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RIGIDITY OF AMALGAMATED PRODUCT IN NEGATIVE CURVATURE

GÉRARD BESSON, GILLES COURTOIS, SYLVAIN GALLOT

ABSTRACT. Let Γ be the fundamental group of a compact riemannian manifold X of sectional curvature $K \leq -1$ and dimension $n \geq 3$. We suppose that $\Gamma = A *_C B$ is the free product of its subgroups A and B over the amalgamated subgroup C . We prove that the critical exponent $\delta(C)$ of C satisfies $\delta(C) \geq n - 2$. The equality happens if and only if there exist an embedded compact hypersurface $Y \subset X$, totally geodesic, of constant sectional curvature -1 , whose fundamental group is C and which separates X in two connected components whose fundamental groups are A and B . Similar results hold if Γ is an HNN extension, or more generally if Γ acts on a simplicial tree without fixed point.

1. INTRODUCTION

In [15], Y. Shalom proved the following theorem which says that for every lattice Γ in the hyperbolic space and for any decomposition of Γ as an amalgamated product $\Gamma = A *_C B$, the group C has to be “big”. In order to measure how “big” C is, let us define the critical exponent of a discrete group C acting on a Cartan Hadamard manifold by

$$\delta(C) = \inf\{s > 0 \mid \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < +\infty\}.$$

Theorem 1.1. *Let Γ be a lattice in $PO(n, 1)$. Assume that Γ is an amalgamated product of its subgroups A and B over C . Then, the critical exponent $\delta(C)$ of C satisfy $\delta(C) \geq n - 2$.*

An example is given by any n -dimensional hyperbolic manifold X which contains a compact separating connected totally geodesic hypersurface Y . The Van Kampen theorem then says that the fundamental group Γ of X is isomorphic to the free product of the fundamental groups of the two halves of $X - Y$ amalgamated over the fundamental group C of the incompressible hypersurface Y . Such examples do exist in dimension 3 thanks to the W.Thurston’s hyperbolization theorem. In any dimension, A. Lubotsky showed that any standard arithmetic lattice of $PO(n, 1)$ has a finite cover whose fundamental group is an amalgamated product, cf. [11]. In fact, A. Lubotsky proved that any standard arithmetic lattice Γ has a finite index subgroup Γ_0 which is mapped onto a nonabelian free group. A nonabelian free group can be written in infinitely many ways as an amalgamated product, so one get infinitely many decomposition of Γ_0 as an amalgamated

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product by pulling back the amalgamated decomposition of the nonabelian free group.

In these cases there is equality in theorem 1.1, ie. $\delta(C) = n - 2$ where C is the fundamental group of Y , and Y . Shalom suggested in [15] that the equality case in the theorem 1.1 happens only in that case.

The aim of this paper is to show that the theorem 1.1 still holds when Γ is the fundamental group of a compact riemannian manifold of variable sectional curvature less than or equal to -1 , and characterize the equality case.

Theorem 1.2. *Let X be an n -dimensional compact riemannian manifold of sectional curvature $K \leq -1$. We assume that the fundamental group Γ of X is an amalgamated product of its subgroups A and B over C and that neither A nor B equals Γ . Then, the critical exponent $\delta(C)$ of C satisfy $\delta(C) \geq n - 2$. Equality $\delta(C) = n - 2$ happens if and only if C cocompactly preserves a totally geodesic isometrically embedded copy \mathbb{H}^{n-1} of the hyperbolic space of dimension $n - 1$. Moreover, in the equality case, the hypersurface $Y^{n-1} := \mathbb{H}^{n-1}/C$ is embedded in X and separates X in two connected components whose fundamental groups are respectively A and B .*

Remark 1.3. (i) *By the assumption on A or B not being equal to Γ we exclude the trivial decomposition $\Gamma = \Gamma *_C C$ where $A = \Gamma$ and $C = B$ can be an arbitrary subgroup of Γ , for example any cyclic subgroup, in which case the conclusion of theorem 1.2 fails. Also note that because of this assumption on A and B , we have $A \neq C$ and $B \neq C$.*

(ii) *Let us recall that standard arithmetic lattices in $PO(n, 1)$ have finite index subgroup with infinitely many non equivalent decompositions as amalgamated products, cf. [11]. In fact, among these decompositions, all but finitely many of them are such that $\delta(C) > n - 2$. Indeed, by theorem 1.2, if $\delta(C) = n - 2$ then C is the fundamental group of an embedded totally geodesic hypersurface in X , but there are only finitely many totally geodesic hypersurfaces by [21].*

When a group is an amalgamated product, it acts on a simplicial tree without fixed point and theorem 1.1 is a particular case of the

Theorem 1.4. ([15], theorem 1.6). *Let $\Gamma \subset SO(n, 1)$, $n \geq 3$, be a lattice. Suppose Γ acts on a simplicial tree T without fixed vertex. Then there is an edge of T whose stabilizer C satisfies $\delta(C) \geq n - 2$.*

In the case Γ is cocompact, the conclusion of theorem 1.4 holds for the stabilizer of any edge which separates the tree T in two unbounded components, and the proof of this is exactly the same as the proof of theorem 1.2. In particular, when the action of Γ on T is minimal, (ie. there is no proper subtree of T invariant by Γ), the conclusion of theorem 1.4 holds for every edge of T , in the variable curvature setting, and we are able to handle the equality case.

Theorem 1.5. *Let Γ be the fundamental group of an n -dimensional compact riemannian manifold X of sectional curvature less than or equal to -1 . Suppose Γ acts minimally on a simplicial tree T without fixed point. Then, the stabilizer C of every edge of T satisfies $\delta(C) \geq n - 2$. The equality $\delta(C) = n - 2$ happens if and only if there exist a compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric has constant sectional curvature -1 .*

Another interesting case contained in theorem 1.5 is the case of *HNN* extension. Let us recall the definition of an *HNN* extension. Let A and C be groups and $f_1 : C \rightarrow A$, $f_2 : C \rightarrow A$ two injective morphisms of C into A . The *HNN* extension $A*_C$ is the group generated by A and an element t with the relations $tf_1(\gamma)t^{-1} = f_2(\gamma)$. For example, let X be a compact manifold containing a non separating compact incompressible hypersurface $Y \subset X$. Let A be the fundamental group of the manifold with boundary $X - Y$ obtained by cutting X along Y and let C be the fundamental group of Y . The boundary of $X - Y$ consists in two connected components $Y_1 \subset X - Y$ and $Y_2 \subset X - Y$ homeomorphic to Y . By the incompressibility assumption, these inclusions give rise to two embeddings of C into A , and the fundamental group of X is the associated *HNN* extension $A*_C$.

Theorem 1.6. *Let Γ be the fundamental group of a compact riemannian manifold X of dimension n and sectional curvature less than or equal to -1 . Suppose that $\Gamma = A*_C$ where A is a proper subgroup of Γ . Then, we have $\delta(C) \geq n - 2$ and equality $\delta(C) = n - 2$ if and only if there exist a non separating compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, Y with its induced metric is of constant sectional curvature -1 , and the *HNN* decomposition arising from Y is the one we started with.*

Let us summarize the ideas of the proof of theorem 1.2. We work on \tilde{X}/C . The amalgamation assumption provides an essential hypersurface Z in \tilde{X}/C , namely Z is homologically non trivial in \tilde{X}/C . The volume of all hypersurfaces homologous to Z is bounded below by a positive constant because their systole are bounded away from zero. We then construct a smooth map $F : \tilde{X}/C \rightarrow \tilde{X}/C$, homotopic to the identity which contracts the volume of all compact hypersurfaces Y by the factor $\left(\frac{\delta(C)}{n-2}\right)^{n-1}$, namely $vol_{n-1}F(Y) \leq \left(\frac{\delta(C)}{n-2}\right)^{n-1}vol_{n-1}Y$. This contracting property together with the lower bound of the volume of hypersurfaces in the homology class of Z gives the inequality $\delta(C) \geq n - 2$. This map is different from the map constructed in [3], in particular it can be defined under the single condition that the limit set of C is not reduced to one point. Moreover, its derivative has an upper bound depending only on the critical exponent of C .

The equality case goes as follows. When $\delta(C) = n - 2$, the map $F : \tilde{X}/C \rightarrow \tilde{X}/C$ contracts the $(n - 1)$ -dimensional volumes, ie. $|Jac_{n-1}F| \leq 1$.

This contracting property is infinitesimally rigid in the following sense. Let us consider a lift \tilde{F} of F . If $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$ at some point $x \in \tilde{X}$, then $\tilde{F}(x) = x$, there exists a tangent hyperplane $E \subset T_x\tilde{X}$ such that $D\tilde{F}(x)$ is the orthogonal projector of $T_x\tilde{X}$ onto E and the limit set Λ_C is contained in the *topological equator* $E(\infty) \subset \partial\tilde{X}$ associated to E . By topological equator $E(\infty) \subset \partial\tilde{X}$ associated to E , we mean the set of end points of those geodesic rays starting at x tangentially to E .

We then prove the existence of a point $x \in \tilde{X}$ such that

$$(1.1) \quad |\text{Jac}_{n-1}\tilde{F}(x)| = 1.$$

If there would exist a minimizing cycle in the homology class of Z in \tilde{X}/C , any point of such a cycle would satisfy (1.1). As no such minimizing cycle a priori exists because of non compactness of \tilde{X}/C , we prove instead the existence of a L^2 harmonic $(n-1)$ -form dual to Z , which is enough to prove existence of a point x such that (1.1) holds.

At this stage of the proof, there is a big difference between the constant curvature case and the variable curvature case.

In the constant curvature case, any topological equator bounds a totally geodesic hyperbolic hypersurface \mathbb{H}^{n-1} , and therefore, as the group C preserves $\Lambda_C \subset E(\infty) = \partial\mathbb{H}^{n-1}$, it is not hard to see that C also preserves \mathbb{H}^{n-1} and acts cocompactly on it, and the hypersurface of the equality case in theorem 1.2 is \mathbb{H}^{n-1}/C , [2].

In the variable curvature case, we first show the existence of a C -invariant totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ whose boundary at infinity coincides with $\Lambda(C)$, and then we show that \tilde{Z}_∞ is isometric to the real hyperbolic space. We then show that $Y =: \mathbb{H}_{\mathbb{R}}^{n-1}/C$, which is compact, injects in $X = \tilde{X}/\Gamma$ and separates X in two connected components whose fundamental groups are A and B respectively.

In order to show the existence of such a totally geodesic hypersurface \tilde{Z}_∞ , we first prove that C is a convex cocompact group, ie. the convex hull of the limit set of C in \tilde{X} has a compact quotient under the action of C , and that the limit set of C is homeomorphic to an $(n-2)$ -dimensional sphere.

The convex cocompactness property of C and the fact that the limit set $\Lambda(C)$ of C is homeomorphic to an $(n-2)$ -dimensional topological sphere are the two key points in the equality case.

This compactness property then allows us to prove the existence of a minimizing current in the homology class of the essential hypersurface $Z \subset \tilde{X}/C$. By regularity theorem this minimizing current \tilde{Z}_∞ is a smooth manifold except at a singular set of codimension at least 8. By the contracting properties of our map F , \tilde{Z}_∞ is fixed by F and the geometric properties of F at fixed points where the $(n-1)$ -jacobian of F equals 1 allows us to prove that \tilde{Z}_∞ is totally geodesic and isometric to the hyperbolic space.

Let us now briefly describe the proof of the convex cocompactness property of C in the equality case.

The group C (or a finite index subgroup of it) actually globally preserves a smooth cocompact hypersurface $\tilde{Z} \subset \tilde{X}$ which separates \tilde{X} into two connected components and whose boundary $\partial\tilde{Z} \subset \partial\tilde{X}$ coincides with $\Lambda_C \subset E(\infty)$. In the case where C wouldn't be convex cocompact, we are able to find an horoball $HB(\theta_0)$ centered at some point $\theta_0 \in \Lambda_C$ in the complementary of which lies the hypersurface \tilde{Z} .

The contradiction then comes from the following.

Consider a sequence of points $\theta_i \in \partial\tilde{X}$ converging to θ_0 and geodesic rays α_i starting from the point $x \in \tilde{X}$ at which $|Jac_{n-1}(x)| = 1$ and ending up at θ_i . These geodesic rays have to cross \tilde{Z} at points z_i which are at bounded distance from the orbit Cx of x , therefore the shadows \mathcal{O}_i of balls centered at these z_i enlightened from x have to contain points of Λ_C by the shadow lemma of D. Sullivan. On the other hand, we show that it is possible to choose the sequence θ_i in such a way that these shadows \mathcal{O}_i don't meet Λ_C . This property $\mathcal{O}_i \cap \Lambda_C = \emptyset$ comes from a choice of θ_i such that the distance between z_i and the set H of all geodesics rays at x tangent to $E \subset T_x\tilde{X}$ tends to ∞ . Intuitively, in order to chose z_i as far as possible from H , the points θ_i have to be chosen transversally to Λ_C . This transversality condition is not well defined because the limit set Λ_C might be highly non regular. Thus, in order to prove that such a choice is possible, we argue again by contradiction. If for any choice of a sequence θ_i converging to θ_0 , the distance between z_i and H stays bounded, then the Gromov distances $d(\theta_i, \theta_0)$ between θ_i and θ_0 satisfy $d(\theta_i, \Lambda_C) = o(d(\theta_i, \theta_0))$, and therefore any tangent cone of Λ_C at θ_0 would coincide with a tangent cone of $\partial\tilde{X}$ at θ_0 , which is known to be topologically \mathbb{R}^{n-1} . But on the other hand, the existence of a point x such that $|Jac_{n-1}(x)| = 1$ and the fact that C acts uniformly quasiconformally with respect to the Gromov distance on $\partial\tilde{X}$ imply that the Alexandroff compactification of the above tangent cone of Λ_C at θ_0 is homeomorphic to Λ_C which is contained in a topological sphere S^{n-2} , leading to a contradiction.

From convex cocompactness of C and the fact that the limit set of C is a topological $(n-2)$ -dimensional sphere, there is an alternative proof of the existence of a totally geodesic C -invariant copy of the hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n-1} \subset \tilde{X}$ which consists in observing that the topological dimension and the Hausdorff dimension of the limit set $\Lambda(C)$ are equal to $n-2$ and then use the following result of M. Bonk and B. Kleiner (which we quote in the riemannian manifold setting although it remains true for $CAT(-1)$ spaces) instead of the (simpler) minimal current argument.

Theorem 1.7. [4] *Let X be a Cartan Hadamard n -dimensional manifold whose sectional curvature satisfy $K \leq -1$, and C a convex cocompact discrete subgroup of isometries of X with limit set Λ_C . Let us assume that the topological dimension and the Hausdorff dimension (with respect to the Gromov distance on $\partial\tilde{X}$) of Λ_C coincide and are equal to an integer p . Then, C*

preserves a totally geodesic embedded copy of the real hyperbolic space \mathbb{H}^{p+1} , with $\partial\mathbb{H}^{p+1} = \Lambda_C$.

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2. ESSENTIAL HYPERSURFACES

Let Γ be a discrete cocompact group of isometries of a n -dimensional Cartan-Hadamard manifold (\tilde{X}, \tilde{g}) whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. Let us assume that the compact manifold $X = \tilde{X}/\Gamma$ is orientable. Let us also assume that $\Gamma = A *_C B$ is an amalgamated product of its subgroups A and B over C .

We first reduce to the case where $[\Gamma : C]$ is infinite.

Namely, if $[\Gamma : C] < \infty$, then the critical exponent $\delta(C) = \delta(\Gamma) \geq n - 1$, and the equality in theorem 1.2 holds.

We then can assume that $[\Gamma : C] = \infty$.

Lemma 2.1. *Let $\Gamma = A *_C B$ be as above, with $[\Gamma : C] = \infty$. If neither A nor B equals Γ , then $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$.*

Proof : The Mayer-Vietoris sequence coming from the decomposition $\Gamma = A *_C B$ writes cf. [6], Corollary 7.7,

$$\begin{aligned} H_n(\tilde{X}/C, \mathbb{Z}) &\rightarrow H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z}) \rightarrow H_n(\tilde{X}/\Gamma, \mathbb{Z}) \rightarrow \dots \\ &\dots \rightarrow H_{n-1}(\tilde{X}/C, \mathbb{Z}) \rightarrow \dots \end{aligned}$$

As $[\Gamma : C] = \infty$, $H_n(\tilde{X}/C, \mathbb{Z}) = 0$ thus, if $H_{n-1}(\tilde{X}/C, \mathbb{Z}) = 0$, we deduce from the Mayer-Vietoris sequence that $H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z})$ is isomorphic to $H_n(\tilde{X}/\Gamma, \mathbb{Z})$. As $H_{n-1}(\tilde{X}/\Gamma, \mathbb{Z}) = \mathbb{Z}$, we then deduce that either $[\Gamma : A] = \infty$ and $B = \Gamma$, or $[\Gamma : B] = \infty$ and $A = \Gamma$. \square

In fact in the sequel of the paper we will make use of a smooth essential hypersurface Z in \tilde{X}/C .

Definition 2.2. *A compact smooth orientable hypersurface Z of an n -dimensional manifold Y is essential in Y if $i_*([Z]) \neq 0$ where $[Z] \in H_{n-1}(Z, \mathbb{R})$ denotes the fundamental class of Z and $i_* : H_{n-1}(Z, \mathbb{R}) \rightarrow H_{n-1}(Y, \mathbb{R})$ the morphism induced by the inclusion $i : Z \hookrightarrow Y$.*

The end of this section is devoted to finding such an hypersurface Z in \tilde{X}/C .

Let us recall a few facts about amalgamated products and their actions on trees, following [14]. Let $\Gamma = A *_C B$ be an amalgamated products of

its subgroups A and B over C . Then, Γ acts on a simplicial tree \tilde{T} with a fundamental domain $T \subset \tilde{T}$ being a segment, ie. an edge joining two vertices. Let us describe this tree \tilde{T} . There are two orbits of vertices Γv_A and Γv_B , the stabilizer of the edge v_A (resp. v_B) being A , (resp. B). There is one orbit of edges Γe_C , the stabilizer of the edge e_C being C . The fundamental domain T can be chosen as the edge e_C joining the two vertices v_A and v_B . The set of vertices adjacent to v_A , (resp. v_B), is in one to one correspondance with A/C , (resp. B/C). Note that as neither A nor B are equal to Γ , then $[A : C] \neq 1$ and $[B : C] \neq 1$, therefore for an arbitrary point t_0 on the edge e_C we see that $\tilde{T} - t_0$ is a disjoint union of two unbounded connected components. This fact will be used later on.

Let us consider a continuous Γ -equivariant map $\tilde{f} : \tilde{X} \rightarrow T$ where T is the Bass-Serre tree associated to the amalgamation $\Gamma = A *_C B$. One regularizes \tilde{f} such that it is smooth in restriction to the complementary of the inverse image of the set of vertices of T . Let t_0 a regular value of \tilde{f} contained in that edge of T which is fixed by the subgroup C and define $\tilde{Z} = \tilde{f}^{-1}(t_0)$. \tilde{Z} is a smooth orientable possibly not connected hypersurface in \tilde{X} , globally C -invariant. Let us write $Z = \tilde{Z}/C$. We will show $Z \subset \tilde{X}/C$ is compact and that one of the connected components of Z is essential.

