theta_0))$ , and therefore any tangent cone of  $\Lambda_C$  at  $\theta_0$  would coincide with a tangent cone of  $\partial\tilde{X}$  at  $\theta_0$ , which is known to be topologically  $\mathbb{R}^{n-1}$ . But on the other hand, the existence of a point  $x$  such that  $|Jac_{n-1}(x)| = 1$  and the fact that  $C$  acts uniformly quasiconformally with respect to the Gromov distance on  $\partial\tilde{X}$  imply that the Alexandroff compactification of the above tangent cone of  $\Lambda_C$  at  $\theta_0$  is homeomorphic to  $\Lambda_C$  which is contained in a topological sphere  $S^{n-2}$ , leading to a contradiction.

From convex cocompactness of  $C$  and the fact that the limit set of  $C$  is a topological  $(n-2)$ -dimensional sphere, there is an alternative proof of the existence of a totally geodesic  $C$ -invariant copy of the hyperbolic space  $\mathbb{H}_{\mathbb{R}}^{n-1} \subset \tilde{X}$  which consists in observing that the topological dimension and the Hausdorff dimension of the limit set  $\Lambda(C)$  are equal to  $n-2$  and then use the following result of M. Bonk and B. Kleiner (which we quote in the riemannian manifold setting although it remains true for  $CAT(-1)$  spaces) instead of the (simpler) minimal current argument.

**Theorem 1.7.** [4] *Let  $X$  be a Cartan Hadamard  $n$ -dimensional manifold whose sectional curvature satisfy  $K \leq -1$ , and  $C$  a convex cocompact discrete subgroup of isometries of  $X$  with limit set  $\Lambda_C$ . Let us assume that the topological dimension and the Hausdorff dimension (with respect to the Gromov distance on  $\partial\tilde{X}$ ) of  $\Lambda_C$  coincide and are equal to an integer  $p$ . Then,  $C$*

preserves a totally geodesic embedded copy of the real hyperbolic space  $\mathbb{H}^{p+1}$ , with  $\partial\mathbb{H}^{p+1} = \Lambda_C$ .

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## 2. ESSENTIAL HYPERSURFACES

Let  $\Gamma$  be a discrete cocompact group of isometries of a  $n$ -dimensional Cartan-Hadamard manifold  $(\tilde{X}, \tilde{g})$  whose sectional curvature satisfies  $K_{\tilde{g}} \leq -1$ . Let us assume that the compact manifold  $X = \tilde{X}/\Gamma$  is orientable. Let us also assume that  $\Gamma = A *_C B$  is an amalgamated product of its subgroups  $A$  and  $B$  over  $C$ .

We first reduce to the case where  $[\Gamma : C]$  is infinite.

Namely, if  $[\Gamma : C] < \infty$ , then the critical exponent  $\delta(C) = \delta(\Gamma) \geq n - 1$ , and the equality in theorem 1.2 holds.

We then can assume that  $[\Gamma : C] = \infty$ .

**Lemma 2.1.** *Let  $\Gamma = A *_C B$  be as above, with  $[\Gamma : C] = \infty$ . If neither  $A$  nor  $B$  equals  $\Gamma$ , then  $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$ .*

**Proof :** The Mayer-Vietoris sequence coming from the decomposition  $\Gamma = A *_C B$  writes cf. [6], Corollary 7.7,

$$\begin{aligned} H_n(\tilde{X}/C, \mathbb{Z}) &\rightarrow H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z}) \rightarrow H_n(\tilde{X}/\Gamma, \mathbb{Z}) \rightarrow \dots \\ &\dots \rightarrow H_{n-1}(\tilde{X}/C, \mathbb{Z}) \rightarrow \dots \end{aligned}$$

As  $[\Gamma : C] = \infty$ ,  $H_n(\tilde{X}/C, \mathbb{Z}) = 0$  thus, if  $H_{n-1}(\tilde{X}/C, \mathbb{Z}) = 0$ , we deduce from the Mayer-Vietoris sequence that  $H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z})$  is isomorphic to  $H_n(\tilde{X}/\Gamma, \mathbb{Z})$ . As  $H_{n-1}(\tilde{X}/\Gamma, \mathbb{Z}) = \mathbb{Z}$ , we then deduce that either  $[\Gamma : A] = \infty$  and  $B = \Gamma$ , or  $[\Gamma : B] = \infty$  and  $A = \Gamma$ .  $\square$

In fact in the sequel of the paper we will make use of a smooth essential hypersurface  $Z$  in  $\tilde{X}/C$ .

**Definition 2.2.** *A compact smooth orientable hypersurface  $Z$  of an  $n$ -dimensional manifold  $Y$  is essential in  $Y$  if  $i_*([Z]) \neq 0$  where  $[Z] \in H_{n-1}(Z, \mathbb{R})$  denotes the fundamental class of  $Z$  and  $i_* : H_{n-1}(Z, \mathbb{R}) \rightarrow H_{n-1}(Y, \mathbb{R})$  the morphism induced by the inclusion  $i : Z \hookrightarrow Y$ .*

The end of this section is devoted to finding such an hypersurface  $Z$  in  $\tilde{X}/C$ .

Let us recall a few facts about amalgamated products and their actions on trees, following [14]. Let  $\Gamma = A *_C B$  be an amalgamated products of

its subgroups  $A$  and  $B$  over  $C$ . Then,  $\Gamma$  acts on a simplicial tree  $\tilde{T}$  with a fundamental domain  $T \subset \tilde{T}$  being a segment, ie. an edge joining two vertices. Let us describe this tree  $\tilde{T}$ . There are two orbits of vertices  $\Gamma v_A$  and  $\Gamma v_B$ , the stabilizer of the edge  $v_A$  (resp.  $v_B$ ) being  $A$ , (resp.  $B$ ). There is one orbit of edges  $\Gamma e_C$ , the stabilizer of the edge  $e_C$  being  $C$ . The fundamental domain  $T$  can be chosen as the edge  $e_C$  joining the two vertices  $v_A$  and  $v_B$ . The set of vertices adjacent to  $v_A$ , (resp.  $v_B$ ), is in one to one correspondance with  $A/C$ , (resp.  $B/C$ ). Note that as neither  $A$  nor  $B$  are equal to  $\Gamma$ , then  $[A : C] \neq 1$  and  $[B : C] \neq 1$ , therefore for an arbitrary point  $t_0$  on the edge  $e_C$  we see that  $\tilde{T} - t_0$  is a disjoint union of two unbounded connected components. This fact will be used later on.

Let us consider a continuous  $\Gamma$ -equivariant map  $\tilde{f} : \tilde{X} \rightarrow T$  where  $T$  is the Bass-Serre tree associated to the amalgamation  $\Gamma = A *_C B$ . One regularizes  $\tilde{f}$  such that it is smooth in restriction to the complementary of the inverse image of the set of vertices of  $T$ . Let  $t_0$  a regular value of  $\tilde{f}$  contained in that edge of  $T$  which is fixed by the subgroup  $C$  and define  $\tilde{Z} = \tilde{f}^{-1}(t_0)$ .  $\tilde{Z}$  is a smooth orientable possibly not connected hypersurface in  $\tilde{X}$ , globally  $C$ -invariant. Let us write  $Z = \tilde{Z}/C$ . We will show  $Z \subset \tilde{X}/C$  is compact and that one of the connected components of  $Z$  is essential.

**Lemma 2.3.**  *$Z \subset \tilde{X}/C$  is compact.*

**Proof :** Let us show that for any sequence  $z_n \in \tilde{Z}$ , there exists a subsequence  $z_{n_k}$  and  $\gamma_k \in C$  such that  $\gamma_k z_{n_k}$  converges. As  $\Gamma$  is cocompact, there exists  $g_n \in \Gamma$  such that the set  $(g_n z_n)$  is relatively compact. Let  $g_{n_k} z_{n_k}$  a subsequence which converges to a point  $z \in \tilde{X}$ . By continuity, the sequence  $\tilde{f}(g_{n_k} z_{n_k})$  converges to  $\tilde{f}(z)$ , and by equivariance we get

$$g_{n_k} \tilde{f}(z_{n_k}) = g_{n_k} t_0 \rightarrow \tilde{f}(z)$$

when  $k$  tends to  $\infty$ . As  $\Gamma$  acts in a simplicial way on the tree  $T$  and transitively on the set of edges, the sequence  $g_{n_k} t_0$  is stationary, ie  $g_{n_k} t_0 = t'_0 = g t_0$  for  $k$  large enough. Thus  $g^{-1} g_{n_k} = \gamma_k \in C$  for  $k$  large enough since it fixes  $t_0$  and  $\gamma_k z_{n_k} = g^{-1} g_{n_k} z_{n_k}$  converges to  $g^{-1}(z)$   $\square$

The smooth compact hypersurface  $Z$  we constructed might be not connected. Let us write  $Z = Z_1 \cup Z_2 \cup \dots \cup Z_k$  where the  $Z_j$ 's are the connected components of  $Z$ . Each  $Z_j$  is a compact smooth oriented hypersurface of  $\tilde{X}/C$ .

The aim of what follows is to prove that at least one component  $Z_i$  of  $Z$  is essential.

**Lemma 2.4.** *There exists  $i \in [1, k]$  such that  $Z_i$  is essential in  $\tilde{X}/C$ .*

**Proof :** If there exists a  $Z_i$  which doesn't separate  $\tilde{X}/C$  in two connected components, then  $Z_i$  is essential in  $\tilde{X}/C$ . So we can assume that every  $Z_j$ ,  $j = 1, \dots, k$ , does separate  $\tilde{X}/C$  in two connected components. In that case



we will show that there exists a  $Z_i$  which separates  $\tilde{X}/C$  in two unbounded connected components which easily implies that  $Z_i$  is essential.

Let us denote  $U_l$ ,  $l = 1, 2, \dots, p$ , the connected components of  $\tilde{X}/C - \bigcup_{j=1}^k Z_j$ .

Claim : at least two components  $U_m, U_{m'}$  are unbounded.

Assuming the claim let us finish the proof of the lemma. For each  $Z_j$  we denote  $V_j, V'_j$  the two connected components of  $\tilde{X}/C - Z_j$ . Then  $U_m = W_1 \cap W_2 \cap \dots \cap W_k$  where for each  $j$ ,  $W_j = V_j$  or  $W_j = V'_j$ . In the same way,  $U_{m'} = W'_1 \cap W'_2 \cap \dots \cap W'_k$ . As  $U_m \cap U_{m'} = \emptyset$ , there exists  $i \in [1, k]$  such that  $W_i \cap W'_i = \emptyset$ , thus  $U_m \subset V_i$  and  $U_{m'} \subset V'_i$  or  $U_m \subset V'_i$  and  $U_{m'} \subset V_i$  so  $Z_i$  separates  $\tilde{X}/C$  into two unbounded components. This proves the lemma.

Let us prove the claim.

We have already noticed that  $T - \{t_0\}$  is the disjoint union of two unbounded connected components  $T_1$  and  $T_2$ . As  $C$  acts on  $T$  isometrically and simplicially then  $T/C - \{t_0\} = T_1/C \cup T_2/C$  is the disjoint union of two unbounded connected components. Let  $\bar{f} : \tilde{X}/C \rightarrow T/C$  the quotient map of  $\bar{f}$ . For each component  $U_i$ , we have

$\bar{f}(U_i) \subset T_1/C$  or  $\bar{f}(U_i) \subset T_2/C$ , thus we can conclude the claim because  $\bar{f}$  is onto  $\square$

Let  $\pi : \tilde{X} \rightarrow \tilde{X}/C$  be the natural projection. For any  $i = 1, 2, \dots, k$ , let us denote  $\{\tilde{Z}_i^j\}_{j \in J}$  the set of connected components of  $\tilde{Z}_i =: \pi^{-1}(Z_i)$ .

For each  $i \in [1, k]$ , we claim that  $C$  acts transitively on the set  $\{\tilde{Z}_i^j\}_{j \in J}$ . Namely, let us consider  $\tilde{Z}_i^j, \tilde{Z}_i^{j'}$ ,  $\tilde{z} \in \tilde{Z}_i^j$ ,  $\tilde{z}' \in \tilde{Z}_i^{j'}$ , and write  $z = \pi\tilde{z} \in Z_i$  and  $z' = \pi\tilde{z}' \in Z_i$ . Let  $\alpha$  be a continuous path on  $Z_i$  such that  $\alpha(0) = z$  and  $\alpha(1) = z'$ , and  $\tilde{\alpha}$  the lift of  $\alpha$  such that  $\tilde{\alpha}(0) = \tilde{z}$ . We have  $\pi\tilde{\alpha}(1) = z'$  and  $\tilde{\alpha}(1) \in \tilde{Z}_i^j$  for some  $j$ , thus, there exists  $c \in C$  such that  $c(\tilde{\alpha}(1)) = \tilde{z}'$  and therefore  $c\tilde{Z}_i^j = \tilde{Z}_i^{j'}$ .  $\square$

Let us denote  $C_i^j$  the stabilizer of  $\tilde{Z}_i^j$ , and  $Z_i^j = \tilde{Z}_i^j/C_i^j \subset \tilde{X}/C_i^j$ . Let us write  $p : \tilde{X}/C_i^j \rightarrow \tilde{X}/C$  the natural projection.

**Lemma 2.5.** *The restriction of  $p$  to  $Z_i^j$  is a diffeomorphism onto  $Z_i$ . In particular,  $Z_i^j$  is compact.*

**proof :** Let  $z$  and  $z'$  be two points in  $Z_i^j$  such that  $p(z) = p(z')$ . Let  $\tilde{z}$  and  $\tilde{z}'$  be lifts of  $z$  and  $z'$  in  $\tilde{X}$ . These two points  $\tilde{z}$  and  $\tilde{z}'$  which are in  $\tilde{Z}_i^j$  actually belong to the same connected component  $\tilde{Z}_i^j$  because for  $j \neq j'$ ,  $\tilde{Z}_i^{j'}/C_i^j \cap \tilde{Z}_i^j/C_i^j = \emptyset$ . As  $p(z) = p(z')$ , there exists  $c \in C$  such that  $\tilde{z}' = c\tilde{z}$ , thus  $c \in C_i^j$ , and  $z = z'$ , therefore the restriction of  $p$  to  $Z_i^j$  is injective. The surjectivity comes from the fact that  $\pi^{-1}Z_i = \bigcup_{j \in J} \tilde{Z}_i^j$  and  $C$  acts transitively on the set  $\{\tilde{Z}_i^j\}_{j \in J}$ .  $\square$

Let us consider the integer  $i \in [1, k]$  as in lemma 2.4, ie. such that  $Z_i \hookrightarrow \tilde{X}/C$  is essential, and choose  $\tilde{Z}_i^l$  one component of  $\pi^{-1}(Z_i)$ .

After possibly replacing  $C_i^l$  by an index two subgroup, we may assume that  $C_i^l$  globally preserves each of the two connected components  $U_i^l$  and  $V_i^l$  of  $\tilde{X} - \tilde{Z}_i^l$ .

**Lemma 2.6.** *Let  $i, l$  and  $C_i^l$  be chosen as above. The compact hypersurface  $Z_i^l = \tilde{Z}_i^l/C_i^l$  is essential in  $\tilde{X}/C_i^l$ . Moreover the two connected components  $U_i^l/C_i^l$  and  $V_i^l/C_i^l$  of  $\tilde{X}/C_i^l - Z_i^l$  are unbounded.*

**Proof :** Let us consider  $p : \tilde{X}/C_i^l \rightarrow \tilde{X}/C$ . By lemma 2.5, the restriction of  $p$  to  $Z_i^l$  is a diffeomorphism onto  $Z_i$ , therefore  $Z_i^l$  is essential in  $\tilde{X}/C_i^l$  because  $Z_i$  is essential in  $\tilde{X}/C$ . As  $C_i^l$  preserves  $U_i^l$  and  $V_i^l$ ,  $Z_i^l$  separates  $\tilde{X}/C_i^l$  into two connected components  $U_i^l/C_i^l$  and  $V_i^l/C_i^l$ . and as  $Z_i^l$  is essential in  $\tilde{X}/C_i^l$ ,  $U_i^l/C_i^l$  and  $V_i^l/C_i^l$  are unbounded.  $\square$

**In the sequel of the paper we will denote  $\tilde{Z}' = \tilde{Z}_i^l$ ,  $C' = C_i^l$  and  $Z' = Z_i^l = \tilde{Z}_i^l/C_i^l$ .**

### 3. ISOSYSTOLIC INEQUALITY

In this section we summarize facts and results due to M.Gromov, [9]. Let  $Z$  be a  $p$ -dimensional compact orientable manifold and  $i : Z \hookrightarrow Y$  an embedding of  $Z$  into  $Y$  where  $Y$  is an aspherical space. We suppose that  $i_*([Z]) \neq 0$  where  $i_* : H_p(Z, \mathbb{R}) \rightarrow H_p(Y, \mathbb{R})$  is the morphism induced by the embedding  $Z \hookrightarrow Y$ . Let us fix a riemannian metric  $g$  on  $Z$ . For each  $z \in Z$  we consider the set  $\mathcal{C}_z$  of those loops  $\alpha$  at  $z$  such that  $i \circ \alpha$  is homotopically non trivial in  $Y$ .

Let us define the systole of  $(Z, g, i)$  at the point  $z$  by

**Definition 3.1.**  $sys_i(Z, g, z) = \inf\{length(\alpha), \alpha \in \mathcal{C}_z\}$

and the systole of  $(Z, g, i)$  by

**Definition 3.2.**  $sys_i(Z, g) = \inf\{sys_i(Z, g, z) \mid z \in Z\}$ .

The following isosystolic inequality, due to M.Gromov says that the volume of any essential submanifold  $Z$  of an aspherical space  $Y$  relatively to any riemannian metric on  $Z$  is universally bounded below by its systole.

**Theorem 3.3.** [9] *There exists a constant  $C_p$  such that for each  $p$ -dimensional riemannian manifold  $(Z, g)$  and any embedding  $Z \hookrightarrow Y$  into an aspherical space  $Y$  such that  $i_*([Z]) \neq 0$  where  $i_* : H_p(Z, \mathbb{R}) \rightarrow H_p(Y, \mathbb{R})$  is the induced morphism in homology, then  $vol_p(Z, g) \geq C_p(sys_i(Z, g))^p$*

We will apply this volume estimates to the essential hypersurface  $i : Z \hookrightarrow \tilde{X}/C$  that we constructed in lemma 2.6.

The following lemma is immediate.

**Lemma 3.4.** *Let  $C$  be a discrete group acting on a simply connected manifold  $\tilde{X}$ ,  $\tilde{Z}$  a  $C$ -invariant hypersurface of  $\tilde{X}$  and  $i : Z = \tilde{Z}/C \hookrightarrow \tilde{X}/C$  the natural inclusion. Let  $g$  any riemannian metric on  $Z$  and  $\tilde{g}$  the lift of  $g$  to  $\tilde{Z}$ . Then, for any  $z \in Z$  we have,*

$$\text{sys}_i(Z, g, z) = \inf\{d_{\tilde{g}}(\tilde{z}, \gamma\tilde{z}), \gamma \in C\}$$

where  $\tilde{z} \in \tilde{Z}$  is a lift of  $z \in Z$  and  $d_{\tilde{g}}$  is the distance induced by  $\tilde{g}$  on  $\tilde{Z}$ .

**Proof :** Let  $\alpha \in \mathcal{C}_z$  a loop based at  $z \in Z$ . As  $i \circ \alpha$  is an homotopically non trivial loop at  $i(z) = z$  in  $\tilde{X}/C$ , its lift  $\tilde{i \circ \alpha}$  at some  $\tilde{z} \in \tilde{Z}$  ends up at  $\gamma\tilde{z}$  for some  $\gamma \in C$ .  $\square$

#### 4. VOLUME OF HYPERSURFACES IN $\tilde{X}/C$

Let  $(\tilde{X}, \tilde{g})$  be a  $n$ -dimensional Cartan-Hadamard manifold whose sectional curvature satisfies  $K_{\tilde{g}} \leq -1$  and  $C$  a discrete group of isometries of  $(\tilde{X}, \tilde{g})$ . We assume that the group  $C$  is non elementary, namely  $C$  fixes neither one nor two points in the geometric boundary  $\partial\tilde{X}$  of  $(\tilde{X}, \tilde{g})$ .