Lemma 2.3. *$Z \subset \tilde{X}/C$ is compact.*

Proof : Let us show that for any sequence $z_n \in \tilde{Z}$, there exists a subsequence z_{n_k} and $\gamma_k \in C$ such that $\gamma_k z_{n_k}$ converges. As Γ is cocompact, there exists $g_n \in \Gamma$ such that the set $(g_n z_n)$ is relatively compact. Let $g_{n_k} z_{n_k}$ a subsequence which converges to a point $z \in \tilde{X}$. By continuity, the sequence $\tilde{f}(g_{n_k} z_{n_k})$ converges to $\tilde{f}(z)$, and by equivariance we get

$$g_{n_k} \tilde{f}(z_{n_k}) = g_{n_k} t_0 \rightarrow \tilde{f}(z)$$

when k tends to ∞ . As Γ acts in a simplicial way on the tree T and transitively on the set of edges, the sequence $g_{n_k} t_0$ is stationary, ie $g_{n_k} t_0 = t'_0 = g t_0$ for k large enough. Thus $g^{-1} g_{n_k} = \gamma_k \in C$ for k large enough since it fixes t_0 and $\gamma_k z_{n_k} = g^{-1} g_{n_k} z_{n_k}$ converges to $g^{-1}(z)$ \square

The smooth compact hypersurface Z we constructed might be not connected. Let us write $Z = Z_1 \cup Z_2 \cup \dots \cup Z_k$ where the Z_j 's are the connected components of Z . Each Z_j is a compact smooth oriented hypersurface of \tilde{X}/C .

The aim of what follows is to prove that at least one component Z_i of Z is essential.

Lemma 2.4. *There exists $i \in [1, k]$ such that Z_i is essential in \tilde{X}/C .*

Proof : If there exists a Z_i which doesn't separate \tilde{X}/C in two connected components, then Z_i is essential in \tilde{X}/C . So we can assume that every Z_j , $j = 1, \dots, k$, does separate \tilde{X}/C in two connected components. In that case

we will show that there exists a Z_i which separates \tilde{X}/C in two unbounded connected components which easily implies that Z_i is essential.

Let us denote U_l , $l = 1, 2, \dots, p$, the connected components of $\tilde{X}/C - \bigcup_{j=1}^k Z_j$.

Claim : at least two components $U_m, U_{m'}$ are unbounded.

Assuming the claim let us finish the proof of the lemma. For each Z_j we denote V_j, V'_j the two connected components of $\tilde{X}/C - Z_j$. Then $U_m = W_1 \cap W_2 \cap \dots \cap W_k$ where for each j , $W_j = V_j$ or $W_j = V'_j$. In the same way, $U_{m'} = W'_1 \cap W'_2 \cap \dots \cap W'_k$. As $U_m \cap U_{m'} = \emptyset$, there exists $i \in [1, k]$ such that $W_i \cap W'_i = \emptyset$, thus $U_m \subset V_i$ and $U_{m'} \subset V'_i$ or $U_m \subset V'_i$ and $U_{m'} \subset V_i$ so Z_i separates \tilde{X}/C into two unbounded components. This proves the lemma.

Let us prove the claim.

We have already noticed that $T - \{t_0\}$ is the disjoint union of two unbounded connected components T_1 and T_2 . As C acts on T isometrically and simplicially then $T/C - \{t_0\} = T_1/C \cup T_2/C$ is the disjoint union of two unbounded connected components. Let $\bar{f} : \tilde{X}/C \rightarrow T/C$ the quotient map of \bar{f} . For each component U_i , we have

$\bar{f}(U_i) \subset T_1/C$ or $\bar{f}(U_i) \subset T_2/C$, thus we can conclude the claim because \bar{f} is onto \square

Let $\pi : \tilde{X} \rightarrow \tilde{X}/C$ be the natural projection. For any $i = 1, 2, \dots, k$, let us denote $\{\tilde{Z}_i^j\}_{j \in J}$ the set of connected components of $\tilde{Z}_i =: \pi^{-1}(Z_i)$.

For each $i \in [1, k]$, we claim that C acts transitively on the set $\{\tilde{Z}_i^j\}_{j \in J}$. Namely, let us consider $\tilde{Z}_i^j, \tilde{Z}_i^{j'}$, $\tilde{z} \in \tilde{Z}_i^j$, $\tilde{z}' \in \tilde{Z}_i^{j'}$, and write $z = \pi\tilde{z} \in Z_i$ and $z' = \pi\tilde{z}' \in Z_i$. Let α be a continuous path on Z_i such that $\alpha(0) = z$ and $\alpha(1) = z'$, and $\tilde{\alpha}$ the lift of α such that $\tilde{\alpha}(0) = \tilde{z}$. We have $\pi\tilde{\alpha}(1) = z'$ and $\tilde{\alpha}(1) \in \tilde{Z}_i^j$ for some j , thus, there exists $c \in C$ such that $c(\tilde{\alpha}(1)) = \tilde{z}'$ and therefore $c\tilde{Z}_i^j = \tilde{Z}_i^{j'}$. \square

Let us denote C_i^j the stabilizer of \tilde{Z}_i^j , and $Z_i^j = \tilde{Z}_i^j/C_i^j \subset \tilde{X}/C_i^j$. Let us write $p : \tilde{X}/C_i^j \rightarrow \tilde{X}/C$ the natural projection.

Lemma 2.5. *The restriction of p to Z_i^j is a diffeomorphism onto Z_i . In particular, Z_i^j is compact.*

proof : Let z and z' be two points in Z_i^j such that $p(z) = p(z')$. Let \tilde{z} and \tilde{z}' be lifts of z and z' in \tilde{X} . These two points \tilde{z} and \tilde{z}' which are in \tilde{Z}_i^j actually belong to the same connected component \tilde{Z}_i^j because for $j \neq j'$, $\tilde{Z}_i^{j'}/C_i^j \cap \tilde{Z}_i^j/C_i^j = \emptyset$. As $p(z) = p(z')$, there exists $c \in C$ such that $\tilde{z}' = c\tilde{z}$, thus $c \in C_i^j$, and $z = z'$, therefore the restriction of p to Z_i^j is injective. The surjectivity comes from the fact that $\pi^{-1}Z_i = \bigcup_{j \in J} \tilde{Z}_i^j$ and C acts transitively on the set $\{\tilde{Z}_i^j\}_{j \in J}$. \square

Let us consider the integer $i \in [1, k]$ as in lemma 2.4, ie. such that $Z_i \hookrightarrow \tilde{X}/C$ is essential, and choose \tilde{Z}_i^l one component of $\pi^{-1}(Z_i)$.

After possibly replacing C_i^l by an index two subgroup, we may assume that C_i^l globally preserves each of the two connected components U_i^l and V_i^l of $\tilde{X} - \tilde{Z}_i^l$.

Lemma 2.6. *Let i, l and C_i^l be chosen as above. The compact hypersurface $Z_i^l = \tilde{Z}_i^l/C_i^l$ is essential in \tilde{X}/C_i^l . Moreover the two connected components U_i^l/C_i^l and V_i^l/C_i^l of $\tilde{X}/C_i^l - Z_i^l$ are unbounded.*

Proof : Let us consider $p : \tilde{X}/C_i^l \rightarrow \tilde{X}/C$. By lemma 2.5, the restriction of p to Z_i^l is a diffeomorphism onto Z_i , therefore Z_i^l is essential in \tilde{X}/C_i^l because Z_i is essential in \tilde{X}/C . As C_i^l preserves U_i^l and V_i^l , Z_i^l separates \tilde{X}/C_i^l into two connected components U_i^l/C_i^l and V_i^l/C_i^l . and as Z_i^l is essential in \tilde{X}/C_i^l , U_i^l/C_i^l and V_i^l/C_i^l are unbounded. \square

In the sequel of the paper we will denote $\tilde{Z}' = \tilde{Z}_i^l$, $C' = C_i^l$ and $Z' = Z_i^l = \tilde{Z}_i^l/C_i^l$.

3. ISOSYSTOLIC INEQUALITY

In this section we summarize facts and results due to M.Gromov, [9]. Let Z be a p -dimensional compact orientable manifold and $i : Z \hookrightarrow Y$ an embedding of Z into Y where Y is an aspherical space. We suppose that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \rightarrow H_p(Y, \mathbb{R})$ is the morphism induced by the embedding $Z \hookrightarrow Y$. Let us fix a riemannian metric g on Z . For each $z \in Z$ we consider the set \mathcal{C}_z of those loops α at z such that $i \circ \alpha$ is homotopically non trivial in Y .

Let us define the systole of (Z, g, i) at the point z by

Definition 3.1. $sys_i(Z, g, z) = \inf\{length(\alpha), \alpha \in \mathcal{C}_z\}$

and the systole of (Z, g, i) by

Definition 3.2. $sys_i(Z, g) = \inf\{sys_i(Z, g, z) \mid z \in Z\}$.

The following isosystolic inequality, due to M.Gromov says that the volume of any essential submanifold Z of an aspherical space Y relatively to any riemannian metric on Z is universally bounded below by its systole.

Theorem 3.3. [9] *There exists a constant C_p such that for each p -dimensional riemannian manifold (Z, g) and any embedding $Z \hookrightarrow Y$ into an aspherical space Y such that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \rightarrow H_p(Y, \mathbb{R})$ is the induced morphism in homology, then $vol_p(Z, g) \geq C_p(sys_i(Z, g))^p$*

We will apply this volume estimates to the essential hypersurface $i : Z \hookrightarrow \tilde{X}/C$ that we constructed in lemma 2.6.

The following lemma is immediate.

Lemma 3.4. *Let C be a discrete group acting on a simply connected manifold \tilde{X} , \tilde{Z} a C -invariant hypersurface of \tilde{X} and $i : Z = \tilde{Z}/C \hookrightarrow \tilde{X}/C$ the natural inclusion. Let g any riemannian metric on Z and \tilde{g} the lift of g to \tilde{Z} . Then, for any $z \in Z$ we have,*

$$\text{sys}_i(Z, g, z) = \inf\{d_{\tilde{g}}(\tilde{z}, \gamma\tilde{z}), \gamma \in C\}$$

where $\tilde{z} \in \tilde{Z}$ is a lift of $z \in Z$ and $d_{\tilde{g}}$ is the distance induced by \tilde{g} on \tilde{Z} .

Proof : Let $\alpha \in \mathcal{C}_z$ a loop based at $z \in Z$. As $i \circ \alpha$ is an homotopically non trivial loop at $i(z) = z$ in \tilde{X}/C , its lift $\tilde{i \circ \alpha}$ at some $\tilde{z} \in \tilde{Z}$ ends up at $\gamma\tilde{z}$ for some $\gamma \in C$. \square

4. VOLUME OF HYPERSURFACES IN \tilde{X}/C

Let (\tilde{X}, \tilde{g}) be a n -dimensional Cartan-Hadamard manifold whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$ and C a discrete group of isometries of (\tilde{X}, \tilde{g}) . We assume that the group C is non elementary, namely C fixes neither one nor two points in the geometric boundary $\partial\tilde{X}$ of (\tilde{X}, \tilde{g}) .

The aim of this section is to construct a map $F : \tilde{X}/C \rightarrow \tilde{X}/C$ such that for any compact hypersurface Z of \tilde{X}/C , we have

$$\text{vol}_{n-1}(F(Z)) \leq \left(\frac{\delta+1}{n-1}\right)^{n-1} \text{vol}_{n-1}(Z)$$

where δ is the critical exponent of C and $\text{vol}_{n-1}(Z)$ stands for the $(n-1)$ -dimensional volume of the metric on Z induced from g . For every subgroup $C' \subset C$ and any hypersurface Z' of \tilde{X}/C' the lift $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$ of F will also verify

$$\text{vol}_{n-1}(F'(Z')) \leq \left(\frac{\delta+1}{n-1}\right)^{n-1} \text{vol}_{n-1}(Z').$$

In order to construct the map F we need a few preliminaries. We consider a finite positive Borel measure μ on the boundary $\partial\tilde{X}$ whose support contains at least two points. Let us fix an origin $o \in \tilde{X}$ and denote $B(x, \theta)$ the Busemann function defined for each $x \in \tilde{X}$ and $\theta \in \partial\tilde{X}$ by

$$B(x, \theta) = \lim_{t \rightarrow \infty} \text{dist}(x, c(t)) - t$$

where $c(t)$ is the geodesic ray such that $c(0) = o$ and $c(+\infty) = \theta$.

Let $\mathcal{D}_\mu : \tilde{X} \rightarrow \mathbb{R}$ the function defined by

$$(4.1) \quad \mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} e^{B(y, \theta)} d\mu(\theta)$$

A computation shows that

$$(4.2) \quad Dd\mathcal{D}_\mu(y) = \int_{\partial\tilde{X}} (DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta)) e^{B(y, \theta)} d\mu(\theta).$$

When $K_{\tilde{g}} \leq -1$ the Rauch comparison theorem says that for every $y \in \tilde{X}$, and $\theta \in \partial\tilde{X}$,

$$(4.3) \quad DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta) \geq \tilde{g}.$$

We then get

$$(4.4) \quad Dd\mathcal{D}_\mu(y) \geq \mathcal{D}_\mu(y)\tilde{g},$$

thus $Dd\mathcal{D}_\mu(y)$ is positive definite and \mathcal{D}_μ is strictly convex.

Lemma 4.1. *We have $\lim_{y_k \rightarrow \partial\tilde{X}} \mathcal{D}_\mu(y) = +\infty$.*

proof : Let $y_k \in \tilde{X}$ a sequence such that

$$(4.5) \quad \lim_{k \rightarrow \infty} y_k = \theta_0 \in \partial\tilde{X}.$$

As $\text{supp}(\mu) \cap (\partial\tilde{X} - \{\theta_0\}) \neq \emptyset$, there exists a compact subset $K \subset \partial\tilde{X} - \{\theta_0\}$ such that $\mu(K) > 0$ thus,

$$(4.6) \quad \int_{\partial\tilde{X}} e^{B(y_k, \theta)} d\mu \geq \int_K e^{B(y_k, \theta)} d\mu \rightarrow +\infty.$$

□

Corollary 4.2. *Let μ a finite borel measure on $\partial\tilde{X}$ whose support contains at least two points. The function \mathcal{D}_μ has a unique minimum. This minimum will be denoted by $\mathcal{C}(\mu)$.*

Let us now consider some discrete subgroup $C \subset \text{Isom}(\tilde{X}, \tilde{g})$. Recall that a family of Patterson measures $(\mu_x)_{x \in \tilde{X}}$ associated to C is a set of positive finite measures μ_x on $\partial\tilde{X}$, $x \in \tilde{X}$, such that the following holds for all $x \in \tilde{X}$, $\gamma \in C$,

$$(4.7) \quad \mu_{\gamma x} = \gamma_* \mu_x$$

$$(4.8) \quad \mu_x = e^{-\delta B(x, \theta)} \mu_o,$$

where $o \in \tilde{X}$ is a fixed origin, B the Busemann function associated to o and δ the critical exponent of C .

We assume now that $\text{supp}(\mu_o)$ contains at least two points and define the map $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$ for $x \in \tilde{X}$ by

$$(4.9) \quad \tilde{F}(x) = \mathcal{C}(e^{-B(x, \theta)} \mu_x).$$

Here are a few notations. For a subspace E of $T_x \tilde{X}$, we will write $\text{Jac}_E \tilde{F}(x)$ the determinant of the matrix of the restriction of $D\tilde{F}(x)$ to E with respect to orthonormal bases of E and $D\tilde{F}(x)E$. For an integer p ,

we denote by $Jac_p \tilde{F}(x)$ the supremum of $|Jac_E \tilde{F}(x)|$ as E runs through the set of p -dimensional subspaces of $T_x \tilde{X}$.

Lemma 4.3. *The map \tilde{F} is smooth, homotopic to the Identity and verifies for all $x \in \tilde{X}$, $\gamma \in C$ and $p \in [2, n = \dim(X)]$,*

$$(i) \quad \tilde{F}(\gamma x) = \gamma \tilde{F}(x)$$

$$(ii) \quad |Jac_p \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

Proof :

The map

$$(x, y) \rightarrow \int_{\partial \tilde{X}} e^{B(y, \theta) - B(x, \theta)} d\mu_x(\theta) = \int_{\partial \tilde{X}} e^{B(y, \theta) - (\delta+1)B(x, \theta)} d\mu_o(\theta)$$

is smooth because $y \rightarrow B(y, \theta)$ is smooth.

For all x the map $y \rightarrow \int_{\partial \tilde{X}} e^{B(y, \theta) - (\delta+1)B(x, \theta)} d\mu_o(\theta)$ is strictly convex by (4.4) and tends to infinity when y tend to $\partial \tilde{X}$ (cf. lemma 4.1), thus the unique minimum $\tilde{F}(x)$ is a smooth function. The equivariance of \tilde{F} comes from the cocycle relation $B(\gamma y, \gamma \theta) - B(\gamma x, \gamma \theta) = B(y, \theta) - B(x, \theta)$.

For each $x \in \tilde{X}$ let c_x be the geodesic in \tilde{X} such that $c_x(0) = x$, $c_x(1) = \tilde{F}(x)$ and which is parametrized with constant speed. The map $\tilde{F}_t : \tilde{X} \rightarrow \tilde{X}$ defined by $\tilde{F}_t(x) = c_x(t)$ is a C -equivariant homotopy between $Id_{\tilde{X}}$ and \tilde{F} .

It remains to prove (ii).

The point $\tilde{F}(x)$ is characterized by

$$(4.10) \quad \int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0.$$

In order to simplify the notations we will write $B_{(x, \theta)}$ instead of $B(x, \theta)$ and we will denote ν_x the measure $e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$. We will also write $D\tilde{F}(u)$ instead of $D\tilde{F}(x)(u)$.

The differential of \tilde{F} is characterized by the following: for $u \in T_x \tilde{X}$ and $v \in T_{\tilde{F}(x)} \tilde{X}$, one has

$$(4.11) \quad \int_{\partial \tilde{X}} [DdB_{(\tilde{F}(x), \theta)}(D\tilde{F}(u), v) + DB_{(\tilde{F}(x), \theta)}(v)DB_{(\tilde{F}(x), \theta)}(D\tilde{F}(u))] d\nu_x(\theta) \\ = (\delta + 1) \int_{\partial \tilde{X}} DB_{(\tilde{F}(x), \theta)}(v)DB_{(x, \theta)}(u) d\nu_x(\theta).$$

We define the quadratic forms k and h for $v \in T_{\tilde{F}(x)} \tilde{X}$ by

$$(4.12) \quad k(v, v) = \int_{\partial \tilde{X}} [DdB_{(\tilde{F}(x), \theta)}(v, v) + (DB_{(\tilde{F}(x), \theta)}(v))^2] d\nu_x(\theta).$$

and

$$(4.13) \quad h(v, v) = \int_{\partial\tilde{X}} DB_{(\tilde{F}(x), \theta)}(v)^2 d\nu_x(\theta).$$

The relation (4.11) writes, for $u \in T_x\tilde{X}$ and $v \in T_{\tilde{F}(x)}\tilde{X}$:

$$(4.14) \quad k(D\tilde{F}(u), v) = (\delta + 1) \int_{\partial\tilde{X}} DB_{(\tilde{F}(x), \theta)}(v) DB_{(x, \theta)}(u) d\nu_x(\theta).$$

We defines the quadratic form h' on $T_x\tilde{X}$ for $u \in T_x\tilde{X}$ by

$$(4.15) \quad h'(u, u) = \int_{\partial\tilde{X}} DB_{(x, \theta)}(u)^2 d\nu_x(\theta),$$

and one derives from (4.14)

$$(4.16) \quad |k(D\tilde{F}(x)(u), v)| \leq (\delta + 1)h(v, v)^{1/2}h'(u, u)^{1/2}.$$

One now can estimate $Jac_P\tilde{F}(x)$. Let $P \subset T_x\tilde{X}$, $dim P = p$. If $D\tilde{F}(P)$ has dimension lower than p , then there is nothing to be proven. Let us assume that $dim D\tilde{F}(P) = p$. Denote by the same letters H' [resp. H and K] the selfadjoint operators (with respect to \tilde{g}) associated to the quadratic forms h' [resp. h , k] restricted to P [resp. $D\tilde{F}(P)$].