The aim of this section is to construct a map  $F : \tilde{X}/C \rightarrow \tilde{X}/C$  such that for any compact hypersurface  $Z$  of  $\tilde{X}/C$ , we have

$$\text{vol}_{n-1}(F(Z)) \leq \left(\frac{\delta+1}{n-1}\right)^{n-1} \text{vol}_{n-1}(Z)$$

where  $\delta$  is the critical exponent of  $C$  and  $\text{vol}_{n-1}(Z)$  stands for the  $(n-1)$ -dimensional volume of the metric on  $Z$  induced from  $g$ . For every subgroup  $C' \subset C$  and any hypersurface  $Z'$  of  $\tilde{X}/C'$  the lift  $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$  of  $F$  will also verify

$$\text{vol}_{n-1}(F'(Z')) \leq \left(\frac{\delta+1}{n-1}\right)^{n-1} \text{vol}_{n-1}(Z').$$

In order to construct the map  $F$  we need a few preliminaries. We consider a finite positive Borel measure  $\mu$  on the boundary  $\partial\tilde{X}$  whose support contains at least two points. Let us fix an origin  $o \in \tilde{X}$  and denote  $B(x, \theta)$  the Busemann function defined for each  $x \in \tilde{X}$  and  $\theta \in \partial\tilde{X}$  by

$$B(x, \theta) = \lim_{t \rightarrow \infty} \text{dist}(x, c(t)) - t$$

where  $c(t)$  is the geodesic ray such that  $c(0) = o$  and  $c(+\infty) = \theta$ .

Let  $\mathcal{D}_\mu : \tilde{X} \rightarrow \mathbb{R}$  the function defined by

$$(4.1) \quad \mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} e^{B(y, \theta)} d\mu(\theta)$$

A computation shows that

$$(4.2) \quad Dd\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} (DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta)) e^{B(y, \theta)} d\mu(\theta).$$

When  $K_{\tilde{g}} \leq -1$  the Rauch comparison theorem says that for every  $y \in \tilde{X}$ , and  $\theta \in \partial\tilde{X}$ ,

$$(4.3) \quad DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta) \geq \tilde{g}.$$

We then get

$$(4.4) \quad Dd\mathcal{D}_\mu(y) \geq \mathcal{D}_\mu(y)\tilde{g},$$

thus  $Dd\mathcal{D}_\mu(y)$  is positive definite and  $\mathcal{D}_\mu$  is strictly convex.

**Lemma 4.1.** *We have  $\lim_{y_k \rightarrow \partial\tilde{X}} \mathcal{D}_\mu(y) = +\infty$ .*

**proof :** Let  $y_k \in \tilde{X}$  a sequence such that

$$(4.5) \quad \lim_{k \rightarrow \infty} y_k = \theta_0 \in \partial\tilde{X}.$$

As  $\text{supp}(\mu) \cap (\partial\tilde{X} - \{\theta_0\}) \neq \emptyset$ , there exists a compact subset  $K \subset \partial\tilde{X} - \{\theta_0\}$  such that  $\mu(K) > 0$  thus,

$$(4.6) \quad \int_{\partial\tilde{X}} e^{B(y_k, \theta)} d\mu \geq \int_K e^{B(y_k, \theta)} d\mu \rightarrow +\infty.$$

□

**Corollary 4.2.** *Let  $\mu$  a finite borel measure on  $\partial\tilde{X}$  whose support contains at least two points. The function  $\mathcal{D}_\mu$  has a unique minimum. This minimum will be denoted by  $\mathcal{C}(\mu)$ .*

Let us now consider some discrete subgroup  $C \subset \text{Isom}(\tilde{X}, \tilde{g})$ . Recall that a family of Patterson measures  $(\mu_x)_{x \in \tilde{X}}$  associated to  $C$  is a set of positive finite measures  $\mu_x$  on  $\partial\tilde{X}$ ,  $x \in \tilde{X}$ , such that the following holds for all  $x \in \tilde{X}$ ,  $\gamma \in C$ ,

$$(4.7) \quad \mu_{\gamma x} = \gamma_* \mu_x$$

$$(4.8) \quad \mu_x = e^{-\delta B(x, \theta)} \mu_o,$$

where  $o \in \tilde{X}$  is a fixed origin,  $B$  the Busemann function associated to  $o$  and  $\delta$  the critical exponent of  $C$ .

We assume now that  $\text{supp}(\mu_o)$  contains at least two points and define the map  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$  for  $x \in \tilde{X}$  by

$$(4.9) \quad \tilde{F}(x) = \mathcal{C}(e^{-B(x, \theta)} \mu_x).$$

Here are a few notations. For a subspace  $E$  of  $T_x \tilde{X}$ , we will write  $\text{Jac}_E \tilde{F}(x)$  the determinant of the matrix of the restriction of  $D\tilde{F}(x)$  to  $E$  with respect to orthonormal bases of  $E$  and  $D\tilde{F}(x)E$ . For an integer  $p$ ,

we denote by  $Jac_p \tilde{F}(x)$  the supremum of  $|Jac_E \tilde{F}(x)|$  as  $E$  runs through the set of  $p$ -dimensional subspaces of  $T_x \tilde{X}$ .

**Lemma 4.3.** *The map  $\tilde{F}$  is smooth, homotopic to the Identity and verifies for all  $x \in \tilde{X}$ ,  $\gamma \in C$  and  $p \in [2, n = \dim(X)]$ ,*

$$(i) \quad \tilde{F}(\gamma x) = \gamma \tilde{F}(x)$$

$$(ii) \quad |Jac_p \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

**Proof :**

The map

$$(x, y) \rightarrow \int_{\partial \tilde{X}} e^{B(y, \theta) - B(x, \theta)} d\mu_x(\theta) = \int_{\partial \tilde{X}} e^{B(y, \theta) - (\delta+1)B(x, \theta)} d\mu_o(\theta)$$

is smooth because  $y \rightarrow B(y, \theta)$  is smooth.

For all  $x$  the map  $y \rightarrow \int_{\partial \tilde{X}} e^{B(y, \theta) - (\delta+1)B(x, \theta)} d\mu_o(\theta)$  is strictly convex by (4.4) and tends to infinity when  $y$  tend to  $\partial \tilde{X}$  (cf. lemma 4.1), thus the unique minimum  $\tilde{F}(x)$  is a smooth function. The equivariance of  $\tilde{F}$  comes from the cocycle relation  $B(\gamma y, \gamma \theta) - B(\gamma x, \gamma \theta) = B(y, \theta) - B(x, \theta)$ .

For each  $x \in \tilde{X}$  let  $c_x$  be the geodesic in  $\tilde{X}$  such that  $c_x(0) = x$ ,  $c_x(1) = \tilde{F}(x)$  and which is parametrized with constant speed. The map  $\tilde{F}_t : \tilde{X} \rightarrow \tilde{X}$  defined by  $\tilde{F}_t(x) = c_x(t)$  is a  $C$ -equivariant homotopy between  $Id_{\tilde{X}}$  and  $\tilde{F}$ .

It remains to prove (ii).

The point  $\tilde{F}(x)$  is characterized by

$$(4.10) \quad \int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0.$$

In order to simplify the notations we will write  $B_{(x, \theta)}$  instead of  $B(x, \theta)$  and we will denote  $\nu_x$  the measure  $e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$ . We will also write  $D\tilde{F}(u)$  instead of  $D\tilde{F}(x)(u)$ .

The differential of  $\tilde{F}$  is characterized by the following: for  $u \in T_x \tilde{X}$  and  $v \in T_{\tilde{F}(x)} \tilde{X}$ , one has

$$(4.11) \quad \begin{aligned} & \int_{\partial \tilde{X}} [DdB_{(\tilde{F}(x), \theta)}(D\tilde{F}(u), v) + DB_{(\tilde{F}(x), \theta)}(v)DB_{(\tilde{F}(x), \theta)}(D\tilde{F}(u))] d\nu_x(\theta) \\ &= (\delta + 1) \int_{\partial \tilde{X}} DB_{(\tilde{F}(x), \theta)}(v)DB_{(x, \theta)}(u) d\nu_x(\theta). \end{aligned}$$

We define the quadratic forms  $k$  and  $h$  for  $v \in T_{\tilde{F}(x)} \tilde{X}$  by

$$(4.12) \quad k(v, v) = \int_{\partial \tilde{X}} [DdB_{(\tilde{F}(x), \theta)}(v, v) + (DB_{(\tilde{F}(x), \theta)}(v))^2] d\nu_x(\theta).$$

and

$$(4.13) \quad h(v, v) = \int_{\partial\tilde{X}} DB_{(\tilde{F}(x), \theta)}(v)^2 d\nu_x(\theta).$$

The relation (4.11) writes, for  $u \in T_x\tilde{X}$  and  $v \in T_{\tilde{F}(x)}\tilde{X}$  :

$$(4.14) \quad k(D\tilde{F}(u), v) = (\delta + 1) \int_{\partial\tilde{X}} DB_{(\tilde{F}(x), \theta)}(v) DB_{(x, \theta)}(u) d\nu_x(\theta).$$

We defines the quadratic form  $h'$  on  $T_x\tilde{X}$  for  $u \in T_x\tilde{X}$  by

$$(4.15) \quad h'(u, u) = \int_{\partial\tilde{X}} DB_{(x, \theta)}(u)^2 d\nu_x(\theta),$$

and one derives from (4.14)

$$(4.16) \quad |k(D\tilde{F}(x)(u), v)| \leq (\delta + 1)h(v, v)^{1/2}h'(u, u)^{1/2}.$$

One now can estimate  $Jac_P\tilde{F}(x)$ . Let  $P \subset T_x\tilde{X}$ ,  $dim P = p$ . If  $D\tilde{F}(P)$  has dimension lower than  $p$ , then there is nothing to be proven. Let us assume that  $dim D\tilde{F}(P) = p$ . Denote by the same letters  $H'$  [resp.  $H$  and  $K$ ] the selfadjoint operators (with respect to  $\tilde{g}$ ) associated to the quadratic forms  $h'$  [resp.  $h$ ,  $k$ ] restricted to  $P$  [resp.  $D\tilde{F}(P)$ ].

Let  $(v_i)_{i=1}^p$  an orthonormal basis of  $D\tilde{F}(P)$  which diagonalizes  $H$  and  $(u_i)_{i=1}^p$  an orthonormal basis of  $P$  such that the matrix of  $K \circ D\tilde{F}(x) : P \rightarrow D\tilde{F}(P)$  is triangular. Then,

$$(4.17) \quad \det K \cdot |Jac_P\tilde{F}(x)| \leq (\delta + 1)^p (\prod_{i=1}^p h(v_i, v_i)^{1/2}) (\prod_{i=1}^p h'(u_i, u_i)^{1/2})$$

thus,

$$(4.18) \quad \det K \cdot |Jac_P\tilde{F}(x)| \leq (\delta + 1)^p \left( \frac{Trace H}{p} \right)^{p/2} \left( \frac{Trace H'}{p} \right)^{p/2}.$$

In these inequalities one can normalize the measures

$$\nu_x = e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$$

such that their total mass equals one, which gives

$$(4.19) \quad trace H = \sum_{i=1}^p h(v_i, v_i) \leq 1,$$

the last inequality coming from the fact that for all  $\theta \in \partial\tilde{X}$ ,

$$(4.20) \quad \sum_{i=1}^p DB_{(\tilde{F}(x), \theta)}(v_i)^2 \leq \|\nabla B(\tilde{F}(x), \theta)\|^2 = 1$$

and from the previous normalization.

Similarly,

$$(4.21) \quad \text{trace}H' = \sum_{i=1}^p h'(u_i, u_i) \leq 1.$$

We then obtain with (4.18)

$$(4.22) \quad \det K \cdot |\text{Jac}_P \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

Thanks to (4.3), we have  $\det K \geq 1$ , so

$$(4.23) \quad |\text{Jac}_P \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

We get (ii) by taking the supremum in  $P$ .  $\square$

As the map  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$  is  $C$ -equivariant, then for every subgroup  $C' \subset C$ ,  $\tilde{F}$  gives rise to a map  $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$  and so does the homotopy  $\tilde{F}_t$  between  $\tilde{F}$  and  $\text{Id}_{\tilde{X}}$ .

**Corollary 4.4.** *The map  $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$  is homotopic to the Identity map and verifies for all  $x \in \tilde{X}/C'$  and  $p \in [2, n = \dim X]$*

$$|\text{Jac}F'_p(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

Let  $C \subset \text{Isom}(\tilde{X}, \tilde{g})$  as above, ie such that the support of the Patterson-Sullivan measures contains at least two points and with critical exponent  $\delta$ . Let  $C' \subset C$  be a subgroup.

Let us consider an compact hypersurface  $Z' \subset \tilde{X}/C'$ .

Denote  $F'^k = F' \circ F' \circ \dots \circ F'$  the composition of  $F'$   $k$ -times.

Let us write  $g_k = (F'^k)^*g$ , where  $g$  is the metric on  $\tilde{X}/C'$  induced by  $\tilde{g}$ . The symmetric 2-tensor  $g_k$  may not be a riemannian metric on  $\tilde{X}/C'$  nor its restriction to  $Z'$ , so we have to modify it. For  $\epsilon > 0$ , the following symmetric 2-tensor  $g_{\epsilon,k}$  is a riemannian metric on  $\tilde{X}/C'$  and so is its restriction  $h_{\epsilon,k}$  to  $Z'$ .

$$(4.24) \quad g_{\epsilon,k} = g_k + \epsilon^2 g.$$

**Lemma 4.5.** *Let  $h_{\epsilon,k}$  be the restriction of  $g_{\epsilon,k}$  to the hypersurface  $Z'$  and  $g_{Z'}$  the restriction of  $g$  to  $Z'$ . Let  $\Phi_{\epsilon,k} : Z' \rightarrow \mathbb{R}$  the density defined for all  $x \in Z'$  by  $dv_{h_{\epsilon,k}}(x) = \Phi_{\epsilon,k}(x) dv_{g_{Z'}}(x)$ . For any sequence  $\epsilon_k$  such that  $\lim_{k \rightarrow \infty} \epsilon_k = 0$ , there exists a sequence  $\epsilon'_k$ ,  $\lim_{k \rightarrow \infty} \epsilon'_k = 0$ , such that for all  $x \in Z'$ ,*

$$0 < \Phi_{\epsilon'_k,k}(x) \leq |\text{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

In particular,

$$\Phi_{\epsilon'_k,k}(x) \leq \left(\frac{\delta+1}{n-1}\right)^{k(n-1)} + \epsilon_k$$

and

$$\text{vol}(Z', h_{\epsilon'_k, k}) \leq \left[ \left( \frac{\delta + 1}{n - 1} \right)^{k(n-1)} + \epsilon_k \right] \text{vol}(Z', g_{Z'}).$$

**Corollary 4.6.** *Under the above assumptions, if  $\delta < n - 2$  there exists a sequence  $\epsilon'_k$  such that  $\lim_{k \rightarrow \infty} \epsilon'_k = 0$ , and  $\lim_{k \rightarrow \infty} \text{vol}(Z', h_{\epsilon'_k, k}) = 0$ .*

**Proof of lemma 4.5 :**

Let us fix  $k$  an integer. Let  $x \in Z'$  and  $u \in T_x Z'$ . We have  $g_{\epsilon, k}(u, u) = h_{\epsilon, k}(u, u) = g(DF'^k(x)(u), DF'^k(x)(u)) + \epsilon^2 g(u, u)$  thus  $h_{\epsilon, k}(u, u) = g(A_{x, \epsilon} u, u)$  where  $A_{x, \epsilon} \in \text{End}(T_x Z')$  is the self adjoint operator  $A_x = DF'^k(x)^* \circ DF'^k(x) + \epsilon^2 \text{Id}$ , with  $DF'^k(x)^*$  the adjoint of  $DF'^k(x) : (T_x Z', g(x)) \rightarrow (DF'^k(x)(T_x Z'), g(F'^k(x)))$ .

By compactness of  $Z'$  and continuity of  $A_{x, \epsilon}$ , there exist  $\epsilon'_k$  such that

$$\Phi_{\epsilon'_k, k}(x) = \det A_{x, \epsilon'_k}^{1/2} \leq \det A_{x, 0} + \epsilon_k,$$

thus

$$\Phi_{\epsilon'_k, k}(x) \leq |\text{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

The lemma then follows from Corollary 4.4.  $\square$

## 5. PROOF OF THEOREM 1.2

This section is devoted to the proof of the Theorem 1.2. Let  $\Gamma$  be a discrete cocompact group of isometries of a  $n$ -dimensional Cartan-Hadamard manifold  $(\tilde{X}, \tilde{g})$  whose sectional curvature satisfies  $K_{\tilde{g}} \leq -1$ . We assume that  $\Gamma = A *_C B$ . At the end of section 2 we constructed a subgroup  $C' \subset C$  and an orientable hypersurface  $\tilde{Z}' \subset \tilde{X}$  such that  $C' \cdot \tilde{Z}' = \tilde{Z}'$  and  $Z' = \tilde{Z}'/C'$  is compact in  $\tilde{X}/C'$ . Moreover  $Z'$  is essential in  $\tilde{X}/C'$  ie  $i_*([Z']) \neq 0$  where  $i_* : H_{n-1}(Z', \mathbb{R}) \rightarrow H_{n-1}(\tilde{X}/C', \mathbb{R})$  is the morphism induced on homology groups by the inclusion  $i : Z' \rightarrow \tilde{X}/C'$  and  $[Z']$  the fundamental class of  $Z'$ .

### 5.1 Proof of the inequality

We now prove the inequality in the theorem 1.2. Let us assume that  $\delta < n - 2$  and derive a contradiction. Let  $h_{\epsilon'_k, k}$  the sequence of metric defined on  $Z'$  in lemma 4.5, then by corollary 4.6 we have

$$(5.1) \quad \lim_{k \rightarrow \infty} \text{vol}(Z', h_{\epsilon'_k, k}) = 0.$$

We now show that the systole of the metric  $h_{\epsilon'_k, k}$  on  $Z'$  is bounded below independently of  $k$ . Recall that the systole of  $i : Z' \rightarrow \tilde{X}/C'$  at a point  $z \in Z'$  with respect to a metric  $h_{\epsilon, k}$  can be defined by

$$(5.2) \quad \text{sys}_1(Z', h_{\epsilon, k}, z) = \inf_{\gamma \in C'} \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon, k})}(\tilde{z}, \gamma \tilde{z})$$

where  $\tilde{z}$  is any lift of  $z$  and  $\tilde{h}_{\epsilon, k}$  the lift on  $\tilde{Z}'$  of  $h_{\epsilon, k}$ , (cf. lemma 3.4).



Let  $\alpha(t)$  be a minimizing geodesic between  $\tilde{z}$  and  $\gamma\tilde{z}$  on  $(Z', \tilde{h}_{\epsilon,k})$ . By definition of  $\tilde{h}_{\epsilon,k}$  we have

$$(5.3) \quad \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq l_{\tilde{g}}(\tilde{F}^k \circ \alpha)$$

where  $l_{\tilde{g}}$  stands for the length with respect to  $\tilde{g}$  on  $\tilde{X}$ .  
We get

$$(5.4) \quad \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq \text{dist}_{(\tilde{X}, \tilde{g})}(\tilde{F}^k(\tilde{z}), \gamma\tilde{F}^k(\tilde{z})) \geq \rho$$

where  $\rho$  is the injectivity radius of  $\tilde{X}/C'$ .  
We then have

$$(5.5) \quad \text{sys}_i(Z', h_{\epsilon,k}) \geq \rho,$$

and thanks to the Theorem 3.3 we obtain

$$(5.6) \quad \text{vol}(Z', h_{\epsilon',k}) \geq C_n \rho^{n-1}$$

which contradicts (5.1).

□

## 5.2 Proof of the equality case

There will be several steps.

**Step 1:** The limit set of  $C$  is contained in a topological equator.

**Step 2:** The weak tangent to  $\partial\tilde{X}$  and  $\Lambda_{C'}$ .

**Step 3:** The limit set  $\Lambda_{C'}$  of  $C'$  and the limit set  $\Lambda_C$  of  $C$  are equal to a topological equator.

**Step 4:**  $C'$  and  $C$  are convex cocompact.

**Step 5:**  $C$  preserves a copy of the real hyperbolic space  $\mathbb{H}_{\mathbb{R}}^{n-1}$  totally geodesically embedded in  $\tilde{X}$ .

**Step 6:** Conclusion

**Step 1: The limit set  $\Lambda_C$  of  $C$  is contained in a topological equator.**

Let  $x \in \tilde{X}$  and  $E \subset T_x\tilde{X}$  a codimension one subspace. For each  $u \in T_x\tilde{X}$ ,  $\tilde{g}(u, u) = 1$ , one considers the geodesic  $c_u$  defined by  $c_u(0) = x$  and  $\dot{c}_u(0) = u$ . We define the equator  $E(\infty)$  associated to  $E$  as the subset of  $\partial\tilde{X}$

$$(5.7) \quad E(\infty) = \{c_u(+\infty)/u \in E\}$$

Our goal is to prove the existence of a point  $x \in \tilde{X}$  and an hyperplane  $E \subset T_x \tilde{X}$  such that the limit set  $\Lambda_C$  satisfies  $\Lambda_C \subset E(\infty)$ .

Recall that  $C' \subset C$  globally preserves an hypersurface  $\tilde{Z}'$  such that  $\tilde{Z}'/C' \subset \tilde{X}/C'$  is compact and essential.

Let us also recall that we have constructed a  $C$ -equivariant map  $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$  such that, for all  $x \in \tilde{X}$ ,

$$(5.8) \quad |Jac_{n-1} \tilde{F}(x)| \leq \left( \frac{\delta + 1}{n - 1} \right)^{n-1}$$

where the critical exponent  $\delta$  of  $C$  satisfies  $\delta = n - 2$ , thus

$$(5.9) \quad |Jac_{n-1} \tilde{F}(x)| \leq 1.$$

The step 1 follows from the two following Propositions.

**Proposition 5.1.** *Let  $x \in \tilde{X}$  such that  $|Jac_{n-1} \tilde{F}(x)| = 1$ . Then there exists  $E \subset T_x \tilde{X}$  such that the limit set  $\Lambda_C$  satisfies  $\Lambda_C \subset E(\infty)$ . Moreover,  $\tilde{F}(x) = x$  and  $D\tilde{F}(x)$  is the orthogonal projector onto  $E$ .*

**Proposition 5.2.** *There exists  $x \in \tilde{X}$  such that  $|Jac_{n-1} \tilde{F}(x)| = 1$ .*

### Proof of Proposition 5.1

Let  $x \in \tilde{X}$  such that  $|Jac_{n-1} \tilde{F}(x)| = 1$ . By definition we have a subspace  $E \subset T_x \tilde{X}$  such that  $|Jac_E \tilde{F}(x)| = 1$ . By (4.18) and  $\det K \geq 1$  we have,

$$(5.10) \quad \begin{aligned} |Jac_E \tilde{F}(x)| &\leq (n-1)^{n-1} \left( \frac{\text{trace} H}{n-1} \right)^{\frac{n-1}{2}} \left( \frac{\text{trace} H'}{n-1} \right)^{\frac{n-1}{2}} \\ &\leq (n-1)^{n-1} \left( \frac{1}{n-1} \right)^{n-1}. \end{aligned}$$

In particular as  $|Jac_E \tilde{F}(x)| = 1$ , we have equality in the inequalities (5.10), thus,  $\text{trace} H = \text{trace}(h) = 1$ , and

$$(5.11) \quad H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}.$$

Let us recall that the quadratic form  $h$  is defined by

$$h(v, v) = \int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta)(v)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)$$

where  $\mu_x$  is the Patterson-Sullivan measure of  $C$  normalized by

$$(5.12) \quad \int_{\partial \tilde{X}} e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 1.$$

We then have

$$\begin{aligned} 1 &= \text{trace}(h) = \text{trace}H = \sum_{i=1}^{n-1} h(v_i, v_i) = \\ &= \int_{\partial\tilde{X}} \sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \\ &\leq \int_{\partial\tilde{X}} \|DB(\tilde{F}(x), \theta)\|^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \leq 1, \end{aligned}$$

because  $\sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 \leq \|DB(\tilde{F}(x), \theta)\|^2 = 1$   
for all  $\theta \in \partial\tilde{X}$ .

Therefore for  $\mu_x$ -almost all  $\theta \in \text{supp}(e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)) = \text{supp}(\mu_x)$ , we have

$$(5.13) \quad \sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 = \|DB(\tilde{F}(x), \theta)\|^2 = 1.$$

In (5.12),  $\sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2$  represents the square of the norm of the projection of  $\nabla B(\tilde{F}(x), \theta)$  on  $E$ .

By continuity of  $B(x, \theta)$  in  $\theta$  one then gets  $\Lambda_C = \text{supp}(\mu_x) \subset E(\infty)$ .

Let us now prove that  $\tilde{F}(x) = x$ . When  $\text{Jac}\tilde{F}_E(x) = 1$ , we have equality in the Cauchy-Schwarz inequality (4.16), therefore for each  $i = 1, \dots, n-1$  and  $\theta \in \Lambda_C$  we get  $DB(\tilde{F}(x), \theta)(v_i) = DB(x, \theta)(u_i)$ . Therefore we deduces from (5.13) that  $\nabla B(x, \theta) = \sum_{i=1}^{n-1} DB(x, \theta)(u_i)u_i$ , which imply with (4.10) that

$$(5.14) \quad \int_{\partial\tilde{X}} DB(x, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0.$$

On the other hand, as  $\int_{\partial\tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0$  and  $H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}$ , the support of  $\mu_x$  cannot be just a pair of points, therefore the barycenter of the measure  $e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$  defined in [2] is well defined and characterized as the point  $z \in \tilde{X}$  such that

$$\int_{\partial\tilde{X}} DB(z, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0,$$

thus (5.14) and (4.10) imply  $x = \tilde{F}(x)$ .  $\square$

**Proof of Proposition 5.2 :**

If we knew that there exists a minimizing hypersurface  $Z_0$  in the homology class of  $Z$ , then every points  $x \in Z_0$  would verify  $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$ . We unfortunately don't know if there exists such a minimizing hypersurface nor a minimizing current in the homology class of  $Z$ . Instead we will consider an  $L^2(\tilde{X}/C')$  harmonic  $(n-1)$ -form dual to the homology class of  $Z$ .

We need the following lemmas in order to prove the existence of such a dual form.

Let  $\lambda_1(\tilde{X}/C')$  be the bottom of the spectrum of the Laplacian on  $(\tilde{X}, \tilde{g})$ , ie.

$$(5.15) \quad \lambda_1(\tilde{X}/C') = \inf_{u \in C_0^\infty(\tilde{X}/C')} \left\{ \frac{\int_{\tilde{X}/C'} |du|^2}{\int_{\tilde{X}/C'} u^2} \right\}.$$

**Lemma 5.3.** *Let  $C \subset \text{Isom}(\tilde{X}, \tilde{g})$  a discrete group of isometries with critical exponent  $\delta = n - 2$  where  $(\tilde{X}, \tilde{g})$  is an  $n$ -dimensional Cartan-Hadamard manifold of sectional curvature  $K_{\tilde{g}} \leq -1$ . Then for any subgroup  $C' \subset C$  we have  $\lambda_1(\tilde{X}/C') \geq n - 2$ .*

**Proof :**

Thanks to a theorem of Barta, cf.[17] Theorem 2.1, the lemma boils down to finding a positive function  $c : \tilde{X}/C' \rightarrow \mathbb{R}_+$  such that  $\Delta c(x) \geq (n-2)c(x)$ . Here, the laplacian  $\Delta$  is the positive operator ie.  $\Delta c = -\text{trace} Ddc$ . We consider the smooth function  $\tilde{c} : \tilde{X} \rightarrow \mathbb{R}_+$  defined by  $\tilde{c}(x) = \mu_x(\partial\tilde{X})$  where  $\{\mu_x\}_{x \in \tilde{X}}$  is a family of Patterson-Sullivan measure of  $C$ . The function  $\tilde{c}$  is  $C$ -equivariant therefore it defines a map  $c : \tilde{X}/C' \rightarrow \mathbb{R}_+$  for any subgroup  $C' \subset C$ . Let us show

$$(5.16) \quad \Delta c(x) \geq \delta(n-1-\delta)c(x) = n-2.$$

We have

$$\tilde{c}(x) = \int_{\partial\tilde{X}} e^{-\delta B(x,\theta)} d\mu_o(\theta)$$

therefore

$$\Delta \tilde{c}(x) = \int_{\partial\tilde{X}} [-\delta \Delta B(x,\theta) - \delta^2] d\mu_x(\theta).$$

The sectional curvature  $K_{\tilde{g}}$  of  $(\tilde{X}, \tilde{g})$  satisfies  $K_{\tilde{g}} \leq -1$  we thus have  $-\Delta B(x,\theta) \geq n-1$  and as  $\delta = n-2$  we get

$$\Delta \tilde{c}(x) \geq [\delta(n-1) - \delta^2] \tilde{c}(x) = (n-2)\tilde{c}(x).$$

□

The following lemma is due to G.Carron and E.Pedon, [8]. For a complete riemannian manifold  $Y$ , we denote  $H_c^1(Y, \mathbb{R})$  the first cohomology group generated by differential forms with compact support.

**Lemma 5.4** ([8], Lemme 5.1). *Let  $Y$  be a complete riemannian manifold all ends of whose having infinite volume and such that  $\lambda_1(Y) > 0$ , then the natural morphism*

$$H_c^1(Y, \mathbb{R}) \rightarrow H_{L^2}^1(Y, \mathbb{R})$$

*is injective. In particular any  $\alpha \in H_c^1(Y, \mathbb{R})$  admits a representative  $\bar{\alpha}$  which is in  $L^2(Y, \mathbb{R})$ .*

**Corollary 5.5.** *Let  $C'$  be as above and assume that there exists a compact essential hypersurface  $Z' \subset \tilde{X}/C'$ . Then there exists an harmonic  $n-1$ -form  $\omega$  in  $L^2(\tilde{X}/C')$  such that  $\int_{Z'} \omega \neq 0$ .*

**Proof :**

Let  $\alpha \in H_c^1(\tilde{X}/C', \mathbb{R})$  a Poincaré dual of  $[Z'] \in H_{n-1}(\tilde{X}/C', \mathbb{R})$ . By definition of  $\alpha$ , for any  $\beta \in H^{n-1}(\tilde{X}/C', \mathbb{R})$ , one has

$$(5.17) \quad \int_{Z'} \beta = \int_{\tilde{X}/C'} \beta \wedge \alpha,$$

([5] p.51, note that  $\tilde{X}/C'$  has a "finite good cover").

After Lemma 5.4,  $\alpha$  admits a non trivial harmonic representative  $\bar{\alpha}$  in  $L^2(\tilde{X}/C')$ . (In order to apply the Lemma 5.4, one has to check that all ends of  $\tilde{X}/C'$  have infinite volume, ie for a compact  $K \subset \tilde{X}/C'$  each unbounded connected component of  $\tilde{X}/C' - K$  has infinite volume: this comes from the fact that the injectivity radius of  $\tilde{X}/C'$  is bounded below by the injectivity radius of  $X = \tilde{X}/\Gamma$  and the sectional curvature bounded above by  $-1$ .) The  $(n-1)$ -harmonic form  $\omega = *\bar{\alpha}$ , where  $*$  is the Hodge operator, is in  $L^2(\tilde{X}/C')$  and verifies after (5.17)

$$(5.18) \quad \int_{Z'} \omega = \int_{\tilde{X}/C'} \omega \wedge \bar{\alpha} = \int_{\tilde{X}/C'} \omega \wedge *\omega = \|\omega\|_{L^2(\tilde{X}/C')}^2 \neq 0.$$

□

We can now prove the proposition 5.2. Let us briefly describe the idea. We consider the iterates  $F'^k$  of  $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$ . As  $F'$  is homotopic to the identity map,  $F'^k(Z')$  is homologous to  $Z'$  and if  $\omega$  is the harmonic form of the corollary 5.5 we have

$$(5.19) \quad \int_{Z'} (F'^k)^*(\omega) = \int_{Z'} \omega = a \neq 0.$$

We don't know if  $F'^k(Z')$  converges or stays in a compact subset of  $\tilde{X}/C'$  but we will show that  $F'^k(Z')$  cannot entirely diverge in  $\tilde{X}/C'$  and that there exists a  $z' \in Z'$  such that  $F'^k(z')$  subconverges to a point  $x \in \tilde{X}/C'$  with  $|Jac_{n-1}F'(x)| = 1$ .

From (5.19) one gets

$$(5.20) \quad 0 < |a| = \left| \int_{Z'} (F'^k)^*(\omega) \right| \leq \int_{Z'} |Jac_{T_{Z'}F'^k}(z)| \cdot \|\omega(F'^k(z))\| dz$$

where

$$|Jac_{T_{Z'}F'^k}(z)| = \|DF'^k(z)(u_1) \wedge DF'^k(z)(u_2) \wedge \dots \wedge DF'^k(z)(u_{n-1})\|$$

and  $(u_1, \dots, u_{n-1})$  is an orthonormal basis of  $T_z(Z')$ .

Let us define  $\mathcal{B} = \{z \in Z', |JacF_{T_{Z'}}^k(z)| \text{ does not converge to } 0\}$ .

For  $z \in Z'$  we define the sequence  $z_k$  by  $z_0 = z$  and  $z_k = F'(z_{k-1}) = F'^k(z) \in \tilde{X}/C'$ .

**Lemma 5.6.** *There exists  $z \in \mathcal{B}$  and a subsequence  $z_{k_j}$  such that  $z_{k_j}$  converges to a point  $x \in \tilde{X}/C'$  with  $|Jac_{n-1}(x)| = 1$ .*

**Proof :**

We first remark that  $\lim_{x \rightarrow \infty} \|\omega(x)\| = 0$ . This follows the following facts:  $\omega$  is harmonic,  $\omega \in L^2(\tilde{X}/C')$  and the injectivity radius of  $\tilde{X}/C'$  is bounded below by a positive constant.

Let us assume that for all  $z \in \mathcal{B}$  the sequence  $z_k$  diverges in  $\tilde{X}/C'$ . Then we have, for all  $z \in \mathcal{B}$ ,

$$(5.21) \quad \|\omega(z_k)\| = \|\omega(F'^k(z))\| \rightarrow 0$$

whenever  $k$  tends to  $\infty$  because of the previous remark.

On the other hand, as  $\|\omega(F'^k(z))\| \leq C$  and  $|Jac_{n-1}F'^k| \leq 1$ , it follows from (5.20)

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \int_{Z'} (F'^k)^*(\omega) \right| \leq \\ & \lim_{k \rightarrow \infty} \left[ \int_{\mathcal{B}} \|\omega(F'^k(z))\| dz + C \int_{Z' - \mathcal{B}} |Jac_{Z'} F'^k(z)| dz \right] = 0 \end{aligned}$$

which contradicts our assumption.

Thus there exists a point  $z \in Z'$  such that

$$(5.22) \quad |Jac_{Z'} F'^k(z)| \rightarrow \alpha \neq 0$$

and such that there exists a subsequence  $z_{k_j} = F'^{k_j}(z)$  with

$$(5.23) \quad \lim_{j \rightarrow \infty} z_{k_j} = x \in \tilde{X}/C'.$$

The property (5.22) comes from the fact that the sequence  $|Jac_{Z'} F'^k(z)|$  doesn't tend to zero and is decreasing (because  $|Jac_{n-1}F'| \leq 1$ ).

Let us define

$$E_0 = T_z Z', E_1 = DF'(z)(E_0)$$

and

$$E_k = DF'(z_{k-1})(E_{k-1}) \subset T_{z_k}(\tilde{X}/C').$$

As  $z_{k_j} \rightarrow x$  we can assume, after extracting again a subsequence, that  $E_{k_j} \rightarrow E \subset T_x(\tilde{X}/C')$ . On the other hand we also have

$$(5.24) \quad |Jac_{Z'} F'^k(z)| = |Jac_{E_{k-1}} F'(z_{k-1})| |Jac_{E_{k-2}} F'(z_{k-2})| \dots |Jac_{E_0} F'(z)|$$

We know that  $|Jac_{E_k} F'(z_k)| = 1 - \epsilon_k$  where  $0 \leq \epsilon_k < 1$ . As  $z \in \mathcal{B}$ , we have

$$\lim_{k \rightarrow \infty} \prod_{j=1}^k (1 - \epsilon_j) = \alpha > 0$$

therefore  $\lim_{k \rightarrow \infty} \epsilon_k = 0$  and by continuity we have  $|Jac_E F'(x)| = 1$ .  $\square$

Now we can finish the proof of the step 1. Let consider a lift of  $x$  in  $\tilde{X}$  and  $E$  in  $T\tilde{X}$  that we again call  $x$  and  $E$ . Then we have  $|Jac_E \tilde{F}'(x)| = 1$ .  $\square$

Let us remark that corollary 4.4 and (5.19) give another proof of the inequality  $\delta \geq n - 2$ , which does not use the isosystolic inequality, ie. Theorem 3.3.

### Step 2 : The weak tangent of $\partial\tilde{X}$ and $\Lambda_C$

We first recall the definition of the Gromov-distance on  $\partial\tilde{X}$ . For two arbitrary points  $\theta$  and  $\theta'$  in  $\partial\tilde{X}$  let us define

$$(5.25) \quad l(\theta, \theta') = \inf\{t > 0 / \text{dist}(\alpha_\theta(t), \alpha_{\theta'}(t)) = 1\}$$

and

$$(5.26) \quad d(\theta, \theta') = e^{-l(\theta, \theta')}$$

then  $d$  is a distance on  $\partial\tilde{X}$ .

We now recall a few definitions following [4]. A complete metric space  $(S, \bar{d})$  is a weak tangent of a metric space  $(Z, d)$  if there exist a point  $0 \in S$ , a sequence of points  $z_k \in Z$  and a sequence of positive real numbers  $\lambda_k \rightarrow \infty$  such that the sequence of pointed metric spaces  $(Z, \lambda_k d, z_k)$  converges in the pointed Gromov-Hausdorff topology to  $(S, \bar{d}, 0)$  where  $(Z, \lambda_k d)$  stands for the set  $Z$  endowed with the rescaled metric  $\lambda_k d$ .

Let us recall that the sequence of metric spaces  $(Z_k, d_k, z_k)$  converges to  $(S, \bar{d}, 0)$  in the pointed Gromov-Hausdorff topology if the following conditions hold, (cf. (B-B-I), definition 8.1.1).

**Definition 5.7.** *We say that the sequence of metric spaces  $(Z_k, d_k, z_k)$  converges to  $(S, \bar{d}, o)$  if for any  $R > 0$ ,  $\epsilon > 0$  there exists  $k_0$  such that for any  $k \geq k_0$  there exists a (non necessary continuous) map  $f : B(z_k, R) \rightarrow S$  such that*

(i)  $f(z_k) = 0$ ,

for any two points  $x$  and  $y$  in  $B(z_k, R)$ ,

(ii)  $|\bar{d}(f(x), f(y)) - d_k(x, y)| \leq \epsilon$ ,

and

(iii) the  $\epsilon$ -neighborhood of the set  $f(B(z_k, R))$  contains  $B(0, R - \epsilon)$ .