Let $(v_i)_{i=1}^p$ an orthonormal basis of $D\tilde{F}(P)$ which diagonalizes H and $(u_i)_{i=1}^p$ an orthonormal basis of P such that the matrix of $K \circ D\tilde{F}(x) : P \rightarrow D\tilde{F}(P)$ is triangular. Then,

$$(4.17) \quad \det K \cdot |Jac_P\tilde{F}(x)| \leq (\delta + 1)^p (\prod_{i=1}^p h(v_i, v_i)^{1/2}) (\prod_{i=1}^p h'(u_i, u_i)^{1/2})$$

thus,

$$(4.18) \quad \det K \cdot |Jac_P\tilde{F}(x)| \leq (\delta + 1)^p \left(\frac{Trace H}{p} \right)^{p/2} \left(\frac{Trace H'}{p} \right)^{p/2}.$$

In these inequalities one can normalize the measures

$$\nu_x = e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$$

such that their total mass equals one, which gives

$$(4.19) \quad trace H = \sum_{i=1}^p h(v_i, v_i) \leq 1,$$

the last inequality coming from the fact that for all $\theta \in \partial\tilde{X}$,

$$(4.20) \quad \sum_{i=1}^p DB_{(\tilde{F}(x), \theta)}(v_i)^2 \leq \|\nabla B(\tilde{F}(x), \theta)\|^2 = 1$$

and from the previous normalization.

Similarly,

$$(4.21) \quad \text{trace}H' = \sum_{i=1}^p h'(u_i, u_i) \leq 1.$$

We then obtain with (4.18)

$$(4.22) \quad \det K \cdot |\text{Jac}_P \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

Thanks to (4.3), we have $\det K \geq 1$, so

$$(4.23) \quad |\text{Jac}_P \tilde{F}(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

We get (ii) by taking the supremum in P . \square

As the map $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$ is C -equivariant, then for every subgroup $C' \subset C$, \tilde{F} gives rise to a map $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$ and so does the homotopy \tilde{F}_t between \tilde{F} and $\text{Id}_{\tilde{X}}$.

Corollary 4.4. *The map $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$ is homotopic to the Identity map and verifies for all $x \in \tilde{X}/C'$ and $p \in [2, n = \dim X]$*

$$|\text{Jac}F'_p(x)| \leq \left(\frac{\delta+1}{p}\right)^p.$$

Let $C \subset \text{Isom}(\tilde{X}, \tilde{g})$ as above, ie such that the support of the Patterson-Sullivan measures contains at least two points and with critical exponent δ . Let $C' \subset C$ be a subgroup.

Let us consider an compact hypersurface $Z' \subset \tilde{X}/C'$.

Denote $F'^k = F' \circ F' \circ \dots \circ F'$ the composition of F' k -times.

Let us write $g_k = (F'^k)^*g$, where g is the metric on \tilde{X}/C' induced by \tilde{g} . The symmetric 2-tensor g_k may not be a riemannian metric on \tilde{X}/C' nor its restriction to Z' , so we have to modify it. For $\epsilon > 0$, the following symmetric 2-tensor $g_{\epsilon,k}$ is a riemannian metric on \tilde{X}/C' and so is its restriction $h_{\epsilon,k}$ to Z' .

$$(4.24) \quad g_{\epsilon,k} = g_k + \epsilon^2 g.$$

Lemma 4.5. *Let $h_{\epsilon,k}$ be the restriction of $g_{\epsilon,k}$ to the hypersurface Z' and $g_{Z'}$ the restriction of g to Z' . Let $\Phi_{\epsilon,k} : Z' \rightarrow \mathbb{R}$ the density defined for all $x \in Z'$ by $dv_{h_{\epsilon,k}}(x) = \Phi_{\epsilon,k}(x) dv_{g_{Z'}}(x)$. For any sequence ϵ_k such that $\lim_{k \rightarrow \infty} \epsilon_k = 0$, there exists a sequence ϵ'_k , $\lim_{k \rightarrow \infty} \epsilon'_k = 0$, such that for all $x \in Z'$,*

$$0 < \Phi_{\epsilon'_k,k}(x) \leq |\text{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

In particular,

$$\Phi_{\epsilon'_k,k}(x) \leq \left(\frac{\delta+1}{n-1}\right)^{k(n-1)} + \epsilon_k$$

and

$$\text{vol}(Z', h_{\epsilon'_k, k}) \leq \left[\left(\frac{\delta + 1}{n - 1} \right)^{k(n-1)} + \epsilon_k \right] \text{vol}(Z', g_{Z'}).$$

Corollary 4.6. *Under the above assumptions, if $\delta < n - 2$ there exists a sequence ϵ'_k such that $\lim_{k \rightarrow \infty} \epsilon'_k = 0$, and $\lim_{k \rightarrow \infty} \text{vol}(Z', h_{\epsilon'_k, k}) = 0$.*

Proof of lemma 4.5 :

Let us fix k an integer. Let $x \in Z'$ and $u \in T_x Z'$. We have $g_{\epsilon, k}(u, u) = h_{\epsilon, k}(u, u) = g(DF'^k(x)(u), DF'^k(x)(u)) + \epsilon^2 g(u, u)$ thus $h_{\epsilon, k}(u, u) = g(A_{x, \epsilon} u, u)$ where $A_{x, \epsilon} \in \text{End}(T_x Z')$ is the self adjoint operator $A_x = DF'^k(x)^* \circ DF'^k(x) + \epsilon^2 \text{Id}$, with $DF'^k(x)^*$ the adjoint of $DF'^k(x) : (T_x Z', g(x)) \rightarrow (DF'^k(x)(T_x Z'), g(F'^k(x)))$.

By compactness of Z' and continuity of $A_{x, \epsilon}$, there exist ϵ'_k such that

$$\Phi_{\epsilon'_k, k}(x) = \det A_{x, \epsilon'_k}^{1/2} \leq \det A_{x, 0} + \epsilon_k,$$

thus

$$\Phi_{\epsilon'_k, k}(x) \leq |\text{Jac}_{n-1} F'^k(x)| + \epsilon_k.$$

The lemma then follows from Corollary 4.4. \square

5. PROOF OF THEOREM 1.2

This section is devoted to the proof of the Theorem 1.2. Let Γ be a discrete cocompact group of isometries of a n -dimensional Cartan-Hadamard manifold (\tilde{X}, \tilde{g}) whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. We assume that $\Gamma = A *_C B$. At the end of section 2 we constructed a subgroup $C' \subset C$ and an orientable hypersurface $\tilde{Z}' \subset \tilde{X}$ such that $C' \cdot \tilde{Z}' = \tilde{Z}'$ and $Z' = \tilde{Z}'/C'$ is compact in \tilde{X}/C' . Moreover Z' is essential in \tilde{X}/C' ie $i_*([Z']) \neq 0$ where $i_* : H_{n-1}(Z', \mathbb{R}) \rightarrow H_{n-1}(\tilde{X}/C', \mathbb{R})$ is the morphism induced on homology groups by the inclusion $i : Z' \rightarrow \tilde{X}/C'$ and $[Z']$ the fundamental class of Z' .

5.1 Proof of the inequality

We now prove the inequality in the theorem 1.2. Let us assume that $\delta < n - 2$ and derive a contradiction. Let $h_{\epsilon'_k, k}$ the sequence of metric defined on Z' in lemma 4.5, then by corollary 4.6 we have

$$(5.1) \quad \lim_{k \rightarrow \infty} \text{vol}(Z', h_{\epsilon'_k, k}) = 0.$$

We now show that the systole of the metric $h_{\epsilon'_k, k}$ on Z' is bounded below independently of k . Recall that the systole of $i : Z' \rightarrow \tilde{X}/C'$ at a point $z \in Z'$ with respect to a metric $h_{\epsilon, k}$ can be defined by

$$(5.2) \quad \text{sys}_1(Z', h_{\epsilon, k}, z) = \inf_{\gamma \in C'} \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon, k})}(\tilde{z}, \gamma \tilde{z})$$

where \tilde{z} is any lift of z and $\tilde{h}_{\epsilon, k}$ the lift on \tilde{Z}' of $h_{\epsilon, k}$, (cf. lemma 3.4).

Let $\alpha(t)$ be a minimizing geodesic between \tilde{z} and $\gamma\tilde{z}$ on $(Z', \tilde{h}_{\epsilon,k})$. By definition of $\tilde{h}_{\epsilon,k}$ we have

$$(5.3) \quad \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq l_{\tilde{g}}(\tilde{F}^k \circ \alpha)$$

where $l_{\tilde{g}}$ stands for the length with respect to \tilde{g} on \tilde{X} .
We get

$$(5.4) \quad \text{dist}_{(\tilde{Z}', \tilde{h}_{\epsilon,k})}(\tilde{z}, \gamma\tilde{z}) \geq \text{dist}_{(\tilde{X}, \tilde{g})}(\tilde{F}^k(\tilde{z}), \gamma\tilde{F}^k(\tilde{z})) \geq \rho$$

where ρ is the injectivity radius of \tilde{X}/C' .
We then have

$$(5.5) \quad \text{sys}_i(Z', h_{\epsilon,k}) \geq \rho,$$

and thanks to the Theorem 3.3 we obtain

$$(5.6) \quad \text{vol}(Z', h_{\epsilon',k}) \geq C_n \rho^{n-1}$$

which contradicts (5.1).

□

5.2 Proof of the equality case

There will be several steps.

Step 1: The limit set of C is contained in a topological equator.

Step 2: The weak tangent to $\partial\tilde{X}$ and $\Lambda_{C'}$.

Step 3: The limit set $\Lambda_{C'}$ of C' and the limit set Λ_C of C are equal to a topological equator.

Step 4: C' and C are convex cocompact.

Step 5: C preserves a copy of the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n-1}$ totally geodesically embedded in \tilde{X} .

Step 6: Conclusion

Step 1: The limit set Λ_C of C is contained in a topological equator.

Let $x \in \tilde{X}$ and $E \subset T_x\tilde{X}$ a codimension one subspace. For each $u \in T_x\tilde{X}$, $\tilde{g}(u, u) = 1$, one considers the geodesic c_u defined by $c_u(0) = x$ and $\dot{c}_u(0) = u$. We define the equator $E(\infty)$ associated to E as the subset of $\partial\tilde{X}$

$$(5.7) \quad E(\infty) = \{c_u(+\infty)/u \in E\}$$

Our goal is to prove the existence of a point $x \in \tilde{X}$ and an hyperplane $E \subset T_x \tilde{X}$ such that the limit set Λ_C satisfies $\Lambda_C \subset E(\infty)$.

Recall that $C' \subset C$ globally preserves an hypersurface \tilde{Z}' such that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact and essential.

Let us also recall that we have constructed a C -equivariant map $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$ such that, for all $x \in \tilde{X}$,

$$(5.8) \quad |Jac_{n-1} \tilde{F}(x)| \leq \left(\frac{\delta + 1}{n - 1} \right)^{n-1}$$

where the critical exponent δ of C satisfies $\delta = n - 2$, thus

$$(5.9) \quad |Jac_{n-1} \tilde{F}(x)| \leq 1.$$

The step 1 follows from the two following Propositions.

Proposition 5.1. *Let $x \in \tilde{X}$ such that $|Jac_{n-1} \tilde{F}(x)| = 1$. Then there exists $E \subset T_x \tilde{X}$ such that the limit set Λ_C satisfies $\Lambda_C \subset E(\infty)$. Moreover, $\tilde{F}(x) = x$ and $D\tilde{F}(x)$ is the orthogonal projector onto E .*

Proposition 5.2. *There exists $x \in \tilde{X}$ such that $|Jac_{n-1} \tilde{F}(x)| = 1$.*

Proof of Proposition 5.1

Let $x \in \tilde{X}$ such that $|Jac_{n-1} \tilde{F}(x)| = 1$. By definition we have a subspace $E \subset T_x \tilde{X}$ such that $|Jac_E \tilde{F}(x)| = 1$. By (4.18) and $\det K \geq 1$ we have,

$$(5.10) \quad \begin{aligned} |Jac_E \tilde{F}(x)| &\leq (n-1)^{n-1} \left(\frac{\text{trace} H}{n-1} \right)^{\frac{n-1}{2}} \left(\frac{\text{trace} H'}{n-1} \right)^{\frac{n-1}{2}} \\ &\leq (n-1)^{n-1} \left(\frac{1}{n-1} \right)^{n-1}. \end{aligned}$$

In particular as $|Jac_E \tilde{F}(x)| = 1$, we have equality in the inequalities (5.10), thus, $\text{trace} H = \text{trace}(h) = 1$, and

$$(5.11) \quad H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}.$$

Let us recall that the quadratic form h is defined by

$$h(v, v) = \int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta)(v)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)$$

where μ_x is the Patterson-Sullivan measure of C normalized by

$$(5.12) \quad \int_{\partial \tilde{X}} e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 1.$$

We then have

$$\begin{aligned} 1 &= \text{trace}(h) = \text{trace}H = \sum_{i=1}^{n-1} h(v_i, v_i) = \\ &= \int_{\partial\tilde{X}} \sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \\ &\leq \int_{\partial\tilde{X}} \|DB(\tilde{F}(x), \theta)\|^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \leq 1, \end{aligned}$$

because $\sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 \leq \|DB(\tilde{F}(x), \theta)\|^2 = 1$
for all $\theta \in \partial\tilde{X}$.

Therefore for μ_x -almost all $\theta \in \text{supp}(e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)) = \text{supp}(\mu_x)$, we have

$$(5.13) \quad \sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 = \|DB(\tilde{F}(x), \theta)\|^2 = 1.$$

In (5.12), $\sum_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2$ represents the square of the norm of the projection of $\nabla B(\tilde{F}(x), \theta)$ on E .

By continuity of $B(x, \theta)$ in θ one then gets $\Lambda_C = \text{supp}(\mu_x) \subset E(\infty)$.

Let us now prove that $\tilde{F}(x) = x$. When $\text{Jac}\tilde{F}_E(x) = 1$, we have equality in the Cauchy-Schwarz inequality (4.16), therefore for each $i = 1, \dots, n-1$ and $\theta \in \Lambda_C$ we get $DB(\tilde{F}(x), \theta)(v_i) = DB(x, \theta)(u_i)$. Therefore we deduces from (5.13) that $\nabla B(x, \theta) = \sum_{i=1}^{n-1} DB(x, \theta)(u_i)u_i$, which imply with (4.10) that

$$(5.14) \quad \int_{\partial\tilde{X}} DB(x, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0.$$

On the other hand, as $\int_{\partial\tilde{X}} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0$ and $H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E)}$, the support of μ_x cannot be just a pair of points, therefore the barycenter of the measure $e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x$ defined in [2] is well defined and characterized as the point $z \in \tilde{X}$ such that

$$\int_{\partial\tilde{X}} DB(z, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0,$$

thus (5.14) and (4.10) imply $x = \tilde{F}(x)$. \square

Proof of Proposition 5.2 :

If we knew that there exists a minimizing hypersurface Z_0 in the homology class of Z , then every points $x \in Z_0$ would verify $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$. We unfortunately don't know if there exists such a minimizing hypersurface nor a minimizing current in the homology class of Z . Instead we will consider an $L^2(\tilde{X}/C')$ harmonic $(n-1)$ -form dual to the homology class of Z .

We need the following lemmas in order to prove the existence of such a dual form.

Let $\lambda_1(\tilde{X}/C')$ be the bottom of the spectrum of the Laplacian on (\tilde{X}, \tilde{g}) , ie.

$$(5.15) \quad \lambda_1(\tilde{X}/C') = \inf_{u \in C_0^\infty(\tilde{X}/C')} \left\{ \frac{\int_{\tilde{X}/C'} |du|^2}{\int_{\tilde{X}/C'} u^2} \right\}.$$

Lemma 5.3. *Let $C \subset \text{Isom}(\tilde{X}, \tilde{g})$ a discrete group of isometries with critical exponent $\delta = n - 2$ where (\tilde{X}, \tilde{g}) is an n -dimensional Cartan-Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$. Then for any subgroup $C' \subset C$ we have $\lambda_1(\tilde{X}/C') \geq n - 2$.*

Proof :

Thanks to a theorem of Barta, cf.[17] Theorem 2.1, the lemma boils down to finding a positive function $c : \tilde{X}/C' \rightarrow \mathbb{R}_+$ such that $\Delta c(x) \geq (n-2)c(x)$. Here, the laplacian Δ is the positive operator ie. $\Delta c = -\text{trace} Ddc$. We consider the smooth function $\tilde{c} : \tilde{X} \rightarrow \mathbb{R}_+$ defined by $\tilde{c}(x) = \mu_x(\partial\tilde{X})$ where $\{\mu_x\}_{x \in \tilde{X}}$ is a family of Patterson-Sullivan measure of C . The function \tilde{c} is C -equivariant therefore it defines a map $c : \tilde{X}/C' \rightarrow \mathbb{R}_+$ for any subgroup $C' \subset C$. Let us show

$$(5.16) \quad \Delta c(x) \geq \delta(n-1-\delta)c(x) = n-2.$$

We have

$$\tilde{c}(x) = \int_{\partial\tilde{X}} e^{-\delta B(x,\theta)} d\mu_o(\theta)$$

therefore

$$\Delta \tilde{c}(x) = \int_{\partial\tilde{X}} [-\delta \Delta B(x,\theta) - \delta^2] d\mu_x(\theta).$$

The sectional curvature $K_{\tilde{g}}$ of (\tilde{X}, \tilde{g}) satisfies $K_{\tilde{g}} \leq -1$ we thus have $-\Delta B(x,\theta) \geq n-1$ and as $\delta = n-2$ we get

$$\Delta \tilde{c}(x) \geq [\delta(n-1) - \delta^2] \tilde{c}(x) = (n-2)\tilde{c}(x).$$

□

The following lemma is due to G.Carron and E.Pedon, [8]. For a complete riemannian manifold Y , we denote $H_c^1(Y, \mathbb{R})$ the first cohomology group generated by differential forms with compact support.