In the previous definition,  $B(z_k, R)$  stands for the ball of radius  $R$  centered at the point  $z_k$  in  $(Z_k, d_k)$ .

For a metric space  $(Z, d)$  we will denote  $WT(Z, d)$  the set of weak tangents of  $(Z, d)$ .

Let  $\Gamma$  a cocompact group of isometries of  $(\tilde{X}, \tilde{g})$  a  $n$ -dimensional Cartan Hadamard manifold of sectional curvature  $K_{\tilde{g}} \leq -1$  and  $C'$  a subgroup of  $\Gamma$ . The limit set  $\Lambda_\Gamma$  of  $\Gamma$  is the full boundary  $\partial\tilde{X}$  namely a topological  $(n-1)$ -dimensional sphere  $S^{n-1}$ . We endow  $\partial\tilde{X}$  with the Gromov distance  $d$  defined in (5.26). In [4] Lemma 5.2, M.Bonk and B.Kleiner show among other properties the following

**Lemma 5.8.** *For any weak tangent space  $(S, \bar{d})$  in  $WT((\partial\tilde{X}, d))$ ,  $S$  is homeomorphic to  $\partial\tilde{X}$  less a point, thus to  $\mathbb{R}^{n-1}$ .*

In fact the crucial assumption in the above lemma, coming from the cocompactness of  $\Gamma$ , is the property that any triple of points in  $\partial\tilde{X}$  can be uniformly separated by an element of  $\Gamma$ , ie. there is  $\delta > 0$  such that for any three points  $\theta_1, \theta_2, \theta_3 \in \partial\tilde{X}$  there exists a  $\gamma \in \Gamma$  such that  $d(\gamma\theta_i, \gamma\theta_j) \geq \delta$  for all  $1 \leq i \neq j \leq 3$ . Following the argument of M.Bonk and B.Kleiner one can show that if  $C'$  is a subgroup of  $\Gamma$  such that one weak tangent  $(S, \bar{d})$  of  $(\Lambda_{C'}, d)$  is in  $WT(\partial\tilde{X}, d)$  and enough triples of points of  $\Lambda_{C'}$  can be uniformly separated by elements of  $C'$ , then  $S$  is homeomorphic to  $\Lambda_{C'}$  less a point. In particular,  $\Lambda_{C'}$  is homeomorphic to  $\partial\tilde{X}$ .

**Lemma 5.9.** *Let  $\mathcal{L} \subset \partial\tilde{X}$  be a closed  $C'$ -invariant set and  $\theta_0 \in \mathcal{L}$ . We assume that there exist a sequence of positive real numbers  $\lambda_k \rightarrow \infty$  such that the sequence of pointed metric spaces  $(\mathcal{L}, \lambda_k d, \theta_0)$  converges in the pointed Gromov-Hausdorff topology to  $(S, \bar{d}, 0)$  where  $(S, \bar{d}, 0)$  is a weak tangent of  $(\partial\tilde{X}, d)$ . We also assume that there exist positive constants  $C$  and  $\delta$ , a sequence of points  $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$  and a sequence of elements  $\gamma_k \in C'$  such that  $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$  and  $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$  for all  $0 \leq i \neq j \leq 2$ . Then,  $S$  is homeomorphic to  $\mathcal{L}$  less a point. In particular  $\mathcal{L}$  is homeomorphic to  $\partial\tilde{X}$ .*

The proof of this lemma is postponed in the Appendix.

**Step 3 : The limit set  $\Lambda_{C'}$  of  $C'$  and the limit set  $\Lambda_C$  of  $C$  are equal to a topological equator.**

We have shown in step 1 that  $\Lambda_C$  is a subset of some topological equator  $E(\infty)$ .

Let  $o \in \tilde{X}$  and  $E \subset T_o\tilde{X}$  be such that  $|Jac_E\tilde{F}(o)| = 1$  and  $E(\infty)$  is the equator associated to  $E$ .

Recall that there exists a subgroup  $C'$  of  $C$  which globally preserves an hypersurface  $\tilde{Z}' \subset \tilde{X}$  and that  $\tilde{Z}'/C' \subset \tilde{X}/C'$  is compact. Furthermore  $\tilde{Z}'$  separates  $\tilde{X}$  into two connected components  $\tilde{U}$  et  $\tilde{V}$ . We can assume that  $\tilde{U}$  and  $\tilde{V}$  are globally invariant by  $C'$  after having replaced  $C'$  by an index 2 subgroup.

The limit set  $\Lambda_{C'}$  of  $C'$  is contained in  $\Lambda_C$ , therefore  $\Lambda_{C'} \subset E(\infty)$ .



We will show that  $\Lambda_{C'} = E(\infty)$ .

For any subset  $W \in \tilde{X}$  we define the boundary at infinity  $\partial W$  of  $W$  by

$$(5.27) \quad \partial W = Cl(W) \cap \partial \tilde{X}$$

where  $Cl(W)$  stands for the closure of  $W$  in  $\tilde{X} \cup \partial \tilde{X}$ .

As  $\tilde{Z}'/C'$  is compact,  $\tilde{Z}'$  is at bounded distance of the orbit  $C'z$  of some point  $z$  in  $\tilde{Z}'$ , thus

$$(5.28) \quad Cl(\tilde{Z}') \cap \partial \tilde{X} = \Lambda_{C'}$$

By definition we have  $\Lambda_{C'} \subset \partial \tilde{U}$  and  $\Lambda_{C'} \subset \partial \tilde{V}$ .

**Lemma 5.10.** *Let us assume that  $\Lambda_{C'} \neq E(\infty)$ , then either  $\partial \tilde{U} = \Lambda_{C'}$  or  $\partial \tilde{V} = \Lambda_{C'}$ .*

**Proof:** We know that  $\Lambda_{C'} \subset \partial \tilde{U}$  and  $\Lambda_{C'} \subset \partial \tilde{V}$ .

Let us assume that the conclusion of the lemma is not true, so there is  $\zeta \in \partial \tilde{U} - \Lambda_{C'}$  and  $\theta \in \partial \tilde{V} - \Lambda_{C'}$ .

As  $\Lambda_{C'} \subset E(\infty)$ ,  $\Lambda_{C'} \neq E(\infty)$  and  $\partial \tilde{X}$  is a sphere, any two points of  $\partial \tilde{X} - \Lambda_{C'}$  can be joined by a continuous path contained in  $\partial \tilde{X} - \Lambda_{C'}$  and so does  $\zeta$  and  $\theta$ , joined by such a path  $\alpha$ .

The set  $Z' \cup \Lambda_{C'}$  is a closed subset of  $\tilde{X} \cup \partial \tilde{X}$  thus there is an open connected neighborhood  $W$  of  $\alpha$  in  $\tilde{X} \cup \partial \tilde{X}$  contained in the complementary of  $Z' \cup \Lambda_{C'}$ .

As  $\zeta$  and  $\theta$  can be approximated by points in  $\tilde{U}$  and  $\tilde{V}$  respectively there exist points  $x \in \tilde{U} \cap W$  and  $y \in \tilde{V} \cap W$  that can be joined by a continuous path by connectedness of  $W$ , which leads to a contradiction.  $\square$

**Remark 5.11.** *In fact, we are going to show that under the assumption  $\delta(C') = n - 2$ , it is impossible to have  $\partial \tilde{U} = \Lambda_{C'}$  or  $\partial \tilde{V} = \Lambda_{C'}$ .*

For any  $x \in \tilde{X}$  and  $\theta \in \partial \tilde{X}$  let us denote  $HB(x, \theta)$  the open horoball centered at  $\theta$  and passing through  $x$ .

**Lemma 5.12.** *Let us assume that  $\partial \tilde{U} = \Lambda_{C'}$ . Then there exist  $\theta_0 \in \Lambda_{C'}$  and  $z' \in \tilde{Z}'$  such that  $HB(z', \theta_0) \subset \tilde{U}$ .*

**Proof:** Let us recall that  $\tilde{X}/C' - Z' = U \cup V$  where  $U = \pi(\tilde{U})$   $V = \pi(\tilde{V})$  and  $\pi : \tilde{X} \rightarrow \tilde{X}/C'$  is the projection .

We know that  $U$  and  $V$  are unbounded. Let  $x_n$  a sequence of points in  $U$  such that  $dist(x_n, Z') \rightarrow \infty$ . Let  $z_n \in Z'$  such that  $dist(x_n, Z') = dist(x_n, z_n)$ . We consider a fundamental domain  $D \subset \tilde{Z}'$  of  $C'$ . There exist lifts  $\tilde{z}_n \in D$  and  $\tilde{x}_n \in \tilde{U}$  such that  $dist(\tilde{x}_n, Z') = dist(\tilde{x}_n, \tilde{z}_n)$  tends to infinity.

By compactness we can assume that a subsequence  $\tilde{x}_{n_j}$  converges to a point  $\theta_0 \in \partial \tilde{X}$  and  $\tilde{z}_{n_j}$  also converges to a point  $\tilde{z} \in \tilde{D}$ . Furthermore

the sequence of open balls  $B(\tilde{x}_{n_j}, \text{dist}(\tilde{x}_{n_j}, \tilde{z}_{n_j})) \subset \tilde{U}$  converges to the open horoball  $HB(\theta_0, \tilde{z}) \in \tilde{U}$ .  $\square$

**Proposition 5.13.**  $\Lambda_{C'} = E(\infty)$  and  $\Lambda_C = E(\infty)$ .

We first describe the idea of the proof and next state some facts we will need in order to do it.

As  $\Lambda_{C'} \subset \Lambda_C \subset E(\infty)$  the proposition boils down to proving that  $\Lambda_{C'} = E(\infty)$ . Let us assume  $\Lambda_{C'} \neq E(\infty)$  and find a contradiction.

We will show that for any sequence  $\theta_i$  converging to  $\theta_0$  the geodesic starting at a point  $o$  such that  $|Jac_{n-1}\tilde{F}(o)| = 1$  and ending at  $\theta_i$  crosses the hypersurface  $\tilde{Z}'$  in a point  $z_i$ . For an **appropriate choice** of such a sequence  $\theta_i$  (roughly speaking, the sequence  $\theta_i$  is chosen to be converging to  $\theta_0$  "transversally to  $\Lambda_{C'}$ "), the shadow (defined below) projected from  $o$  through some geodesic ball  $B(z_i, r)$  will not intersect  $\Lambda_{C'}$ . On the other hand this shadow has to meet the limit set  $\Lambda_{C'}$  because of the shadow lemma of D. Sullivan, which leads to a contradiction.

Precisely, by Lemma (5.10) and Lemma (5.12) we know that  $\partial\tilde{U} = \Lambda_{C'}$  and that there exists an open horoball  $HB(\theta_0, \tilde{z}) \subset \tilde{U}$  centered at a point  $\theta \in \Lambda_{C'}$ , and whose closure contains a point  $\tilde{z} \in \tilde{Z}'$ .

Let  $o \in \tilde{X}$  and  $E \in T_o\tilde{X}$  an hyperplane such that  $\tilde{F}(o) = o$ ,  $|Jac_E\tilde{F}(o)| = 1$ , and  $\Lambda_{C'} \subset E(\infty)$  where  $E(\infty)$  is the topological equator associated to  $E$ .

For each  $\theta \in \partial\tilde{X}$  we denote by  $\alpha_\theta$  the geodesic starting from  $o$  and such that  $\alpha_\theta(+\infty) = \theta$ .

Let  $\theta_i \in \partial\tilde{X} - E(\infty) = \partial\tilde{V} - \partial\tilde{U}$  be a sequence converging to  $\theta_0$ . By continuity, for each  $i$  large enough, the geodesic  $\alpha_{\theta_i}$  spends some time inside the horoball  $HB(\theta_0, \tilde{z}) \subset \tilde{U}$  and ends up inside  $\tilde{V}$  because  $\theta_i$  converges to  $\theta_0$  and  $\theta_i$  belongs to  $\partial\tilde{V} - \partial\tilde{U}$ .

Thus  $\alpha_{\theta_i}$  eventually crosses  $\tilde{Z}'$ . Let  $z_i \in \alpha_{\theta_i} \cap \tilde{Z}'$ . As  $\tilde{Z}'/C'$  is compact, there is an element  $\gamma_i \in C'$  such that  $z_i = \gamma_i(x_i)$  where  $x_i$  is a point in the closure  $\bar{D}$  of a fundamental domain  $D$  for the action of  $C'$  on  $\tilde{Z}'$ . The points  $\gamma_i(x_i)$  and  $\gamma_i(o)$  stay at bounded distance because  $\text{dist}(\gamma_i(x_i), \gamma_i(o)) = \text{dist}(x_i, o) \leq \text{dist}(o, D) + \text{diam}D$ . In particular,  $\lim_{i \rightarrow \infty} \gamma_i(o) = \theta_0$ .

We have proved the

**Lemma 5.14.** *Let  $\theta_i \in \partial\tilde{X} - E(\infty)$  be a sequence which converges to  $\theta_0$ . There exists a constant  $A$  such that for  $i$  large enough there exists  $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$  and  $\gamma_i \in C'$  such that  $\text{dist}(z_i, \gamma_i(o)) \leq A$  and both  $z_i$  and  $\gamma_i(o)$  converge to  $\theta_0$ .*

Let  $x$  and  $y$  two points in  $\tilde{X}$ .

We define the shadow  $\mathcal{O}(x, y, R) \subset \partial\tilde{X}$  of the ball  $B(y, R)$  enlightened from the point  $x$  by

$$(5.29) \quad \mathcal{O}(x, y, R) = \{\alpha(+\infty)\}$$

where  $\alpha$  runs through the set of geodesic rays starting from  $x$  and meeting  $B(y, R)$ .

Let  $\{\mu_x\}_x$  be a family of Patterson measures associated to the discrete group  $C'$  with critical exponent  $\delta' = \delta(C')$ .

The following shadow lemma is due to D.Sullivan.

**Lemma 5.15.** [18], [13], [20]. *There exist positive constants  $C$  and  $R$  such that for any  $y$  in  $\tilde{X}$ ,  $\nu_y(\mathcal{O}(y, \gamma(y), R)) \geq Ce^{\delta'd(y, \gamma(y))}$*

**Corollary 5.16.** *Let  $z_i$  be defined in lemma (5.14), then we have  $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$  for  $i$  large enough.*

We now prove that for a good choice of  $\theta_i$ , the shadow  $\mathcal{O}(o, z_i, R+A)$  (with  $z_i$  associated to  $\theta_i$  as in lemma 5.14) never meet  $\Lambda_{C'}$  for all large  $i$ 's, ie. for any  $\theta \in \Lambda_{C'}$  the geodesic  $\alpha_\theta$  does not cross  $B(z_i, R+A)$ . We have no control on the radius  $R$  coming from the shadow lemma nor on the constant  $A$  but we will show

**Proposition 5.17.** *There exists a sequence  $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$  such that  $\theta_i$  converges to  $\theta_0$  and*

$$\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = +\infty$$

where  $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$  has been constructed in lemma (5.14).

**Corollary 5.18.** *For  $i$  large enough,  $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$ .*

The corollary (5.16) and the corollary (5.18) lead to a contradiction, which ends the proof of the proposition (5.13).

The end of the paragraph is devoted to proving the proposition (5.17).

**Lemma 5.19.** *Let  $\theta_i$  be a sequence of points in  $\partial\tilde{X}$  converging to  $\theta_0$  and  $z_i$  constructed in lemma (5.14). Assume that*

$$\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = C < +\infty \text{ then } \lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0.$$

**Proof :** We first show that

$$(5.30) \quad \lim_{i \rightarrow \infty} \text{dist}(z_i, \alpha_{\theta_0}) = \infty$$

Recall that, for any  $z \in \tilde{X}$  and  $\theta \in \partial\tilde{X}$ ,  $B(z, \theta)$  equals the decreasing limit as  $t$  tends to infinity of  $\text{dist}(z, \alpha_\theta(t)) - \text{dist}(o, \alpha_\theta(t))$  where  $\alpha_\theta(t)$  is the geodesic ray joinging  $o$  to  $\theta$ . Therefore, as the points  $z_i \in \tilde{Z}$  belongs to the complementary of the fixed horoball  $HB(\tilde{z}, \theta_0)$ , we have,

$$(5.31) \quad \text{dist}(z_i, \alpha_{\theta_0}(T_i)) \geq T_i + B(\tilde{z}, \theta_0)$$

where  $\text{dist}(z_i, \alpha_{\theta_0}(T_i)) = \text{dist}(z_i, \alpha_{\theta_0})$ .

On the other hand, as  $z_i$  tends to  $\theta_0$ ,  $T_i$  tends to infinity so (5.30) is proven.

Let  $t_i$  be such that  $z_i = \alpha_{\theta_i}(t_i)$ . By (5.30) we have

$$(5.32) \quad \lim_{i \rightarrow \infty} \text{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) = \infty.$$

Let  $u_i$  be such that

$$(5.33) \quad \text{dist}(\alpha_{\theta_i}(u_i), \alpha_{\theta_0}(u_i)) = 1,$$

then in particular  $u_i \leq t_i$  for  $i$  large enough and by the triangle inequality we have

$$(5.34) \quad \text{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) \leq 2(t_i - u_i) + 1.$$

By (5.32) we get

$$(5.35) \quad \lim_{i \rightarrow \infty} (t_i - u_i) = +\infty.$$

Let us assume there exists a sequence  $\theta'_i \in \Lambda_{C'}$  and a constant  $C$  such that

$$(5.36) \quad \text{dist}(z_i, \alpha_{\theta'_i}) \leq C < +\infty.$$

We can assume that  $C \geq 1$ .

Let  $v_i$  be such that

$$(5.37) \quad \text{dist}(z_i, \alpha_{\theta'_i}) = \text{dist}(z_i, \alpha_{\theta'_i}(v_i)).$$

By triangle inequality,

$$(5.38) \quad |t_i - v_i| \leq C$$

and,

$$(5.39) \quad \text{dist}(\alpha_{\theta'_i}(t_i), \alpha_{\theta_i}(t_i)) \leq 2C.$$

On the other hand, as the curvature of  $\tilde{X}$  is bounded above by  $-1$ , a classical comparison theorem gives for any  $t \in [0, t_i]$ ,

$$(5.40) \quad \sinh\left(\frac{\text{dist}(\alpha_{\theta'_i}(t), \alpha_{\theta_i}(t))}{2}\right) \leq \sinh C \cdot \frac{\sinh t}{\sinh t_i}.$$

Let  $s_i$  be such that

$$(5.41) \quad \text{dist}(\alpha_{\theta'_i}(s_i), \alpha_{\theta_i}(s_i)) = 1.$$

There are two cases. Either  $s_i \geq t_i$  or  $s_i < t_i$ . If  $s_i < t_i$ , we get from (5.39) and (5.40) the existence of a constant  $A$  such that for any  $i$ ,

$$(5.42) \quad s_i \geq t_i - A,$$

and this inequality also holds when  $s_i \geq t_i$ .

From (5.42) we get

$$(5.43) \quad \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = e^{-s_i+u_i} \leq e^A e^{-t_i+u_i},$$

therefore, thanks to (5.35) we obtain

$$(5.44) \quad \lim_{i \rightarrow \infty} \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = 0.$$

which ends the proof of lemma (5.19).  $\square$

**Lemma 5.20.** *Let us assume that for every sequence  $\theta_i$  of points in  $\partial\tilde{X}$  converging to  $\theta_0$ ,  $\lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$ . Let  $\lambda_k \rightarrow \infty$  be such that the sequence of spaces  $(\partial\tilde{X}, \lambda_k d, \theta_0)$  converges to a space  $(S, \bar{d}, 0)$  in the pointed Gromov-Hausdorff topology, then the sequence of spaces  $(\Lambda_{C'}, \lambda_k d, \theta_0)$  also converges to  $(S, \bar{d}, 0)$ .*

**Proof :** Let us define

$$r(\epsilon) =: \sup\left\{\frac{d(\theta, \Lambda_{C'})}{d(\theta, \theta_0)}, \theta \neq \theta_0, d(\theta, \theta_0) \leq \epsilon\right\}.$$

The assumption says that

$$(5.45) \quad \lim_{\epsilon \rightarrow 0} r(\epsilon) = 0.$$

For an arbitrary metric space  $(Y, d)$  and  $Y'$  a subset of  $Y$ , let us denote  $B_{(Y,d)}(y, R)$  the closed ball of  $(Y, d)$  of radius  $R$  centered at  $y \in Y$ , and  $\mathcal{U}_\epsilon^{(Y,d)}(Y')$  the  $\epsilon$ -neighborhood of  $Y'$  in  $(Y, d)$ . For a metric space  $(Y, d)$  and a positive number  $\lambda$ , let us denote  $\lambda Y$  the rescaled space  $(Y, \lambda d)$ .

By definition of the function  $r$ , we have for any  $R$ ,

$$(5.46) \quad \begin{aligned} B_{\lambda_k \partial\tilde{X}}(\theta_0, R) &\subset \mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k) \\ &\subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R + 2\epsilon_k). \end{aligned}$$

where  $\epsilon_k =: Rr(R/\lambda_k)$ .

Let us fix  $\alpha > 0$ . By definition 5.7, for any  $R > 0$ ,  $\epsilon > 0$ , there exist a map  $f : B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha) \rightarrow S$  such that for  $k \geq k_0$ ,

(i)  $f(\theta_0) = 0$ ,

for any two points  $x$  and  $y$  in  $B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha)$ ,

- (ii)  $|\bar{d}(f(x), f(y)) - \lambda_k d(x, y)| \leq \epsilon$ ,  
and  
(iii)  $B_{(S, \bar{d})}(0, R + \alpha - \epsilon) \subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha))$ .

Moreover let us prove:

- (iv)  $B_{(S, \bar{d})}(0, R - 2\epsilon) \subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R))$ .

Indeed, let  $z \in B_{(S, \bar{d})}(0, R - 2\epsilon)$ . By (iii), there exists  $\theta \in B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha)$  such that  $\bar{d}(z, f(\theta)) \leq \epsilon$ . Since  $\bar{d}(z, 0) \leq R - 2\epsilon$ , we thus deduce from triangle inequality  $\bar{d}(f(\theta), 0) \leq R - \epsilon$ , and therefore we get, thanks to (i) and (ii),  $\lambda_k d(\theta, \theta_0) \leq R$ .  $\square$

By (5.45), for  $\epsilon$  small enough, there exists  $k_1 \geq k_0$  such that for any  $k \geq k_1$ , then  $2\epsilon_k \leq \epsilon$  and

$$B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha).$$

Therefore, by (5.46) and the above properties (i),(ii),(iii) and (iv) of the map  $f$ , and the triangle inequality we get,

$$\begin{aligned} B_{(S, \bar{d})}(0, R - 2\epsilon) &\subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R)) \\ &\subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(\mathcal{U}_{\epsilon_k}^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)) \\ &\subset \mathcal{U}_{2\epsilon + \epsilon_k}^{(S, \bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)). \end{aligned}$$

About the second inclusion above let us remark that the set  $\mathcal{U}_{\epsilon_k}^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)$  is contained in  $B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda \partial \tilde{X}}(\theta_0, R + \alpha)$ , so that we can apply  $f$  to this set.

From the above inclusions we obtain

$$B_{(S, \bar{d})}(0, R - 3\epsilon) \subset \mathcal{U}_{2\epsilon + \epsilon_k}^{(S, \bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R))$$

which implies the convergence of  $(\Lambda_{C'}, \lambda_k d, \theta_0)$  to  $(S, \bar{d}, 0)$ .

$\square$

**Corollary 5.21.** *Let us assume that for every sequence  $\theta_i$  of points in  $\partial \tilde{X}$  converging to  $\theta_0$  and  $z_i$  the sequence of points constructed in lemma (5.14),  $\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha \theta) < +\infty$ . Let  $\lambda_k \rightarrow \infty$  be such that the sequence of spaces  $(\partial \tilde{X}, \lambda_k d, \theta_0)$  converges to the space  $(S, \delta, 0)$  in the pointed Gromov-Hausdorff topology, then the sequence of spaces  $(\Lambda_{C'}, \lambda_k d, \theta_0)$  also converges to  $(S, \delta, 0)$ .*

We will show now that there exist a sequence of points  $\theta_1^k, \theta_2^k \in \Lambda_{C'}$  converging to  $\theta_0$ , such that the mutual distances  $d(\theta_1^k, \theta_2^k)$ ,  $d(\theta_1^k, \theta_0)$ ,  $d(\theta_2^k, \theta_0)$  is tending to zero at the same rate, and the triple  $\theta_1^k, \theta_2^k, \theta_0$  can be uniformly separated by elements  $\gamma_k \in C'$ .

**Lemma 5.22.** *Assume that every weak tangent of  $(\partial\tilde{X}, d)$  at  $\theta_0$  belongs to  $WT(\Lambda_{C'}, d)$ , then there exist positive constants  $c, \delta$ , a sequence  $\epsilon_k$  tending to 0 when  $k$  tends to  $\infty$ , a sequence  $\gamma_k \in C'$ , a sequence of points  $\theta_1^k, \theta_2^k \in \Lambda_{C'}$  such that for  $i = 1, 2$ ,*

$$c^{-1}\epsilon_k \leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k,$$

$$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_0) \leq c\epsilon_k \text{ and}$$

$$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \geq \delta, d(\gamma_k\theta_i^k, \gamma_k\theta_0) \geq \delta.$$

**Proof :** For any  $x \in \tilde{X} \cup \partial\tilde{X}$  and  $y \in \tilde{X} \cup \partial\tilde{X}$  let us define  $\alpha_{x,y}$  the geodesic ray joining  $x$  and  $y$ . Let  $o \in \tilde{X}$  and  $E \in T_o\tilde{X}$  be such that  $|Jac_E\tilde{F}(o)| = 1$  and  $E(\infty)$  the equator associated to  $E$ . Let  $\gamma_k \in C'$  be a sequence such that  $\gamma_k(o)$  converges to the point  $\theta_0$  where  $\theta_0 \in \Lambda_{C'}$  is the point coming from lemma 5.12. In particular, according to that lemma, there exist a point  $\tilde{z} \in \tilde{Z}'$  such that the hypersurface  $\tilde{Z}'$  is contained in the complementary of the open horoball  $HB(\tilde{z}, \theta_0)$ . We define  $D := dist(\tilde{z}, o)$ . As  $\tilde{Z}'$  lies outside the open horoball  $HB(\tilde{z}, \theta_0)$ , the points  $\gamma_k(o)$  belong to the complementary of the open horoball  $HB(\alpha_{\tilde{z}, \theta_0}(D), \theta_0)$ . By standard triangle comparison argument (comparison with the hyperbolic case) the angle  $Angle(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o})$  between the two geodesic rays  $\alpha_{\gamma_k(o), \theta_0}$  and  $\alpha_{\gamma_k(o), o}$  satisfies :

$$(5.47) \quad \lim_{k \rightarrow \infty} Angle(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o}) = 0.$$

By equivariance we have  $\Lambda_{C'} \subset (\gamma_k E)(\infty)$  where  $\gamma_k E \subset T_{\gamma_k(o)}\tilde{X}$ . For any  $v \in T\tilde{X}$  let  $\alpha_v$  be the geodesic ray such that  $\dot{\alpha}_v(0) = v$ . Let us denote by  $u_k$  the unit vector in  $\gamma_k E$  such that  $\alpha_{u_k}(+\infty) = \theta_0$  and let us choose some  $w_k \in \gamma_k E$  such that  $\langle u_k, w_k \rangle = 0$  (this is possible because  $n-1 \geq 2$ ).

We claim now that there exist  $v_k \in \gamma_k E$  such that the angle between  $v_k$  and  $w_k$  is not too far from 0 or  $\pi$ , namely

$$(5.48) \quad |\langle v_k, w_k \rangle| \geq \frac{1}{(n-1)^{1/2}},$$

and  $\alpha_{v_k}(+\infty) \in \Lambda_{C'}$  or  $\alpha_{v_k}(-\infty) \in \Lambda_{C'}$ .

Let us prove this claim.

According to Proposition 5.1 and to (5.11), the restriction to  $\gamma_k E$  of the quadratic form  $h(u) = \int DB(\gamma_k(o), \theta)(u)^2 d\mu_{\gamma_k(o)}(\theta)$  verifies

$$(5.49) \quad h_{\gamma_k E}(u) = \frac{\|u\|^2}{n-1}.$$

Therefore, if for all  $u \in \gamma_k(E)$  such that  $\alpha_u(+\infty) = \theta \in \Lambda_{C'}$  we had  $|\langle u, w_k \rangle| < \frac{1}{(n-1)^{1/2}}$ , then one would get  $h(w_k) < \frac{1}{n-1}$ , which contradicts (5.49) and proves the claim.

In particular the angle between  $u_k$  and  $v_k$  is not too far from  $\pi/2$  for  $k$  large enough, ie.

$$(5.50) \quad | \langle u_k, v_k \rangle | \leq \left( \frac{n-2}{n-1} \right)^{1/2},$$

and thanks to (5.47), we have for  $k$  large enough

$$(5.51) \quad | \langle \dot{\alpha}_{\gamma_k(0),o}(0), v_k \rangle | \leq \left( \frac{n-\frac{3}{2}}{n-1} \right)^{1/2}.$$

Let us now assume for example that  $\theta_k = \alpha_{v_k}(+\infty) \in \Lambda_{C'}$ . Let us show that

$$(5.52) \quad \lim_{k \rightarrow \infty} d(\theta_0, \theta_k) = 0.$$

Assume that (5.52) is not true. Then, one can assume after extracting a subsequence that  $\theta_k$  converges to  $\theta \neq \theta_0$ . Therefore the geodesic rays  $\alpha_{\gamma_k(0),o}$  and  $\alpha_{v_k}$  would converge to the geodesics  $\alpha_{\theta_0,o}$  and  $\alpha_{\theta_0,\theta}$  and thus the angle  $\text{Angle}(\alpha_{\gamma_k(0),o}, \alpha_{v_k})$  would converge to 0. But this would contradict (5.51).

Let us now denote  $\epsilon_k =: d(\theta_k, \theta_0)$ . According to (5.52),  $\lim_{k \rightarrow \infty} \epsilon_k = 0$ . We now consider the following sequence of pointed metric space  $(\partial \tilde{X}, \epsilon_k^{-1} d, \theta_0)$ , a subsequence of which being converging to some metric space  $(S, \delta)$ , cf[]]. For convenience we still denote by the same index  $k$  the subsequence. By the corollary 5.21, the sequence  $(\Lambda_{C'}, \epsilon_k^{-1} d, \theta_0)$  also converges to  $(S, \delta)$ . According to lemma 5.8, the space  $S$  is homeomorphic to  $\mathbb{R}^{n-1}$ . In particular there exist a sequence of points  $\theta'_k \in \Lambda_{C'}$  and a constant  $c$  such that

$$(5.53) \quad c^{-1} \epsilon_k \leq d(\theta_k, \theta'_k) \leq c \epsilon_k,$$

$$(5.54) \quad c^{-1} \epsilon_k \leq d(\theta'_k, \theta_0) \leq c \epsilon_k.$$

Thus, the points  $\theta_1^k = \theta_k$  and  $\theta_2^k = \theta'_k$  satisfy the two first properties of lemma 5.22.

In order to complete the proof of lemma 5.22, we will show that the elements  $\eta_k =: \gamma_k^{-1}$  uniformly separate  $\theta_0$ ,  $\theta_1^k$  and  $\theta_2^k$ .

Thanks to (5.50) the angle at  $\gamma_k(o)$  between  $\theta_1^k$  and  $\theta_0$  is uniformly bounded away from 0 and  $\pi$  and so does the angle at  $o$  between  $\gamma_k^{-1}(\theta_1^k)$  and  $\gamma_k^{-1}(\theta_0)$ . Therefore, as the angle is Hölder-equivalent to the distance  $d$ , cf. [10], there is a constant  $c$  such that

$$(5.55) \quad d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_0)) \geq c.$$



Now the cocompact group  $\Gamma$  acts uniformly quasi-conformally on  $(\partial\tilde{X}, d)$ , ([4] and [19] Theorem 5.2), and so does  $C' \subset \Gamma$ , therefore

$$(5.56) \quad d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_2^k)) \geq c,$$

and

$$(5.57) \quad d(\gamma_k^{-1}(\theta_2^k), \gamma_k^{-1}(\theta_0)) \geq c.$$

which ends the proof of lemma 5.22.  $\square$

**Proof of Proposition 5.17 :**

Let us assume that for every sequence  $\theta_i$  of points in  $\partial\tilde{X}$  converging to  $\theta_0$ ,  $\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) < +\infty$ , then by corollary 5.21 and lemma 5.22 there exist a positive constant  $c$ , a sequence  $\epsilon_k$  tending to 0 when  $k$  tends to  $\infty$ , a sequence  $\gamma_k \in C'$ , a sequence of points  $\theta_1^k, \theta_2^k \in \Lambda_{C'}$  such that

$$\begin{aligned} c^{-1}\epsilon_k &\leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k, \\ c^{-1}\epsilon_k &\leq d(\theta_i^k, \theta_0) \leq c\epsilon_k \text{ and} \\ d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) &\geq \delta, \quad d(\gamma_k\theta_i^k, \gamma_k\theta_0) \geq \delta. \end{aligned}$$

Applying lemma 5.9 for  $\mathcal{L} = \Lambda_{C'}$  and  $\lambda_k = \epsilon_k^{-1}$  we conclude that  $\Lambda_{C'}$  is homeomorphic to  $\partial\tilde{X}$ , which is impossible because  $\Lambda_{C'}$  is contained in a topological equator  $E(\infty)$ .  $\square$

**Step 4 :  $C'$  and  $C$  are convex cocompact.**

We first define convex cocompactness. For a discrete group  $C$  of isometries acting on a Cartan Hadamard manifold of negative sectional curvature with limit set  $\Lambda_C$ , one defines the geodesic hull  $\mathcal{G}(\Lambda_C)$  of  $\Lambda_C$  as the set of all geodesics both ends of whose belong to  $\Lambda_C$ .

The geodesic hull of  $\Lambda_C$  is a  $C$  invariant set. One says that  $C$  is convex cocompact if  $\mathcal{G}(\Lambda_C)/C$  is compact.

**Lemma 5.23.**  *$C'$  is convex cocompact.*

**Proof :** Let us denote  $\pi : \tilde{X} \rightarrow \tilde{X}/C'$  the projection. Assume that  $C'$  is not convex cocompact. Then, there exist a sequence  $x_n \in \mathcal{G}(C')$  such that  $x_n$  tends to infinity. In particular  $\text{dist}(x_n, Z') \rightarrow +\infty$ , where  $Z' = \tilde{Z}'/C'$  is the compact hypersurface which separates  $\tilde{X}/C'$  in two unbounded connected components. There exist lifts  $\tilde{x}_n$  of  $x_n$  such that

$$(5.58) \quad \tilde{x}_n \rightarrow \theta_0 \in \Lambda_{C'}$$

$$(5.59) \quad \text{dist}(\tilde{x}_n, \tilde{Z}') = \text{dist}(\tilde{x}_n, \tilde{z}_n)$$

where  $\tilde{z}_n \in \tilde{Z}'$  is bounded. Therefore there exist  $\tilde{z} \in \tilde{Z}'$  such that  $HB(\tilde{z}, \theta_0) \subset \tilde{U}$ , where  $\tilde{U}$  is one of the two connected components of  $\tilde{X} - \tilde{Z}'$ , the other being  $\tilde{V}$ .

We recall that  $\mathcal{M}$ ,  $\mathcal{N}$  are the two connected components of  $\partial\tilde{X} - \Lambda_{C'}$ . We also have  $\partial\tilde{Z}' = \Lambda_{C'} = E(\infty)$ , and after possibly replacing  $C'$  by an index two subgroup, we can assume that  $C'$  preserves  $\tilde{U}$  and  $\tilde{V}$ .

**Claim :** There are the two following cases.

Either one of the two boundaries  $\partial\tilde{U}$  or  $\partial\tilde{V}$  is equal to  $\Lambda_{C'}$  (in this case the other boundary is equal to  $\partial\tilde{X}$ ), or  $\partial\tilde{U} = \bar{\mathcal{M}}$  and  $\partial\tilde{V} = \bar{\mathcal{N}}$ , where  $\bar{\mathcal{M}}$  and  $\bar{\mathcal{N}}$  are the closure of  $\mathcal{M}$  and  $\mathcal{N}$ .

Let us prove the claim. We first remark that if there exist  $\theta \in \partial\tilde{U} \cap \mathcal{M}$ , then  $\mathcal{M} \subset \partial\tilde{U}$ . Namely, let  $\xi$  be any other point in  $\mathcal{M}$  and  $\alpha$  a continuous path in  $\mathcal{M}$  joining  $\theta$  and  $\xi$ . Since the set  $\tilde{Z}' \cup \Lambda_{C'}$  is a closed subset in  $\tilde{X} \cup \partial\tilde{X}$ , there exist an open connected neighborhood  $W$  of  $\alpha$  in  $\tilde{X} \cup \partial\tilde{X}$  contained in the complementary of  $\tilde{Z}' \cup \Lambda_{C'}$ . Therefore, as  $W \cap \tilde{U} \neq \emptyset$ , we have  $W \cap \tilde{X} \subset \tilde{U}$  and  $\xi \in \partial\tilde{U}$ . Let us assume that neither  $\partial\tilde{U}$  nor  $\partial\tilde{V}$  is equal to  $\Lambda_{C'}$ . Then, each boundary  $\partial\tilde{U}$  and  $\partial\tilde{V}$  contains  $\mathcal{M}$  or  $\mathcal{N}$ . But on the other hand, since the set  $\tilde{Z}' \cup \Lambda_{C'}$  is closed,  $(\partial\tilde{U} - \Lambda_{C'}) \cap (\partial\tilde{V} - \Lambda_{C'}) = \emptyset$  thus we have  $\partial\tilde{U} = \mathcal{M}$  and  $\partial\tilde{V} = \mathcal{N}$  or the other way around and the claim is proved.

**Case 1 :**  $\partial\tilde{U} = \Lambda_{C'}$  and  $\partial\tilde{V} = \partial\tilde{X}$  or the other way around.

In this case, we are in the situation of the step 3, which leads to a contradiction, cf. remark 5.11.

**Case 2 :**  $\partial\tilde{U} = \bar{\mathcal{M}}$  and  $\partial\tilde{V} = \bar{\mathcal{N}}$ .

In that case, assuming  $C'$  is not convex-cocompact, there exist an open horoball  $HB(\theta_0, \tilde{z}) \subset \tilde{U}$  where  $\theta_0 \in \Lambda_{C'}$ ,  $\tilde{z} \in \tilde{Z}'$ ,  $\partial\tilde{U} = \bar{\mathcal{M}}$  and  $\partial\tilde{V} = \bar{\mathcal{N}}$ . We will find a contradiction in a similar way as in case 1, ie. step 3. We consider a point  $o \in \tilde{X}$  and an hyperplane  $E \subset T_o\tilde{X}$  such that  $|Jac_E\tilde{F}(o)| = 1$  and  $\Lambda_{C'} = E(\infty)$ .

Let  $\theta_i \in \mathcal{N}$  be a sequence which converge to  $\theta_0$ . By continuity, for  $i$  large enough, the geodesic ray  $\alpha_{o,\theta_i}$  spends some time in  $HB(\theta_0, \tilde{z}) \subset \tilde{U}$  and ends up in  $\tilde{V}$  because  $\theta_i$  converges to  $\theta_0$  and  $\theta_i$  belongs to  $\mathcal{N} = \partial\tilde{V} - \Lambda_{C'}$ . Therefore,  $\alpha_{o,\theta_i}$  eventually crosses  $\tilde{Z}'$ . Let  $z_i$  be some point in  $\tilde{Z}' \cap \alpha_{o,\theta_i}$ .