Lemma 5.4 ([8], Lemme 5.1). *Let Y be a complete riemannian manifold all ends of whose having infinite volume and such that $\lambda_1(Y) > 0$, then the natural morphism*

$$H_c^1(Y, \mathbb{R}) \rightarrow H_{L^2}^1(Y, \mathbb{R})$$

is injective. In particular any $\alpha \in H_c^1(Y, \mathbb{R})$ admits a representative $\bar{\alpha}$ which is in $L^2(Y, \mathbb{R})$.

Corollary 5.5. *Let C' be as above and assume that there exists a compact essential hypersurface $Z' \subset \tilde{X}/C'$. Then there exists an harmonic $n-1$ -form ω in $L^2(\tilde{X}/C')$ such that $\int_{Z'} \omega \neq 0$.*

Proof :

Let $\alpha \in H_c^1(\tilde{X}/C', \mathbb{R})$ a Poincaré dual of $[Z'] \in H_{n-1}(\tilde{X}/C', \mathbb{R})$. By definition of α , for any $\beta \in H^{n-1}(\tilde{X}/C', \mathbb{R})$, one has

$$(5.17) \quad \int_{Z'} \beta = \int_{\tilde{X}/C'} \beta \wedge \alpha,$$

([5] p.51, note that \tilde{X}/C' has a "finite good cover").

After Lemma 5.4, α admits a non trivial harmonic representative $\bar{\alpha}$ in $L^2(\tilde{X}/C')$. (In order to apply the Lemma 5.4, one has to check that all ends of \tilde{X}/C' have infinite volume, ie for a compact $K \subset \tilde{X}/C'$ each unbounded connected component of $\tilde{X}/C' - K$ has infinite volume: this comes from the fact that the injectivity radius of \tilde{X}/C' is bounded below by the injectivity radius of $X = \tilde{X}/\Gamma$ and the sectional curvature bounded above by -1 .) The $(n-1)$ -harmonic form $\omega = *\bar{\alpha}$, where $*$ is the Hodge operator, is in $L^2(\tilde{X}/C')$ and verifies after (5.17)

$$(5.18) \quad \int_{Z'} \omega = \int_{\tilde{X}/C'} \omega \wedge \bar{\alpha} = \int_{\tilde{X}/C'} \omega \wedge *\omega = \|\omega\|_{L^2(\tilde{X}/C')}^2 \neq 0.$$

□

We can now prove the proposition 5.2. Let us briefly describe the idea. We consider the iterates F'^k of $F' : \tilde{X}/C' \rightarrow \tilde{X}/C'$. As F' is homotopic to the identity map, $F'^k(Z')$ is homologous to Z' and if ω is the harmonic form of the corollary 5.5 we have

$$(5.19) \quad \int_{Z'} (F'^k)^*(\omega) = \int_{Z'} \omega = a \neq 0.$$

We don't know if $F'^k(Z')$ converges or stays in a compact subset of \tilde{X}/C' but we will show that $F'^k(Z')$ cannot entirely diverge in \tilde{X}/C' and that there exists a $z' \in Z'$ such that $F'^k(z')$ subconverges to a point $x \in \tilde{X}/C'$ with $|Jac_{n-1}F'(x)| = 1$.

From (5.19) one gets

$$(5.20) \quad 0 < |a| = \left| \int_{Z'} (F'^k)^*(\omega) \right| \leq \int_{Z'} |Jac_{T_{Z'}F'^k}(z)| \cdot \|\omega(F'^k(z))\| dz$$

where

$$|Jac_{T_{Z'}F'^k}(z)| = \|DF'^k(z)(u_1) \wedge DF'^k(z)(u_2) \wedge \dots \wedge DF'^k(z)(u_{n-1})\|$$

and (u_1, \dots, u_{n-1}) is an orthonormal basis of $T_z(Z')$.

Let us define $\mathcal{B} = \{z \in Z', |JacF'_{T_{Z'}}(z)| \text{ does not converge to } 0\}$.

For $z \in Z'$ we define the sequence z_k by $z_0 = z$ and $z_k = F'(z_{k-1}) = F'^k(z) \in \tilde{X}/C'$.

Lemma 5.6. *There exists $z \in \mathcal{B}$ and a subsequence z_{k_j} such that z_{k_j} converges to a point $x \in \tilde{X}/C'$ with $|Jac_{n-1}(x)| = 1$.*

Proof :

We first remark that $\lim_{x \rightarrow \infty} \|\omega(x)\| = 0$. This follows the following facts: ω is harmonic, $\omega \in L^2(\tilde{X}/C')$ and the injectivity radius of \tilde{X}/C' is bounded below by a positive constant.

Let us assume that for all $z \in \mathcal{B}$ the sequence z_k diverges in \tilde{X}/C' . Then we have, for all $z \in \mathcal{B}$,

$$(5.21) \quad \|\omega(z_k)\| = \|\omega(F'^k(z))\| \rightarrow 0$$

whenever k tends to ∞ because of the previous remark.

On the other hand, as $\|\omega(F'^k(z))\| \leq C$ and $|Jac_{n-1}F'^k| \leq 1$, it follows from (5.20)

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| \int_{Z'} (F'^k)^*(\omega) \right| \leq \\ & \lim_{k \rightarrow \infty} \left[\int_{\mathcal{B}} \|\omega(F'^k(z))\| dz + C \int_{Z' - \mathcal{B}} |Jac_{Z'} F'^k(z)| dz \right] = 0 \end{aligned}$$

which contradicts our assumption.

Thus there exists a point $z \in Z'$ such that

$$(5.22) \quad |Jac_{Z'} F'^k(z)| \rightarrow \alpha \neq 0$$

and such that there exists a subsequence $z_{k_j} = F'^{k_j}(z)$ with

$$(5.23) \quad \lim_{j \rightarrow \infty} z_{k_j} = x \in \tilde{X}/C'.$$

The property (5.22) comes from the fact that the sequence $|Jac_{Z'} F'^k(z)|$ doesn't tend to zero and is decreasing (because $|Jac_{n-1}F'| \leq 1$).

Let us define

$$E_0 = T_z Z', E_1 = DF'(z)(E_0)$$

and

$$E_k = DF'(z_{k-1})(E_{k-1}) \subset T_{z_k}(\tilde{X}/C').$$

As $z_{k_j} \rightarrow x$ we can assume, after extracting again a subsequence, that $E_{k_j} \rightarrow E \subset T_x(\tilde{X}/C')$. On the other hand we also have

$$(5.24) \quad |Jac_{Z'} F'^k(z)| = |Jac_{E_{k-1}} F'(z_{k-1})| |Jac_{E_{k-2}} F'(z_{k-2})| \dots |Jac_{E_0} F'(z)|$$

We know that $|Jac_{E_k} F'(z_k)| = 1 - \epsilon_k$ where $0 \leq \epsilon_k < 1$. As $z \in \mathcal{B}$, we have

$$\lim_{k \rightarrow \infty} \prod_{j=1}^k (1 - \epsilon_j) = \alpha > 0$$

therefore $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and by continuity we have $|Jac_E F'(x)| = 1$. \square

Now we can finish the proof of the step 1. Let consider a lift of x in \tilde{X} and E in $T\tilde{X}$ that we again call x and E . Then we have $|Jac_E \tilde{F}'(x)| = 1$. \square

Let us remark that corollary 4.4 and (5.19) give another proof of the inequality $\delta \geq n - 2$, which does not use the isosystolic inequality, ie. Theorem 3.3.

Step 2 : The weak tangent of $\partial\tilde{X}$ and Λ_C

We first recall the definition of the Gromov-distance on $\partial\tilde{X}$. For two arbitrary points θ and θ' in $\partial\tilde{X}$ let us define

$$(5.25) \quad l(\theta, \theta') = \inf\{t > 0 / \text{dist}(\alpha_\theta(t), \alpha_{\theta'}(t)) = 1\}$$

and

$$(5.26) \quad d(\theta, \theta') = e^{-l(\theta, \theta')}$$

then d is a distance on $\partial\tilde{X}$.

We now recall a few definitions following [4]. A complete metric space (S, \bar{d}) is a weak tangent of a metric space (Z, d) if there exist a point $0 \in S$, a sequence of points $z_k \in Z$ and a sequence of positive real numbers $\lambda_k \rightarrow \infty$ such that the sequence of pointed metric spaces $(Z, \lambda_k d, z_k)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(Z, \lambda_k d)$ stands for the set Z endowed with the rescaled metric $\lambda_k d$.

Let us recall that the sequence of metric spaces (Z_k, d_k, z_k) converges to $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology if the following conditions hold, (cf. (B-B-I), definition 8.1.1).

Definition 5.7. *We say that the sequence of metric spaces (Z_k, d_k, z_k) converges to (S, \bar{d}, o) if for any $R > 0$, $\epsilon > 0$ there exists k_0 such that for any $k \geq k_0$ there exists a (non necessary continuous) map $f : B(z_k, R) \rightarrow S$ such that*

(i) $f(z_k) = 0$,

for any two points x and y in $B(z_k, R)$,

(ii) $|\bar{d}(f(x), f(y)) - d_k(x, y)| \leq \epsilon$,

and

(iii) the ϵ -neighborhood of the set $f(B(z_k, R))$ contains $B(0, R - \epsilon)$.

In the previous definition, $B(z_k, R)$ stands for the ball of radius R centered at the point z_k in (Z_k, d_k) .

For a metric space (Z, d) we will denote $WT(Z, d)$ the set of weak tangents of (Z, d) .

Let Γ a cocompact group of isometries of (\tilde{X}, \tilde{g}) a n -dimensional Cartan Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$ and C' a subgroup of Γ . The limit set Λ_Γ of Γ is the full boundary $\partial\tilde{X}$ namely a topological $(n-1)$ -dimensional sphere S^{n-1} . We endow $\partial\tilde{X}$ with the Gromov distance d defined in (5.26). In [4] Lemma 5.2, M.Bonk and B.Kleiner show among other properties the following

Lemma 5.8. *For any weak tangent space (S, \bar{d}) in $WT((\partial\tilde{X}, d))$, S is homeomorphic to $\partial\tilde{X}$ less a point, thus to \mathbb{R}^{n-1} .*

In fact the crucial assumption in the above lemma, coming from the cocompactness of Γ , is the property that any triple of points in $\partial\tilde{X}$ can be uniformly separated by an element of Γ , ie. there is $\delta > 0$ such that for any three points $\theta_1, \theta_2, \theta_3 \in \partial\tilde{X}$ there exists a $\gamma \in \Gamma$ such that $d(\gamma\theta_i, \gamma\theta_j) \geq \delta$ for all $1 \leq i \neq j \leq 3$. Following the argument of M.Bonk and B.Kleiner one can show that if C' is a subgroup of Γ such that one weak tangent (S, \bar{d}) of $(\Lambda_{C'}, d)$ is in $WT(\partial\tilde{X}, d)$ and enough triples of points of $\Lambda_{C'}$ can be uniformly separated by elements of C' , then S is homeomorphic to $\Lambda_{C'}$ less a point. In particular, $\Lambda_{C'}$ is homeomorphic to $\partial\tilde{X}$.

Lemma 5.9. *Let $\mathcal{L} \subset \partial\tilde{X}$ be a closed C' -invariant set and $\theta_0 \in \mathcal{L}$. We assume that there exist a sequence of positive real numbers $\lambda_k \rightarrow \infty$ such that the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(S, \bar{d}, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and δ , a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, S is homeomorphic to \mathcal{L} less a point. In particular \mathcal{L} is homeomorphic to $\partial\tilde{X}$.*

The proof of this lemma is postponed in the Appendix.

Step 3 : The limit set $\Lambda_{C'}$ of C' and the limit set Λ_C of C are equal to a topological equator.

We have shown in step 1 that Λ_C is a subset of some topological equator $E(\infty)$.

Let $o \in \tilde{X}$ and $E \subset T_o\tilde{X}$ be such that $|Jac_E\tilde{F}(o)| = 1$ and $E(\infty)$ is the equator associated to E .

Recall that there exists a subgroup C' of C which globally preserves an hypersurface $\tilde{Z}' \subset \tilde{X}$ and that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact. Furthermore \tilde{Z}' separates \tilde{X} into two connected components \tilde{U} et \tilde{V} . We can assume that \tilde{U} and \tilde{V} are globally invariant by C' after having replaced C' by an index 2 subgroup.

The limit set $\Lambda_{C'}$ of C' is contained in Λ_C , therefore $\Lambda_{C'} \subset E(\infty)$.

We will show that $\Lambda_{C'} = E(\infty)$.

For any subset $W \in \tilde{X}$ we define the boundary at infinity ∂W of W by

$$(5.27) \quad \partial W = Cl(W) \cap \partial \tilde{X}$$

where $Cl(W)$ stands for the closure of W in $\tilde{X} \cup \partial \tilde{X}$.

As \tilde{Z}'/C' is compact, \tilde{Z}' is at bounded distance of the orbit $C'z$ of some point z in \tilde{Z}' , thus

$$(5.28) \quad Cl(\tilde{Z}') \cap \partial \tilde{X} = \Lambda_{C'}$$

By definition we have $\Lambda_{C'} \subset \partial \tilde{U}$ and $\Lambda_{C'} \subset \partial \tilde{V}$.

Lemma 5.10. *Let us assume that $\Lambda_{C'} \neq E(\infty)$, then either $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$.*

Proof: We know that $\Lambda_{C'} \subset \partial \tilde{U}$ and $\Lambda_{C'} \subset \partial \tilde{V}$.

Let us assume that the conclusion of the lemma is not true, so there is $\zeta \in \partial \tilde{U} - \Lambda_{C'}$ and $\theta \in \partial \tilde{V} - \Lambda_{C'}$.

As $\Lambda_{C'} \subset E(\infty)$, $\Lambda_{C'} \neq E(\infty)$ and $\partial \tilde{X}$ is a sphere, any two points of $\partial \tilde{X} - \Lambda_{C'}$ can be joined by a continuous path contained in $\partial \tilde{X} - \Lambda_{C'}$ and so does ζ and θ , joined by such a path α .

The set $Z' \cup \Lambda_{C'}$ is a closed subset of $\tilde{X} \cup \partial \tilde{X}$ thus there is an open connected neighborhood W of α in $\tilde{X} \cup \partial \tilde{X}$ contained in the complementary of $Z' \cup \Lambda_{C'}$.

As ζ and θ can be approximated by points in \tilde{U} and \tilde{V} respectively there exist points $x \in \tilde{U} \cap W$ and $y \in \tilde{V} \cap W$ that can be joined by a continuous path by connectedness of W , which leads to a contradiction. \square

Remark 5.11. *In fact, we are going to show that under the assumption $\delta(C') = n - 2$, it is impossible to have $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$.*

For any $x \in \tilde{X}$ and $\theta \in \partial \tilde{X}$ let us denote $HB(x, \theta)$ the open horoball centered at θ and passing through x .

Lemma 5.12. *Let us assume that $\partial \tilde{U} = \Lambda_{C'}$. Then there exist $\theta_0 \in \Lambda_{C'}$ and $z' \in \tilde{Z}'$ such that $HB(z', \theta_0) \subset \tilde{U}$.*

Proof: Let us recall that $\tilde{X}/C' - Z' = U \cup V$ where $U = \pi(\tilde{U})$ $V = \pi(\tilde{V})$ and $\pi : \tilde{X} \rightarrow \tilde{X}/C'$ is the projection .

We know that U and V are unbounded. Let x_n a sequence of points in U such that $dist(x_n, Z') \rightarrow \infty$. Let $z_n \in Z'$ such that $dist(x_n, Z') = dist(x_n, z_n)$. We consider a fundamental domain $D \subset \tilde{Z}'$ of C' . There exist lifts $\tilde{z}_n \in D$ and $\tilde{x}_n \in \tilde{U}$ such that $dist(\tilde{x}_n, Z') = dist(\tilde{x}_n, \tilde{z}_n)$ tends to infinity.

By compactness we can assume that a subsequence \tilde{x}_{n_j} converges to a point $\theta_0 \in \partial \tilde{X}$ and \tilde{z}_{n_j} also converges to a point $\tilde{z} \in \tilde{D}$. Furthermore

the sequence of open balls $B(\tilde{x}_{n_j}, \text{dist}(\tilde{x}_{n_j}, \tilde{z}_{n_j})) \subset \tilde{U}$ converges to the open horoball $HB(\theta_0, \tilde{z}) \in \tilde{U}$. \square

Proposition 5.13. $\Lambda_{C'} = E(\infty)$ and $\Lambda_C = E(\infty)$.

We first describe the idea of the proof and next state some facts we will need in order to do it.

As $\Lambda_{C'} \subset \Lambda_C \subset E(\infty)$ the proposition boils down to proving that $\Lambda_{C'} = E(\infty)$. Let us assume $\Lambda_{C'} \neq E(\infty)$ and find a contradiction.

We will show that for any sequence θ_i converging to θ_0 the geodesic starting at a point o such that $|Jac_{n-1}\tilde{F}(o)| = 1$ and ending at θ_i crosses the hypersurface \tilde{Z}' in a point z_i . For an **appropriate choice** of such a sequence θ_i (roughly speaking, the sequence θ_i is chosen to be converging to θ_0 "transversally to $\Lambda_{C'}$ "), the shadow (defined below) projected from o through some geodesic ball $B(z_i, r)$ will not intersect $\Lambda_{C'}$. On the other hand this shadow has to meet the limit set $\Lambda_{C'}$ because of the shadow lemma of D. Sullivan, which leads to a contradiction.

Precisely, by Lemma (5.10) and Lemma (5.12) we know that $\partial\tilde{U} = \Lambda_{C'}$ and that there exists an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ centered at a point $\theta \in \Lambda_{C'}$, and whose closure contains a point $\tilde{z} \in \tilde{Z}'$.

Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ an hyperplane such that $\tilde{F}(o) = o$, $|Jac_E\tilde{F}(o)| = 1$, and $\Lambda_{C'} \subset E(\infty)$ where $E(\infty)$ is the topological equator associated to E .

For each $\theta \in \partial\tilde{X}$ we denote by α_θ the geodesic starting from o and such that $\alpha_\theta(+\infty) = \theta$.

Let $\theta_i \in \partial\tilde{X} - E(\infty) = \partial\tilde{V} - \partial\tilde{U}$ be a sequence converging to θ_0 . By continuity, for each i large enough, the geodesic α_{θ_i} spends some time inside the horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up inside \tilde{V} because θ_i converges to θ_0 and θ_i belongs to $\partial\tilde{V} - \partial\tilde{U}$.

Thus α_{θ_i} eventually crosses \tilde{Z}' . Let $z_i \in \alpha_{\theta_i} \cap \tilde{Z}'$. As \tilde{Z}'/C' is compact, there is an element $\gamma_i \in C'$ such that $z_i = \gamma_i(x_i)$ where x_i is a point in the closure \bar{D} of a fundamental domain D for the action of C' on \tilde{Z}' . The points $\gamma_i(x_i)$ and $\gamma_i(o)$ stay at bounded distance because $\text{dist}(\gamma_i(x_i), \gamma_i(o)) = \text{dist}(x_i, o) \leq \text{dist}(o, D) + \text{diam}D$. In particular, $\lim_{i \rightarrow \infty} \gamma_i(o) = \theta_0$.

We have proved the

Lemma 5.14. *Let $\theta_i \in \partial\tilde{X} - E(\infty)$ be a sequence which converges to θ_0 . There exists a constant A such that for i large enough there exists $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ and $\gamma_i \in C'$ such that $\text{dist}(z_i, \gamma_i(o)) \leq A$ and both z_i and $\gamma_i(o)$ converge to θ_0 .*

Let x and y two points in \tilde{X} .

We define the shadow $\mathcal{O}(x, y, R) \subset \partial\tilde{X}$ of the ball $B(y, R)$ enlightened from the point x by

$$(5.29) \quad \mathcal{O}(x, y, R) = \{\alpha(+\infty)\}$$

where α runs through the set of geodesic rays starting from x and meeting $B(y, R)$.

Let $\{\mu_x\}_x$ be a family of Patterson measures associated to the discrete group C' with critical exponent $\delta' = \delta(C')$.

The following shadow lemma is due to D.Sullivan.

Lemma 5.15. [18], [13], [20]. *There exist positive constants C and R such that for any y in \tilde{X} , $\nu_y(\mathcal{O}(y, \gamma(y), R)) \geq Ce^{\delta'd(y, \gamma(y))}$*

Corollary 5.16. *Let z_i be defined in lemma (5.14), then we have $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$ for i large enough.*

We now prove that for a good choice of θ_i , the shadow $\mathcal{O}(o, z_i, R+A)$ (with z_i associated to θ_i as in lemma 5.14) never meet $\Lambda_{C'}$ for all large i 's, ie. for any $\theta \in \Lambda_{C'}$ the geodesic α_θ does not cross $B(z_i, R+A)$. We have no control on the radius R coming from the shadow lemma nor on the constant A but we will show

Proposition 5.17. *There exists a sequence $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$ such that θ_i converges to θ_0 and*

$$\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = +\infty$$

where $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$ has been constructed in lemma (5.14).

Corollary 5.18. *For i large enough, $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$.*

The corollary (5.16) and the corollary (5.18) lead to a contradiction, which ends the proof of the proposition (5.13).

The end of the paragraph is devoted to proving the proposition (5.17).

Lemma 5.19. *Let θ_i be a sequence of points in $\partial\tilde{X}$ converging to θ_0 and z_i constructed in lemma (5.14). Assume that*

$$\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = C < +\infty \text{ then } \lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0.$$

Proof : We first show that

$$(5.30) \quad \lim_{i \rightarrow \infty} \text{dist}(z_i, \alpha_{\theta_0}) = \infty$$

Recall that, for any $z \in \tilde{X}$ and $\theta \in \partial\tilde{X}$, $B(z, \theta)$ equals the decreasing limit as t tends to infinity of $\text{dist}(z, \alpha_\theta(t)) - \text{dist}(o, \alpha_\theta(t))$ where $\alpha_\theta(t)$ is the geodesic ray joinging o to θ . Therefore, as the points $z_i \in \tilde{Z}$ belongs to the complementary of the fixed horoball $HB(\tilde{z}, \theta_0)$, we have,

$$(5.31) \quad \text{dist}(z_i, \alpha_{\theta_0}(T_i)) \geq T_i + B(\tilde{z}, \theta_0)$$

where $\text{dist}(z_i, \alpha_{\theta_0}(T_i)) = \text{dist}(z_i, \alpha_{\theta_0})$.

On the other hand, as z_i tends to θ_0 , T_i tends to infinity so (5.30) is proven.

Let t_i be such that $z_i = \alpha_{\theta_i}(t_i)$. By (5.30) we have

$$(5.32) \quad \lim_{i \rightarrow \infty} \text{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) = \infty.$$

Let u_i be such that

$$(5.33) \quad \text{dist}(\alpha_{\theta_i}(u_i), \alpha_{\theta_0}(u_i)) = 1,$$

then in particular $u_i \leq t_i$ for i large enough and by the triangle inequality we have

$$(5.34) \quad \text{dist}(\alpha_{\theta_i}(t_i), \alpha_{\theta_0}(t_i)) \leq 2(t_i - u_i) + 1.$$

By (5.32) we get

$$(5.35) \quad \lim_{i \rightarrow \infty} (t_i - u_i) = +\infty.$$

Let us assume there exists a sequence $\theta'_i \in \Lambda_{C'}$ and a constant C such that

$$(5.36) \quad \text{dist}(z_i, \alpha_{\theta'_i}) \leq C < +\infty.$$

We can assume that $C \geq 1$.

Let v_i be such that

$$(5.37) \quad \text{dist}(z_i, \alpha_{\theta'_i}) = \text{dist}(z_i, \alpha_{\theta'_i}(v_i)).$$

By triangle inequality,

$$(5.38) \quad |t_i - v_i| \leq C$$

and,

$$(5.39) \quad \text{dist}(\alpha_{\theta'_i}(t_i), \alpha_{\theta_i}(t_i)) \leq 2C.$$

On the other hand, as the curvature of \tilde{X} is bounded above by -1 , a classical comparison theorem gives for any $t \in [0, t_i]$,

$$(5.40) \quad \sinh\left(\frac{\text{dist}(\alpha_{\theta'_i}(t), \alpha_{\theta_i}(t))}{2}\right) \leq \sinh C \cdot \frac{\sinh t}{\sinh t_i}.$$

Let s_i be such that

$$(5.41) \quad \text{dist}(\alpha_{\theta'_i}(s_i), \alpha_{\theta_i}(s_i)) = 1.$$

There are two cases. Either $s_i \geq t_i$ or $s_i < t_i$. If $s_i < t_i$, we get from (5.39) and (5.40) the existence of a constant A such that for any i ,

$$(5.42) \quad s_i \geq t_i - A,$$

and this inequality also holds when $s_i \geq t_i$.

From (5.42) we get

$$(5.43) \quad \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = e^{-s_i+u_i} \leq e^A e^{-t_i+u_i},$$

therefore, thanks to (5.35) we obtain

$$(5.44) \quad \lim_{i \rightarrow \infty} \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = 0.$$

which ends the proof of lemma (5.19). \square

Lemma 5.20. *Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0 , $\lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$. Let $\lambda_k \rightarrow \infty$ be such that the sequence of spaces $(\partial\tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology, then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \bar{d}, 0)$.*

Proof : Let us define

$$r(\epsilon) =: \sup\left\{\frac{d(\theta, \Lambda_{C'})}{d(\theta, \theta_0)}, \theta \neq \theta_0, d(\theta, \theta_0) \leq \epsilon\right\}.$$

The assumption says that

$$(5.45) \quad \lim_{\epsilon \rightarrow 0} r(\epsilon) = 0.$$

For an arbitrary metric space (Y, d) and Y' a subset of Y , let us denote $B_{(Y,d)}(y, R)$ the closed ball of (Y, d) of radius R centered at $y \in Y$, and $\mathcal{U}_\epsilon^{(Y,d)}(Y')$ the ϵ -neighborhood of Y' in (Y, d) . For a metric space (Y, d) and a positive number λ , let us denote λY the rescaled space $(Y, \lambda d)$.

By definition of the function r , we have for any R ,

$$(5.46) \quad \begin{aligned} B_{\lambda_k \partial\tilde{X}}(\theta_0, R) &\subset \mathcal{U}_{\epsilon_k}^{\lambda_k \partial\tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k) \\ &\subset B_{\lambda_k \partial\tilde{X}}(\theta_0, R + 2\epsilon_k). \end{aligned}$$

where $\epsilon_k =: Rr(R/\lambda_k)$.

Let us fix $\alpha > 0$. By definition 5.7, for any $R > 0$, $\epsilon > 0$, there exist a map $f : B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha) \rightarrow S$ such that for $k \geq k_0$,

(i) $f(\theta_0) = 0$,

for any two points x and y in $B_{\lambda_k \partial\tilde{X}}(\theta_0, R + \alpha)$,

- (ii) $|\bar{d}(f(x), f(y)) - \lambda_k d(x, y)| \leq \epsilon$,
and
(iii) $B_{(S, \bar{d})}(0, R + \alpha - \epsilon) \subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha))$.

Moreover let us prove:

- (iv) $B_{(S, \bar{d})}(0, R - 2\epsilon) \subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R))$.

Indeed, let $z \in B_{(S, \bar{d})}(0, R - 2\epsilon)$. By (iii), there exists $\theta \in B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha)$ such that $\bar{d}(z, f(\theta)) \leq \epsilon$. Since $\bar{d}(z, 0) \leq R - 2\epsilon$, we thus deduce from triangle inequality $\bar{d}(f(\theta), 0) \leq R - \epsilon$, and therefore we get, thanks to (i) and (ii), $\lambda_k d(\theta, \theta_0) \leq R$. \square

By (5.45), for ϵ small enough, there exists $k_1 \geq k_0$ such that for any $k \geq k_1$, then $2\epsilon_k \leq \epsilon$ and

$$B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha).$$

Therefore, by (5.46) and the above properties (i),(ii),(iii) and (iv) of the map f , and the triangle inequality we get,

$$\begin{aligned} B_{(S, \bar{d})}(0, R - 2\epsilon) &\subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R)) \\ &\subset \mathcal{U}_\epsilon^{(S, \bar{d})} f(\mathcal{U}_{\epsilon_k}^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)) \\ &\subset \mathcal{U}_{2\epsilon + \epsilon_k}^{(S, \bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)). \end{aligned}$$

About the second inclusion above let us remark that the set $\mathcal{U}_{\epsilon_k}^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)$ is contained in $B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda \partial \tilde{X}}(\theta_0, R + \alpha)$, so that we can apply f to this set.

From the above inclusions we obtain

$$B_{(S, \bar{d})}(0, R - 3\epsilon) \subset \mathcal{U}_{2\epsilon + \epsilon_k}^{(S, \bar{d})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R))$$

which implies the convergence of $(\Lambda_{C'}, \lambda_k d, \theta_0)$ to $(S, \bar{d}, 0)$.

\square

Corollary 5.21. *Let us assume that for every sequence θ_i of points in $\partial \tilde{X}$ converging to θ_0 and z_i the sequence of points constructed in lemma (5.14), $\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha \theta) < +\infty$. Let $\lambda_k \rightarrow \infty$ be such that the sequence of spaces $(\partial \tilde{X}, \lambda_k d, \theta_0)$ converges to the space $(S, \delta, 0)$ in the pointed Gromov-Hausdorff topology, then the sequence of spaces $(\Lambda_{C'}, \lambda_k d, \theta_0)$ also converges to $(S, \delta, 0)$.*

We will show now that there exist a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ converging to θ_0 , such that the mutual distances $d(\theta_1^k, \theta_2^k)$, $d(\theta_1^k, \theta_0)$, $d(\theta_2^k, \theta_0)$ is tending to zero at the same rate, and the triple $\theta_1^k, \theta_2^k, \theta_0$ can be uniformly separated by elements $\gamma_k \in C'$.

Lemma 5.22. *Assume that every weak tangent of $(\partial\tilde{X}, d)$ at θ_0 belongs to $WT(\Lambda_{C'}, d)$, then there exist positive constants c, δ , a sequence ϵ_k tending to 0 when k tends to ∞ , a sequence $\gamma_k \in C'$, a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ such that for $i = 1, 2$,*

$$c^{-1}\epsilon_k \leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k,$$

$$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_0) \leq c\epsilon_k \text{ and}$$

$$d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) \geq \delta, d(\gamma_k\theta_i^k, \gamma_k\theta_0) \geq \delta.$$

Proof : For any $x \in \tilde{X} \cup \partial\tilde{X}$ and $y \in \tilde{X} \cup \partial\tilde{X}$ let us define $\alpha_{x,y}$ the geodesic ray joining x and y . Let $o \in \tilde{X}$ and $E \in T_o\tilde{X}$ be such that $|Jac_E\tilde{F}(o)| = 1$ and $E(\infty)$ the equator associated to E . Let $\gamma_k \in C'$ be a sequence such that $\gamma_k(o)$ converges to the point θ_0 where $\theta_0 \in \Lambda_{C'}$ is the point coming from lemma 5.12. In particular, according to that lemma, there exist a point $\tilde{z} \in \tilde{Z}'$ such that the hypersurface \tilde{Z}' is contained in the complementary of the open horoball $HB(\tilde{z}, \theta_0)$. We define $D := dist(\tilde{z}, o)$. As \tilde{Z}' lies outside the open horoball $HB(\tilde{z}, \theta_0)$, the points $\gamma_k(o)$ belong to the complementary of the open horoball $HB(\alpha_{\tilde{z}, \theta_0}(D), \theta_0)$. By standard triangle comparison argument (comparison with the hyperbolic case) the angle $Angle(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o})$ between the two geodesic rays $\alpha_{\gamma_k(o), \theta_0}$ and $\alpha_{\gamma_k(o), o}$ satisfies :

$$(5.47) \quad \lim_{k \rightarrow \infty} Angle(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o}) = 0.$$

By equivariance we have $\Lambda_{C'} \subset (\gamma_k E)(\infty)$ where $\gamma_k E \subset T_{\gamma_k(o)}\tilde{X}$. For any $v \in T\tilde{X}$ let α_v be the geodesic ray such that $\dot{\alpha}_v(0) = v$. Let us denote by u_k the unit vector in $\gamma_k E$ such that $\alpha_{u_k}(+\infty) = \theta_0$ and let us choose some $w_k \in \gamma_k E$ such that $\langle u_k, w_k \rangle = 0$ (this is possible because $n-1 \geq 2$).

We claim now that there exist $v_k \in \gamma_k E$ such that the angle between v_k and w_k is not too far from 0 or π , namely

$$(5.48) \quad |\langle v_k, w_k \rangle| \geq \frac{1}{(n-1)^{1/2}},$$

and $\alpha_{v_k}(+\infty) \in \Lambda_{C'}$ or $\alpha_{v_k}(-\infty) \in \Lambda_{C'}$.

Let us prove this claim.

According to Proposition 5.1 and to (5.11), the restriction to $\gamma_k E$ of the quadratic form $h(u) = \int DB(\gamma_k(o), \theta)(u)^2 d\mu_{\gamma_k(o)}(\theta)$ verifies

$$(5.49) \quad h_{\gamma_k E}(u) = \frac{\|u\|^2}{n-1}.$$

Therefore, if for all $u \in \gamma_k(E)$ such that $\alpha_u(+\infty) = \theta \in \Lambda_{C'}$ we had $|\langle u, w_k \rangle| < \frac{1}{(n-1)^{1/2}}$, then one would get $h(w_k) < \frac{1}{n-1}$, which contradicts (5.49) and proves the claim.

In particular the angle between u_k and v_k is not too far from $\pi/2$ for k large enough, ie.

$$(5.50) \quad | \langle u_k, v_k \rangle | \leq \left(\frac{n-2}{n-1} \right)^{1/2},$$

and thanks to (5.47), we have for k large enough

$$(5.51) \quad | \langle \dot{\alpha}_{\gamma_k(0),o}(0), v_k \rangle | \leq \left(\frac{n-\frac{3}{2}}{n-1} \right)^{1/2}.$$

Let us now assume for example that $\theta_k = \alpha_{v_k}(+\infty) \in \Lambda_{C'}$. Let us show that

$$(5.52) \quad \lim_{k \rightarrow \infty} d(\theta_0, \theta_k) = 0.$$

Assume that (5.52) is not true. Then, one can assume after extracting a subsequence that θ_k converges to $\theta \neq \theta_0$. Therefore the geodesic rays $\alpha_{\gamma_k(0),o}$ and α_{v_k} would converge to the geodesics $\alpha_{\theta_0,o}$ and $\alpha_{\theta_0,\theta}$ and thus the angle $\text{Angle}(\alpha_{\gamma_k(0),o}, \alpha_{v_k})$ would converge to 0. But this would contradict (5.51).

Let us now denote $\epsilon_k =: d(\theta_k, \theta_0)$. According to (5.52), $\lim_{k \rightarrow \infty} \epsilon_k = 0$. We now consider the following sequence of pointed metric space $(\partial \tilde{X}, \epsilon_k^{-1} d, \theta_0)$, a subsequence of which being converging to some metric space (S, δ) , cf[]]. For convenience we still denote by the same index k the subsequence. By the corollary 5.21, the sequence $(\Lambda_{C'}, \epsilon_k^{-1} d, \theta_0)$ also converges to (S, δ) . According to lemma 5.8, the space S is homeomorphic to \mathbb{R}^{n-1} . In particular there exist a sequence of points $\theta'_k \in \Lambda_{C'}$ and a constant c such that

$$(5.53) \quad c^{-1} \epsilon_k \leq d(\theta_k, \theta'_k) \leq c \epsilon_k,$$

$$(5.54) \quad c^{-1} \epsilon_k \leq d(\theta'_k, \theta_0) \leq c \epsilon_k.$$

Thus, the points $\theta_1^k = \theta_k$ and $\theta_2^k = \theta'_k$ satisfy the two first properties of lemma 5.22.

In order to complete the proof of lemma 5.22, we will show that the elements $\eta_k =: \gamma_k^{-1}$ uniformly separate θ_0 , θ_1^k and θ_2^k .

Thanks to (5.50) the angle at $\gamma_k(o)$ between θ_1^k and θ_0 is uniformly bounded away from 0 and π and so does the angle at o between $\gamma_k^{-1}(\theta_1^k)$ and $\gamma_k^{-1}(\theta_0)$. Therefore, as the angle is Hölder-equivalent to the distance d , cf. [10], there is a constant c such that

$$(5.55) \quad d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_0)) \geq c.$$

Now the cocompact group Γ acts uniformly quasi-conformally on $(\partial\tilde{X}, d)$, ([4] and [19] Theorem 5.2), and so does $C' \subset \Gamma$, therefore

$$(5.56) \quad d(\gamma_k^{-1}(\theta_1^k), \gamma_k^{-1}(\theta_2^k)) \geq c,$$

and

$$(5.57) \quad d(\gamma_k^{-1}(\theta_2^k), \gamma_k^{-1}(\theta_0)) \geq c.$$

which ends the proof of lemma 5.22. \square

Proof of Proposition 5.17 :

Let us assume that for every sequence θ_i of points in $\partial\tilde{X}$ converging to θ_0 , $\liminf_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) < +\infty$, then by corollary 5.21 and lemma 5.22 there exist a positive constant c , a sequence ϵ_k tending to 0 when k tends to ∞ , a sequence $\gamma_k \in C'$, a sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ such that

$$\begin{aligned} c^{-1}\epsilon_k &\leq d(\theta_1^k, \theta_2^k) \leq c\epsilon_k, \\ c^{-1}\epsilon_k &\leq d(\theta_i^k, \theta_0) \leq c\epsilon_k \text{ and} \\ d(\gamma_k\theta_1^k, \gamma_k\theta_2^k) &\geq \delta, \quad d(\gamma_k\theta_i^k, \gamma_k\theta_0) \geq \delta. \end{aligned}$$

Applying lemma 5.9 for $\mathcal{L} = \Lambda_{C'}$ and $\lambda_k = \epsilon_k^{-1}$ we conclude that $\Lambda_{C'}$ is homeomorphic to $\partial\tilde{X}$, which is impossible because $\Lambda_{C'}$ is contained in a topological equator $E(\infty)$. \square

Step 4 : C' and C are convex cocompact.

We first define convex cocompactness. For a discrete group C of isometries acting on a Cartan Hadamard manifold of negative sectional curvature with limit set Λ_C , one defines the geodesic hull $\mathcal{G}(\Lambda_C)$ of Λ_C as the set of all geodesics both ends of whose belong to Λ_C .