We will prove the following Proposition, similar to the Proposition 5.18,

**Proposition 5.24.** *There exist a sequence  $\theta_i \in \mathcal{N}$  such that  $\theta_i$  converges to  $\theta_0$  and*

$$\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = +\infty$$

where  $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ .

**Remark 5.25.** *The difference between the propositions 5.24 and 5.18 is that we are looking for a sequence  $\theta_i \in \mathcal{N}$  instead of  $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$ .*

Assuming the Proposition 5.24 we find a contradiction in the same way as in step 3. Namely, as  $\tilde{Z}'/C'$  is compact, the points  $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$  stay at

bounded distance from the  $C'$ -orbit of a fixed point, say,  $o$ , thus there exist a constant  $A > 0$  and elements  $\gamma_i \in C'$  such that for any  $i$ ,

$$(5.60) \quad \text{dist}(z_i, \gamma_i o) \leq A.$$

From (5.60) and the shadow lemma 5.15, we obtain  $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$ , and on the other hand, from the proposition 5.24, we have  $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$ , which gives the contradiction. It remains to prove the Proposition 5.24.

**Proof of the proposition 5.24 :** We argue by contradiction, like in the proof of the proposition 5.17. Let us assume that there exist a constant  $C > 0$  such that for any sequence of points  $\theta_i \in \mathcal{N}$  converging to  $\theta_0$ ,  $\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) \leq C$ , then by lemma 5.19, we have for any such sequence  $\theta_i \in \mathcal{N}$

$$(5.61) \quad \lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0.$$

The proof of the following lemma is the same as the proof of lemma 5.20.

**Lemma 5.26.** *Let us assume that for any sequence  $\theta_i \in \mathcal{N}$  converging to  $\theta_0$ ,  $\lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$ . Let  $\{\lambda_k\}$  be a sequence of positive numbers tending to  $+\infty$  such that the sequence of spaces  $(\partial \tilde{X}, \lambda_k d, \theta_0)$  converges to a space  $(S, \delta, 0)$  in the pointed Gromov-Hausdorff topology, then,  $(\mathcal{M}, \lambda_k d, \theta_0)$  also converges to  $(S, \delta, 0)$ .*

**Proof :** Since  $\Lambda_{C'} \subset \bar{\mathcal{M}}$ , the assumption implies that  $\lim_{\epsilon \rightarrow 0} r(\epsilon) = 0$  where

$$r(\epsilon) = \sup \left\{ \frac{d(\theta, \mathcal{M})}{d(\theta, \theta_0)}, \theta \neq \theta_0, \theta \in \mathcal{N}, d(\theta, \theta_0) \leq \epsilon \right\}$$

and the proof goes the same way as in lemma 5.20 replacing  $\Lambda_{C'}$  by  $\mathcal{M}$ .

□

Similarly to the lemma 5.22, we have the

**Lemma 5.27.** *Let us assume that every weak tangent of  $(\partial \tilde{X}, d)$  at  $\theta_0$  belongs to  $WT((\mathcal{M}, d))$ . There exist positive constant  $c, \delta$ , a sequence  $\epsilon_k$  tending to 0 when  $k$  tends to  $+\infty$ , a sequence of  $\gamma_k \in C'$ , a sequence of points  $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$  such that for  $i \neq j \in \{0, 1, 2\}$ ,*

$$c^{-1} \epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c \epsilon_k \text{ and} \\ d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta.$$

We can now end the proof of the proposition 5.24. Let us assume that there exist a constant  $C > 0$  such that for every sequence  $\theta_i$  of points in  $\mathcal{N}$  converging to  $\theta_0$ ,  $\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(\theta_i, \alpha_\theta) \leq C$ , then by (5.61), lemma 5.26 and lemma 5.27, there exist a sequence  $\epsilon_k$  tending to 0 when  $k$  tends to  $\infty$ , a sequence  $\gamma_k \in C'$ , a sequence of points  $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$  such that for  $i \neq j \in \{0, 1, 2\}$ ,

$$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k \text{ and} \\ d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta.$$

Applying the lemma 5.9 for  $\mathcal{L} = \mathcal{M}$  and  $\lambda_k = \epsilon_k^{-1}$ , we conclude that  $\mathcal{M}$  is homeomorphic to  $\partial\tilde{X}$ , which is impossible because  $\partial\tilde{X}$  is a sphere, and  $\mathcal{M}$  is homeomorphic to an hemisphere. This ends the proof of the proposition 5.24.  $\square$

**Corollary 5.28.** *C is convex cocompact.*

**Proof :** The subgroup  $C'$  of  $C$  is convex cocompact and the limit sets of  $C'$  and  $C$  coincide by step 3, therefore  $C$  is convex cocompact.  $\square$ .

**Step 5: C preserves a copy of the  $(n-1)$ -dimensional hyperbolic space  $\mathbb{H}^{n-1}$  totally geodesically embedded in  $\tilde{X}$ .**

From the steps 1-4, we know that the groups  $C$  and  $C'$  are convex cocompact, and that their limit set  $\Lambda_C$  and  $\Lambda_{C'}$  are equal to a topological equator  $E(\infty)$ .

Let us consider the essential hypersurface  $Z' \subset \tilde{X}/C'$ . We will show that there exist a minimizing current representing the class of  $Z'$  in  $H_{n-1}(\tilde{X}/C', \mathbb{R})$  and that this minimizing current lifts to a totally geodesic hypersurface embedded in  $\tilde{X}$ . We will then show that this totally geodesic hypersurface is eventually hyperbolic.

We work in  $\tilde{X}/C'$  and consider the essential hypersurface  $Z' \subset \tilde{X}/C'$ . We will now prove that there exist a minimal current representing the class of  $Z'$  in  $H_{n-1}(\tilde{X}/C', \mathbb{R})$ . Let  $\{Z_k\}$  be a minimizing sequence of currents homologous to  $Z'$ . The orthogonal projection onto the convex core of  $\tilde{X}/C'$  is distance nonincreasing and thus volume nonincreasing. Therefore we can assume that the  $Z_k$ 's are in the the convex core of  $\tilde{X}/C'$ , which is compact. By [12] (5.5), the sequence  $\{Z_k\}$  subconverges to a minimal current  $Z_\infty$  in  $\tilde{X}/C'$ . By [12] (8.2),  $Z_\infty$  is a manifold with possible singularities of codimension greater than or equal to 8. By corollary (4.4) and minimality we get that  $|Jac_{n-1}\tilde{F}(x)| = 1$  at every regular points  $x \in Z_\infty$ . We will use the fact that  $|Jac_{n-1}\tilde{F}(x)| = 1$  at every regular points  $x \in Z_\infty$  in order to prove that  $Z_\infty$  is a totally geodesic hypersurface.

**Lemma 5.29.** *Let  $x$  and  $y$  two distinct points in  $\tilde{X}$  and  $E_x \subset T_x\tilde{X}$ ,  $E_y \subset T_y\tilde{X}$  be such that  $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$  and  $Jac_{n-1}\tilde{F}'(y) = Jac_{E_y}\tilde{F}'(y) = 1$ . Then, the geodesic  $\alpha_{x,y}$  (resp.  $\alpha_{y,x}$ ) joining  $x$  and  $y$  (resp.  $y$  and  $x$ ) satisfies  $\dot{\alpha}_{x,y}(0) \in E_x$ , (resp.  $\dot{\alpha}_{y,x}(0) \in E_y$ ). In particular,  $\alpha_{x,y}(+\infty)$  and  $\alpha_{y,x}(+\infty)$  belong to  $\Lambda_{C'}$ .*

**Proof:** Let  $S_x$  and  $S_y$  be the unit spheres of  $E_x$  and  $E_y$ . For any unit tangent vector  $u \in T_z\tilde{X}$  at some point  $z$ , we define  $\theta_u \in \partial\tilde{X}$  by  $\dot{\alpha}_{z,\theta_u}(0) = u$ . By step 3,  $\Lambda_{C'} = E_x(\infty) = E_y(\infty)$ , therefore for every  $u \in S_x$ ,  $\theta_u \in \Lambda_{C'}$  and there exist  $v \in E_y$  such that  $\theta_u = \theta_v$ . As  $E_y$  is a vector space,  $\theta_{-v}$  belongs to

$\Lambda_{C'}$  therefore there exist  $w \in E_x$  such that  $\theta_w = \theta_{-v}$ . The map  $f : S_x \rightarrow S_x$  defined by  $f(u) = w$  is a continuous map. The lemma then boils down to proving that there exist  $u \in S_x$  such that  $f(u) = -u$  because in that case,  $x, y$  and  $\theta_u$  are on the same geodesic  $\alpha_{x, \theta_u}$ .

The following properties of  $f$  are obvious.

- (i) For every  $u \in S_x$ ,  $f(u) \neq u$ .
- (ii)  $f \circ f = Id$ .

So  $f$  is an involution of the sphere without fixed point and for any such map, we claim that there exist  $u$  in the sphere such that  $f(u) = -u$ . In order to prove the claim, we follow a very similar argument in [16], theorem 1. We argue by contradiction. Let us assume that for every  $u \in S_x$ ,  $f(u) \neq -u$ . The map  $g : S_x \rightarrow S_x$  defined by  $g(u) = \frac{f(u)+u}{\|f(u)+u\|}$ , is then well defined and continuous. Let us remark that as for every  $u \in S_x$ ,  $f(u) \neq -u$ , then  $f$  is homotopic to the Identity, and so is  $g$ . Moreover by (ii) we clearly have  $g \circ f = g$ , thus the map  $g$  factorizes through  $S_x/G_f$  where  $G_f$  is the group generated by the involution  $f$ . By (i)  $f$  has no fixed point thus  $S_x/G_f$  is a manifold and the projection  $p : S_x \rightarrow S_x/G_f$  is a degree 2 map. Therefore, the induced endomorphism  $g_*$  on  $H_{n-1}(S_x, \mathbb{Z}_2)$  is trivial, which contradicts the fact that  $g$  is homotopic to the Identity.  $\square$

**Corollary 5.30.** *Let  $\mathcal{H}^{n-1} \subset \tilde{X}$  be an hypersurface with possibly non empty boundary  $\partial\mathcal{H}^{n-1}$ , such that for any  $x \in \mathcal{H}^{n-1}$ ,  $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$ , where  $E_x$  is the tangent space of  $\mathcal{H}^{n-1}$  at  $x$ . Let us consider  $x \in \mathcal{H}^{n-1}$  such that  $dist_{\tilde{X}}(x, \partial\mathcal{H}^{n-1}) = r > 0$ . Then, for any  $x' \in \mathcal{H}^{n-1}$  with  $dist_{\tilde{X}}(x, x') < r$ , the geodesic  $\alpha_{x, x'}$  joining  $x$  and  $x'$  is contained in  $\mathcal{H}^{n-1}$ . In particular,  $\mathcal{H}^{n-1}$  is locally convex.*

**Proof of the corollary:** Let us fix  $\theta \in \Lambda_{C'}$  and consider the vector field  $\nabla B(y, \theta)$  in  $\tilde{X}$ . Let  $x \in \mathcal{H}^{n-1}$ . As  $Jac_{E_x}\tilde{F}'(x) = 1$ , we have by step 3  $\Lambda_{C'} = E_x(\infty)$ . Then, for any  $x \in \mathcal{H}^{n-1}$ ,  $\nabla B(x, \theta)$  is tangent to  $\mathcal{H}^{n-1}$ , therefore the geodesic  $\alpha_{x, \theta}$  satisfies  $\alpha_{x, \theta}(t) \in \mathcal{H}^{n-1}$  for all  $t \in [0, r)$ . Let  $x' \in \mathcal{H}^{n-1}$ . By lemma 5.29,  $\dot{\alpha}_{x, x'}(0) \in E_x$ , therefore  $\alpha_{x, x'} = \alpha_{x, \theta}$  and  $\alpha_{x, x'}(t) \in \mathcal{H}^{n-1}$  for all  $t \in [0, r)$ .  $\square$

We now prove that  $Z_\infty$  is a totally geodesic hypersurface in  $\tilde{X}/C'$ . Let us recall that  $Z_\infty$  is a manifold which is smooth except at a singular subset of codimension at least 7. Let us consider a lift  $\tilde{Z}_\infty \subset \tilde{X}$  of  $Z_\infty$  and denote  $\tilde{Z}_\infty^{reg}$  (resp.  $\tilde{Z}_\infty^{sing}$ ) the set of regular (resp.) singular points of  $\tilde{Z}_\infty$ .

**Lemma 5.31.**  *$\tilde{Z}_\infty$  is a totally geodesic hypersurface in  $\tilde{X}$ .*

**Proof:** Let us consider a regular point  $x \in \tilde{Z}_\infty^{reg}$ . We shall show that for every point  $x' \in \tilde{Z}_\infty^{reg}$  the geodesic segment joining  $x$  and  $x'$  is contained in  $\tilde{Z}_\infty$ , and as the set of regular points is dense in  $\tilde{Z}_\infty$  (as the complementary of a subset of codimension at least 8), this will show that  $\tilde{Z}_\infty$  is totally geodesic.

We claim that there exist a sequence  $y_k \in \tilde{Z}_\infty^{reg}$  such that  $\lim_{k \rightarrow \infty} y_k = x'$  and the geodesic segment joining  $x$  and  $y_k$  is contained in  $\tilde{Z}_\infty$ .

The claim immediately implies that the geodesic segment joining  $x$  and  $x'$  is contained in  $\tilde{Z}_\infty$ .

Let us prove the claim.

For  $y \in \tilde{Z}_\infty^{reg}$  we consider  $\alpha_{x,y}$  the geodesic joining  $x$  and  $y$  and define

$$(5.62) \quad t_y = \inf\{t > 0, \alpha_{x,y}(t) \notin \tilde{Z}_\infty\}$$

As  $x$  is a regular point, by corollary 5.30, there exist  $\epsilon > 0$  such that  $t_y > \epsilon$ .

In order to prove the claim, we argue by contradiction. Let us assume that there exist  $r > 0$  such that for any  $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg}$ ,  $t_y < \text{dist}(x, y)$ . By corollary 5.30 applied to  $\tilde{Z}_\infty^{reg}$ , we have  $\alpha_{x,y}(t_y) \in \tilde{Z}_\infty^{sing}$ . As the set of regular points is an open subset of  $\tilde{Z}_\infty$ , if  $r$  is small enough we have  $B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg} = B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$ . We choose such an  $r$  and we consider the set  $S$  of all singular points contained in the union of all geodesic segments joining  $x$  to a point  $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$ . Let us consider the map defined on  $S$  by

$$p(y) = \alpha_{x,y}(\epsilon).$$

As we already saw, for any  $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$ , we have  $t_y > \epsilon$ , therefore the map  $p$  is distance decreasing and by assumption  $p$  is surjective onto an open subset of the sphere and  $p(S)$  is homeomorphic to an open subset of  $\mathbb{R}^{n-1}$ , therefore the Hausdorff dimension of  $S$  is greater than or equal to  $n - 1$ , which contradicts the fact that the singular set has codimension at least 8 in  $\tilde{Z}_\infty$ .  $\square$

The totally geodesic hypersurface  $\tilde{Z}_\infty \subset \tilde{X}$  is preserved by  $C$ , and  $\tilde{Z}_\infty/C$  is of minimal volume in its homology class.

Let us prove that  $\tilde{Z}_\infty$  is isometric to the hyperbolic space  $\mathbb{H}_\mathbb{R}^{n-1}$ .

**Lemma 5.32.**  $\tilde{Z}_\infty$  is isometric to the hyperbolic space  $\mathbb{H}_\mathbb{R}^{n-1}$ .

**Proof :**

As  $\tilde{Z}_\infty/C$  is of minimal volume in its homology class, we have by Proposition 5.1, for all  $x \in \tilde{Z}_\infty$ ,  $\text{Jac}_{E_x} \tilde{F}(x) = 1$  and  $\tilde{F}(x) = x$ , where  $E_x$  is the tangent space of  $\tilde{Z}_\infty$  at  $x$ . Moreover, we saw in the proof of proposition 5.1 that

$$H = \frac{1}{n-1} \text{Id}_{D\tilde{F}(x)(E_x)} = \frac{1}{n-1} \text{Id}_{E_x},$$

therefore we get from (4.11) and  $\tilde{F}(x) = x$ , that for all  $u, v \in T_x \tilde{Z}_\infty$ ,

$$(5.63) \quad \int_{\partial \tilde{X}} [DdB_{(x,\theta)}(u, v) + DB_{(x,\theta)}(u)DB_{(x,\theta)}(v)] d\nu_x(\theta) \\ = \tilde{g}(u, v)$$

where  $\tilde{g}$  is the metric on  $\tilde{X}$ . As  $\tilde{Z}_\infty$  is totally geodesic, the relation (5.62) remains true with the Busemann function  $B^{\tilde{Z}_\infty}$  of  $\tilde{Z}_\infty$  instead of the Busemann function  $B$  of  $\tilde{X}$ :

$$(5.64) \quad \int_{\partial\tilde{X}} [DdB_{(x,\theta)}^{\tilde{Z}_\infty}(u, v) + DB_{(x,\theta)}^{\tilde{Z}_\infty}(u)DB_{(x,\theta)}^{\tilde{Z}_\infty}(v)] \\ = \tilde{g}(u, v).$$

On the other hand, as  $\tilde{Z}_\infty$  is totally geodesic, its sectional curvature is less than or equal to  $-1$ , thus by Rauch comparison theorem, we have

$$(5.65) \quad DdB^{\tilde{Z}_\infty}(x, \theta) + DB^{\tilde{Z}_\infty}(x, \theta) \otimes DB^{\tilde{Z}_\infty}(x, \theta) \geq \tilde{g}_{\tilde{Z}_\infty}$$

for all  $\theta \in \partial\tilde{Z}_\infty = \Lambda_C$ , where  $\tilde{g}_{\tilde{Z}_\infty}$  is the restriction of  $\tilde{g}$  to  $\tilde{Z}_\infty$ .

As the support of the measure  $\nu_x$  is  $\partial\tilde{Z}_\infty = \Lambda_C$  (by convex cocompactness of  $C$ ) and the Busemann function is continuous, we get from (5.64) and (5.65) that for all  $x \in \tilde{Z}_\infty$  and all  $\theta \in \tilde{Z}_\infty$

$$(5.66) \quad DdB^{\tilde{Z}_\infty}(x, \theta) + DB^{\tilde{Z}_\infty}(x, \theta) \otimes DB^{\tilde{Z}_\infty}(x, \theta) \\ = \tilde{g}_{\tilde{Z}_\infty}(x).$$

and this last relation is characteristic of the hyperbolic space.  $\square$

### Step 6: Conclusion

So far we have shown that  $C$  preserves a totally geodesic copy of the hyperbolic space  $\mathbb{H}_{\mathbb{R}}^{n-1} \subset \tilde{X}$  such that  $\mathbb{H}_{\mathbb{R}}^{n-1}/C$  is compact.

Our goal now is to show that  $Y =: \mathbb{H}_{\mathbb{R}}^{n-1}/C$  injects diffeomorphically in  $X = \tilde{X}/\Gamma$  and separates  $X$  in two connected components  $R$  and  $S$  such that  $A = \pi_1(R)$  and  $B = \pi_1(S)$ .

In order to do this, we will consider the  $\Gamma$  orbit of  $\mathbb{H}_{\mathbb{R}}^{n-1}$  in  $\tilde{X}$  and the two connected components  $U$  and  $V$  of  $\tilde{X} - \Gamma\mathbb{H}_{\mathbb{R}}^{n-1}$  which are adjacent to  $\mathbb{H}_{\mathbb{R}}^{n-1}$ . The stabilizers  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  of  $U$ ,  $V$  and  $\mathbb{H}_{\mathbb{R}}^{n-1}$  contain respectively  $A$ ,  $B$  and  $C$  and the hypersurface  $\mathbb{H}_{\mathbb{R}}^{n-1}/C$  injects in  $X = \tilde{X}/\Gamma$  and separates  $X$  in two connected components  $R$  and  $S$  such that  $\pi_1(R) = \bar{A}$  and  $\pi_1(S) = \bar{B}$ . We then show that  $\bar{C} = C$ ,  $\bar{A} = A$  and  $\bar{B} = B$ .