The geodesic hull of Λ_C is a C invariant set. One says that C is convex cocompact if $\mathcal{G}(\Lambda_C)/C$ is compact.

Lemma 5.23. *C' is convex cocompact.*

Proof : Let us denote $\pi : \tilde{X} \rightarrow \tilde{X}/C'$ the projection. Assume that C' is not convex cocompact. Then, there exist a sequence $x_n \in \mathcal{G}(C')$ such that x_n tends to infinity. In particular $\text{dist}(x_n, Z') \rightarrow +\infty$, where $Z' = \tilde{Z}'/C'$ is the compact hypersurface which separates \tilde{X}/C' in two unbounded connected components. There exist lifts \tilde{x}_n of x_n such that

$$(5.58) \quad \tilde{x}_n \rightarrow \theta_0 \in \Lambda_{C'}$$

$$(5.59) \quad \text{dist}(\tilde{x}_n, \tilde{Z}') = \text{dist}(\tilde{x}_n, \tilde{z}_n)$$

where $\tilde{z}_n \in \tilde{Z}'$ is bounded. Therefore there exist $\tilde{z} \in \tilde{Z}'$ such that $HB(\tilde{z}, \theta_0) \subset \tilde{U}$, where \tilde{U} is one of the two connected components of $\tilde{X} - \tilde{Z}'$, the other being \tilde{V} .

We recall that \mathcal{M} , \mathcal{N} are the two connected components of $\partial\tilde{X} - \Lambda_{C'}$. We also have $\partial\tilde{Z}' = \Lambda_{C'} = E(\infty)$, and after possibly replacing C' by an index two subgroup, we can assume that C' preserves \tilde{U} and \tilde{V} .

Claim : There are the two following cases.

Either one of the two boundaries $\partial\tilde{U}$ or $\partial\tilde{V}$ is equal to $\Lambda_{C'}$ (in this case the other boundary is equal to $\partial\tilde{X}$), or $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$, where $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ are the closure of \mathcal{M} and \mathcal{N} .

Let us prove the claim. We first remark that if there exist $\theta \in \partial\tilde{U} \cap \mathcal{M}$, then $\mathcal{M} \subset \partial\tilde{U}$. Namely, let ξ be any other point in \mathcal{M} and α a continuous path in \mathcal{M} joining θ and ξ . Since the set $\tilde{Z}' \cup \Lambda_{C'}$ is a closed subset in $\tilde{X} \cup \partial\tilde{X}$, there exist an open connected neighborhood W of α in $\tilde{X} \cup \partial\tilde{X}$ contained in the complementary of $\tilde{Z}' \cup \Lambda_{C'}$. Therefore, as $W \cap \tilde{U} \neq \emptyset$, we have $W \cap \tilde{X} \subset \tilde{U}$ and $\xi \in \partial\tilde{U}$. Let us assume that neither $\partial\tilde{U}$ nor $\partial\tilde{V}$ is equal to $\Lambda_{C'}$. Then, each boundary $\partial\tilde{U}$ and $\partial\tilde{V}$ contains \mathcal{M} or \mathcal{N} . But on the other hand, since the set $\tilde{Z}' \cup \Lambda_{C'}$ is closed, $(\partial\tilde{U} - \Lambda_{C'}) \cap (\partial\tilde{V} - \Lambda_{C'}) = \emptyset$ thus we have $\partial\tilde{U} = \mathcal{M}$ and $\partial\tilde{V} = \mathcal{N}$ or the other way around and the claim is proved.

Case 1 : $\partial\tilde{U} = \Lambda_{C'}$ and $\partial\tilde{V} = \partial\tilde{X}$ or the other way around.

In this case, we are in the situation of the step 3, which leads to a contradiction, cf. remark 5.11.

Case 2 : $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$.

In that case, assuming C' is not convex-cocompact, there exist an open horoball $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ where $\theta_0 \in \Lambda_{C'}$, $\tilde{z} \in \tilde{Z}'$, $\partial\tilde{U} = \bar{\mathcal{M}}$ and $\partial\tilde{V} = \bar{\mathcal{N}}$. We will find a contradiction in a similar way as in case 1, ie. step 3. We consider a point $o \in \tilde{X}$ and an hyperplane $E \subset T_o\tilde{X}$ such that $|Jac_E\tilde{F}(o)| = 1$ and $\Lambda_{C'} = E(\infty)$.

Let $\theta_i \in \mathcal{N}$ be a sequence which converge to θ_0 . By continuity, for i large enough, the geodesic ray α_{o,θ_i} spends some time in $HB(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up in \tilde{V} because θ_i converges to θ_0 and θ_i belongs to $\mathcal{N} = \partial\tilde{V} - \Lambda_{C'}$. Therefore, α_{o,θ_i} eventually crosses \tilde{Z}' . Let z_i be some point in $\tilde{Z}' \cap \alpha_{o,\theta_i}$.

We will prove the following Proposition, similar to the Proposition 5.18,

Proposition 5.24. *There exist a sequence $\theta_i \in \mathcal{N}$ such that θ_i converges to θ_0 and*

$$\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = +\infty$$

where $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$.

Remark 5.25. *The difference between the propositions 5.24 and 5.18 is that we are looking for a sequence $\theta_i \in \mathcal{N}$ instead of $\theta_i \in \partial\tilde{X} - \Lambda_{C'}$.*

Assuming the Proposition 5.24 we find a contradiction in the same way as in step 3. Namely, as \tilde{Z}'/C' is compact, the points $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ stay at

bounded distance from the C' -orbit of a fixed point, say, o , thus there exist a constant $A > 0$ and elements $\gamma_i \in C'$ such that for any i ,

$$(5.60) \quad \text{dist}(z_i, \gamma_i o) \leq A.$$

From (5.60) and the shadow lemma 5.15, we obtain $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} \neq \emptyset$, and on the other hand, from the proposition 5.24, we have $\mathcal{O}(o, z_i, R+A) \cap \Lambda_{C'} = \emptyset$, which gives the contradiction. It remains to prove the Proposition 5.24.

Proof of the proposition 5.24 : We argue by contradiction, like in the proof of the proposition 5.17. Let us assume that there exist a constant $C > 0$ such that for any sequence of points $\theta_i \in \mathcal{N}$ converging to θ_0 , $\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) \leq C$, then by lemma 5.19, we have for any such sequence $\theta_i \in \mathcal{N}$

$$(5.61) \quad \lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0.$$

The proof of the following lemma is the same as the proof of lemma 5.20.

Lemma 5.26. *Let us assume that for any sequence $\theta_i \in \mathcal{N}$ converging to θ_0 , $\lim_{i \rightarrow \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$. Let $\{\lambda_k\}$ be a sequence of positive numbers tending to $+\infty$ such that the sequence of spaces $(\partial \tilde{X}, \lambda_k d, \theta_0)$ converges to a space $(S, \delta, 0)$ in the pointed Gromov-Hausdorff topology, then, $(\mathcal{M}, \lambda_k d, \theta_0)$ also converges to $(S, \delta, 0)$.*

Proof : Since $\Lambda_{C'} \subset \bar{\mathcal{M}}$, the assumption implies that $\lim_{\epsilon \rightarrow 0} r(\epsilon) = 0$ where

$$r(\epsilon) = \sup \left\{ \frac{d(\theta, \mathcal{M})}{d(\theta, \theta_0)}, \theta \neq \theta_0, \theta \in \mathcal{N}, d(\theta, \theta_0) \leq \epsilon \right\}$$

and the proof goes the same way as in lemma 5.20 replacing $\Lambda_{C'}$ by \mathcal{M} .

□

Similarly to the lemma 5.22, we have the

Lemma 5.27. *Let us assume that every weak tangent of $(\partial \tilde{X}, d)$ at θ_0 belongs to $WT((\mathcal{M}, d))$. There exist positive constant c, δ , a sequence ϵ_k tending to 0 when k tends to $+\infty$, a sequence of $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\}$,*

$$c^{-1} \epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c \epsilon_k \text{ and} \\ d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta.$$

We can now end the proof of the proposition 5.24. Let us assume that there exist a constant $C > 0$ such that for every sequence θ_i of points in \mathcal{N} converging to θ_0 , $\lim_{i \rightarrow \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(\theta_i, \alpha_\theta) \leq C$, then by (5.61), lemma 5.26 and lemma 5.27, there exist a sequence ϵ_k tending to 0 when k tends to ∞ , a sequence $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\}$,

$$c^{-1}\epsilon_k \leq d(\theta_i^k, \theta_j^k) \leq c\epsilon_k \text{ and} \\ d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta.$$

Applying the lemma 5.9 for $\mathcal{L} = \mathcal{M}$ and $\lambda_k = \epsilon_k^{-1}$, we conclude that \mathcal{M} is homeomorphic to $\partial\tilde{X}$, which is impossible because $\partial\tilde{X}$ is a sphere, and \mathcal{M} is homeomorphic to an hemisphere. This ends the proof of the proposition 5.24. \square

Corollary 5.28. *C is convex cocompact.*

Proof : The subgroup C' of C is convex cocompact and the limit sets of C' and C coincide by step 3, therefore C is convex cocompact. \square .

Step 5: C preserves a copy of the $(n-1)$ -dimensional hyperbolic space \mathbb{H}^{n-1} totally geodesically embedded in \tilde{X} .

From the steps 1-4, we know that the groups C and C' are convex cocompact, and that their limit set Λ_C and $\Lambda_{C'}$ are equal to a topological equator $E(\infty)$.

Let us consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will show that there exist a minimizing current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$ and that this minimizing current lifts to a totally geodesic hypersurface embedded in \tilde{X} . We will then show that this totally geodesic hypersurface is eventually hyperbolic.

We work in \tilde{X}/C' and consider the essential hypersurface $Z' \subset \tilde{X}/C'$. We will now prove that there exist a minimal current representing the class of Z' in $H_{n-1}(\tilde{X}/C', \mathbb{R})$. Let $\{Z_k\}$ be a minimizing sequence of currents homologous to Z' . The orthogonal projection onto the convex core of \tilde{X}/C' is distance nonincreasing and thus volume nonincreasing. Therefore we can assume that the Z_k 's are in the the convex core of \tilde{X}/C' , which is compact. By [12] (5.5), the sequence $\{Z_k\}$ subconverges to a minimal current Z_∞ in \tilde{X}/C' . By [12] (8.2), Z_∞ is a manifold with possible singularities of codimension greater than or equal to 8. By corollary (4.4) and minimality we get that $|Jac_{n-1}\tilde{F}(x)| = 1$ at every regular points $x \in Z_\infty$. We will use the fact that $|Jac_{n-1}\tilde{F}(x)| = 1$ at every regular points $x \in Z_\infty$ in order to prove that Z_∞ is a totally geodesic hypersurface.

Lemma 5.29. *Let x and y two distinct points in \tilde{X} and $E_x \subset T_x\tilde{X}$, $E_y \subset T_y\tilde{X}$ be such that $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$ and $Jac_{n-1}\tilde{F}'(y) = Jac_{E_y}\tilde{F}'(y) = 1$. Then, the geodesic $\alpha_{x,y}$ (resp. $\alpha_{y,x}$) joining x and y (resp. y and x) satisfies $\dot{\alpha}_{x,y}(0) \in E_x$, (resp. $\dot{\alpha}_{y,x}(0) \in E_y$). In particular, $\alpha_{x,y}(+\infty)$ and $\alpha_{y,x}(+\infty)$ belong to $\Lambda_{C'}$.*

Proof: Let S_x and S_y be the unit spheres of E_x and E_y . For any unit tangent vector $u \in T_z\tilde{X}$ at some point z , we define $\theta_u \in \partial\tilde{X}$ by $\dot{\alpha}_{z,\theta_u}(0) = u$. By step 3, $\Lambda_{C'} = E_x(\infty) = E_y(\infty)$, therefore for every $u \in S_x$, $\theta_u \in \Lambda_{C'}$ and there exist $v \in E_y$ such that $\theta_u = \theta_v$. As E_y is a vector space, θ_{-v} belongs to

$\Lambda_{C'}$ therefore there exist $w \in E_x$ such that $\theta_w = \theta_{-v}$. The map $f : S_x \rightarrow S_x$ defined by $f(u) = w$ is a continuous map. The lemma then boils down to proving that there exist $u \in S_x$ such that $f(u) = -u$ because in that case, x, y and θ_u are on the same geodesic α_{x, θ_u} .

The following properties of f are obvious.

- (i) For every $u \in S_x$, $f(u) \neq u$.
- (ii) $f \circ f = Id$.

So f is an involution of the sphere without fixed point and for any such map, we claim that there exist u in the sphere such that $f(u) = -u$. In order to prove the claim, we follow a very similar argument in [16], theorem 1. We argue by contradiction. Let us assume that for every $u \in S_x$, $f(u) \neq -u$. The map $g : S_x \rightarrow S_x$ defined by $g(u) = \frac{f(u)+u}{\|f(u)+u\|}$, is then well defined and continuous. Let us remark that as for every $u \in S_x$, $f(u) \neq -u$, then f is homotopic to the Identity, and so is g . Moreover by (ii) we clearly have $g \circ f = g$, thus the map g factorizes through S_x/G_f where G_f is the group generated by the involution f . By (i) f has no fixed point thus S_x/G_f is a manifold and the projection $p : S_x \rightarrow S_x/G_f$ is a degree 2 map. Therefore, the induced endomorphism g_* on $H_{n-1}(S_x, \mathbb{Z}_2)$ is trivial, which contradicts the fact that g is homotopic to the Identity. \square

Corollary 5.30. *Let $\mathcal{H}^{n-1} \subset \tilde{X}$ be an hypersurface with possibly non empty boundary $\partial\mathcal{H}^{n-1}$, such that for any $x \in \mathcal{H}^{n-1}$, $Jac_{n-1}\tilde{F}'(x) = Jac_{E_x}\tilde{F}'(x) = 1$, where E_x is the tangent space of \mathcal{H}^{n-1} at x . Let us consider $x \in \mathcal{H}^{n-1}$ such that $dist_{\tilde{X}}(x, \partial\mathcal{H}^{n-1}) = r > 0$. Then, for any $x' \in \mathcal{H}^{n-1}$ with $dist_{\tilde{X}}(x, x') < r$, the geodesic $\alpha_{x, x'}$ joining x and x' is contained in \mathcal{H}^{n-1} . In particular, \mathcal{H}^{n-1} is locally convex.*

Proof of the corollary: Let us fix $\theta \in \Lambda_{C'}$ and consider the vector field $\nabla B(y, \theta)$ in \tilde{X} . Let $x \in \mathcal{H}^{n-1}$. As $Jac_{E_x}\tilde{F}'(x) = 1$, we have by step 3 $\Lambda_{C'} = E_x(\infty)$. Then, for any $x \in \mathcal{H}^{n-1}$, $\nabla B(x, \theta)$ is tangent to \mathcal{H}^{n-1} , therefore the geodesic $\alpha_{x, \theta}$ satisfies $\alpha_{x, \theta}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. Let $x' \in \mathcal{H}^{n-1}$. By lemma 5.29, $\dot{\alpha}_{x, x'}(0) \in E_x$, therefore $\alpha_{x, x'} = \alpha_{x, \theta}$ and $\alpha_{x, x'}(t) \in \mathcal{H}^{n-1}$ for all $t \in [0, r)$. \square

We now prove that Z_∞ is a totally geodesic hypersurface in \tilde{X}/C' . Let us recall that Z_∞ is a manifold which is smooth except at a singular subset of codimension at least 7. Let us consider a lift $\tilde{Z}_\infty \subset \tilde{X}$ of Z_∞ and denote \tilde{Z}_∞^{reg} (resp. \tilde{Z}_∞^{sing}) the set of regular (resp.) singular points of \tilde{Z}_∞ .

Lemma 5.31. *\tilde{Z}_∞ is a totally geodesic hypersurface in \tilde{X} .*

Proof: Let us consider a regular point $x \in \tilde{Z}_\infty^{reg}$. We shall show that for every point $x' \in \tilde{Z}_\infty^{reg}$ the geodesic segment joining x and x' is contained in \tilde{Z}_∞ , and as the set of regular points is dense in \tilde{Z}_∞ (as the complementary of a subset of codimension at least 8), this will show that \tilde{Z}_∞ is totally geodesic.

We claim that there exist a sequence $y_k \in \tilde{Z}_\infty^{reg}$ such that $\lim_{k \rightarrow \infty} y_k = x'$ and the geodesic segment joining x and y_k is contained in \tilde{Z}_∞ .

The claim immediately implies that the geodesic segment joining x and x' is contained in \tilde{Z}_∞ .

Let us prove the claim.

For $y \in \tilde{Z}_\infty^{reg}$ we consider $\alpha_{x,y}$ the geodesic joining x and y and define

$$(5.62) \quad t_y = \inf\{t > 0, \alpha_{x,y}(t) \notin \tilde{Z}_\infty\}$$

As x is a regular point, by corollary 5.30, there exist $\epsilon > 0$ such that $t_y > \epsilon$.

In order to prove the claim, we argue by contradiction. Let us assume that there exist $r > 0$ such that for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg}$, $t_y < \text{dist}(x, y)$. By corollary 5.30 applied to \tilde{Z}_∞^{reg} , we have $\alpha_{x,y}(t_y) \in \tilde{Z}_\infty^{sing}$. As the set of regular points is an open subset of \tilde{Z}_∞ , if r is small enough we have $B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty^{reg} = B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$. We choose such an r and we consider the set S of all singular points contained in the union of all geodesic segments joining x to a point $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$. Let us consider the map defined on S by

$$p(y) = \alpha_{x,y}(\epsilon).$$

As we already saw, for any $y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty$, we have $t_y > \epsilon$, therefore the map p is distance decreasing and by assumption p is surjective onto an open subset of the sphere and $p(S)$ is homeomorphic to an open subset of \mathbb{R}^{n-1} , therefore the Hausdorff dimension of S is greater than or equal to $n - 1$, which contradicts the fact that the singular set has codimension at least 8 in \tilde{Z}_∞ . \square

The totally geodesic hypersurface $\tilde{Z}_\infty \subset \tilde{X}$ is preserved by C , and \tilde{Z}_∞/C is of minimal volume in its homology class.

Let us prove that \tilde{Z}_∞ is isometric to the hyperbolic space $\mathbb{H}_\mathbb{R}^{n-1}$.

Lemma 5.32. \tilde{Z}_∞ is isometric to the hyperbolic space $\mathbb{H}_\mathbb{R}^{n-1}$.

Proof :

As \tilde{Z}_∞/C is of minimal volume in its homology class, we have by Proposition 5.1, for all $x \in \tilde{Z}_\infty$, $\text{Jac}_{E_x} \tilde{F}(x) = 1$ and $\tilde{F}(x) = x$, where E_x is the tangent space of \tilde{Z}_∞ at x . Moreover, we saw in the proof of proposition 5.1 that

$$H = \frac{1}{n-1} \text{Id}_{D\tilde{F}(x)(E_x)} = \frac{1}{n-1} \text{Id}_{E_x},$$

therefore we get from (4.11) and $\tilde{F}(x) = x$, that for all $u, v \in T_x \tilde{Z}_\infty$,

$$(5.63) \quad \int_{\partial \tilde{X}} [DdB_{(x,\theta)}(u, v) + DB_{(x,\theta)}(u)DB_{(x,\theta)}(v)] d\nu_x(\theta) = \tilde{g}(u, v)$$

where \tilde{g} is the metric on \tilde{X} . As \tilde{Z}_∞ is totally geodesic, the relation (5.62) remains true with the Busemann function $B^{\tilde{Z}_\infty}$ of \tilde{Z}_∞ instead of the Busemann function B of \tilde{X} :

$$(5.64) \quad \int_{\partial\tilde{X}} [DdB_{(x,\theta)}^{\tilde{Z}_\infty}(u, v) + DB_{(x,\theta)}^{\tilde{Z}_\infty}(u)DB_{(x,\theta)}^{\tilde{Z}_\infty}(v)] \\ = \tilde{g}(u, v).$$

On the other hand, as \tilde{Z}_∞ is totally geodesic, its sectional curvature is less than or equal to -1 , thus by Rauch comparison theorem, we have

$$(5.65) \quad DdB^{\tilde{Z}_\infty}(x, \theta) + DB^{\tilde{Z}_\infty}(x, \theta) \otimes DB^{\tilde{Z}_\infty}(x, \theta) \geq \tilde{g}_{\tilde{Z}_\infty}$$

for all $\theta \in \partial\tilde{Z}_\infty = \Lambda_C$, where $\tilde{g}_{\tilde{Z}_\infty}$ is the restriction of \tilde{g} to \tilde{Z}_∞ .

As the support of the measure ν_x is $\partial\tilde{Z}_\infty = \Lambda_C$ (by convex cocompactness of C) and the Busemann function is continuous, we get from (5.64) and (5.65) that for all $x \in \tilde{Z}_\infty$ and all $\theta \in \tilde{Z}_\infty$

$$(5.66) \quad DdB^{\tilde{Z}_\infty}(x, \theta) + DB^{\tilde{Z}_\infty}(x, \theta) \otimes DB^{\tilde{Z}_\infty}(x, \theta) \\ = \tilde{g}_{\tilde{Z}_\infty}(x).$$

and this last relation is characteristic of the hyperbolic space. \square

Step 6: Conclusion

So far we have shown that C preserves a totally geodesic copy of the hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n-1} \subset \tilde{X}$ such that $\mathbb{H}_{\mathbb{R}}^{n-1}/C$ is compact.

Our goal now is to show that $Y =: \mathbb{H}_{\mathbb{R}}^{n-1}/C$ injects diffeomorphically in $X = \tilde{X}/\Gamma$ and separates X in two connected components R and S such that $A = \pi_1(R)$ and $B = \pi_1(S)$.

In order to do this, we will consider the Γ orbit of $\mathbb{H}_{\mathbb{R}}^{n-1}$ in \tilde{X} and the two connected components U and V of $\tilde{X} - \Gamma\mathbb{H}_{\mathbb{R}}^{n-1}$ which are adjacent to $\mathbb{H}_{\mathbb{R}}^{n-1}$. The stabilizers \bar{A} , \bar{B} and \bar{C} of U , V and $\mathbb{H}_{\mathbb{R}}^{n-1}$ contain respectively A , B and C and the hypersurface $\mathbb{H}_{\mathbb{R}}^{n-1}/C$ injects in $X = \tilde{X}/\Gamma$ and separates X in two connected components R and S such that $\pi_1(R) = \bar{A}$ and $\pi_1(S) = \bar{B}$. We then show that $\bar{C} = C$, $\bar{A} = A$ and $\bar{B} = B$.

Let \bar{C} be the stabilizer of \mathbb{H}^{n-1} , namely $\bar{C} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} = \mathbb{H}^{n-1}\}$. We have $C \subset \bar{C}$ and as \mathbb{H}^{n-1}/C is compact, so is \mathbb{H}^{n-1}/\bar{C} and thus $[\bar{C} : C] < \infty$.

Let $p : \tilde{X}/C \rightarrow X = \tilde{X}/\Gamma$ and $\bar{p} : \tilde{X}/\bar{C} \rightarrow X = \tilde{X}/\Gamma$ the natural projections. We now show that the restriction of p to \mathbb{H}^{n-1}/C is an embedding, thus $Y := p(\mathbb{H}^{n-1}/C)$ is a compact totally geodesic hypersurface of X .

In the section 2, we constructed a C -invariant hypersurface $\tilde{Z} \subset \tilde{X}$ such that $Z = \tilde{Z}/C \subset \tilde{X}/C$ is compact. The hypersurface is defined as $\tilde{Z} = \tilde{f}^{-1}(t_0)$ where $\tilde{f} : \tilde{X} \rightarrow T$ is an equivariant map onto the Bass-Serre tree associated to the amalgamation $A *_C B$ and t_0 belongs to that edge of T which is fixed by C .

Let us first show two lemmas.

Lemma 5.33. *The restriction of p to \tilde{Z}/C is an embedding into $X = \tilde{X}/\Gamma$.*

Proof : Let $\gamma \in \Gamma$, z, z' in \tilde{Z} such that $z' = \gamma z$. By equivariance,

$$\tilde{f}(\gamma z) = \gamma \tilde{f}(z) = \gamma t_0 = \tilde{f}(z') = t_0,$$

thus $\gamma \in C$. \square

Lemma 5.34. *The restriction of \bar{p} to \mathbb{H}^{n-1}/\bar{C} is an embedding into $X = \tilde{X}/\Gamma$.*

Proof : Let us assume that there is a $\gamma \in \Gamma - \bar{C}$ such that $\gamma \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq \emptyset$ and choose an $x \in \gamma \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1}$. As $\gamma \notin \bar{C}$, there exist $u \in T_x \gamma \mathbb{H}^{n-1} - T_x \mathbb{H}^{n-1}$. We consider c_u the geodesic ray such that $\dot{c}_u(0) = u$. We know that \tilde{Z} is contained in an ϵ -neighbourhood $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ of \mathbb{H}^{n-1} . The ϵ -neighbourhood $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ of \mathbb{H}^{n-1} separates \tilde{X} in two connected components U and V and for $t > 0$ large enough, we have, say, $c_u(t) \in U$ and $c_u(-t) \in V$.

Let \tilde{Z}' be the connected component of \tilde{Z} that we constructed at the end of section 2, whose stabilizer (or an index two subgroup of it) C' is such that \tilde{Z}'/C' separates \tilde{X}/C' in two unbounded connected components U'/C' and V'/C' where U' and V' are the two connected components of $\tilde{X} - \tilde{Z}'$.

We claim that $U \subset U'$ and $V \subset V'$ or the other way around. Indeed if not, U and V would be both contained in, say, U' . But in that case, V' would be contained in $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and therefore V'/C' would be bounded, which is a contradiction.

As $\gamma \tilde{Z}$ lies in the ϵ neighborhood of $\gamma \mathbb{H}^{n-1}$, there exist sequences z_k, z'_k in $\gamma \tilde{Z}$ such that $\text{dist}(z_k, c_u(k)) \leq \epsilon$ and $\text{dist}(z'_k, c_u(-k)) \leq \epsilon$. By proposition 5.13 and lemma 5.23, C' also acts cocompactly on \mathbb{H}^{n-1} , thus C' is of finite index in C , and therefore there are finitely many connected components of \tilde{Z} and the same holds for $\gamma \tilde{Z}$. We thus can assume that the z_k 's and z'_k 's belong to a single connected component of $\gamma \tilde{Z}$. Let us consider a continuous path $\alpha \subset \gamma \tilde{Z}$ joining z_k and z'_k .

By construction the distance between $c_u(k)$ [resp. $c_u(-k)$] and \mathbb{H}^{n-1} tends to infinity and thus, for k large enough, $z_k \in U$ and $z'_k \in V$ or the other way around. By the claim, we then have $z_k \in U'$ and $z'_k \in V'$, therefore the path α has to cross \tilde{Z}' which contradicts the lemma (5.29) and ends the proof of the lemma 5.30. \square

As we already saw, \tilde{Z} has finitely many connected components, and so does $\tilde{X} - \tilde{Z}$. Let us write $\{W_j\}_{j=1, \dots, m}$ the connected components of $\tilde{X} - \tilde{Z}$. As C acts cocompactly on \tilde{Z} and \mathbb{H}^{n-1} there exist $\epsilon > 0$ such that $\mathbb{H}^{n-1} \subset$

$\mathcal{U}_\epsilon \tilde{Z}$ and $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. Moreover $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ separates \tilde{X} in two connected components U and V .

Lemma 5.35. *Let us consider ϵ such that $\tilde{Z} \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and U and V the two connected components of $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$. There are two distinct connected components W_1 and W_2 of $\tilde{X} - \tilde{Z}$ such that $U \subset W_1$ and $V \subset W_2$. Moreover, $\tilde{f}(W_1) \subset \tilde{T}_1$ and $\tilde{f}(W_2) \subset \tilde{T}_2$, where \tilde{T}_1 and \tilde{T}_2 are the two connected components of $\tilde{T} - \{t_0\}$.*

Proof : We argue by contradiction. Let us assume that U and V are contained in the same connected component W_1 of $\tilde{X} - \tilde{Z}$. Then, all other components W_j , $j \neq 1$, satisfy $W_j \subset \mathcal{U}_\epsilon \mathbb{H}^{n-1} \subset \mathcal{U}_{2\epsilon} \tilde{Z}$. Therefore, as C acts cocompactly on $\mathcal{U}_{2\epsilon} \tilde{Z}$, there exist a constant D such that for any $j \neq 1$, $\max_{w \in W_j} \text{dist}_{\tilde{T}}(\tilde{f}(w), t_0) \leq D$. Thus, $\tilde{f}(W_1)$ is contained in one connected component of $\tilde{T} - \{t_0\}$ and $\tilde{f}(\cup_{j \neq 1} W_j)$, contained in the ball $B_{\tilde{T}}(t_0, D)$ of \tilde{T} of radius D centered at t_0 , is bounded. This is clearly impossible because $\tilde{T} - \{t_0\}$ has two unbounded connected components and \tilde{f} is onto. \square

Let us denote $\mathcal{A} = A\mathbb{H}^{n-1}$ the A -orbit of the C -invariant totally geodesic copy of the real hyperbolic space \mathbb{H}^{n-1} , and \bar{A} the stabilizer of \mathcal{A} , ie. $\bar{A} = \{\gamma \in \Gamma, \gamma\mathcal{A} = \mathcal{A}\}$. We define in a similar way $\mathcal{B} = B\mathbb{H}^{n-1}$ and $\bar{B} = \{\gamma \in \Gamma, \gamma\mathcal{B} = \mathcal{B}\}$.

Let us recall that \bar{C} is the stabilizer of \mathbb{H}^{n-1} in Γ . We now prove the following

Lemma 5.36. *We have $\bar{A} = A\bar{C}$ and $\bar{B} = B\bar{C}$. Moreover, \bar{A} and \bar{B} are characterized by $\bar{A} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{A}\}$ and $\bar{B} = \{\gamma \in \Gamma, \gamma\mathbb{H}^{n-1} \in \mathcal{B}\}$.*

Proof : Let $\gamma' \in \bar{A}$, then $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$ and thus there exist $\gamma \in A$ such that $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$, therefore $\gamma^{-1}\gamma' \in \bar{C}$, which proves the first part of the lemma.

Let us prove the second part of the lemma.

Let $\gamma' \in \Gamma$ be such that $\gamma'\mathbb{H}^{n-1} \in \mathcal{A}$. Then there exist $\gamma \in A$ such that $\gamma'\mathbb{H}^{n-1} = \gamma\mathbb{H}^{n-1}$, thus $\gamma^{-1}\gamma' \in \bar{C}$ and therefore $\gamma' \in \gamma\bar{C} \subset A\bar{C} = \bar{A}$. This proves one inclusion, the other inclusion being obvious. \square

For each $\gamma \in \Gamma$, $\gamma\mathbb{H}^{n-1}$ separates \tilde{X} in two connected components U_γ and V_γ .

Let us now prove the following lemma.

Lemma 5.37. *(i) Let $\gamma \in A$, [resp. $\gamma \in B$]. Then, we have $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{A} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$, [resp. $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset U_\gamma$ or $\mathcal{B} - \{\gamma\mathbb{H}^{n-1}\} \subset V_\gamma$.]
(ii) Let γ be an element of $\Gamma - \bar{A}$, [resp. $\Gamma - \bar{B}$]. Then $\mathcal{A} \subset U_\gamma$ or $\mathcal{A} \subset V_\gamma$, [resp. $\mathcal{B} \subset U_\gamma$ or $\mathcal{B} \subset V_\gamma$].*

Proof : (i) We argue by contradiction. Let us consider $\gamma\mathbb{H}^{n-1}$, $\gamma'\mathbb{H}^{n-1}$ and $\gamma''\mathbb{H}^{n-1}$ three distinct elements in \mathcal{A} such that $\gamma'\mathbb{H}^{n-1} \subset U_\gamma$ and $\gamma''\mathbb{H}^{n-1} \subset V_\gamma$. By equivariance we can assume γ is the identity. Let us recall that U and V are the two connected components of $\tilde{X} - \mathcal{U}_\epsilon \mathbb{H}^{n-1}$.

We then have $\gamma'\mathbb{H}^{n-1} \cap U \neq \emptyset$ and $\gamma''\mathbb{H}^{n-1} \cap V \neq \emptyset$, which implies $\gamma'\tilde{Z} \cap U \neq \emptyset$ and $\gamma''\tilde{Z} \cap V \neq \emptyset$.

By lemma (5.31), $U \subset W_1$ and $V \subset W_2$ where W_1 and W_2 are two connected components of $\tilde{X} - \tilde{Z}$ and $\tilde{f}(U) \subset \tilde{T}_1$ and $\tilde{f}(V) \subset \tilde{T}_2$, therefore, $\tilde{f}(U)$ contains $\gamma't_0 \in \tilde{T}_1$ and $\tilde{f}(V)$ contains $\gamma''t_0 \in \tilde{T}_2$. This is impossible because for all elements γ' and γ'' in A , $\gamma't_0$ and $\gamma''t_0$ belong to the same connected component of $\tilde{T} - \{t_0\}$.

(ii) Let us consider $\gamma \in \Gamma - \bar{A}$. We argue by contradiction. Let us assume there exist γ, γ' in A such that

$$\gamma'\mathbb{H}^{n-1} \subset U_\gamma$$

$$(5.67) \quad \gamma''\mathbb{H}^{n-1} \subset V_\gamma$$

Let $\epsilon > 0$ such that $\tilde{Z} \subset \mathcal{U}_\epsilon\mathbb{H}^{n-1}$ and U and V the connected component of $\tilde{X} - \mathcal{U}_\epsilon\mathbb{H}^{n-1}$. By lemma (5.31) we have $\tilde{f}(\gamma U) \subset \gamma\tilde{T}_1$ and $\tilde{f}(\gamma V) \subset \gamma\tilde{T}_2$, where $\gamma\tilde{T}_1$ and $\gamma\tilde{T}_2$ are the two connected components of $\tilde{T} - \{\gamma t_0\}$. By assumption (5.62), we have

$\gamma'\mathbb{H}^{n-1} \cap \gamma U \neq \emptyset$ and $\gamma''\mathbb{H}^{n-1} \cap \gamma V \neq \emptyset$, which implies $\gamma'\tilde{Z} \cap \gamma U \neq \emptyset$ and $\gamma''\tilde{Z} \cap \gamma V \neq \emptyset$, therefore $\gamma't_0 \in \gamma\tilde{T}_1$ and $\gamma''t_0 \in \gamma\tilde{T}_2$, which is impossible because in the tree \tilde{T} , the points $\gamma't_0$ and $\gamma''t_0$ belong to two adjacent edges.

□

By lemma (5.33) (i), for every γ in \bar{A} , [resp. \bar{B}], we can define U_γ as the connected component of $\tilde{X} - \mathbb{H}^{n-1}$ which contains all $\gamma'\mathbb{H}^{n-1}$ for all γ' in \bar{A} , [resp. \bar{B}], and $\gamma'\mathbb{H}^{n-1} \neq \gamma\mathbb{H}^{n-1}$.

Let us define

$$(5.68) \quad U_A := \bigcap_{\gamma \in \bar{A}} U_\gamma.$$

By definition, U_A [resp. U_B] is a convex set in \tilde{X} whose boundary is the collection \mathcal{A} , [resp. \mathcal{B}], of $\gamma\mathbb{H}^{n-1}$, γ in \bar{A} [resp. \bar{B}] and by lemma (5.33) (ii), U_A and U_B are two disjoint connected components of $\tilde{X} - \Gamma\mathbb{H}^{n-1}$.

In fact, U_A [resp. U_B], is the convex hull of \mathcal{A} , [resp. \mathcal{B}], and \bar{A} , [resp. \bar{B}], is the stabilizer of U_A , [resp. U_B].

Lemma 5.38. *The closures of U_A and U_B intersect along \mathbb{H}^{n-1} and $\bar{A} \cap \bar{B} = \bar{C}$. Moreover, no element of Γ sends U_A on U_B nor the other way around.*

Proof : The convex set U_A is the intersection of open half spaces U_γ , $\gamma \in \bar{A}$, and is delimited by the disjoint union of hyperplanes $\gamma\mathbb{H}^{n-1}$, for some $\gamma \in \bar{A}$. The same is true for U_B and as $U_A \cap U_B = \emptyset$, the closures of U_A and U_B can intersect only along one of the connected components of their boundaries, thus along \mathbb{H}^{n-1} which is obviously in both closures. This proves the first part of the lemma, let us prove the second part. By lemma 5.32, $\bar{C} \subset \bar{A} \cap \bar{B}$. Conversely, let us take $\gamma \in \bar{A} \cap \bar{B}$, then γ preserves

the closures of U_A and U_B , thus it preserves their intersection \mathbb{H}^{n-1} , and therefore $\gamma \in \bar{C}$.

Let us prove the last part of the lemma. Let γ be an element such that $\gamma U_A = U_B$. As \mathbb{H}^{n-1} is one component of the boundary \mathcal{B} of U_B , there exist one component $\gamma' \mathbb{H}^{n-1} \in \mathcal{A}$, γ' being in \bar{A} , such that $\gamma(\gamma' \mathbb{H}^{n-1}) = \mathbb{H}^{n-1}$. Therefore, $\gamma\gamma' \in \bar{C}$, thus $\gamma \in \bar{A}$. The same argument yields $\gamma^{-1} \in \bar{B}$, so $\gamma \in \bar{A} \cap \bar{B} = \bar{C}$ and γ preserves U_A and U_B , which contradicts our choice of γ .

□

The Γ -orbit of the closure of $U_A \cup U_B$ covers \tilde{X} . Let us construct a tree \bar{T} embedded in \tilde{X} in the following way: the set of vertices is the set of the connected components of $\tilde{X} - \Gamma \mathbb{H}^{n-1}$ and two vertices are joined by an edge if the boundaries of their corresponding connected components intersect non trivially in \tilde{X} . By construction Γ acts on \bar{T} , the stabilizers of the vertices a and b corresponding to U_A and U_B are \bar{A} and \bar{B} , the stabilizer of the edge between a and b is \bar{C} and a fundamental domain for this action is the segment joining a and b . By [14], I, 4, Theorem 6, the group Γ is the amalgamated product of \bar{A} and \bar{B} over \bar{C} .