Let  $\bar{C}$  be the stabilizer of  $\mathbb{H}^{n-1}$ , namely  $\bar{C} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} = \mathbb{H}^{n-1}\}$ . We have  $C \subset \bar{C}$  and as  $\mathbb{H}^{n-1}/C$  is compact, so is  $\mathbb{H}^{n-1}/\bar{C}$  and thus  $[\bar{C} : C] < \infty$ .

Let  $p : \tilde{X}/C \rightarrow X = \tilde{X}/\Gamma$  and  $\bar{p} : \tilde{X}/\bar{C} \rightarrow X = \tilde{X}/\Gamma$  the natural projections. We now show that the restriction of  $p$  to  $\mathbb{H}^{n-1}/C$  is an embedding, thus  $Y := p(\mathbb{H}^{n-1}/C)$  is a compact totally geodesic hypersurface of  $X$ .

In the section 2, we constructed a  $C$ -invariant hypersurface  $\tilde{Z} \subset \tilde{X}$  such that  $Z = \tilde{Z}/C \subset \tilde{X}/C$  is compact. The hypersurface is defined as  $\tilde{Z} = \tilde{f}^{-1}(t_0)$  where  $\tilde{f} : \tilde{X} \rightarrow T$  is an equivariant map onto the Bass-Serre tree associated to the amalgamation  $A *_C B$  and  $t_0$  belongs to that edge of  $T$  which is fixed by  $C$ .

Let us first show two lemmas.

**Lemma 5.33.** *The restriction of  $p$  to  $\tilde{Z}/C$  is an embedding into  $X = \tilde{X}/\Gamma$ .*

**Proof :** Let  $\gamma \in \Gamma$ ,  $z, z'$  in  $\tilde{Z}$  such that  $z' = \gamma z$ . By equivariance,

$$\tilde{f}(\gamma z) = \gamma \tilde{f}(z) = \gamma t_0 = \tilde{f}(z') = t_0,$$

thus  $\gamma \in C$ .  $\square$

**Lemma 5.34.** *The restriction of  $\bar{p}$  to  $\mathbb{H}^{n-1}/\bar{C}$  is an embedding into  $X = \tilde{X}/\Gamma$ .*

**Proof :** Let us assume that there is a  $\gamma \in \Gamma - \bar{C}$  such that  $\gamma \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq \emptyset$  and choose an  $x \in \gamma \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1}$ . As  $\gamma \notin \bar{C}$ , there exist  $u \in T_x \gamma \mathbb{H}^{n-1} - T_x \mathbb{H}^{n-1}$ . We consider  $c_u$  the geodesic ray such that  $\dot{c}_u(0) = u$ . We know that  $\tilde{Z}$  is contained in an  $\epsilon$ -neighbourhood  $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$  of  $\mathbb{H}^{n-1}$ . The  $\epsilon$ -neighbourhood  $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$  of  $\mathbb{H}^{n-1}$  separates  $\tilde{X}$  in two connected components  $U$  and  $V$  and for  $t > 0$  large enough, we have, say,  $c_u(t) \in U$  and  $c_u(-t) \in V$ .

Let  $\tilde{Z}'$  be the connected component of  $\tilde{Z}$  that we constructed at the end of section 2, whose stabilizer (or an index two subgroup of it)  $C'$  is such that  $\tilde{Z}'/C'$  separates  $\tilde{X}/C'$  in two unbounded connected components  $U'/C'$  and  $V'/C'$  where  $U'$  and  $V'$  are the two connected components of  $\tilde{X} - \tilde{Z}'$ .

We claim that  $U \subset U'$  and  $V \subset V'$  or the other way around. Indeed if not,  $U$  and  $V$  would be both contained in, say,  $U'$ . But in that case,  $V'$  would be contained in  $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$  and therefore  $V'/C'$  would be bounded, which is a contradiction.

As  $\gamma \tilde{Z}$  lies in the  $\epsilon$  neighborhood of  $\gamma \mathbb{H}^{n-1}$ , there exist sequences  $z_k, z'_k$  in  $\gamma \tilde{Z}$  such that  $\text{dist}(z_k, c_u(k)) \leq \epsilon$  and  $\text{dist}(z'_k, c_u(-k)) \leq \epsilon$ . By proposition 5.13 and lemma 5.23,  $C'$  also acts cocompactly on  $\mathbb{H}^{n-1}$ , thus  $C'$  is of finite index in  $C$ , and therefore there are finitely many connected components of  $\tilde{Z}$  and the same holds for  $\gamma \tilde{Z}$ . We thus can assume that the  $z_k$ 's and  $z'_k$ 's belong to a single connected component of  $\gamma \tilde{Z}$ . Let us consider a continuous path  $\alpha \subset \gamma \tilde{Z}$  joining  $z_k$  and  $z'_k$ .

By construction the distance between  $c_u(k)$  [resp.  $c_u(-k)$ ] and  $\mathbb{H}^{n-1}$  tends to infinity and thus, for  $k$  large enough,  $z_k \in U$  and  $z'_k \in V$  or the other way around. By the claim, we then have  $z_k \in U'$  and  $z'_k \in V'$ , therefore the path  $\alpha$  has to cross  $\tilde{Z}'$  which contradicts the lemma (5.29) and ends the proof of the lemma 5.30.  $\square$

As we already saw,  $\tilde{Z}$  has finitely many connected components, and so does  $\tilde{X} - \tilde{Z}$ . Let us write  $\{W_j\}_{j=1, \dots, m}$  the connected components of  $\tilde{X} - \tilde{Z}$ . As  $C$  acts cocompactly on  $\tilde{Z}$  and  $\mathbb{H}^{n-1}$  there exist  $\epsilon > 0$  such that  $\mathbb{H}^{n-1} \subset$



$\mathcal{U}_\epsilon \tilde{Z}$  and  $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ . Moreover  $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$  separates  $\tilde{X}$  in two connected components  $U$  and  $V$ .

**Lemma 5.35.** *Let us consider  $\epsilon$  such that  $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$  and  $U$  and  $V$  the two connected components of  $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ . There are two distinct connected components  $W_1$  and  $W_2$  of  $\tilde{X} - \tilde{Z}$  such that  $U \subset W_1$  and  $V \subset W_2$ . Moreover,  $\tilde{f}(W_1) \subset \tilde{T}_1$  and  $\tilde{f}(W_2) \subset \tilde{T}_2$ , where  $\tilde{T}_1$  and  $\tilde{T}_2$  are the two connected components of  $\tilde{T} - \{t_0\}$ .*

**Proof :** We argue by contradiction. Let us assume that  $U$  and  $V$  are contained in the same connected component  $W_1$  of  $\tilde{X} - \tilde{Z}$ . Then, all other components  $W_j$ ,  $j \neq 1$ , satisfy  $W_j \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1} \subset \mathcal{U}_{2\epsilon} \tilde{Z}$ . Therefore, as  $C$  acts cocompactly on  $\mathcal{U}_{2\epsilon} \tilde{Z}$ , there exist a constant  $D$  such that for any  $j \neq 1$ ,  $\max_{w \in W_j} \text{dist}_{\tilde{T}}(\tilde{f}(w), t_0) \leq D$ . Thus,  $\tilde{f}(W_1)$  is contained in one connected component of  $\tilde{T} - \{t_0\}$  and  $\tilde{f}(\cup_{j \neq 1} W_j)$ , contained in the ball  $B_{\tilde{T}}(t_0, D)$  of  $\tilde{T}$  of radius  $D$  centered at  $t_0$ , is bounded. This is clearly impossible because  $\tilde{T} - \{t_0\}$  has two unbounded connected components and  $\tilde{f}$  is onto.  $\square$

Let us denote  $\mathcal{A} = A\mathbb{H}^{n-1}$  the  $A$ -orbit of the  $C$ -invariant totally geodesic copy of the real hyperbolic space  $\mathbb{H}^{n-1}$ , and  $\bar{A}$  the stabilizer of  $\mathcal{A}$ , ie.  $\bar{A} = \{\gamma \in \Gamma, \gamma\mathcal{A} = \mathcal{A}\}$ . We define in a similar way  $\mathcal{B} = B\mathbb{H}^{n-1}$  and  $\bar{B} = \{\gamma \in \Gamma, \gamma\mathcal{B} = \mathcal{B}\}$ .

Let us recall that  $\bar{C}$  is the stabilizer of  $\mathbb{H}^{n-1}$  in  $\Gamma$ . We now prove the following

**Lemma 5.36.** *We have  $\bar{A} = A\bar{C}$  and  $\bar{B} = B\bar{C}$ . Moreover,  $\bar{A}$  and  $\bar{B}$  are characterized by  $\bar{A} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{A}\}$  and  $\bar{B} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{B}\}$ .*

**Proof :** Let  $\gamma' \in \bar{A}$ , then  $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$  and thus there exist  $\gamma \in A$  such that  $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$ , therefore  $\gamma^{-1}\gamma' \in \bar{C}$ , which proves the first part of the lemma.

Let us prove the second part of the lemma.

Let  $\gamma' \in \Gamma$  be such that  $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$ . Then there exist  $\gamma \in A$  such that  $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$ , thus  $\gamma^{-1}\gamma' \in \bar{C}$  and therefore  $\gamma' \in \gamma\bar{C} \subset A\bar{C} = \bar{A}$ . This proves one inclusion, the other inclusion being obvious.  $\square$

For each  $\gamma \in \Gamma$ ,  $\gamma\mathbb{H}^{n-1}$  separates  $\tilde{X}$  in two connected components  $U_\gamma$  and  $V_\gamma$ .

Let us now prove the following lemma.

**Lemma 5.37.** *(i) Let  $\gamma \in A$ , [resp.  $\gamma \in B$ ]. Then, we have  $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$  or  $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$ , [resp.  $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$  or  $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$ .]  
(ii) Let  $\gamma$  be an element of  $\Gamma - \bar{A}$ , [resp.  $\Gamma - \bar{B}$ ]. Then  $\mathcal{A} \subset U_\gamma$  or  $\mathcal{A} \subset V_\gamma$ , [resp.  $\mathcal{B} \subset U_\gamma$  or  $\mathcal{B} \subset V_\gamma$ ].*

**Proof :** (i) We argue by contradiction. Let us consider  $\gamma\mathbb{H}^{n-1}$ ,  $\gamma'\mathbb{H}^{n-1}$  and  $\gamma''\mathbb{H}^{n-1}$  three distinct elements in  $\mathcal{A}$  such that  $\gamma'\mathbb{H}^{n-1} \subset U_\gamma$  and  $\gamma''\mathbb{H}^{n-1} \subset V_\gamma$ . By equivariance we can assume  $\gamma$  is the identity. Let us recall that  $U$  and  $V$  are the two connected components of  $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ .

We then have  $\gamma'\mathbb{H}^{n-1} \cap U \neq \emptyset$  and  $\gamma''\mathbb{H}^{n-1} \cap V \neq \emptyset$ , which implies  $\gamma'\tilde{Z} \cap U \neq \emptyset$  and  $\gamma''\tilde{Z} \cap V \neq \emptyset$ .

By lemma (5.31),  $U \subset W_1$  and  $V \subset W_2$  where  $W_1$  and  $W_2$  are two connected components of  $\tilde{X} - \tilde{Z}$  and  $\tilde{f}(U) \subset \tilde{T}_1$  and  $\tilde{f}(V) \subset \tilde{T}_2$ , therefore,  $\tilde{f}(U)$  contains  $\gamma't_0 \in \tilde{T}_1$  and  $\tilde{f}(V)$  contains  $\gamma''t_0 \in \tilde{T}_2$ . This is impossible because for all elements  $\gamma'$  and  $\gamma''$  in  $A$ ,  $\gamma't_0$  and  $\gamma''t_0$  belong to the same connected component of  $\tilde{T} - \{t_0\}$ .

(ii) Let us consider  $\gamma \in \Gamma - \bar{A}$ . We argue by contradiction. Let us assume there exist  $\gamma, \gamma'$  in  $A$  such that

$$\gamma'\mathbb{H}^{n-1} \subset U_\gamma$$

$$(5.67) \quad \gamma''\mathbb{H}^{n-1} \subset V_\gamma$$

Let  $\epsilon > 0$  such that  $\tilde{Z} \subset \mathcal{U}_\epsilon\mathbb{H}^{n-1}$  and  $U$  and  $V$  the connected component of  $\tilde{X} - \mathcal{U}_\epsilon\mathbb{H}^{n-1}$ . By lemma (5.31) we have  $\tilde{f}(\gamma U) \subset \gamma\tilde{T}_1$  and  $\tilde{f}(\gamma V) \subset \gamma\tilde{T}_2$ , where  $\gamma\tilde{T}_1$  and  $\gamma\tilde{T}_2$  are the two connected components of  $\tilde{T} - \{\gamma t_0\}$ . By assumption (5.62), we have

$\gamma'\mathbb{H}^{n-1} \cap \gamma U \neq \emptyset$  and  $\gamma''\mathbb{H}^{n-1} \cap \gamma V \neq \emptyset$ , which implies  $\gamma'\tilde{Z} \cap \gamma U \neq \emptyset$  and  $\gamma''\tilde{Z} \cap \gamma V \neq \emptyset$ , therefore  $\gamma't_0 \in \gamma\tilde{T}_1$  and  $\gamma''t_0 \in \gamma\tilde{T}_2$ , which is impossible because in the tree  $\tilde{T}$ , the points  $\gamma't_0$  and  $\gamma''t_0$  belong to two adjacent edges.

□

By lemma (5.33) (i), for every  $\gamma$  in  $\bar{A}$ , [resp.  $\bar{B}$ ], we can define  $U_\gamma$  as the connected component of  $\tilde{X} - \mathbb{H}^{n-1}$  which contains all  $\gamma'\mathbb{H}^{n-1}$  for all  $\gamma'$  in  $\bar{A}$ , [resp.  $\bar{B}$ ], and  $\gamma'\mathbb{H}^{n-1} \neq \gamma\mathbb{H}^{n-1}$ .

Let us define

$$(5.68) \quad U_A := \bigcap_{\gamma \in \bar{A}} U_\gamma.$$

By definition,  $U_A$  [resp.  $U_B$ ] is a convex set in  $\tilde{X}$  whose boundary is the collection  $\mathcal{A}$ , [resp.  $\mathcal{B}$ ], of  $\gamma\mathbb{H}^{n-1}$ ,  $\gamma$  in  $\bar{A}$  [resp.  $\bar{B}$ ] and by lemma (5.33) (ii),  $U_A$  and  $U_B$  are two disjoint connected components of  $\tilde{X} - \Gamma\mathbb{H}^{n-1}$ .

In fact,  $U_A$  [resp.  $U_B$ ], is the convex hull of  $\mathcal{A}$ , [resp.  $\mathcal{B}$ ], and  $\bar{A}$ , [resp.  $\bar{B}$ ], is the stabilizer of  $U_A$ , [resp.  $U_B$ ].

**Lemma 5.38.** *The closures of  $U_A$  and  $U_B$  intersect along  $\mathbb{H}^{n-1}$  and  $\bar{A} \cap \bar{B} = \bar{C}$ . Moreover, no element of  $\Gamma$  sends  $U_A$  on  $U_B$  nor the other way around.*

**Proof :** The convex set  $U_A$  is the intersection of open half spaces  $U_\gamma$ ,  $\gamma \in \bar{A}$ , and is delimited by the disjoint union of hyperplanes  $\gamma\mathbb{H}^{n-1}$ , for some  $\gamma \in \bar{A}$ . The same is true for  $U_B$  and as  $U_A \cap U_B = \emptyset$ , the closures of  $U_A$  and  $U_B$  can intersect only along one of the connected components of their boundaries, thus along  $\mathbb{H}^{n-1}$  which is obviously in both closures. This proves the first part of the lemma, let us prove the second part. By lemma 5.32,  $\bar{C} \subset \bar{A} \cap \bar{B}$ . Conversely, let us take  $\gamma \in \bar{A} \cap \bar{B}$ , then  $\gamma$  preserves

the closures of  $U_A$  and  $U_B$ , thus it preserves their intersection  $\mathbb{H}^{n-1}$ , and therefore  $\gamma \in \bar{C}$ .

Let us prove the last part of the lemma. Let  $\gamma$  be an element such that  $\gamma U_A = U_B$ . As  $\mathbb{H}^{n-1}$  is one component of the boundary  $\mathcal{B}$  of  $U_B$ , there exist one component  $\gamma' \mathbb{H}^{n-1} \in \mathcal{A}$ ,  $\gamma'$  being in  $\bar{A}$ , such that  $\gamma(\gamma' \mathbb{H}^{n-1}) = \mathbb{H}^{n-1}$ . Therefore,  $\gamma\gamma' \in \bar{C}$ , thus  $\gamma \in \bar{A}$ . The same argument yields  $\gamma^{-1} \in \bar{B}$ , so  $\gamma \in \bar{A} \cap \bar{B} = \bar{C}$  and  $\gamma$  preserves  $U_A$  and  $U_B$ , which contradicts our choice of  $\gamma$ .

□

The  $\Gamma$ -orbit of the closure of  $U_A \cup U_B$  covers  $\tilde{X}$ . Let us construct a tree  $\bar{T}$  embedded in  $\tilde{X}$  in the following way: the set of vertices is the set of the connected components of  $\tilde{X} - \Gamma \mathbb{H}^{n-1}$  and two vertices are joined by an edge if the boundaries of their corresponding connected components intersect non trivially in  $\tilde{X}$ . By construction  $\Gamma$  acts on  $\bar{T}$ , the stabilizers of the vertices  $a$  and  $b$  corresponding to  $U_A$  and  $U_B$  are  $\bar{A}$  and  $\bar{B}$ , the stabilizer of the edge between  $a$  and  $b$  is  $\bar{C}$  and a fundamental domain for this action is the segment joining  $a$  and  $b$ . By [14], I, 4, Theorem 6, the group  $\Gamma$  is the amalgamated product of  $\bar{A}$  and  $\bar{B}$  over  $\bar{C}$ .

We now claim that  $\bar{A} = A$ ,  $\bar{B} = B$  and  $\bar{C} = C$ .

As  $A$ ,  $B$  and  $C$  are subgroups of  $\bar{A}$ ,  $\bar{B}$ , and  $\bar{C}$ , the corresponding Mayer-Vietoris sequences of  $A *_C B$  and  $\bar{A} *_C \bar{B}$  are related by the following commutative diagram

$$\begin{array}{ccccc} H_n(A, \mathbb{R}) \oplus H_n(B, \mathbb{R}) & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(C, \mathbb{R}) \\ & & \downarrow & & \downarrow \\ H_n(\bar{A}, \mathbb{R}) \oplus H_n(\bar{B}, \mathbb{R}) & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C}, \mathbb{R}). \end{array}$$

We know that the index  $[\bar{C} : C]$  is finite, and by the lemma 5.32, the indices of  $A$ ,  $B$ , and  $C$  in  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  are finite and equal. On the other hand, the indices  $[\Gamma : A]$  and  $[\Gamma : B]$  are infinite by assumption, thus the previous diagram becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(C, \mathbb{R}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C}, \mathbb{R}). \end{array}$$

Moreover, the map  $H_n(\Gamma, \mathbb{R}) \rightarrow H_{n-1}(\bar{C}, \mathbb{R})$  is bijective. Namely, the injectivity comes from the above diagram and the surjectivity from the fact that the hypersurface  $\mathbb{H}^{n-1}/\bar{C}$  bounds in  $\mathbb{H}^n/A$  and  $\mathbb{H}^n/B$  so that the map  $H_{n-1}(C, \mathbb{R}) \rightarrow H_{n-1}(A, \mathbb{R}) \oplus H_{n-1}(B, \mathbb{R})$  is trivial. Therefore the index  $[\bar{C} : C] = 1$  and we get  $\bar{A} = A$ ,  $\bar{B} = B$  and  $\bar{C} = C$ . □

## 6. PROOF OF THE THEOREMS 1.5 AND 1.6.

The proof of theorem 1.5 is exactly the same as the proof of theorem 1.2. The actions of  $\Gamma$  on  $\tilde{X}$  and  $T$  give rise to a continuous  $\Gamma$ -equivariant map  $\tilde{f} : \tilde{X} \rightarrow T$ . Like in section 2, we build an hypersurface  $\tilde{f}^{-1}(t_0)$  where  $t_0$  is

a regular value of  $\tilde{f}$  belonging the interior of an edge. As the edge separates the tree in two unbounded components, the section 2 applies and we get a subgroup  $C'$  of  $C$ , and an hypersurface  $\tilde{Z}' \subset \tilde{X}/C'$  which is essential. Now, if the action of  $\Gamma$  is minimal, every edge separates  $T$  in two unbounded components.  $\square$

## 7. APPENDIX

The goal of this section is to give a proof of lemma 5.9. This lemma is contained in lemma 2.1, 5.1 and 5.2 of [4], but our situation being not exactly the same, we reproduce it down here for sake of completeness.