We now claim that $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$.

As A , B and C are subgroups of \bar{A} , \bar{B} , and \bar{C} , the corresponding Mayer-Vietoris sequences of $A *_C B$ and $\bar{A} *_C \bar{B}$ are related by the following commutative diagram

$$\begin{array}{ccccc} H_n(A, \mathbb{R}) \oplus H_n(B, \mathbb{R}) & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(C, \mathbb{R}) \\ & & \downarrow & & \downarrow \\ H_n(\bar{A}, \mathbb{R}) \oplus H_n(\bar{B}, \mathbb{R}) & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C}, \mathbb{R}). \end{array}$$

We know that the index $[\bar{C} : C]$ is finite, and by the lemma 5.32, the indices of A , B , and C in \bar{A} , \bar{B} and \bar{C} are finite and equal. On the other hand, the indices $[\Gamma : A]$ and $[\Gamma : B]$ are infinite by assumption, thus the previous diagram becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(C, \mathbb{R}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(\Gamma, \mathbb{R}) & \longrightarrow & H_{n-1}(\bar{C}, \mathbb{R}). \end{array}$$

Moreover, the map $H_n(\Gamma, \mathbb{R}) \rightarrow H_{n-1}(\bar{C}, \mathbb{R})$ is bijective. Namely, the injectivity comes from the above diagram and the surjectivity from the fact that the hypersurface \mathbb{H}^{n-1}/\bar{C} bounds in \mathbb{H}^n/A and \mathbb{H}^n/B so that the map $H_{n-1}(C, \mathbb{R}) \rightarrow H_{n-1}(A, \mathbb{R}) \oplus H_{n-1}(B, \mathbb{R})$ is trivial. Therefore the index $[\bar{C} : C] = 1$ and we get $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$. □

6. PROOF OF THE THEOREMS 1.5 AND 1.6.

The proof of theorem 1.5 is exactly the same as the proof of theorem 1.2. The actions of Γ on \tilde{X} and T give rise to a continuous Γ -equivariant map $\tilde{f} : \tilde{X} \rightarrow T$. Like in section 2, we build an hypersurface $\tilde{f}^{-1}(t_0)$ where t_0 is

a regular value of \tilde{f} belonging the interior of an edge. As the edge separates the tree in two unbounded components, the section 2 applies and we get a subgroup C' of C , and an hypersurface $\tilde{Z}' \subset \tilde{X}/C'$ which is essential. Now, if the action of Γ is minimal, every edge separates T in two unbounded components. \square

7. APPENDIX

The goal of this section is to give a proof of lemma 5.9. This lemma is contained in lemma 2.1, 5.1 and 5.2 of [4], but our situation being not exactly the same, we reproduce it down here for sake of completeness.

Let us restate the lemma 5.9.

Lemma 7.1. *Let $\mathcal{L} \subset \partial\tilde{X}$ be a closed C' -invariant subset and $\theta_0 \in \mathcal{L}$. We assume that there exist a sequence of positive real numbers $\lambda_k \rightarrow \infty$ such that the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \delta, 0)$ where $(S, \delta, 0)$ is a weak tangent of $(\partial\tilde{X}, d)$. We also assume that there exist positive constants C and δ , a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, \mathcal{L} is homeomorphic to the one point compactification \hat{S} of S . In particular \mathcal{L} is homeomorphic to $\partial\tilde{X}$.*

We first give a definition of pointed Hausdorff-Gromov convergence which is equivalent to the definition 5.7. We follow [4], paragraph 4.

A sequence of metric spaces (Z_k, d_k, z_k) converges to the metric space $(S, \delta, 0)$ if for every $R > 0$, and every $\epsilon > 0$, there exist an integer N , a subset $D \subset B_S(0, R)$, subsets $D_k \subset B_{Z_k}(z_k, R)$ and bijections $f_k : D_k \rightarrow D$ such that for $k \geq N$,

- (i) $f_k(z_k) = 0$,
- (ii) the set D is ϵ -dense in $B_S(0, R)$, and the sets D_k are ϵ -dense in $B_{Z_k}(z_k, R)$,
- (iii) $|d_{Z_k}(x, y) - d_Z(f_k(x), f_k(y))| < \epsilon$,
where x, y belong to D_k .

Let us describe now the lemmas 2.1 and 5.1 following [4].

For a metric space (Z, d) the cross ratio of four points $\{z_i\}$, $i = 1, \dots, 4$, is the quantity

$$(7.1) \quad [z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}$$

Given two metric spaces X and Y , an homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$, and an injective map $f : X \rightarrow Y$, we say that f is an η -quasi-Möbius map if for any four points $\{x_i\}$, $i = 1, \dots, 4$, in X , we have

$$(7.2) \quad [f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]).$$

For example, any discrete cocompact group of isometries of \tilde{X} , where \tilde{X} is a Cartan-Hadamard manifold with sectional curvature $K \leq -1$, is acting on the ideal boundary $(\partial\tilde{X}, d)$ endowed with the Gromov distance by η -quasi-Möbius transformations for some η .

Lemma 7.2 ([4], Lemma 2.1). *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and for any integer k , $g_k : \tilde{D}_k \rightarrow Y$ an η -quasi-Möbius map defined on a subset \tilde{D}_k of X . We assume that the Hausdorff distance between \tilde{D}_k and X satisfies*

$$\lim_{k \rightarrow \infty} \text{dist}_H(\tilde{D}_k, X) = 0$$

and that for any integer k , there exist points (x_1^k, x_2^k, x_3^k) in D_k and (y_1^k, y_2^k, y_3^k) in Y , such that $g_k(x_i^k) = y_i^k$ for $i \in \{1, 2, 3\}$, $d_X(x_i^k, x_j^k) \geq \delta$ and $d_Y(y_i^k, y_j^k) \geq \delta$ for $i, j \in \{1, 2, 3\}, i \neq j$, where δ is independant of k . Then a subsequence of g_k converges uniformly to a quasi-Möbius map $f : X \rightarrow Y$, ie. $\lim_{k_j \rightarrow \infty} \text{dist}_H(g_{k_j}, f|_{\tilde{D}_{k_j}}) = 0$. If in addition, we suppose that

$$\lim_{k \rightarrow \infty} \text{dist}_H(g_k(\tilde{D}_k), Y) = 0,$$

then the sequence $\{g_{k_j}\}$ converges uniformly to a quasi-Möbius homeomorphism $f : X \rightarrow Y$.

Before stating the second lemma, let us define a metric space Z to be uniformly perfect if there exist a constant $\lambda \geq 1$ such that for every $z \in Z$ and $0 < R < \text{diam}Z$, we have $\bar{B}(z, R) - B(z, \frac{R}{\lambda}) \neq \emptyset$.

Lemma 7.3 ([4], lemma 5.1). *Let Z be a compact uniformly perfect metric space and G an η -quasi-Möbius action on Z . Suppose that for each integer k we are given a set D_k in a ball $B_k = B(z, R_k) \subset Z$ that is $(\epsilon_k R_k)$ -dense in B_k , where $\epsilon_k > 0$, distinct points $x_1^k, x_2^k, x_3^k \in B(z, \lambda_k R_k)$, where $\lambda_k > 0$, with*

$$d_Z(x_i^k, x_j^k) \geq \delta_k R_k$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where $\delta_k > 0$, and groups elements $\gamma_k \in G$ such that for $y_i^k := \gamma_k(x_i^k)$ we have,

$$d_Z(y_i^k, y_j^k) \geq \delta'$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where δ' is independant of k . Let $D'_k = \gamma_k(D_k)$, and suppose that $\lambda_k \rightarrow 0$ when $k \rightarrow \infty$, and the sequence $\frac{\epsilon_k}{\delta_k^2}$ is bounded. Then $\lim_{k \rightarrow \infty} \text{dist}_H(D'_k, Z) = 0$.

Let us go back to the proof of lemma 6.1. By definition of convergence, there exist a subsequence of $\{\lambda_k\}$, which we still denote by $\{\lambda_k\}$, subsets $\tilde{D}_k \subset B_S(0, k)$, $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$, where \tilde{D}_k and D_k are minimal $1/k$ -dense

subsets of $B_S(0, k)$ and $B_{(\mathcal{L}, \lambda_k d)}(\theta_0, k)$, and bijections $f_k : \tilde{D}_k \rightarrow D_k$ such that for all $x, y \in \tilde{D}_k$,

$$(7.3) \quad \frac{1}{2}\delta(x, y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\delta(x, y),$$

cf. [4], (5.4).

We can suppose that the points $\theta_0^k := \theta_0$, θ_1^k , and θ_2^k in lemma 6.1 belong to the set D_k . By assumption there exist elements $\gamma_k \in C'$ and a constant δ such that

$$(7.4) \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta$$

for all $i, j \in \{0, 1, 2\}$.

The lemma 6.1 is a direct consequence of the lemma 6.2 applied to $(X, d_X) = (\hat{S}, \hat{\delta})$ and $(Y, d_Y) = (\mathcal{L}, d)$ and to the sequence of maps $g_k := \gamma_k \circ f_k$, where \hat{S} is the one point compactification of S and $\hat{\delta}$ the distance on \hat{S} associated to δ , cf. [4] Lemma 2.2.

Let us denote x_0^k, x_1^k, x_2^k be the points in S such that $f_k(x_i^k) = \theta_i^k$, for $i \in \{0, 1, 2\}$.

Let us check that the assumptions of lemma 6.2 are verified.

The fact that $\lim_{k \rightarrow \infty} \text{dist}_H(\tilde{D}_k, \hat{S}) = 0$ comes the same way as in [4], (5.5).

By (6.3), we have, $\delta(x_i^k, x_j^k) \geq \frac{\lambda_k}{2} d(\theta_i^k, \theta_j^k)$ and by assumption we then get

$$(7.5) \quad \delta(x_i^k, x_j^k) \geq \frac{1}{2C}.$$

We then get the separation assumption on triples of points by choosing $\delta := \inf\{D, \frac{1}{2C}\}$.

It remains to check the assumption on $g_k(\tilde{D}_k) = \gamma_k \circ f_k(\tilde{D}_k) = \gamma_k(D_k)$, namely,

$$(7.6) \quad \lim_{k \rightarrow \infty} \text{dist}_H(\gamma_k(D_k), \Lambda_{C'}) = 0.$$

In order to prove the property (6.6), we want to apply the lemma 6.3, but as the set (\mathcal{L}, d) is a priori not uniformly perfect, we shall replace the uniform perfectness by the fact that $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to a space $(S, \delta, 0)$, which is uniformly perfect, cf. ().

We will show the

Lemma 7.4. *We consider the subsets $\tilde{D}_k \subset B_S(0, k)$ and $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$, where \tilde{D}_k and D_k are $1/k$ -dense subsets of $B_S(0, k)$ and $B_{(\mathcal{L}, \lambda_k d)}(\theta_0, k)$, and the bijections $f_k : \tilde{D}_k \rightarrow D_k$ coming from the convergence of the sequence of pointed metric spaces $(\mathcal{L}, \lambda_k d, \theta_0)$ to $(S, \delta, 0)$ where $(S, \delta, 0)$ is a weak tangent of $(\partial \tilde{X}, d)$. We also assume that there exist positive constants C and δ , a*

sequence of points $\theta_1^k, \theta_2^k \in \Lambda_{C'}$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k\theta_i^k, \gamma_k\theta_j^k) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, the Hausdorff distance $\text{dist}_H(\gamma_k D_k, \mathcal{L})$ tends to 0 as k tends to infinity.

Proof : The proof is word by word the same as the proof of lemma 6.3, ie. lemma 5.1 (i) of [4] with a difference in case 2).

We have $B_{\lambda_k \mathcal{L}}(\theta_0, k) = B_{\mathcal{L}}(\theta_0, \frac{k}{\lambda_k})$ and $D_k \subset B_{\lambda_k \mathcal{L}}(\theta_0, k)$ an $\frac{1}{k}$ -dense subset, for the metric $\lambda_k d$. In term of the distance d , the set D_k is $(\epsilon_k R_k)$ -dense in $B_{\mathcal{L}}(\theta_0, R_k)$, where $R_k := \frac{k}{\lambda_k}$ and $\epsilon_k := \frac{1}{k^2}$. By assumption, the points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k$ belong to $B_{\mathcal{L}}(\theta_0, \mu_k R_k)$, and satisfy

$$(7.7) \quad d(\theta_i^k, \theta_j^k) \geq \delta_k R_k$$

where $\delta_k := \frac{1}{Ck}$, and $\mu_k := \frac{C}{k}$.

The points $\gamma_k \theta_i^k$ satisfy

$$(7.8) \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta,$$

and $\frac{\epsilon_k}{\delta_k^2} = C^2$ is bounded.

Let us consider a point $\theta \in \mathcal{L}$. We want to approximate it by a point of $\gamma_k D_k$.

We can write $\theta = \gamma_k \theta_k$, for some $\theta_k \in \Lambda_{C'}$. There are two cases.

Case 1). For infinitely many indices k , $\theta_k \in B_{\mathcal{L}}(\theta_0, R_k)$. We work in that case for these indices k , thus there are points $\theta'_k \in D_k \cap B_{\mathcal{L}}(\theta_0, R_k)$, with $d(\theta_k, \theta'_k) \leq \epsilon_k R_k$.

Since the distance between the θ_i^k 's is bounded below by $\delta_k R_k$, we can find at least two of them which we call a_k and b_k , such that

$$d(\theta_k, b_k) \geq \frac{\delta_k R_k}{2}$$

and,

$$(7.9) \quad d(\theta'_k, a_k) \geq \frac{\delta_k R_k}{2}.$$

As C' is contained in the cocompact group Γ , it acts in a quasi-Möbius way on $(\partial \tilde{X}, d)$

thus,

$$(7.10) \quad \frac{d(\gamma_k \theta'_k, \gamma_k \theta_k) d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k \theta'_k, \gamma_k b_k) d(\gamma_k a_k, \gamma_k \theta_k)} \leq \eta \left(\frac{d(\theta'_k, \theta_k) d(a_k, b_k)}{d(\theta'_k, b_k) d(\theta_k, a_k)} \right)$$

for some homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$. This implies

$$(7.11) \quad d(\gamma_k \theta'_k, \gamma_k \theta_k) \leq \frac{(\text{diam} \mathcal{L})^2 \eta(8\epsilon_k \mu_k / \delta_k^2)}{\delta},$$

therefore $d(\gamma_k \theta'_k, \gamma_k \theta_k)$ tends to zero as k tends to infinity.

Case 2) . For all but finitely many indices k , $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$.

We work with these indices k such that $\theta_k \notin B_{\mathcal{L}}(\theta_0, R_k)$.

We know that ϵ_k/δ_k^2 is bounded above independantly of k , and by assumption, $\delta_k \leq 2\mu_k$.

We claim that there exist $\xi_k \in D_k$ and a positive constant c_0 such that for all k ,

$$(7.12) \quad \frac{d(\xi_k, \theta_0)}{R_k} \geq c_0$$

let us prove the claim.

On one hand, as $(\partial\tilde{X}, d)$ is uniformly perfect, and so is it's weak tangent (S, δ) because the one point compactification $(\hat{S}, \hat{\delta})$ of (S, δ) is quasi-Möbius homeomorphic to $(\partial\tilde{X}, d)$, therefore there exist a constant $C_0 \in [0, 1)$ such that for every $x \in S$ and $0 < R < \text{diam}S$, we have

$$(7.13) \quad \bar{B}_{(S, \delta)}(0, R) - B_{(S, \delta)}(0, C_0 R) \neq \emptyset.$$

On the other hand, $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to $(S, \delta, 0)$. After reindexing the sequence $\{\lambda_k\}$, we have for each $\epsilon > 0$ a map $g_k : B_{\lambda_k \mathcal{L}}(\theta_0, k) \rightarrow S$ such that

(i) $g_k(\theta_0) = 0$,

for any two points θ and θ' in $B_{\lambda_k \mathcal{L}}(\theta_0, k)$,

(ii) $|\delta(g_k(\theta), g_k(\theta')) - \lambda_k d(\theta, \theta')| \leq \epsilon$,

(iii) the ϵ -neighborhood of $g_k(B_{\lambda_k \mathcal{L}}(\theta_0, k))$ contains $B_{(S, \delta)}(0, k - \epsilon)$.

By (iii), we have

$$(7.14) \quad \bar{B}_{(S, \delta)}(0, k - \epsilon) \subset \mathcal{U}_\epsilon^{(S, \delta)} g_k(\bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)).$$

By (6.13) there exist $y_k \in \bar{B}_{(S, \delta)}(0, k - \epsilon) - B_{(S, \delta)}(0, C_0(k - \epsilon))$, and by (6.14) there exist $\xi'_k \in \bar{B}_{\lambda_k \mathcal{L}}(\theta_0, k)$ such that

$$(7.15) \quad \delta(y_k, g_k(\xi'_k)) \leq \epsilon.$$

We now evaluate $d(y_k, g_k(\xi'_k))$. By the above properties (i), (ii), (6.15) and the triangle inequality we have

$$\lambda_k d(\xi'_k, \theta_0) \geq \delta(g_k(\xi'_k), 0) - \epsilon \geq \delta(y_k, 0) - \delta(y_k, g_k(\xi'_k)) - \epsilon$$

$$(7.16) \quad \geq C_0(k - \epsilon) - 2\epsilon.$$

As D_k is $\epsilon_k R_k$ -dense in $B_{(\mathcal{L}, d)}(\theta_0, k/\lambda_k)$, there exist $\xi_k \in D_k$ such that $d(\xi_k, \xi'_k) \leq \epsilon_k R_k = \frac{k\epsilon_k}{\lambda_k}$.

Let us denote $c_0 = C_0/2$. For k large enough we have $\frac{C_0(k-\epsilon)-2\epsilon-k\epsilon_k}{\lambda_k} \geq \frac{c_0 k}{\lambda_k}$, therefore by (6.16) we get

$$(7.17) \quad d(\xi_k, \theta_0) \geq d(\xi'_k, \theta_0) - d(\xi'_k, \xi_k) \geq c_0 R_k,$$

which proves the claim.

We can assume that for k large enough, $\mu_k < c_0/2 < 1/2$.

We choose $a_k = \theta_1^k$ and $b_k = \theta_2^k$, and we get

$$(7.18) \quad \begin{aligned} & \frac{d(\gamma_k \xi_k, \gamma_k \theta_k) d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k \xi_k, \gamma_k b_k) d(\gamma_k a_k, \gamma_k \theta_k)} \leq \eta \left(\frac{d(\xi_k, \theta_k) d(a_k, b_k)}{d(\xi_k, b_k) d(\theta_k, a_k)} \right) \\ & \leq \eta \left(\frac{4\mu_k d(\theta_k, \theta_0^k)}{(d(\theta_k, \theta_0^k) - \mu_k R_k)(c_0 - \mu_k)} \right) \\ & \leq \eta(16\mu_k/c_0). \end{aligned}$$

We get

$$d(\gamma_k \theta_k, \gamma_k \xi_k) \leq (\text{diam} \Lambda_{C'})^2 \eta(16\mu_k/c_0)/\delta.$$

□

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