Let us restate the lemma 5.9.

**Lemma 7.1.** *Let  $\mathcal{L} \subset \partial\tilde{X}$  be a closed  $C'$ -invariant subset and  $\theta_0 \in \mathcal{L}$ . We assume that there exist a sequence of positive real numbers  $\lambda_k \rightarrow \infty$  such that the sequence of pointed metric spaces  $(\mathcal{L}, \lambda_k d, \theta_0)$  converges in the pointed Gromov-Hausdorff topology to  $(S, \delta, 0)$  where  $(S, \delta, 0)$  is a weak tangent of  $(\partial\tilde{X}, d)$ . We also assume that there exist positive constants  $C$  and  $\delta$ , a sequence of points  $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$  and a sequence of elements  $\gamma_k \in C'$  such that  $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$  and  $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$  for all  $0 \leq i \neq j \leq 2$ . Then,  $\mathcal{L}$  is homeomorphic to the one point compactification  $\hat{S}$  of  $S$ . In particular  $\mathcal{L}$  is homeomorphic to  $\partial\tilde{X}$ .*

We first give a definition of pointed Hausdorff-Gromov convergence which is equivalent to the definition 5.7. We follow [4], paragraph 4.

A sequence of metric spaces  $(Z_k, d_k, z_k)$  converges to the metric space  $(S, \delta, 0)$  if for every  $R > 0$ , and every  $\epsilon > 0$ , there exist an integer  $N$ , a subset  $D \subset B_S(0, R)$ , subsets  $D_k \subset B_{Z_k}(z_k, R)$  and bijections  $f_k : D_k \rightarrow D$  such that for  $k \geq N$ ,

- (i)  $f_k(z_k) = 0$ ,
- (ii) the set  $D$  is  $\epsilon$ -dense in  $B_S(0, R)$ , and the sets  $D_k$  are  $\epsilon$ -dense in  $B_{Z_k}(z_k, R)$ ,
- (iii)  $|d_{Z_k}(x, y) - d_Z(f_k(x), f_k(y))| < \epsilon$ ,  
where  $x, y$  belong to  $D_k$ .

Let us describe now the lemmas 2.1 and 5.1 following [4].

For a metric space  $(Z, d)$  the cross ratio of four points  $\{z_i\}$ ,  $i = 1, \dots, 4$ , is the quantity

$$(7.1) \quad [z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}$$

Given two metric spaces  $X$  and  $Y$ , an homeomorphism  $\eta : [0, \infty) \rightarrow [0, \infty)$ , and an injective map  $f : X \rightarrow Y$ , we say that  $f$  is an  $\eta$ -quasi-Möbius map if for any four points  $\{x_i\}$ ,  $i = 1, \dots, 4$ , in  $X$ , we have

$$(7.2) \quad [f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]).$$

For example, any discrete cocompact group of isometries of  $\tilde{X}$ , where  $\tilde{X}$  is a Cartan-Hadamard manifold with sectional curvature  $K \leq -1$ , is acting on the ideal boundary  $(\partial\tilde{X}, d)$  endowed with the Gromov distance by  $\eta$ -quasi-Möbius transformations for some  $\eta$ .

**Lemma 7.2** ([4], Lemma 2.1 ). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be two compact metric spaces, and for any integer  $k$ ,  $g_k : \tilde{D}_k \rightarrow Y$  an  $\eta$ -quasi-Möbius map defined on a subset  $\tilde{D}_k$  of  $X$ . We assume that the Hausdorff distance between  $\tilde{D}_k$  and  $X$  satisfies*

$$\lim_{k \rightarrow \infty} \text{dist}_H(\tilde{D}_k, X) = 0$$

and that for any integer  $k$ , there exist points  $(x_1^k, x_2^k, x_3^k)$  in  $D_k$  and  $(y_1^k, y_2^k, y_3^k)$  in  $Y$ , such that  $g_k(x_i^k) = y_i^k$  for  $i \in \{1, 2, 3\}$ ,  $d_X(x_i^k, x_j^k) \geq \delta$  and  $d_Y(y_i^k, y_j^k) \geq \delta$  for  $i, j \in \{1, 2, 3\}, i \neq j$ , where  $\delta$  is independant of  $k$ . Then a subsequence of  $g_k$  converges uniformly to a quasi-Möbius map  $f : X \rightarrow Y$ , ie.  $\lim_{k_j \rightarrow \infty} \text{dist}_H(g_{k_j}, f|_{\tilde{D}_{k_j}}) = 0$ . If in addition, we suppose that

$$\lim_{k \rightarrow \infty} \text{dist}_H(g_k(\tilde{D}_k), Y) = 0,$$

then the sequence  $\{g_{k_j}\}$  converges uniformly to a quasi-Möbius homeomorphism  $f : X \rightarrow Y$ .

Before stating the second lemma, let us define a metric space  $Z$  to be uniformly perfect if there exist a constant  $\lambda \geq 1$  such that for every  $z \in Z$  and  $0 < R < \text{diam}Z$ , we have  $\bar{B}(z, R) - B(z, \frac{R}{\lambda}) \neq \emptyset$ .

**Lemma 7.3** ([4], lemma 5.1 ). *Let  $Z$  be a compact uniformly perfect metric space and  $G$  an  $\eta$ -quasi-Möbius action on  $Z$ . Suppose that for each integer  $k$  we are given a set  $D_k$  in a ball  $B_k = B(z, R_k) \subset Z$  that is  $(\epsilon_k R_k)$ -dense in  $B_k$ , where  $\epsilon_k > 0$ , distinct points  $x_1^k, x_2^k, x_3^k \in B(z, \lambda_k R_k)$ , where  $\lambda_k > 0$ , with*

$$d_Z(x_i^k, x_j^k) \geq \delta_k R_k$$

for  $i, j \in \{1, 2, 3\}, i \neq j$ , where  $\delta_k > 0$ , and groups elements  $\gamma_k \in G$  such that for  $y_i^k := \gamma_k(x_i^k)$  we have,

$$d_Z(y_i^k, y_j^k) \geq \delta'$$

for  $i, j \in \{1, 2, 3\}, i \neq j$ , where  $\delta'$  is independant of  $k$ . Let  $D'_k = \gamma_k(D_k)$ , and suppose that  $\lambda_k \rightarrow 0$  when  $k \rightarrow \infty$ , and the sequence  $\frac{\epsilon_k}{\delta_k^2}$  is bounded. Then  $\lim_{k \rightarrow \infty} \text{dist}_H(D'_k, Z) = 0$ .

Let us go back to the proof of lemma 6.1. By definition of convergence, there exist a subsequence of  $\{\lambda_k\}$ , which we still denote by  $\{\lambda_k\}$ , subsets  $\tilde{D}_k \subset B_S(0, k)$ ,  $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$ , where  $\tilde{D}_k$  and  $D_k$  are minimal  $1/k$ -dense

subsets of  $B_S(0, k)$  and  $B_{(\mathcal{L}, \lambda_k d)}(\theta_0, k)$ , and bijections  $f_k : \tilde{D}_k \rightarrow D_k$  such that for all  $x, y \in \tilde{D}_k$ ,

$$(7.3) \quad \frac{1}{2}\delta(x, y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\delta(x, y),$$

cf. [4], (5.4).

We can suppose that the points  $\theta_0^k := \theta_0$ ,  $\theta_1^k$ , and  $\theta_2^k$  in lemma 6.1 belong to the set  $D_k$ . By assumption there exist elements  $\gamma_k \in C'$  and a constant  $\delta$  such that

$$(7.4) \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta$$

for all  $i, j \in \{0, 1, 2\}$ .

The lemma 6.1 is a direct consequence of the lemma 6.2 applied to  $(X, d_X) = (\hat{S}, \hat{\delta})$  and  $(Y, d_Y) = (\mathcal{L}, d)$  and to the sequence of maps  $g_k := \gamma_k \circ f_k$ , where  $\hat{S}$  is the one point compactification of  $S$  and  $\hat{\delta}$  the distance on  $\hat{S}$  associated to  $\delta$ , cf. [4] Lemma 2.2.

Let us denote  $x_0^k, x_1^k, x_2^k$  be the points in  $S$  such that  $f_k(x_i^k) = \theta_i^k$ , for  $i \in \{0, 1, 2\}$ .

Let us check that the assumptions of lemma 6.2 are verified.

The fact that  $\lim_{k \rightarrow \infty} \text{dist}_H(\tilde{D}_k, \hat{S}) = 0$  comes the same way as in [4], (5.5).

By (6.3), we have,  $\delta(x_i^k, x_j^k) \geq \frac{\lambda_k}{2} d(\theta_i^k, \theta_j^k)$  and by assumption we then get

$$(7.5) \quad \delta(x_i^k, x_j^k) \geq \frac{1}{2C}.$$

We then get the separation assumption on triples of points by choosing  $\delta := \inf\{D, \frac{1}{2C}\}$ .

It remains to check the assumption on  $g_k(\tilde{D}_k) = \gamma_k \circ f_k(\tilde{D}_k) = \gamma_k(D_k)$ , namely,

$$(7.6) \quad \lim_{k \rightarrow \infty} \text{dist}_H(\gamma_k(D_k), \Lambda_{C'}) = 0.$$

In order to prove the property (6.6), we want to apply the lemma 6.3, but as the set  $(\mathcal{L}, d)$  is a priori not uniformly perfect, we shall replace the uniform perfectness by the fact that  $(\mathcal{L}, \lambda_k d, \theta_0)$  converges to a space  $(S, \delta, 0)$ , which is uniformly perfect, cf. ().

We will show the

**Lemma 7.4.** *We consider the subsets  $\tilde{D}_k \subset B_S(0, k)$  and  $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$ , where  $\tilde{D}_k$  and  $D_k$  are  $1/k$ -dense subsets of  $B_S(0, k)$  and  $B_{(\mathcal{L}, \lambda_k d)}(\theta_0, k)$ , and the bijections  $f_k : \tilde{D}_k \rightarrow D_k$  coming from the convergence of the sequence of pointed metric spaces  $(\mathcal{L}, \lambda_k d, \theta_0)$  to  $(S, \delta, 0)$  where  $(S, \delta, 0)$  is a weak tangent of  $(\partial \tilde{X}, d)$ . We also assume that there exist positive constants  $C$  and  $\delta$ , a*

sequence of points  $\theta_1^k, \theta_2^k \in \Lambda_{C'}$  and a sequence of elements  $\gamma_k \in C'$  such that  $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$  and  $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$  for all  $0 \leq i \neq j \leq 2$ . Then, the Hausdorff distance  $\text{dist}_H(\gamma_k D_k, \mathcal{L})$  tends to 0 as  $k$  tends to infinity.

**Proof :** The proof is word by word the same as the proof of lemma 6.3, ie. lemma 5.1 (i) of [4] with a difference in case 2).

We have  $B_{\lambda_k \mathcal{L}}(\theta_0, k) = B_{\mathcal{L}}(\theta_0, \frac{k}{\lambda_k})$  and  $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$  an  $\frac{1}{k}$ -dense subset, for the metric  $\lambda_k d$ . In term of the distance  $d$ , the set  $D_k$  is  $(\epsilon_k R_k)$ -dense in  $B_{\mathcal{L}}(\theta_0, R_k)$ , where  $R_k := \frac{k}{\lambda_k}$  and  $\epsilon_k := \frac{1}{k^2}$ . By assumption, the points  $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k$  belong to  $B_{\mathcal{L}}(\theta_0, \mu_k R_k)$ , and satisfy

$$(7.7) \quad d(\theta_i^k, \theta_j^k) \geq \delta_k R_k$$

where  $\delta_k := \frac{1}{Ck}$ , and  $\mu_k := \frac{C}{k}$ .

The points  $\gamma_k \theta_i^k$  satisfy

$$(7.8) \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta,$$

and  $\frac{\epsilon_k}{\delta_k^2} = C^2$  is bounded.

Let us consider a point  $\theta \in \mathcal{L}$ . We want to approximate it by a point of  $\gamma_k D_k$ .

We can write  $\theta = \gamma_k \theta_k$ , for some  $\theta_k \in \Lambda_{C'}$ . There are two cases.

*Case 1).* For infinitely many indices  $k$ ,  $\theta_k \in B_{\mathcal{L}}(\theta_0, R_k)$ . We work in that case for these indices  $k$ , thus there are points  $\theta'_k \in D_k \cap B_{\mathcal{L}}(\theta_0, R_k)$ , with  $d(\theta_k, \theta'_k) \leq \epsilon_k R_k$ .

Since the distance between the  $\theta_i^k$ 's is bounded below by  $\delta_k R_k$ , we can find at least two of them which we call  $a_k$  and  $b_k$ , such that

$$d(\theta_k, b_k) \geq \frac{\delta_k R_k}{2}$$

and,

$$(7.9) \quad d(\theta'_k, a_k) \geq \frac{\delta_k R_k}{2}.$$

As  $C'$  is contained in the cocompact group  $\Gamma$ , it acts in a quasi-Möbius way on  $(\partial \tilde{X}, d)$

thus,

$$(7.10) \quad \frac{d(\gamma_k \theta'_k, \gamma_k \theta_k) d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k \theta'_k, \gamma_k b_k) d(\gamma_k a_k, \gamma_k \theta_k)} \leq \eta \left( \frac{d(\theta'_k, \theta_k) d(a_k, b_k)}{d(\theta'_k, b_k) d(\theta_k, a_k)} \right)$$

for some homeomorphism  $\eta : [0, \infty) \rightarrow [0, \infty)$ . This implies

$$(7.11) \quad d(\gamma_k \theta'_k, \gamma_k \theta_k) \leq \frac{(\text{diam} \mathcal{L})^2 \eta(8\epsilon_k \mu_k / \delta_k^2)}{\delta},$$

therefore  $d(\gamma_k \theta'_k, \gamma_k \theta_k)$  tends to zero as  $k$  tends to infinity.

*Case 2)* . For all but finitely many indices  $k$ ,  $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$ .

We work with these indices  $k$  such that  $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$ .

We know that  $\epsilon_k/\delta_k^2$  is bounded above independantly of  $k$ , and by assumption,  $\delta_k \leq 2\mu_k$ .

We claim that there exist  $\xi_k \in D_k$  and a positive constant  $c_0$  such that for all  $k$ ,

$$(7.12) \quad \frac{d(\xi_k, \theta_0)}{R_k} \geq c_0$$

let us prove the claim.

On one hand, as  $(\partial\tilde{X}, d)$  is uniformly perfect, and so is it's weak tangent  $(S, \delta)$  because the one point compactification  $(\hat{S}, \hat{\delta})$  of  $(S, \delta)$  is quasi-Möbius homeomorphic to  $(\partial\tilde{X}, d)$ , therefore there exist a constant  $C_0 \in [0, 1]$  such that for every  $x \in S$  and  $0 < R < \text{diam}S$ , we have

$$(7.13) \quad \bar{B}_{(S, \delta)}(0, R) - B_{(S, \delta)}(0, C_0 R) \neq \emptyset.$$

On the other hand,  $(\mathcal{L}, \lambda_k d, \theta_0)$  converges to  $(S, \delta, 0)$ . After reindexing the sequence  $\{\lambda_k\}$ , we have for each  $\epsilon > 0$  a map  $g_k : B_{\lambda_k \mathcal{L}}(\theta_0, k) \rightarrow S$  such that

(i)  $g_k(\theta_0) = 0$ ,

for any two points  $\theta$  and  $\theta'$  in  $B_{\lambda_k \mathcal{L}}(\theta_0, k)$ ,

(ii)  $|\delta(g_k(\theta), g_k(\theta')) - \lambda_k d(\theta, \theta')| \leq \epsilon$ ,

(iii) the  $\epsilon$ -neighborhood of  $g_k(B_{\lambda_k \mathcal{L}}(\theta_0, k))$  contains  $B_{(S, \delta)}(0, k - \epsilon)$ .

By (iii), we have

$$(7.14) \quad \bar{B}_{(S, \delta)}(0, k - \epsilon) \subset \mathcal{U}_\epsilon^{(S, \delta)} g_k(\bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)).$$

By (6.13) there exist  $y_k \in \bar{B}_{(S, \delta)}(0, k - \epsilon) - B_{(S, \delta)}(0, C_0(k - \epsilon))$ , and by (6.14) there exist  $\xi'_k \in \bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)$  such that

$$(7.15) \quad \delta(y_k, g_k(\xi'_k)) \leq \epsilon.$$

We now evaluate  $d(y_k, g_k(\xi'_k))$ . By the above properties (i), (ii), (6.15) and the triangle inequality we have

$$\lambda_k d(\xi'_k, \theta_0) \geq \delta(g_k(\xi'_k), 0) - \epsilon \geq \delta(y_k, 0) - \delta(y_k, g_k(\xi'_k)) - \epsilon$$

$$(7.16) \quad \geq C_0(k - \epsilon) - 2\epsilon.$$

As  $D_k$  is  $\epsilon_k R_k$ -dense in  $B_{(\mathcal{L}, d)}(\theta_0, k/\lambda_k)$ , there exist  $\xi_k \in D_k$  such that  $d(\xi_k, \xi'_k) \leq \epsilon_k R_k = \frac{k\epsilon_k}{\lambda_k}$ .



Let us denote  $c_0 = C_0/2$ . For  $k$  large enough we have  $\frac{C_0(k-\epsilon)-2\epsilon-k\epsilon_k}{\lambda_k} \geq \frac{c_0 k}{\lambda_k}$ , therefore by (6.16) we get

$$(7.17) \quad d(\xi_k, \theta_0) \geq d(\xi'_k, \theta_0) - d(\xi'_k, \xi_k) \geq c_0 R_k,$$

which proves the claim.

We can assume that for  $k$  large enough,  $\mu_k < c_0/2 < 1/2$ .

We choose  $a_k = \theta_1^k$  and  $b_k = \theta_2^k$ , and we get

$$(7.18) \quad \begin{aligned} & \frac{d(\gamma_k \xi_k, \gamma_k \theta_k) d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k \xi_k, \gamma_k b_k) d(\gamma_k a_k, \gamma_k \theta_k)} \leq \eta \left( \frac{d(\xi_k, \theta_k) d(a_k, b_k)}{d(\xi_k, b_k) d(\theta_k, a_k)} \right) \\ & \leq \eta \left( \frac{4\mu_k d(\theta_k, \theta_0^k)}{(d(\theta_k, \theta_0^k) - \mu_k R_k)(c_0 - \mu_k)} \right) \\ & \leq \eta(16\mu_k/c_0). \end{aligned}$$

We get

$$d(\gamma_k \theta_k, \gamma_k \xi_k) \leq (\text{diam} \Lambda_{C'})^2 \eta(16\mu_k/c_0)/\delta.$$

□

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